ADVANCES IN MATHEMATICS **102**, 202–229 (1993)

Standard Basis Theorem for Quantum Linear Groups

ROSA Q. HUANG*

Mathematics Department, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

AND

JAMES J. ZHANG†

Mathematics Department, University of Michigan, Ann Arbor, Michigan 48109

A linear basis indexed by standard bitableaux is given for quantum linear semigroups. As a consequence, one can easily derive a standard basis for quantum linear groups. From a combinatorial point of view, quantum linear semigroups are identical with what results from quantization of the letterplace algebra Rota and co-workers, *Adv. Math.* **27** (1978), 63–92. The standard basis theorem proved here can be viewed as a quantum straightening formula. The present paper is written in the language of supersymmetric algebra. In doing so, we actually have obtained a standard basis for Manin's quantum linear supersemigroups.

Contents

INTRODUCTION

The theory of the letterplace algebra and its supersymmetric extensions has been systematically developed in [DKR] and [GRS] by Rota and his co-workers. Algebraically, the supersymmetric letterplace algebra extends the notion of the coordinate ring of matrices. The idea of letters and places gives the letterplace algebra a wide range of applicability, as well as a

* Research supported in part by NSA grant 93Y-118.
† Research supported in part by an NSF grant.

0001-8708/93 $9.00

rich combinatorial structure. For this and other reasons, the letterplace algebra has found many applications, and has proved to be an effective algebraic–combinatorial tool in disparate areas, such as classical invariant theory [GRS, KR, Hu], representation theory [BPT, BT], resolutions of algebras and modules [AR, BR], projective geometry [RS, Wh], rigidity theory [WW], etc.

In the present paper we use Manin's approach to quantum groups and develop a quantum analogue of the supersymmetric letterplace algebra.

Such quantized letterplace algebra coincides with Manin's quantum linear supersemigroup E_q [Ma2] when the set of letters is the same as the set of places. Generalizing Manin's definition of quantum general linear supergroups, we define the notion of supersymmetric quantum letterplace algebra $\mathrm{Super}[L \mid P]_q$. Our definition consists in requiring that the left co-representation T_l from the quantum letter algebra $\mathrm{Super}[L]_q$ to $\mathrm{Super}[L \mid P]_q \otimes \mathrm{Super}[P]_q$ as well as the right co-representation T_r from the quantum place algebra $\mathrm{Super}[P]_q$ to $\mathrm{Super}[L]_q \otimes \mathrm{Super}[L \mid P]_q$ should be algebra homomorphisms.

The quantum letterplace algebra thus defined is significantly different than the ordinary letterplace algebra. For example, "left" and "right" quantum minors turn out to be different, although only by a scalar multiple; only left-sided (resp. right-sided) Laplace expansion holds for left (resp. right) quantum minors (Proposition 4). We have found so far no clear relation between products of (left or right) quantum minors in different orders. Worst of all, the exchange identity [GRS, Proposition 10], which is one of the crucial identities in the theory of the ordinary letterplace algebra, no longer holds in the quantum case, and has to be replaced by a weaker identity (Lemma 10), which can be viewed as a quantum generalization of Garnir relations (called Young symmetry relations by Taft and Towber [TT]. Despite these difficulties, we show that the quantum letterplace algebra still possesses an amazing combinatorial structure: standard quantum left (resp. right) bitableaux form a linear basis of the quantum letterplace algebra (Theorem 9), which reduces to the well-known straightening formula in the classical case. However, an explicit algorithm of expressing a quantum (left or right) bitableau as a linear combination of standard ones becomes more mysterious in the quantum setup, because the lack of commutativity of the algebra.

The paper is organized as follows. In Sections 1 and 2, we define supersymmetric quantum algebra $\mathrm{Super}[L]_q$ and supersymmetric quantum letterplace algebra $\mathrm{Super}[L \mid P]_q$. Some basic properties and explicit relations holding in the algebra $\mathrm{Super}[L \mid P]_q$ are given. In Section 3, we define left and right quantum minors and prove the corresponding Laplace expansions. Relationships between left and right quantum minors are studied. In Section 4, we state and prove the quantum straightening formula for Super$[L|P]_q$ by first deriving a weaker form of the exchange identity.

We would like to thank Naihuan Jing for several enlightening conversations on the subject.

1. Supersymmetric Quantum Algebras

Throughout this paper K is a fixed field with any characteristic, and algebras are super algebras, namely, $\mathbf{Z}_2$-graded associative K-algebras with identity 1. To recall the concepts, an algebra A is called *$\mathbf{Z}_2$-graded* if $A = A_0 \oplus A_1$, where both A_0 and A_1 are K-subvector spaces of A such that $A_0 A_0 + A_1 A_1 \subseteq A_0$ and $A_0 A_1 + A_1 A_0 \subseteq A_1$. Any non-zero element a in A_0 has $\mathbf{Z}_2$-degree 0 modulo 2; we write $|a| = 0$. Any non-zero element a in A_1 has $\mathbf{Z}_2$-degree 1 modulo 2; we write $|a| = 1$. For convenience, we omit "modulo 2" if no confusion occurs. Any element in A_0 or A_1 is called *$\mathbf{Z}_2$-homogeneous*. In the language of super algebras, we say non-zero elements in A_0 are *positive* and non-zero elements in A_1 are *negative*. The identity element 1 has $\mathbf{Z}_2$-degree 0 and hence it is positive. All algebra homomorphisms are $\mathbf{Z}_2$-graded, namely, the homomorphisms preserve the $\mathbf{Z}_2$-degrees (or the signs) of elements. Given two $\mathbf{Z}_2$-graded algebras R and S, the usual tensor product of the algebras R and S is also a $\mathbf{Z}_2$-graded algebra. However, the multiplication of the usual tensor product of algebras is not used in this paper. Instead, we use the multiplication with sign as follows. Let $R \otimes S$ be the tensor product of R and S as K-vector spaces. The multiplication in the *super tensor product* $R \otimes S$ is defined by

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{|b|\,|c|}\, ac \otimes bd$$

for all $\mathbf{Z}_2$-homogeneous elements $a, c \in R$ and $b, d \in S$. (The multiplication of the usual tensor product of two algebras has the above form without the sign $(-1)^{|b|\,|c|}$.) The super tensor product $R \otimes S$ is an associative $\mathbf{Z}_2$-graded algebra. For all homogeneous elements a and b, the $\mathbf{Z}_2$-degree $|a \otimes b|$ of $a \otimes b$ is $|a| + |b|$ modulo 2. From now on, $\otimes$ means the super tensor product.

Let $L = L^+ \cup L^-$ be a $\mathbf{Z}_2$-graded set with a linear order $<$. We have $|a| = 0$ for all $a \in L^+$ and $|a| = 1$ for all $a \in L^-$. Elements in L^+ are called *positive variables* and elements in L^- *negative variables*, and sometimes we indicate a positive variable by a^+ and a negative variable by a^-. The *tensor algebra* Tens$[L]$, generated by the elements of L, is a $\mathbf{Z}_2$-graded algebra with $|a_1 \cdots a_n| = |a_1| + \cdots + |a_n|$ modulo 2; any $\mathbf{Z}_2$-graded algebra is a homomorphic image of a tensor algebra Tens$[L]$ with a suitable choice of L^+ and L^-. Let q be a non-zero element in K. The *supersymmetric quantum algebra* generated by L (with a linear order $<$), denoted by Super$[L]_q$, is defined to be the qûotient associative algebra of Tens$[L]$ subject to the following relations:

(i) $ba = (-1)^{|a||b|} q^{-1} ab$, whenever $a < b$,

(ii) $a^- a^- = 0$.

