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Qualitative dynamics of planar chains

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In this paper we derive the equations of motion and analyze some aspects of the dynamics of planar chains. We show that (a) for any integer vector $\boldsymbol{n} = (n_1, \ldots, n_N)$ (where N is the number of joints) there exists a periodic motion such that the angle of jth joint changes by $2\pi n_j$ during one period; (b) the straight configuration is (nonlinearly) stable and that any "folded" rectilinear configuration is unstable; (c) the Coriolis effects play no role in stability of these rectilinear configurations, in contrast to other problems such as the stabilizing effect on the spinning top.

1. Introduction and results

In this paper we derive the equations of motion and study the dynamics of planar chains consisting of coupled rigid bodies, in the absence of external forces. More precisely, we consider $N+2$ equal point masses $m = 1/(N+2)$ of total mass one, coupled by $N+1$ massless links of length l.

Our subsequent discussion carries over almost verbatim to the case of unequal masses and lengths. We can allow torsional coupling between the links, in particular the nearest neighbor coupling. In the latter case we obtain precisely the discretization of the beam, and thus the exact equations for such a beam. It is a crucial point that we derive the equations in intrinsic coordinates of the chain, i.e. without any reference to an inertial frame. It is very useful to have such reduced equations for numerical simulations, since the numerical damping does not affect the angular momentum of the chain as it does when the non-reduced equations are used.

We use some standard methods in the calculus of variations in the large to describe an interesting class of periodic motions of the system. The variational methods have been used in showing the existence of periodic solutions in many settings; of the more recent examples we mention, without an attempt at completeness, refs. [8, 9, 11, 13]. We will also prove nonlinear stability of some simple motions for all time.

The problem of dynamics of coupled rigid bodies has been studied at least since the last century, but in recent years it has received renewed attention due to its relevance in understanding the dynamics of flexible space structures as well as in some applications in robotics; see refs. [2, 4, 5, 10, 14, 17, 18].

The system of compound pendula that was studied extensively [1, 12, 13] is somewhat similar

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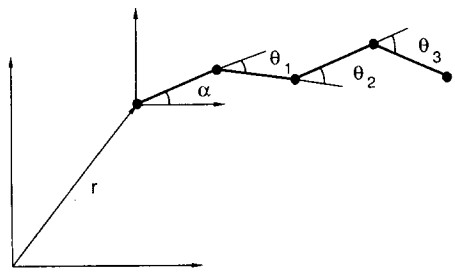

Fig. 1. The Chain.

to ours in that it is just a chain suspended by one end in a gravitational field. The crucial difference between this system and the free chain considered here is the lack of a preferred inertial frame for the free chain. It is therefore desirable to reduce the dimension of the problem by eliminating the reference to an inertial frame; the reduced equations will be in terms of angles θ_j between the neighboring links (fig. 1) and will not contain any variables characterizing the orientation or position with respect to an inertial frame.

We will show that the evolution of the angles θ_j is governed by the Euler–Lagrange equations with a Lagrangian of the form $L = L_2 + L_1 + L_0$, where L_i is homogeneous of degree i in terms of the relative angular velocities $\dot{\theta}_j$, $j = 1, \ldots, N$. The term L_1 gives rise to the gyroscopic (or Coriolis) forces. In the case when $L_1 = 0$, i.e. when $L = L_2 + L_0$, the Maupertuis least action principle states that the trajectories with energy E connecting two points θ^0 and θ^1 in the configuration space are the extremals of

$$\int_0^1 \sqrt{(L_0 + E) L_2}\, \mathrm{d}t,$$

and thus the trajectories are the geodesics for the Riemannian metric given by $\mathrm{d}J^2 = (L_0 + E) L_2\, \mathrm{d}t^2$; it is known as the Jacobi metric. In the presence of gyroscopic forces, i.e. when $L_1 \neq 0$, the Maupertuis principle is modified to

$$\delta \int_0^1 \left[2\sqrt{(L_0 + E) L_2} + L_1 \right] \mathrm{d}t = 0,$$

see refs. [5, 13] and $\mathrm{d}J^2 = [2\sqrt{(L_0 + E) L} + L_1]\mathrm{d}t^2$ is no longer a metric since L_1 changes sign when $\dot{\theta}$ is replaced by $-\dot{\theta}$. This can be restated by saying that the indicatrix

$$\left\{ \dot{\theta} \colon 2\sqrt{(L_0 + E) L_2} + L_1 = \text{const.}, \ \theta \text{ fixed} \right\}$$

is non-symmetric with respect to the origin $\dot{\theta} = 0$ in the tangent space, and the dynamical system is no longer reversible.