Remark. The supersymmetric quantum algebra Super$[L]_{q^{-1}}$ coincides with "quantum superspace" A_q (with all $q_{ij} = q$) defined in [Ma2, p. 136].

Let Mon(L) be the set of words, namely, monomials on L. Given a word $u \in \text{Mon}(L)$, let length(u) be the number of variables that occur in u, and let $|u|$ be the $\mathbf{Z}_2$-degree of u, which is the number of negative variables that occur in u modulo 2. For two words $u, v \in \text{Mon}(L)$, one can easily check that

$$uv = (-1)^{|u||v|} q^{-i(uv) + i(vu)} vu,$$

where $i(u)$ denotes the number of inversions in the word u, namely, if $u = a_1 a_2 \cdots a_k$, then $i(u)$ is the number of pairs (a_i, a_j) such that $a_i > a_j$ and $i < j$.

The coproduct Δ, which is a linear (but not algebraic) operator from Super$[L]_q$ to the vector space Super$[L]_q \otimes$ Super$[L]_q$, is defined such that for any monomial $u \in \text{Mon}(L)$,

(i) $\Delta u = 0$ if some negative variable occurs more than once in u; otherwise

(ii) $\Delta u = \sum_{(v,w)} k_{vw} v \otimes w$, where the sum ranges over the ordered partitions (v, w) of u as multisets and each coefficient k_{vw} satisfies

$$u = k_{vw} vw \tag{1}$$

in Super$[L]_q$, where $k_{vw} \in K$. For example, if $a^- < b^-$ and $c^+ < d^+$, then

$$\Delta a^- b^- = 1 \otimes a^- b^- + a^- \otimes b^- - q b^- \otimes a^- + a^- b^- \otimes 1,$$

$$\begin{aligned} \Delta c^+ c^+ d^+ = {} & 1 \otimes c^+ c^+ d^+ + c^+ \otimes c^+ d^+ + q d^+ \otimes c^+ d^+ \\ & + c^+ c^+ \otimes d^+ + q c^+ d^+ \otimes c^+ + c^+ c^+ d^+ \otimes 1. \end{aligned}$$

For convenience, we use the Sweedler notation $\Delta u = \sum_u u_{(1)} \otimes u_{(2)}$ to denote the sum $\sum_{(v,w)} k_{vw} v \otimes w$. It can be verified directly that Δ is well defined and satisfies the coassociativity law

$$(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta$$

as linear maps from $\mathrm{Super}[L]_q$ to $\mathrm{Super}[L]_q^{\otimes 3}$. This map is denoted as $\Delta^{(2)}$. In general, we define the linear map $\Delta^{(k)}$ by setting

$$\Delta^{(k)} = (\mathrm{id} \otimes \Delta^{(k-1)}) \circ \Delta,$$

which is a linear operator from $\mathrm{Super}[L]_q$ to $\mathrm{Super}[L]_q^{\otimes k+1}$ such that

$$\begin{aligned} \Delta u &= \sum_{(u_1, u_2, \ldots, u_{k+1})} k_{u_1 u_2 \cdots u_{k+1}} u_1 \otimes u_2 \otimes \cdots \otimes u_{k+1} \\ &= \sum_u u_{(1)} \otimes u_{(2)} \otimes \cdots \otimes u_{(k+1)} \qquad \text{in Sweedler notation,} \end{aligned}$$

where the first sum ranges over the ordered partitions $(u_1, u_2, \ldots, u_{k+1})$ of u as multisets, and each coefficient $k_{u_1 u_2 \cdots u_{k+1}}$ satisfies $u = k_{u_1 u_2 \cdots u_{k+1}} u_1 u_2 \cdots u_{k+1}$ in $\mathrm{Super}[L]_q$.

Remark. The coproduct defined here cannot be made into an algebra homomorphism, since

$$\Delta(a^+ a^+) = 1 \otimes a^+ a^+ + a^+ \otimes a^+ + a^+ a^+ \otimes 1$$

instead of having $2a^+ \otimes a^+$ in the middle term.

2. Quantum Letter-Place Algebras

Let $L = L^+ \cup L^-$ and $P = P^+ \cup P^-$ be two linearly ordered $\mathbf{Z}_2$-graded finite sets. Elements in L are called *letters* and elements in P *places*. In this case we call $\mathrm{Super}[L]_q$ (resp. $\mathrm{Super}[P]_q$) *supersymmetric quantum letter* (resp. *place*) *algebra*. Let $\mathrm{Tens}[L \mid P]$ denote the tensor algebra generated by all $\mathbf{Z}_2$-graded elements $(a \mid \alpha)$, where $a \in L$, $\alpha \in P$, and $|(a \mid \alpha)| = |a| + |\alpha|$ modulo 2. Given a monomial $m = (a \mid \alpha)(b \mid \beta) \cdots (c \mid \gamma)$, the $\mathbf{Z}_2$-degree of m is given by

$$|m| \equiv |a| + |\alpha| + |b| + |\beta| + \cdots + |c| + |\gamma| \pmod 2.$$

To keep notations simple, we usually write $|(a \mid \alpha)|$ as $|a\alpha|$.

Let $\mathrm{Tens}[L \mid P] \otimes \mathrm{Super}[P]_q$ and $\mathrm{Super}[L]_q \otimes \mathrm{Tens}[L \mid P]$ denote the super tensor products. Then the multiplications of these two algebras satisfy

$$(m \otimes \mu) \cdot (m' \otimes \mu') = (-1)^{|\mu|\,|m'|}\, mm' \otimes \mu\mu',$$

$$(u \otimes m) \cdot (u' \otimes m') = (-1)^{|m|\,|u'|}\, uu' \otimes mm',$$

where m, m', μ, μ', and u, u' are $\mathbf{Z}_2$-homogeneous elements in $\mathrm{Tens}[L \mid P]$, $\mathrm{Super}[P]_q$, and $\mathrm{Super}[L]_q$, respectively. Next, we define the maps

$$T_l : L \to \mathrm{Tens}[L \mid P] \otimes \mathrm{Super}[P]_q,$$

$$T_r : P \to \mathrm{Super}[L]_q \otimes \mathrm{Tens}[L \mid P]$$

by

$$T_l(a) = \sum_{\alpha \in P} (a \mid \alpha) \otimes \alpha,$$

$$T_r(\alpha) = \sum_{a \in L} a \otimes (a \mid \alpha).$$

The *supersymmetric quantum letter–place algebra*, denoted by $\mathrm{Super}[L \mid P]_q$, is defined to be the quotient $\mathbf{Z}_2$-graded associative K-algebra of $\mathrm{Tens}[L \mid P]$ with minimal relations such that the map T_l can be extended to an algebra homomorphism

$$T_l : \mathrm{Super}[L]_q \to \mathrm{Super}[L \mid P]_q \otimes \mathrm{Super}[P]_q$$

and the map T_r can be extended to an algebra homomorphism

$$T_r : \mathrm{Super}[P]_q \to \mathrm{Super}[L]_q \otimes \mathrm{Super}[L \mid P]_q.$$

The algebra homomorphisms T_l and T_r are called a *left co-representation* and a *right co-representation* of $\mathrm{Super}[L \mid P]_q$ respectively. For convenience, we keep the same notation for an element in $\mathrm{Tens}[L \mid P]$ and its image in $\mathrm{Super}[L \mid P]_q$ under the canonical map.

PROPOSITION 1. *The supersymmetric quantum letterplace algebra* $\mathrm{Super}[L \mid P]_q$ *is the quotient algebra of* $\mathrm{Tens}[L \mid P]$ *subject to the following relations*:

(R1) $(a^- \mid \alpha^+)^2 = 0$

(R2) $(a^+ \mid \alpha^-)^2 = 0$

(R3) $(a^- \mid \beta)(a^- \mid \alpha) = (-1)^{|a\beta|\,|a\alpha|}\, q(a^- \mid \alpha)(a^- \mid \beta)$ *for all* $\alpha < \beta$

(R4) $(a^+ \mid \beta)(a^+ \mid \alpha) = (-1)^{|\alpha\beta|\,|a\alpha|}\, q^{-1}(a^+ \mid \alpha)(a^+ \mid \beta)$ *for all* $\alpha < \beta$

(R5) $(b \mid \alpha^-)(a \mid \alpha^-) = (-1)^{|b\alpha|\,|a\alpha|}\, q(a \mid \alpha^-)(b \mid \alpha^-)$ *for all* $a < b$

(R6) $(b \mid \alpha^+(a \mid \alpha^+) = (-1)^{|b\alpha|\,|a\alpha|}\, q^{-1}(a \mid \alpha^+)(b \mid \alpha^+)$ *for all* $a < b$

(R7) $(b \mid \alpha)(a \mid \beta) = (-1)^{|b\alpha|\,|a\beta|}\, (a \mid \beta)(b \mid \alpha)$ *for all* $a < b$, $\alpha < \beta$

(R8) $(b \mid \beta)(a \mid \alpha) - (-1)^{|b\beta|\,|a\alpha|}\, (a \mid \alpha)(b \mid \beta) = (-1)^{|b\beta|\,|a| + |b|\,|\beta|} (q^{-1} - q)(a \mid \beta)(b \mid \alpha)$ *for all* $a < b$, $\alpha < \beta$.

One observes that the sign rules in the above relations agree with the sign rules in the supersymmetric letterplace algebra defined in [GRS].

Proof. The maps T_l and T_r are algebra homomorphisms if and only if they satisfy the following relations:

(T1) $T_l(b)\,T_l(a) = (-1)^{|b|\,|a|}\,q^{-1}T_l(a)\,T_l(b)$ for all $a<b$

(T2) $T_l(a^-)\,T_l(a^-)=0$

(T3) $T_r(\beta)\,T_r(a) = (-1)^{|\beta|\,|\alpha|}\,q^{-1}T_r(\alpha)\,T_r(\beta)$ for all $\alpha<\beta$

(T4) $T_r(\alpha^-)\,T_r(\alpha^-)=0$.

By definition,

$$\begin{aligned}
&T_l(a)\,T_l(b)\\
&\quad=\Big(\sum_{\alpha\in P}(a\,|\,\alpha)\otimes\alpha\Big)\Big(\sum_{\beta\in P}(b\,|\,\beta)\otimes\beta\Big)\\
&\quad=\sum_{\alpha,\beta}(-1)^{|\alpha|\,|b\beta|}\,(a\,|\,\alpha)(b\,|\,\beta)\otimes\alpha\beta\\
&\quad=\sum_{\alpha<\beta}\big((-1)^{|\alpha|\,|b\beta|}\,(a\,|\,\alpha)(b\,|\,\beta)+(-1)^{|\alpha|\,|\beta|+|\beta|\,|b\alpha|}\,q^{-1}(a\,|\,\beta)(b\,|\,\alpha)\big)\otimes\alpha\beta\\
&\qquad+\sum_{\alpha\in P^+}(a\,|\,\alpha)(b\,|\,\alpha)\otimes\alpha^2
\end{aligned}$$

and

$$\begin{aligned}
&T_r(\alpha)\,T_r(\beta)\\
&\quad=\Big(\sum_{a\in L}a\otimes(a\,|\,\alpha)\Big)\Big(\sum_{b\in L}b\otimes(b\,|\,\beta)\Big)\\
&\quad=\sum_{a,b}(-1)^{|a\alpha|\,|b|}\,ab\otimes(a\,|\,\alpha)(b\,|\,\beta)\\
&\quad=\sum_{a<b}ab\otimes\Big((-1)^{|a\alpha|\,|b|}\,(a\,|\,\alpha)(b\,|\,\beta)+(-1)^{|a|\,|b|+|b\alpha|\,|a|}\,q^{-1}(b\,|\,\alpha)(a\,|\,\alpha)\Big)\\
&\qquad+\sum_{a\in L^+}a^2\otimes(a\,|\,\alpha)(a\,|\,\beta).
\end{aligned}$$

One can check directly that

$$(T2) \Leftrightarrow (R1) + (R3),$$

$$(T4) \Leftrightarrow (R2) + (R5),$$

$$(T1) + (T3) \Leftrightarrow (R4) + (R6) + (R7) + (R8). \quad \blacksquare$$

We may re-write the relations (R1) to (R6) in the following way (and keep (R7) and (R8)):

(R1–2) $(a \mid \alpha)^2 = 0$, if $|a\alpha| = 1$

(R3–4) $(a \mid \beta)(a \mid \alpha) = (-1)^{|a\beta|\,|a\alpha|}\, q^{-(-1)^{|a|}}(a \mid \alpha)(a \mid \beta)$ for all $\alpha < \beta$

(R5–6) $(b \mid \alpha)(a \mid \alpha) = (-1)^{|b\alpha|\,|a\alpha|}\, q^{-(-1)^{|\alpha|}}(a \mid \alpha)(b \mid \alpha)$ for all $a < b$.

Remarks. (1) If $L = P$ then $\mathrm{Super}[L \mid L]_{q^{-1}}$ coincides with the "quantum linear super semi-group" E_q (with all $q_{ij} = q$) defined in [Ma2, p. 136]. As a consequence, if $L = L^- = P = P^-$, then $\mathrm{Super}[L \mid L]_q$ is isomorphic to the quantum semi-group $M_n(q)$; if $L = L^+ = P = P^+$, then $\mathrm{Super}[L \mid L]_q$ is isomorphic to the quantum semi-group $M_n(q^{-1})$.

(2) The parameter q can be replaced by $\mathbf{q}$ which is a set $\{q_{ab} = \pm q \mid \forall a < b \text{ in } L\} \cup \{q_{\alpha\beta} = \pm q \mid \forall \alpha < \beta \text{ in } P\}$, where the signs of q_{ab} and $q_{\alpha\beta}$ are independent. Then we define the quantum letter algebra $\mathrm{Super}[L]_{\mathbf{q}}$ to be generated by elements of L subject to the following relations:

(i) $ba = (-1)^{|a|\,|b|}\, q_{ab}^{-1}\, ab$, where $a < b$ in L,

(ii) $a^- a^- = 0$.