In order to obtain periodic solutions one wishes to apply, for instance, a curve shortening process to a noncontractible curve on the configuration torus $\{\theta = (\theta_1 \bmod 2\pi, \ldots, \theta_N \bmod 2\pi)\}$. Precisely this method has been used since Hadamard [6, 20], but only for the case of a Riemannian metric. In this paper we point out that the same ideas and methods go through in the case of a non-reverisble "metric" as well. As a result we obtain the following

Theorem 1.1. For any value of the angular momentum μ there exists an energy value $E_0(\mu)$ such that for any energy value $E > E_0(\mu)$ there are periodic motions of the following kind. Given any N-tuple of integers $\boldsymbol{n} = (n_1, \ldots, n_N)$, there exists a periodic motion $\theta(t)$ of the chain (with energy and momentum E and μ as prescribed above) of period $T = T(E, \mu, \boldsymbol{n})$ such that during time T the ith angle θ_i makes exactly n_i full turns; more precisely, there exists a solution $\theta(t)$ with $\theta(T) - \theta(0) = 2\pi \boldsymbol{n}$, $\dot{\theta}(T) = \dot{\theta}(0)$. In other words, the numbers n_i of relative revolutions between two neighbors can be prescribed arbitrarily and for some initial conditions they will in fact be realized. Viewed from an inertial frame these motions are either periodic or quasiperiodic.

We also prove the following stability result.

Theorem 1.2. For any non-zero value $\mu_0 \neq 0$ of the total angular momentum, the straight line configuration $\theta(t) \equiv 0$ of the chain is stable; any

folded configuration (at least one $\theta_i = \pi$ and $\theta_i = 0$ or π) is unstable. More precisely, given any $\epsilon > 0$ there exists $\delta = \delta(\mu) > 0$ such that for all initial conditions with $|\theta| + |\dot{\theta}| < \delta$ for $t = 0$ we have $|\theta| + |\dot{\theta}| < \epsilon$ for all $t \in \mathbb{R}$. Moreover, ϵ and δ need not be small; they are given by explicit estimates (4.4) and (4.5) below.

The stability of the straight configuration is intuitively rather clear. The proof, however, turns out to be a little less trivial than one might expect, and we provide it for completeness. As a byproduct of the proof, we get quantitative estimates on the stability region (i.e. on ϵ and δ).

2. Derivation of the equations of motion

2.1. The model

We consider $N+2$ masses m linked together by $N+1$ massless links of length l, restricted to a plane. In this section we derive the equations of motion in intrinsic coordinates without reference to an inertial frame. We choose the angles θ_j between the jth and $(j+1)$th segment as such coordinates, see fig. 1.

The equations of motion turn out to be simpler when written in terms of the cumulative angles $\phi_j = \Sigma_{k=1}^{j} \theta_k$; the N-tuple of these angles will be denoted by $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$.

2.2. Equations of motion

In the absence of external forcing the angular momentum μ is conserved. We have

Theorem 2.1. The equation of motion of the system are given by the Lagrangian

$$L(\phi, \dot{\phi}) = \tfrac{1}{2}\big(K(\phi)\,\dot{\phi}, \dot{\phi}\big) + \mu\omega(\phi)\cdot\dot{\phi} - \tfrac{1}{2}\mu^2/I(\phi), \tag{2.1}$$

where $K(\phi)$ is a positive definite matrix, periodic in ϕ, $\omega(\phi)$ is a N-vector and $I(\phi)$ is the moment of inertia relative to the center of mass of the system. All these objects are given via a single matrix Q in (2.2), (2.3) below.

The so-called gyroscopic term ω in the Lagrangian gives rise to the Coriolis force; the notation ω is suggested by the analogous expression $2\omega \times \dot{x}$ for the Coriolis force in the case of a single particle at x in a frame rotating with the angular velocity ω.

2.3. Expressions for $K(\phi)$, $\omega(\phi)$ and $I(\phi)$

The quantities K, ω and I admit an elegant expression in terms of a single $(N+1)\times(N+1)$ matrix Q. For any $\psi \in \mathbb{R}^{N+1}$ define, for $j, k = 0, \ldots, N$,

$$Q_{jk}(\psi) = l^2\left(\frac{N+1-\max(j,k)}{N+2} - \frac{N+2-(N+1-j)(N+1-k)}{(N+2)^2}\right) \times \cos(\psi_j - \psi_k). \tag{2.2}$$

Denoting by Φ the vector $(0, \phi) \in \mathbb{R}^{N+1}$, we will derive the following expressions for $K(\phi)$, $\omega(\phi)$ and $I(\phi)$:

$$\big(K(\phi)\dot{\phi}, \dot{\phi}\big) = \big(Q(\Phi)\dot{\Phi}, \dot{\Phi}\big) - \frac{\big(Q(\Phi)\,\dot{\Phi}, \mathbf{1}\big)^2}{\big(Q(\Phi)\,\mathbf{1}, \mathbf{1}\big)}, \tag{2.3a}$$

$$\omega(\phi)\cdot\dot{\phi} = \frac{\big(Q(\Phi)\,\dot{\Phi}, \mathbf{1}\big)}{\big(Q(\Phi)\,\mathbf{1}, \mathbf{1}\big)} \tag{2.3b}$$

and

$$I(\phi) = \big(Q(\Phi)\,\mathbf{1}, \mathbf{1}\big), \tag{2.3c}$$

where $\mathbf{1} = (1, \ldots, 1)$ ($N+1$ times).

2.4. Derivation of the reduced Lagrangian

We start with the full Lagrangian which is just the kinetic energy expressed in an inertial frame.

Using the complex numbers to denote the position of a point in the plane, we let r be the position of the first mass point relative to a chosen inertial frame. Let α be the angle of the first link with the x-axis, and let

$$z_n = l\,\mathrm{e}^{\mathrm{i}\alpha} \sum_{j=1}^{n} \mathrm{e}^{\mathrm{i}\Phi_{j-1}}, \quad \text{where } \Phi = (0, \phi),$$

be the position of the nth mass point relative to the first mass point. The Lagrangian is then expressed as the sum of the kinetic energies of all masses:

$$\mathscr{L}(\dot{r}, \alpha, \dot{\alpha}, \phi, \dot{\phi}) = \tfrac{1}{2} m \sum_{n=0}^{N+1} |\dot{r} + \dot{z}_n|^2. \tag{2.4}$$

We will obtain the reduced Lagrangian (2.1) by the Routh reduction, eliminating r and α. By Noether's theorem [1], the conservation of linear momentum and the conservation of angular momentum can be written as

$$\mathscr{L}_{\dot{r}} = \lambda, \tag{2.5a}$$

$$r \wedge \mathscr{L}_{\dot{r}} + \mathscr{L}_{\dot{\alpha}} = \mu, \tag{2.5b}$$

where $\wedge$ denotes the scalar cross product: $z \wedge w = \mathrm{Im}(\bar{z} w)$. Without loss of generality, we assume that the linear momentum λ vanishes.

Introducing the positions

$$p_n = \mathrm{e}^{-\mathrm{i}\alpha} z_n \tag{2.6}$$

and their average

$$\langle p \rangle = \frac{1}{N+2} \sum_{n=1}^{N+1} p_n, \tag{2.7}$$

we obtain from the conservation laws (2.5) with $\lambda = 0$

$$\dot{r} = -\mathrm{i}\alpha\,\mathrm{e}^{\mathrm{i}\alpha} \langle p \rangle - \mathrm{e}^{\mathrm{i}\alpha} \langle \dot{p} \rangle, \tag{2.8a}$$

$$\dot{\alpha} = \frac{\mu - (\langle p \wedge \dot{p} \rangle - \langle p \rangle \wedge \langle \dot{p} \rangle)}{\langle |p|^2 \rangle - |\langle p \rangle|^2}. \tag{2.8b}$$

The denominator of (2.8b) is precisely the moment of inertia $I(\phi)$. It never vanishes since it is equal to

$$\frac{1}{N+2} \sum_{n=1}^{N+1} |p_n - \langle p \rangle|^2.$$

We can now use the Routh reduction [19, 21] to obtain the reduced Lagrangian L. Denote by $\tilde{\mathscr{L}}(\phi, \dot{\phi}, \mu)$ the Lagrangian $\mathscr{L}$ given by (2.4) in which $\dot{r}$ and $\dot{\alpha}$ have been replaced according to (2.8); it is important that $\mathscr{L}$ is a α-independent. Using the fact that along a solution $\mathscr{L}_{\dot{r}} = 0$ and $\mathscr{L}_{\dot{\alpha}} = \mu$, one has the identities in t

$$\tilde{\mathscr{L}}_{\phi} = \mathscr{L}_{\phi} + \mu \dot{\alpha}_{\phi}, \quad \tilde{\mathscr{L}}_{\dot{\phi}} = \mathscr{L}_{\dot{\phi}} + \mu \dot{\alpha}_{\dot{\phi}}.$$