Similarly the quantum place algebra $\mathrm{Super}[P]_{\mathbf{q}}$ and the quantum letter-place algebra $\mathrm{Super}[L \mid P]_{\mathbf{q}}$ are defined. All results which hold for $\mathrm{Super}[L \mid P]_q$ in this paper hold for $\mathrm{Super}[L \mid P]_{\mathbf{q}}$.

(3) The base field K can be replaced by any commutative ring with a sub-field containing q, and everything still works.

(4) Given any element $\alpha \in P^+$, the map $a \to (a \mid \alpha)$ defines a unique injective algebra homomorphism from $\mathrm{Super}[L]_q$ to $\mathrm{Super}[L \mid P]_q$. Given any element $a \in L^+$, the map $\alpha \to (a \mid \alpha)$ defines a unique injective algebra homomorphism from $\mathrm{Super}[P]_q$ to $\mathrm{Super}[L \mid P]_q$. Consequently, if $P = P^+ = \{\alpha\}$, then $\mathrm{Super}[L]_q \cong \mathrm{Super}[L \mid P]_q$; if $L = L^+ = \{a\}$, then $\mathrm{Super}[P]_q \cong \mathrm{Super}[L \mid P]_q$.

By the definition of $\mathrm{Super}[L \mid P]_q$, the relations (R1) to (R8) are necessary to make the maps T_l and T_r be algebra homomorphisms. Hence the algebra $\mathrm{Super}[L \mid P]_q$ has a universal property in the following sense.

Let R be a $\mathbf{Z}_2$-graded algebra containing a set of $\mathbf{Z}_2$-homogeneous elements $\{x_{(a|\alpha)} \mid a \in L, \alpha \in P\}$ with either $x_{(a|\alpha)} = 0$ or $|x_{(a|\alpha)}| = |a\alpha|$, and suppose there are two algebra homomorphisms (similar to T_l and T_r),

$$T'_l : \mathrm{Super}[L]_q \to R \otimes \mathrm{Super}[P]_q$$

$$a \to \sum_{\alpha \in P} x_{(a|\alpha)} \otimes \alpha$$

and

$$T'_r : \mathrm{Super}[P]_q \to \mathrm{Super}[L]_q \otimes R$$

$$\alpha \to \sum_{a \in L} a \otimes x_{(a|\alpha)}.$$

As proved in Proposition 1, the relations (R1) to (R8) hold for the set of elements $\{x_{(a|\alpha)} \mid a \in L, \alpha \in P\}$ (by replacing $(a|\alpha)$ by $x_{(a|\alpha)}$ in the relations). Hence there is a unique algebra homomorphism, say F, from $\mathrm{Super}[L|P]_q$ to R defined by $F((a|\alpha)) = x_{(a|\alpha)}$. Consequently, $T'_l = (F \otimes \mathrm{id})\, T_l$ and $T'_r = (\mathrm{id} \otimes F)\, T_r$.

Let L_1, L_2, L_3 be three linearly ordered $\mathbf{Z}_2$-graded finite sets. By definition, we have the following canonical algebra homomorphisms:

$$T_l^{12} : \mathrm{Super}[L_1]_q \to \mathrm{Super}[L_1 | L_2]_q \otimes \mathrm{Super}[L_2]_q,$$

$$T_r^{12} : \mathrm{Super}[L_2]_q \to \mathrm{Super}[L_1]_q \otimes \mathrm{Super}[L_1 | L_2]_q,$$

$$T_l^{23} : \mathrm{Super}[L_2]_q \to \mathrm{Super}[L_2 | L_3]_q \otimes \mathrm{Super}[L_3]_q,$$

$$T_r^{23} : \mathrm{Super}[L_3]_q \to \mathrm{Super}[L_2]_q \otimes \mathrm{Super}[L_2 | L_3]_q,$$

$$T_l^{13} : \mathrm{Super}[L_1]_q \to \mathrm{Super}[L_1 | L_3]_q \otimes \mathrm{Super}[L_3]_q,$$

$$T_r^{13} : \mathrm{Super}[L_3]_q \to \mathrm{Super}[L_1]_q \otimes \mathrm{Super}[L_1 | L_3]_q.$$

Hence the map $(\mathrm{id} \otimes T_l^{23})\, T_l^{12}$ is an algebra homomorphism from $\mathrm{Super}[L_1]_q$ to $\mathrm{Super}[L_1 | L_2]_q \otimes \mathrm{Super}[L_2 | L_3]_q \otimes \mathrm{Super}[L_3]_q$, and the map $(T_r^{12} \otimes \mathrm{id})\, T_r^{23}$ is an algebra homomorphism from $\mathrm{Super}[L_3]_q$ to $\mathrm{Super}[L_1]_q \otimes \mathrm{Super}[L_1 | L_2]_q \otimes \mathrm{Super}[L_2 | L_3]_q$. By the universal property of $\mathrm{Super}[L_1 | L_3]_q$, there is an algebra homomorphism Δ from $\mathrm{Super}[L_1 | L_3]_q$ to $\mathrm{Super}[L_1 | L_2]_q \otimes \mathrm{Super}[L_2 | L_3]_q$ which is uniquely defined by

$$\Delta((a_1 | a_3)) = \sum_{a_2 \in L_2} (a_1 | a_2) \otimes (a_2 | a_3)$$

for all $a_1 \in L_1$ and $a_3 \in L_3$. By the universal property, we have $(\mathrm{id} \otimes T_l^{23})\, T_l^{12} = (\Delta \otimes \mathrm{id})\, T_l^{13}$ and $(T_r^{12} \otimes \mathrm{id})\, T_r^{23} = (\mathrm{id} \otimes \Delta)\, T_r^{13}$. The algebra homomorphism Δ can be regarded as a generalization of T_l and T_r. For example, if $(L_1, L_2, L_3) = (L, P, a^+)$, then $\Delta = T_l$; and if $(L_1, L_2, L_3) = (a^+, L, P)$, then $\Delta = T_r$.

Suppose now that $L_1 = L_2 = L_3 = L$, namely, these sets have the same positive and negative variables and the same linear order. We use letters $a, b, c, \dots$ for the elements of L. The maps T_l and T_r are algebra homomorphisms from $\mathrm{Super}[L]_q$ to $\mathrm{Super}[L \mid L]_q \otimes \mathrm{Super}[L]_q$ and from $\mathrm{Super}[L]_q$ to $\mathrm{Super}[L]_q \otimes \mathrm{Super}[L \mid L]_q$, respectively. Hence $T_l^* := (\mathrm{id} \otimes T_l)\, T_l$ is an algebra homomorphism from $\mathrm{Super}[L]_q$ to $\mathrm{Super}[L \mid L]_q \otimes \mathrm{Super}[L \mid L]_q \otimes \mathrm{Super}[L]_q$, and $T_r^* := (T_r \otimes \mathrm{id})\, T_r$ is an algebra homomorphism from $\mathrm{Super}[L]_q$ to $\mathrm{Super}[L]_q \otimes \mathrm{Super}[L \mid L]_q \otimes \mathrm{Super}[L \mid L]_q$. By definition, we have