Hence solutions with $\lambda = 0$ of the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{L}_{\dot{\phi}} - \mathscr{L}_{\phi} = 0$$

satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\tilde{\mathscr{L}}_{\dot{\phi}} - \mu \dot{\alpha}_{\dot{\phi}} \right) - \left(\tilde{\mathscr{L}}_{\phi} - \mu \alpha_{\phi} \right) = 0,$$

i.e, they satisfy the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{\phi}} - L_{\phi} = 0$$

for the new Lagrangian

$$L(\phi, \dot{\phi}, \mu) = \tilde{\mathscr{L}}(\phi, \dot{\phi}, \mu) - \mu \dot{\alpha}(\phi, \dot{\phi}, \mu). \tag{2.9}$$

With a little algebra one obtains

$$L(\phi, \dot{\phi}, \mu) = \tfrac{1}{2}\left(\langle |\dot{p}|^2 \rangle - |\langle \dot{p} \rangle|^2 \right) - \frac{1}{2} \frac{\left(\mu - (\langle p \wedge \dot{p} \rangle - \langle p \rangle \wedge \langle \dot{p} \rangle) \right)^2}{\langle |p|^2 \rangle - |\langle p \rangle|^2}. \tag{2.10}$$

Using the identities

$$\sum_{n=1}^{N}\sum_{k=1}^{n} a_k = \sum_{k=1}^{N}(N+1-k)a_k, \tag{2.11a}$$

$$\sum_{n=1}^{N}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} = \sum_{j=1}^{N}\sum_{k=1}^{N}[N+1-\max(j,k)]a_{jk},$$

$$a_{jk} = a_{kj}, \tag{2.11b}$$

one immediately obtains that for $\Phi = (0, \phi)$, in the above notation (2.2) for Q (expressed in terms of Φ)

$$\langle |\dot{p}|^2\rangle - |\langle \dot{p}\rangle|^2 = \left(Q(\Phi)\,\dot{\Phi}, \dot{\Phi}\right), \tag{2.12a}$$

$$\langle |p|\rangle - |\langle p\rangle|^2 = I(\Phi) = (Q(\Phi)\,\mathbf{1}, \mathbf{1}), \tag{2.12b}$$

$$\langle p \wedge \dot{p}\rangle - \langle p\rangle \wedge \langle \dot{p}\rangle = \left(Q(\Phi)\,\dot{\Phi}, \mathbf{1}\right). \tag{2.12c}$$

It remains to show that

$$\left(Q(\Phi)\dot{\Phi}, \dot{\Phi}\right) - \frac{\left(Q(\Phi)\,\dot{\Phi}, \mathbf{1}\right)^2}{(Q(\Phi)\,\mathbf{1}, \mathbf{1})} = \mathscr{K}(\Phi, \dot{\Phi})$$

is a positive definite quadratic form in $\dot{\phi}$. Consider the following polynomial in λ:

$$\lambda^2\left(Q(\Phi)\,\dot{\Phi}, \dot{\Phi}\right) - \lambda\left(Q(\Phi)\,\dot{\Phi}, \mathbf{1}\right) + (Q(\Phi)\,\mathbf{1}, \mathbf{1}) = \left(Q(\Phi)\,(\lambda\dot{\Phi} + \mathbf{1}), (\lambda\dot{\Phi} + \mathbf{1})\right).$$

Its discriminant is equal to $-\frac{1}{4}(Q(\Phi)\mathbf{1}, \mathbf{1}) \times \mathscr{K}(\Phi, \dot{\Phi})$. Since Q is positive definite the polynomial can only have roots for $\dot{\Phi} = -(1/\lambda)\mathbf{1}$. Since the first component of $\dot{\Phi}$ is always equal to zero, the last equality cannot hold so that the polynomial has no real roots and hence the discriminant is negative, thus implying the positivity of $\mathscr{K}(\Phi, \dot{\Phi})$. QED