$$T_l^*(a) = \sum_{b, c \in L} (a \mid b) \otimes (b \mid c) \otimes c$$

and

$$T_r^*(c) = \sum_{a, b \in L} a \otimes (a \mid b) \otimes (b \mid c).$$

Let $x_{(a \mid c)}$ denote the element $\sum_{b \in L} (a \mid b) \otimes (b \mid c)$. It is clear that $|x_{(a \mid c)}| = |ac|$ and $\{x_{(a \mid c)} \mid a, c \in L\}$ is a set of elements of the algebra $\mathrm{Super}[L \mid L]_q \otimes \mathrm{Super}[L \mid L]_q$. Hence there is a unique algebra homomorphism Δ from $\mathrm{Super}[L \mid L]_q$ to $\mathrm{Super}[L \mid L]_q \otimes \mathrm{Super}[L \mid L]_q$ defined by

$$\Delta((a \mid c)) = x_{(a \mid c)} = \sum_b (a \mid b) \otimes (b \mid c).$$

PROPOSITION 2 [Ma2, Theorem 3.2] *The algebra* $\mathrm{Super}[L \mid L]_q$ *is a super bialgebra with comultiplication* Δ *defined above and counit* ε *defined by* $\varepsilon((a \mid c)) = \delta_{ac}$, *where* $\delta_{ac} = 0$ *if* $a \neq c$ *and* $\delta_{aa} = 1$. *(A super bialgebra is a* $\mathbf{Z}_2$*-graded algebra satisfying all axioms of bialgebra, where the super tensor product is used instead of the usual tensor product.)*

Proof. It is proved above that Δ is an algebra homomorphism, and it is obvious that ε can be extended to an algebra homomorphism. It remains to prove

(i) coassociativity $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$;

(ii) counit property $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$.

Let us check. The algebra $\mathrm{Super}[L \mid L]_q$ is generated by the elements $(a \mid b)$ for all $a, b \in L$. The maps $(\Delta \otimes \mathrm{id})\Delta$, $(\mathrm{id} \otimes \Delta)\Delta$, $(\varepsilon \otimes \mathrm{id})\Delta$, and $(\mathrm{id} \otimes \varepsilon)\Delta$

are all algebra homomorphisms. Hence we only need to look at the evaluation of these maps on elements $(a|b)$, where $a, b \in L$:

$$(\Delta \otimes \mathrm{id})\,\Delta((a|b)) = (\Delta \otimes \mathrm{id})\Big(\sum_d (a|d) \otimes (d|b)\Big) = \sum_{c,d} (a|c) \otimes (c|d) \otimes (d|b),$$

$$(\mathrm{id} \otimes \Delta)\,\Delta((a|b)) = (\mathrm{id} \otimes \Delta)\Big(\sum_c (a|c) \otimes (c|b)\Big) = \sum_{c,d} (a|c) \otimes (c|d) \otimes (d|b);$$

$$(\varepsilon \otimes \mathrm{id})\,\Delta(a|b) = (\varepsilon \otimes \mathrm{id})\Big(\sum_c (a|c) \otimes (c|b)\Big) = \sum_c \delta_{ac}(c|b) = (a|b),$$

$$(id \otimes \varepsilon)\,\Delta(a|b) = (\mathrm{id} \otimes \varepsilon)\Big(\sum_c (a|c) \otimes (c|b)\Big) = \sum_c (a|c)\delta_{cb} = (a|b).$$

Hence (i) and (ii) hold. ∎

Remark. There are some studies of $E_q = \mathrm{Super}[L|L]_{q^{-1}}$ in [Ma2, Sect. 4.3]. The algebra E_q is not a Hopf super algebra, but the natural Hopf super algebra associated to E_q is the Hopf envelope H_q of E_q (see [Ma2]).

The supersymmetric quantum letter–place algebra $\mathrm{Super}[L|P]_q$ is a continuous deformation of the supersymmetric letterplace $\mathrm{Super}[L|P]$ with the parameter q. If $q=1$, then $\mathrm{Super}[L|P]_1 = \mathrm{Super}[L|P]$. Let $\mathrm{Mon}(L|P)$ be the set of words on the set of letterplace pairs $\{(a|\alpha): a \in L, \alpha \in P\}$. Order the set of letterplace pairs lexicographically:

$$(a|\alpha) < (b|\beta) \text{ if and only if either } a < b \text{ or } a = b \text{ and } \alpha < \beta.$$

A word or monomial $(a_1|\alpha_1)\cdots(a_n|\alpha_n)$ is called *ordered* or *non-decreasing* if $(a_i|\alpha_i) \leqslant (a_{i+1}|\alpha_{i+1})$ for all $i = 1, \ldots, n-1$.

It is known that the set of all ordered monomials (including 1), in which no letterplace pair $(a|\alpha)$ with $\mathbf{Z}_2$-degree one appears more than once, in $\mathrm{Mon}(L|P)$ forms a K-linear basis of $\mathrm{Super}[L|P]$. This statement is also true for $\mathrm{Super}[L|P]_q$ for all non-zero $q \in K$.

THEOREM 3. *The set of ordered monomials, in which no letterplace pair $(a|\alpha)$ with $\mathbf{Z}_2$-degree one appears more than once, in* $\mathrm{Mon}(L|P)$ *is a K-linear basis of* $\mathrm{Super}[L|P]_q$ *for all non-zero $q \in K$.*

Remark. One proof of this theorem is to use Bergman's Diamond Lemma [Be], which was our original proof. The proof given below relies on Manin's result on E_q.

Proof. If $q = 1, -1$, then $\mathrm{Super}[L|P]_q$ is a supersymmetric letterplace algebra and it is easy to verify that the theorem holds. Now we assume that

$q \neq 1, -1$. By using relations (R1) to (R8), any monomial in $\mathrm{Super}[L \mid P]_q$ can be reduced to (and then it is equal to) a linear combination of ordered monomials. Hence the set of non-zero ordered monomials spans $\mathrm{Super}[L \mid P]_q$. It remains to prove that non-zero ordered monomials in $\mathrm{Super}[L \mid P]_q$ are linearly independent. If $L = P$, then $\mathrm{Super}[L \mid L]_q \cong E_{q^{-1}}$. By [Ma2, Theorem 3.12], all non-zero ordered monomials in $\mathrm{Super}[L \mid L]_q$ are linearly independent. In general, let us take $J = L \cup P$ with the extended order $a < \alpha$ for all $a \in L$ and $\alpha \in P$. There is a natural embedding map f from $\mathrm{Super}[L \mid P]_q$ to $\mathrm{Super}[J \mid J]_q$ by sending $(a \mid \alpha)$ to $(a \mid \alpha)$. Since non-zero ordered monomials in $\mathrm{Super}[J \mid J]_q$ are linearly independent and f is an injection, all non-zero monomials in $\mathrm{Super}[L \mid P]_q$ are linearly independent too. Therefore the set of ordered monomials is a K-linear basis of $\mathrm{Super}[L \mid P]_q$. ∎

3. Left and Right Supersymmetric Quantum Minors

In this section we use the left and the right co-representations T_l and T_r to define left and right supersymmetric quantum minors. Given a word $u = a_1 a_2 \cdots a_k \in \mathrm{Mon}(L)$, let $N(u) = \sum_{i<j} |a_i|\, |a_j|$. One can see that $N(u)$ depends only on the *content* of u, which is the multiset of the elements that occur in u; in fact, $N(u) = \frac{1}{2} l(l-1)$, where l is the number of negative variables in u. Given a word $\mu \in \mathrm{Mon}(P)$, $N(\mu)$ is defined in the same way. We define bilinear maps

$$(\cdot \mid \cdot)_r : \mathrm{Super}[L]_{q^{-1}} \otimes \mathrm{Super}[P]_q \to \mathrm{Super}[L \mid P]_q,$$

$$(\cdot \mid \cdot)_l : \mathrm{Super}[L]_q \otimes \mathrm{Super}[P]_{q^{-1}} \to \mathrm{Super}[L \mid P]_q$$

such that

$$T_r(\mu) = \sum (-1)^{N(u)}\, u \otimes (u \mid \mu)_r, \tag{2}$$

$$T_l(u) = \sum (-1)^{N(\mu)}\, (u \mid \mu)_l \otimes \mu, \tag{3}$$

where the sum in (2) ranges over the words u in $\mathrm{Mon}(L)$ of different contents. In other words, we take one and only one word u from each collection of words of the same content to form the summands in (2). For example, the sum in (2) can range over all ordered non-zero monomials $a_1 a_2 \cdots a_k$ in $\mathrm{Mon}(L)$, where $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$. The sum in (3) is similarly defined. If no negative variable occurs more than once in u or in μ, then $(u \mid \mu)_l$ and $(u \mid \mu)_r$ are determined by (2) and (3); otherwise, we set

$$(u \mid \mu)_r = (u \mid \mu)_l = 0. \tag{4}$$

For example, let $L=\{a^+, b^+\}$ and $P=\{\alpha^+, \beta^+\}$ with $a^+ < b^+$ and $\alpha^+ < \beta^+$. Then

$$\begin{aligned}
T_r(\alpha\beta) &= T_r(\alpha)\, T_r(\beta) \\
&= (a\otimes(a\mid\alpha)+b\otimes(b\mid\alpha))\cdot(a\otimes(a\mid\beta)+b\otimes(b\mid\beta)) \\
&= a^2\otimes(a\mid\alpha)(a\mid\beta)+ab\otimes(a\mid\alpha)(b\mid\beta) \\
&\quad + ba\otimes(b\mid\alpha)(a\mid\beta)+b^2\otimes(b\mid\alpha)(b\mid\beta) \\
&= a^2\otimes(a\mid\alpha)(a\mid\beta)+ab\otimes[(a\mid\alpha)(b\mid\beta) \\
&\quad + q^{-1}(b\mid\alpha)(a\mid\beta)]+b^2\otimes(b\mid\alpha)(b\mid\beta) \\
&= a^2\otimes(a\mid\alpha)(a\mid\beta)+ba\otimes[(b\mid\alpha)(a\mid\beta) \\
&\quad + q(a\mid\alpha)(b\mid\beta)]+b^2(b\mid\alpha)(b\mid\beta).
\end{aligned}$$

Therefore, by definition,

$$\begin{aligned}
(aa\mid\alpha\beta)_r &= (a\mid\alpha)(a\mid\beta), \\
(bb\mid\alpha\beta)_r &= (b\mid\alpha)(b\mid\beta), \\
(ab\mid\alpha\beta)_r &= (a\mid\alpha)(b\mid\beta)+q^{-1}(b\mid\alpha)(a\mid\beta), \\
(ba\mid\alpha\beta)_r &= (b\mid\alpha)(a\mid\beta)+q(a\mid\alpha)(b\mid\beta).
\end{aligned}$$

We check next that $(\cdot\mid\cdot)_r$ is well defined. This amounts to showing that

$$(u\mid\mu\beta\alpha v)_r = (-1)^{|\alpha|\,|\beta|}\, q^{-1}(u\mid\mu\alpha\beta v)_r \text{ if } \alpha<\beta, \tag{5}$$

$$(ubav\mid\mu)_r = (-1)^{|a|\,|b|}\, q(uabv\mid\mu)_r \text{ if } a<b. \tag{6}$$

Relation (5) follows from the fact

$$T_r(\mu\beta\alpha v) = (-1)^{|\alpha|\,|\beta|}\, q^{-1}T_r(\mu\alpha\beta v),$$

since T_r is a well-defined algebra homomorphism from $\mathrm{Super}[P]_q$ to $\mathrm{Super}[L]_q\otimes\mathrm{Super}[L\mid P]_q$. Relation (6) is a consequence of the identity

$$(-1)^{N(ubav)}\, ubav\otimes(ubav\mid\mu)_r = (-1)^{N(uabv)}\, uabv\otimes(uabv\mid\mu)_r,$$

which is actually part of the definition of the map $(\cdot\mid\cdot)_r$. Similarly, $(\cdot\mid\cdot)_l$ is also well defined. We call $(u\mid\mu)_r$ and $(u\mid\mu)_l$ a *right* and a *left supersymmetric quantum minor*, respectively. Obviously,

$$(u\mid\mu)_r = (u\mid\mu)_l = 0 \qquad \text{if } \mathrm{length}(u)\neq\mathrm{length}(\mu).$$

Remark. It is clear that $T_r(\mu\alpha^-\alpha^-\nu)=0$ implies $(u\mid\mu\alpha^-\alpha^-\nu)_r=0$ for all $u\in\mathrm{Mon}(L)$. However, we are not able to derive $(ua^-a^-v\mid\mu)_r=0$ by using the same idea; instead, we have to set $(ua^-a^-v\mid\mu)_r=0$.

PROPOSITION 4. *Supersymmetric quantum minors have the Laplace expansions*

$$(u\mid\mu\nu)_r=\sum_u(-1)^{|\mu|\,|u_{(2)}|}(u_{(1)}\mid\mu)_r\cdot(u_{(2)}\mid\nu)_r,\tag{7}$$

$$(uv\mid\mu)_l=\sum_\mu(-1)^{|\mu_{(1)}|\,|v|}(u\mid\mu_{(1)})_l\cdot(v\mid\mu_{(2)})_l,\tag{8}$$

where $\Delta u=\sum_u u_{(1)}\otimes u_{(2)}$ and $\Delta\mu=\sum_\mu \mu_{(1)}\otimes\mu_{(2)}$ are the coproducts defined on $\mathrm{Super}[L]_{q^{-1}}$ *and* $\mathrm{Super}[P]_{q^{-1}}$ *respectively.*

Remark. Laplace expansions for classical quantum groups $GL_n(q)$ are known (see for example [TT, PW]). The following proof is similar to the proof in the classical quantum case.