3. Proof of the existence of periodic solutions

For an arbitrarily chosen integer vector $\boldsymbol{n} = (n_1, \ldots, n_N)$ we choose a smooth oriented loop $\theta(s)$, $0 \le s \le 1$ on the torus $\mathrm{T}^N = \{\theta \bmod 2\pi\}$ with $\theta(1) - \theta(0) = 2\pi\boldsymbol{n}$, and subject this loop to the curve shortening process. This construction is usually done for a Riemannian metric as in ref. [5] or ref. [20] but the arguments carry over verbatim to the case of the Maupertuis pseudo-metric

$$\mathrm{d}J^2 = \left[2\sqrt{(L_0 + E)L_2} + L_1\right]^2 \mathrm{d}t^2, \tag{3.1}$$

where $L = L_0 + L_1 + L_2$ is the Lagrangian given by (2.1) and where the L_j are homogenous of degree j in the velocities. The only difference with the metric case is the non-reversibility. The shortest path (and thus the distance) from A to B differs from that from B to A. In the rest of the discussion, "distance" refers to the directed distance in the direction of the orientation of the curves.

There exists $E_0 = E_0(\mu)$ such that for all $E > E_0(\mu)$ the Maupertuis pseudo-metric $\mathrm{d}J^2$ is positive (i.e. the gyroscopic term L_1 is dominated by the positive term $\sqrt{(L_0 + E)L_2}$). For any such $E > E_0(\mu)$ which we fix from now on, there exists an injectivity radius $\rho > 0$, i.e. the number for which any two points A and B at a distance (measured in J) less than ρ can be connected by a unique minimal from A to B, that is, an extremum of

$$J(\gamma) = \int_\gamma \mathrm{d}J, \qquad \gamma(0) = \mathrm{A}, \quad \gamma(1) = \mathrm{B}. \tag{3.2}$$

We can now carry out the curve shortening procedure. We equipartition the curve θ into q equal segments. We denote all the partition points (on the covering space of T^N) by θ_j, where $\theta_{j+q} = \theta_j + 2\pi\boldsymbol{n}$ and $\mathrm{dist}(\theta_j, \theta_{j+1}) = q^{-1}\,\mathrm{length}(\theta)$. All distances will be counted in the direction of increasing s. We choose q so that the length of θ is less or equal to $q\rho$. One then replaces the curve segments between θ_j and θ_{j+1} by minimals of the variational principle. Note that this will reduce

the length of the curve exept when each segment between θ_j and θ_{j+1} is minimal. We then take the midpoints M_j of the new segments and connect M_j and M_{j+1} by a minimal. This second step also reduces the total length except if the first step yielded an extremal of the variational principle. As in ref. [20], one can show that the curve obtained from θ by this procedure is a continuous deformation of θ, hence of the same homotopy type.

Repeating this procedure yields a sequence of curves $\theta^{(k)}$ of decreasing length and of the same homotopy type. Note that by construction, each of these curves is determined by q points $\theta_j^{(k)}$ since neighbouring points $\theta_j^{(k)}$ and $\theta_{j+1}^{(k)}$ are connected by a unique minimal. Since the set of q-tuples of points of T^n is compact, we can extract an convergent subsequence from the $\theta_j^{(k)}$ yielding a curve $\tilde{\theta}$.

It remains to be shown that $\tilde{\theta}$ is an extremal of the Maupertuis variational principle. This is done as in ref. [20]. Assume it is not. Then in a sufficiently small neighbourhood of $\tilde{\theta}$, all curves are shortened by a finite positive amount. But $\tilde{\theta}$ is the limit of the $\theta^{(k)}$ and thus the amount by which the $\theta^{(k)}$ with large k gets shortened tends to zero. This yields the desired contradiction. The smoothness of $\tilde{\theta}$ can for example be obtained from the fact that it is an extremal of the Maupertuis variational principle and thus the solution of a smooth system of ODE's. This concludes the proof.

4. Proof of nonslinear stability

4.1. Proof of theorem 1.2

We will prove the non-linear stability of the straight line configuration by showing that the energy

$$E(\phi,\dot{\phi}) = \tfrac{1}{2}\big(K(\phi)\dot{\phi},\dot{\phi}\big) + \mu^2 W(\phi),$$

where $W(\phi) = (Q(\phi)\mathbf{1},\mathbf{1})^{-1}$, has a global nondegenerate minimum at $\phi = \dot{\phi} = 0$, corresponding to the straight chain. Theorem 1.2 is an immediate consequence of the following two lemmas.

Lemma 4.1. At $\phi = \dot{\phi} = 0$, $E(\phi,\dot{\phi})$ has a global minimum.

Lemma 4.2. The minimum $\phi = \dot{\phi} = 0$ of the energy $E(\phi,\dot{\phi})$ is non-degenerate.

These lemmas will be proved at the end of this section. It is well known since Lagrange [1, 5] that a non-degenerate minimum of energy leads to stability. Nevertheless, we give full details in order to provide *explicit bounds* for a solution starting near the straight line configuration. We set $x = (\phi,\dot{\phi}) \in \mathbb{R}^{2N}$ and subtract a constant from the energy E

$$V(x) = E(x) - \mu^2 W(0)$$

in order to make $V(0) = 0$. Letting $n = x/|x|$, $t = |x|$ for $x \neq 0$, we have by Taylor's formula

$$V(x) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}V(\tau n)\Big|_{\tau=0} t^2 + \frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}\tau^3}V(\tau n)\, t^3, \tag{4.1}$$

for some $0 \le \tau \le t$. We can rewrite this as

$$V(x) = (Tx, x) + R(x), \tag{4.2}$$

with the matrix

$$T = \begin{pmatrix} \mu^2 \mathrm{D}^2 W(0) & 0 \\ 0 & K(0) \end{pmatrix}$$

and with the remainder

$$R(x) = \frac{\mathrm{d}^3}{\mathrm{d}\tau^3}V(\tau n)t^3 = \sum_{i,j,k} V_{x_i x_j x_k}(\tau n)\, n_i n_j n_k t^3,$$

which satisfies the estimate

$$|R(x)| \le C(N,\mu)|x|^2 \quad \text{for } |x| \le 1, \tag{4.3}$$

where

$$C(N) = N^3 \max_{i,j,k} \sup_{|x| \le 1} \left| V_{x_i x_j x_k}(x) \right|.$$

Denoting by λ the minimal eigenvalue of T and by Λ its maximal eigenvalue, both positive, since $D^2 W$ and K are both positive by lemma 4.2, we obtain

$$\tfrac{1}{2}\lambda |x|^2 \le V(x) \le 2\Lambda |x|^2$$
$$\text{for } |x| \le \min\left(1, \frac{\lambda}{2C(N)}\right).$$

With this estimate it is now possible to prove stability of the straight configuration. Using the above estimate and the fact that $V(x(t))$ is constant along solutions, we see that

$$|x(t)|^2 \le \frac{2}{\lambda} V(x) = \frac{2}{\lambda} V(x(0)) \le 4\frac{\Lambda}{\lambda} |x(0)|^2$$

holds *provided* that $|x(t)| \le \min(1, \lambda/2C(N))$ for all t. The latter is certainly achieved if

$$|x(0)| \le \frac{1}{2}\sqrt{\frac{\lambda}{\Lambda}} \min\left(1, \frac{\lambda}{2C(N)}\right) \equiv \delta_0. \tag{4.4}$$

This gives us a quantitative statement (which implies nonlinear stability). For any $|x(0)| < \delta_0$ given by (4.4) we have

$$|x(t)| < 2\sqrt{\frac{\Lambda}{\lambda}}\, |x(0)| \quad \text{for all } t \in \mathbb{R}. \tag{4.5}$$

4.2. *Proof of lemma 4.1*

It suffices to prove that the *moment of inertia $I(\phi)$ with respect to the center of mass has its unique global maximum at $\phi = 0$* (fig. 2)

Let us mark the point c_0 on a link (or possibly a joint) where the center of mass lies when the chain is straightened out ($\theta = 0$). Having thus defined c_0 we look now at an arbitrary shape with $\phi \ne 0$ and (it suffices to) show that $I(\phi) < I(0)$. To that end we let $I_0(\phi)$ denote the moment of inertia relative to the point c_0 (thought of as a point marked on the chain), and show that

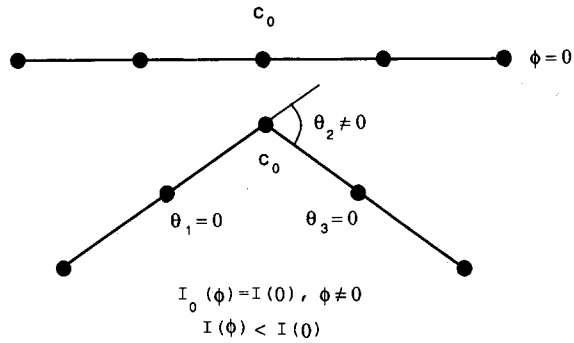

Fig. 2. Proving the straight chain ($\phi = 0$) maximizes the moment of inertia $I(\phi)$.

$$I(\phi) \overset{(A)}{\le} I_0(\phi) \overset{(B)}{\le} I(0),$$

and moreover that at least one inequality (A) or (B) is strict.