Proof. Let us prove (7). We may assume that no negative letter occurs in u more than once. By definition,

$$T_r(\mu\nu)=\sum_u(-1)^{N(u)}u\otimes(u\mid\mu\nu)_r,\qquad\text{and}$$

$$\begin{aligned}T_r(\mu\nu)&=T_r(\mu)\,T_r(\nu)\\&=\Big(\sum_v(-1)^{N(v)}v\otimes(v\mid\mu)_r\Big)\cdot\Big(\sum_w(-1)^{N(w)}w\otimes(w\mid\nu)_r\Big)\\&=\sum_v\sum_w(-1)^{N(v)+N(w)+|v|\,|w|+|\mu|\,|w|}vw\otimes(v\mid\mu)(w\mid\nu),\end{aligned}$$

where each of the sums $\sum_u$, $\sum_v$ and $\sum_w$ ranges over the words in $\mathrm{Mon}(L)$ of different contents. Comparing the coefficients of $u\neq 0$, we obtain

$$(-1)^{N(u)}(u\mid\mu\nu)_r=\sum_{(v,w)}(-1)^{N(v)+N(w)+|v|\,|w|+|\mu|\,|w|}k_{vw}(v\mid\mu)(w\mid\nu),$$

where the sum ranges over the ordered partitions of u as multisets and k_{vw} satisfies the relation $u=k_{vw}vw$ in $\mathrm{Super}[L]_{q^{-1}}$, which is equivalent to $u=k_{vw}^{-1}vw$ in $\mathrm{Super}[L]_q$. The final step follows immediately from

$$N(u)=N(v)+N(w)+|v|\,|w|. \qquad\blacksquare$$

Remark. For right or left supersymmetric quantum minors, there are only one-sided Laplace expansions. In other words, a formula like

$$(uv \mid \mu)_r = \sum_{\mu} (-1)^{|\mu_{(1)}||v|} (u \mid \mu_{(1)})_r \cdot (v \mid \mu_{(2)})_r$$

cannot be made to work, whether $\Delta\mu = \sum_\mu \mu_{(1)} \otimes \mu_{(2)}$ is considered as a coproduct defined on $\mathrm{Super}[P]_q$ or on $\mathrm{Super}[P]_{q^{-1}}$. For example, $(a^+a^+ \mid \alpha^+\beta^+)_r = (a \mid \alpha)(a \mid \beta)$, it equals neither $(a \mid \alpha)(a \mid \beta) + q(a \mid \beta)(a \mid \alpha)$ nor $(a \mid \alpha)(a \mid \beta) + q^{-1}(a \mid \beta)(a \mid \alpha)$.

As a corollary, we have obtained explicit formulas for quantum minors:

PROPOSITION 5. *Let $a_i \in L$ and $\alpha_i \in P$, $i = 1, 2, ..., n$. We have*

$$\begin{aligned}(a_1 a_2 \cdots a_n \mid \alpha_1 a_2 \cdots \alpha_n)_r \\ = \sum_{\sigma} (-1)^{\sum_{i>j} |a_{\sigma_i}||\alpha_j|} k_\sigma(q)(a_{\sigma_1} \mid \alpha_1)(a_{\sigma_2} \mid \alpha_2) \cdots (a_{\sigma_n} \mid \alpha_n) \qquad (9)\end{aligned}$$

if $a_1 a_2 \cdots a_n \neq 0$ in $\mathrm{Super}[L]_{q^{-1}}$, *and*

$$\begin{aligned}(a_1 a_2 \cdots a_n \mid \alpha_1 \alpha_2 \cdots \alpha_n)_l \\ \sum_{\sigma} (-1)^{\sum_{i>j} |a_i||\alpha_{\sigma_j}|} l_\sigma(q)(a_1 \mid \alpha_{\sigma_1})(a_2 \mid \alpha_{\sigma_2}) \cdots (a_n \mid \alpha_{\sigma_n}) \qquad (10)\end{aligned}$$

if $\alpha_1 \alpha_2 \cdots \alpha_n \neq 0$ in $\mathrm{Super}[P]_{q^{-1}}$, *where the sums in* (9) *and* (10) *range over the permutations of the multisets $a_1 a_2 \cdots a_n$ and $\alpha_1 \alpha_2 \cdots \alpha_n$, respectively, and where the coefficients $k_\sigma(q)$ and $l_\sigma(q)$ are defined so that*

$$a_1 a_2 \cdots a_n = k_\sigma(q)\, a_{\sigma_1} a_{\sigma_2} \cdots a_{\sigma_n} \qquad \textit{in } \mathrm{Super}[L]_{q^{-1}}.$$

$$\alpha_1 \alpha_2 \cdots \alpha_n = 1_\sigma(q)\, \alpha_{\sigma_1} \alpha_{\sigma_2} \cdots \alpha_{\sigma_n} \qquad \textit{in } \mathrm{Super}[P]_{q^{-1}}.$$

Remark. If $a_1 a_2 \cdots a_n$ and $\alpha_1 \alpha_2 \cdots \alpha_n$ have repetitions of negative variables, we may take $k_\sigma(q)$ and $l_\sigma(q)$ to be zero for every σ, in order to make identities (9) and (10) still work.

EXAMPLES. 1. If $a_1^- < a_2^- < \cdots < a_n^-$, then

$$\begin{aligned}(a_1^- a_2^- \cdots a_n^- \mid \alpha_1^- \alpha_2^- \cdots \alpha_n^-)_r \\ = (-1)^{n(n-1)/2} \sum_{\sigma \in S_n} (-q)^{-i(\sigma)} (a_{\sigma_1} \mid \alpha_1)(a_{\sigma_2} \mid \alpha_2) \cdots (a_{\sigma_n} \mid \alpha_n),\end{aligned}$$

where $i(\sigma)$ denotes the number of inversions in the permutation σ. This is the ordinary (right) quantum determinant.

2. If $\alpha_1^+ < \alpha_2^+ < \cdots < \alpha_n^+$, then

$$(a_1^+ a_2^+ \cdots a_n^+ \mid \alpha_1^+ \alpha_2^+ \cdots \alpha_n^+)_l = \sum_{\sigma \in S_n} q^{-i(\sigma)} (a_1 \mid \alpha_{\sigma_1})(a_2 \mid \alpha_{\sigma_2}) \cdots (a_n \mid \alpha_{\sigma_n}).$$

This is the ordinary (left) quantum permanent.

3. If $a_1^+ < a_2^+ < \cdots < a_n^+$, then

$$(a_1^+ a_2^+ \cdots a_n^+ \mid \alpha_1^- \alpha_2^- \cdots \alpha_n^-)_r = \sum_{\sigma \in S_n} q^{-i(\sigma)} (a_{\sigma_1} \mid \alpha_1)(a_{\sigma_2} \mid \alpha_2) \cdots (a_{\sigma_n} \mid \alpha_n).$$

4. If $a^- < b^-$, then

$$\begin{aligned}(a^- b^- \mid \alpha^+ \alpha^+)_r &= (a^- \mid \alpha^+)(b^- \mid \alpha^+) - q^{-1}(b^- \mid \alpha^+)(a^- \mid \alpha^+) \\ &= (1 + q^{-2})(a^- \mid \alpha^+)(b^- \mid \alpha^+).\end{aligned}$$

On the other hand $(a^- b^- \mid \alpha^+ \alpha^+)_l = (a^- \mid \alpha^+)(b^- \mid \alpha^+)$. Hence

$$(a^- b^- \mid \alpha^+ \alpha^+)_r = (1 + q^{-2})(a^- b^- \mid \alpha^+ \alpha^+)_l.$$

5. We have

$$(a^+ a^+ \mid \alpha^+ \alpha^+)_r = (a^+ a^+ \mid \alpha^+ \alpha^+)_l = (a^+ \mid \alpha^+)(a^+ \mid \alpha^+).$$

Let L_1, L_2 and L_3 be three $\mathbf{Z}_2$-graded linearly ordered finite sets. For any $u_i \in \mathrm{Mon}(L_i)$, let $(u_i \mid u_j)_l$ (resp. $(u_i \mid u_j)_r$) denote the corresponding left (resp. right) quantum minor in $\mathrm{Super}[L_i \mid L_j]_q$. Let Δ be the map from $\mathrm{Super}[L_1 \mid L_3]_q$ to $\mathrm{Super}[L_1 \mid L_2]_q \otimes \mathrm{Super}[L_2 \mid L_3]_q$ defined in Section 2. The following property can be regarded as a generalization of (2) and (3).