The inequality (A) holds because the moment of inertia $I(\phi)$ relative to the center of mass is less than the moment of inertia $I_0(\phi)$ relative to any other point (c_0 in this case), and strictly less if c_0 is not at the center of mass c. Indeed, denoting the positions of masses in some frame by x_i, we have

$$c = \bar{x} = \frac{1}{N+2} \sum_{k=1}^{N+2} x_i,$$

and

$$\begin{aligned} I_0(\phi) &= m \sum (x_i - c_0)^2 \\ &= m \sum \left[(x_i - \bar{x}) + (\bar{x} - c_0)\right]^2 \\ &= m \sum (x_i - \bar{x})^2 + m \sum (\bar{x} - c_0)^2 \\ &\quad + 2m \sum (x_i - \bar{x})(\bar{x} - c_0) \\ &= I(\phi) + (\bar{x} - c_0)^2, \end{aligned}$$

thus proving (A).

Now (B) holds by the triangle inequality, according to which the distance of each mass from

c_0 is maximized by $\phi = 0$. If (B) is strict, we are done. We therefore assume that (B) is not strict. $I_0(\phi) = I(0)$, $\phi \neq 0$, and prove then that (A) is strict. We proceed as follows. Step 1: $I_0(\phi) = I(0)$ implies that the chain has not more than one bent link (and since $\phi \neq 0$, precisely one), i.e. that $\theta_k \equiv \phi_k - \phi_{k-1} \neq 0$ for one and only one k. Step 2: Precisely one bent link implies that (A) is strict. To carry out step 1 we note that $I(0)$ is the sum of squares of the distances form c_0 to each of the masses, while $I_0(\phi)$ is the sum of squares of distances from the same point c_0 to the masses when the chain is not straight. From the triangle inequality it is clear that the only way the two sums can be equal is when the distance from c_0 to each of the masses is maximized, i.e. each part of the chain lying between c_0 and any of the masses must be straight. Since $\phi \neq 0$, this shows that the only possibility left is that c_0 lies on a joint and that this and only this joint is not straight, as shown in fig. 2. This completes step 1. Step 2 is now very simple: we know now that the center of mass on the chain lies away from c_0, which by the above argument means that $I(\phi) < I_0(\phi)$. QED

4.3. Proof of lemma 4.2 – nondegeneracy of the minimum

Since we already know that $(K(\phi)\dot{\phi}, \dot{\phi})$ is a positive definite quadratic form (see the end of section 2.4), we only need to show that at $\phi = 0$, $\mathrm{D}^2 W(0)$ is positive definite. To that end, using Taylor expansion of $W(\phi) \equiv 1/I(\phi)$ around $\phi = 0$, it suffices to show that $\mathrm{D}^2 I(0)$ is negative definite. We define $\hat{\phi} = (\phi_0, \phi) \in \mathbb{R}^{N+1}$ and

$$\hat{I}(\hat{\phi}) = \left(Q(\hat{\phi})\, 1, 1\right) = \sum_{k,l=0}^{N} c_{kl} \cos(\phi_k - \phi_l). \tag{4.6}$$

One has

$$\hat{I}(0, \phi) = I(\phi). \tag{4.7}$$

Computing the Hessian matrix $(\mathrm{D}^2 \hat{I}(\hat{\phi}))_{n,m} = \partial^2 \hat{I}/\partial\phi_n\, \partial\phi_m$, $n, m = 0, \ldots, N$, yields

$$\left(\mathrm{D}^2 \hat{I}(\hat{\phi})\, \hat{\xi}, \hat{\xi}\right) = 2 \sum_{n,m=0}^{N} c_{nm} \cos(\phi_n - \phi_m)\left(\xi_n \xi_m - \xi_n^2\right),$$

for any $\hat{\xi} \in \mathbb{R}^{N+1}$. Substituting $\hat{\phi} = 0$ and using $c_{nm} > 0$ and $\xi_n \xi_m \leq \frac{1}{2}(\xi_n^2 + \xi_m^2)$ we obtain

$$\left(\mathrm{D}^2 \hat{I}(0)\, \hat{\xi}, \hat{\xi}\right) \leq 0.$$

Equality holds only when $\xi_n = \xi_m$, $n, m = 0, \ldots, N$. Variations of ϕ correspond to $\hat{\xi} = (0, \xi)$, $\xi \in \mathbb{R}^N$, resulting in the identity

$$\left(\mathrm{D}^2 \hat{I}(0)\, \hat{\xi}, \hat{\xi}\right) = \left(\mathrm{D}^2 I(0)\, \xi, \xi\right)$$

and hence $(\mathrm{D}^2 I(0)\, \xi, \xi) \leq 0$. As we have just observed, the equality holds only for $\xi_n = \xi_m$ which together with $\xi_0 = 0$ implies that $\xi_n = 0$, for all n, so that $\phi = 0$ is a nondegenerate minimum of $W(\phi)$.

4.4. Proof of the instability of folded configurations

We now prove that folded configurations (at least one $\theta_j = \pi$) are unstable. As before, it is sufficient to show that for such a configuration, the moment of inertia I can be increased.

For this, we let S_1 be the part of the chain form one end till the first folded joint. Now either the center of mass of the rest of the chain lies to the right or to the left of the joint. (See fig. 3.) That is, either in part A or in part B.

When the center of mass lies in part A, one is in a situation described by fig. 4 where m_1 represents the mass of S_1. Opening the first folded joint increases the distance between the center of masses and thus increases the moment of inertia.

When the center of mass lies in B, and possibly at the boundary between A and B, let S_2 denote the part of the chain between the first folded joint and the next one. Now by assumption, the

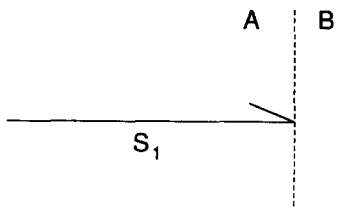

Fig. 3.

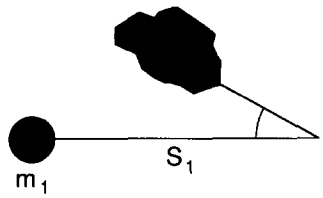

Fig. 4.

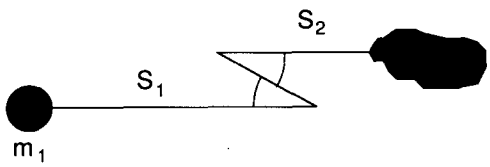

Fig. 5.

center of mass of the rest of the chain will lie in B so that one is in the situation described in fig. 5. It is now easy to see that opening the first two folded joints in opposite directions will increase the moment of inertia. This shows that folded configurations are unstable.

5. A remark on linearization and the Coriolis forces

It is interesting to point out that the linearized equations for a straight line configuration contain no Coriolis forces. Indeed, linearizing the Euler–Lagrange equations around any solution $\phi(t)$, we obtain

$$A(t)\,\ddot{\xi}+B(t)\,\dot{\xi}+C(t)\,\xi=0, \tag{5.1}$$

where $A(t)=K(\phi(t))$, $B(t)=\omega_\phi(\phi(t))-\omega_\phi^{\mathrm{T}}(\phi(t))$, $C(t)=\frac{1}{2}\mu^2W''(\phi(t))$. A and C are symmetric matrices, while $B(t)$ is skew-symmetric; it corresponds to the Coriolis term. Taking now a special solution $\phi=(\phi_1,\ldots,\phi_N)$ with $\phi_j=0$ or π, we obtain $B=0$, according to (2.3b), while $A>0$. If $\phi=0$, then $C\equiv\frac{1}{2}\mu^2W''(\phi)>0$, as we have shown above, and the linearized system (5.1) is stable, as we know already from theorem 1.2. If, however, we consider a folded straight chain, i.e. if at least one of the components $\theta_j=\pi$, then C is not positive definite, as is easy to show using the same method as proving lemma 4.1. It follows than that the linearization

$$A\ddot{\xi}+C\xi=0 \tag{5.2}$$

around such a solution is unstable (and consequently so is the solution itself). To show that (5.2) is unstable, we let $\sqrt{A}>0$ be the positive definite square root of A, and let $\eta=\sqrt{A}\,\xi$; eq. (5.1) becomes

$$\ddot{\eta}+D\eta=0,$$

where $D=(\sqrt{A})^{-1}C(\sqrt{A})^{-1}$. Since $(\sqrt{A})^{-1}$ is a symmetric matrix, D is similar to C and in particular D has a negative eigenvalue, the same as C. This proves that (5.2) is unstable.

Conjecture. The number of negative eigenvalues of C, i.e. the number of positive eigenvalues of the Hessian $I''(\boldsymbol{\theta}_0)$ of the moment of inertia equals the number of folds in the chain, i.e. the number of j for which $\theta_j=\pi\ (\mathrm{mod}\,2\pi)$.

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