PROPOSITION 6. *For any $u_1 \in \mathrm{Mon}(L_1)$ and $u_3 \in \mathrm{Mon}(L_3)$,*

$$\Delta((u_1 \mid u_3)_l) = \sum_{u_2} (-1)^{N(u_2)} (u_1 \mid u_2)_l \otimes (u_2 \mid u_3)_l,$$

where the sum ranges over the words u_2 on $\mathrm{Mon}(L_2)$ of different contents. The same statement is true for right quantum minors.

607 102 2-7

Proof. By the definition of Δ (see Section 2), we have $(\mathrm{id}\otimes T_l^{23})\, T_l^{12} = (\Delta\otimes\mathrm{id})\, T_l^{13}$. Let us apply this to a monomial $u_1 \in \mathrm{Mon}(L_1)$:

$$(\mathrm{id}\otimes T_l^{23})\, T_l^{12}(u_1) = (\mathrm{id}\otimes T_l^{23})\left(\sum_{u_2} (-1)^{N(u_2)} (u_1 \mid u_2)_l \otimes u_2\right)$$

$$= \sum_{u_2, u_3} (-1)^{N(u_2)+N(u_3)} (u_1 \mid u_2)_l \otimes (u_2 \mid u_3)_l \otimes u_3,$$

$$(\Delta\otimes\mathrm{id})\, T_l^{13}(u_1) = (\Delta\otimes\mathrm{id})\left(\sum_{u_3} (-1)^{N(u_3)} (u_1 \mid u_3)_l \otimes u_3\right)$$

$$= \sum_{u_3} (-1)^{N(u_3)} \Delta((u_1 \mid u_3)_l) \otimes u_3.$$

Hence $\Delta((u_1 \mid u_3)_l) = \sum_{u_2} (-1)^{N(u_2)} (u_1 \mid u_2)_l \otimes (u_2 \mid u_3)_l$. ∎

In general a left supersymmetric quantum minor is not equal to the corresponding right one. However, they only differ a scalar multiple. Let us fix some notations first. For a positive integer n, let

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1},$$

$$[n]_q! = [n]_q\, [n-1]_q \cdots [1]_q,$$

$$\begin{bmatrix} n \\ n_1 n_2 \cdots n_i \end{bmatrix}_q = \frac{[n]_q!}{[n_1]_q!\, [n_2]_q! \cdots [n_i]_q!},$$

where $n_1 + n_2 + \cdots + n_i = n$. *From now on we assume that* $[n]_q \neq 0$ *for all* $n > 0$, i.e., q is not a root of unity except $q = 1$. Let S be a multiset $1^{n_1} 2^{n_2} \cdots i^{n_i}$. By [St, Proposition 1.3.17], the following identity holds

$$\sum_\sigma q^{i(\sigma)} = \begin{bmatrix} n \\ n_1 n_2 \cdots n_i \end{bmatrix}_q, \tag{11}$$

where the sum ranges over all permutations of the multiset S. For example, let $S = 1^2 2$; then 112, 121, 211 are the three permutations of S. We have

$$1 + q + q^2 = \begin{bmatrix} 3 \\ 21 \end{bmatrix}_q.$$

THEOREM 7. *Let* $a_1 < a_2 < \cdots < a_s$ *and* $\alpha_1 < \alpha_2 < \cdots < \alpha_t$. *Given positive integers* $n, n_1, n_2, \ldots, n_s$ *and* $m_1, m_2, \ldots, m_t$ *such that* $n = \sum_i n_i = \sum_i m_i$, *we have*

$$\begin{bmatrix} n \\ n_1 n_2 \cdots n_s \end{bmatrix}_{q^{-2}} (a_1^{n_1} a_2^{n_2} \cdots a_s^{n_s} \mid \alpha_1^{m_1} \alpha_2^{m_2} \cdots \alpha_t^{m_t})_l$$

$$= \begin{bmatrix} n \\ m_1 m_2 \cdots m_t \end{bmatrix}_{q^{-2}} (a_1^{n_1} a_2^{n_2} \cdots a_s^{n_s} \mid \alpha_1^{m_1} \alpha_2^{m_2} \cdots \alpha_t^{m_t})_r.$$

Proof. We may assume that no negative variable occurs more than once in $a_1^{n_1}a_2^{n_2}\cdots a_s^{n_2}$ and $\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_t^{m_t}$. Write $a_1^{n_1}a_2^{n_2}\cdots a_s^{n_s}$ as $b_1b_2\cdots b_n$ and $\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_t^{m_t}$ as $\beta_1\beta_2\cdots\beta_n$, where $b_1\leqslant b_2\leqslant\cdots\leqslant b_n$ and $\beta_1\leqslant\beta_2\leqslant\cdots\leqslant\beta_n$. Consider the expression

$$E=\sum_\sigma\sum_\tau(-1)^{i(\sigma^-)+i(\tau^-)+\sum_{i>j}|b_{\sigma_i}|\,|\beta_{\tau_j}|}\,q^{-i(\sigma)-i(\tau)}$$
$$\times(b_{\sigma_1}\mid\beta_{\tau_1})(b_{\sigma_2}\mid\beta_{\tau_2})\cdots(b_{\sigma_n}\mid\beta_{\tau_n}),$$

where σ and τ range over all permutations of the multisets $b_1b_2\cdots b_n$ and $\beta_1\beta_2\cdots\beta_n$, respectively, and where $i(\sigma^-)$ and $i(\tau^-)$ denote the number of inversions involving two negative variables in $b_{\sigma_1}b_{\sigma_2}\cdots b_{\sigma_n}$ and $\beta_{\tau_1}\beta_{\tau_2}\cdots\beta_{\tau_n}$. By Proposition 5, we have

$$\begin{aligned}E&=\sum_\sigma(-1)^{i(\sigma^-)}q^{-i(\sigma)}(b_{\sigma_1}b_{\sigma_2}\cdots b_{\sigma_n}\mid\beta_1\beta_2\cdots\beta_n)_1\\&=\sum_\sigma q^{-2i(\sigma)}(b_1b_2\cdots b_n\mid\beta_1\beta_2\cdots\beta_n)_1\\&=\begin{bmatrix}n\\n_1n_2\cdots n_s\end{bmatrix}_{q^{-2}}(a_1^{n_1}a_2^{n_2}\cdots a_s^{n_s}\mid\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_t^{m_t})_1;\end{aligned}$$

the last step above follows from (11). Similarly,

$$E=\begin{bmatrix}n\\m_1m_2\cdots m_t\end{bmatrix}_{q^{-2}}(a_1^{n_1}a_2^{n_2}\cdots a_s^{n_s}\mid\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_t^{m_t})_r$$

and the result follows. ∎

COROLLARY 8. *If* $a_1<a_2<\cdots<a_n$ *and* $\alpha_1<\alpha_2<\cdots<\alpha_n$, *then*

$$(a_1a_2\cdots a_n\mid\alpha_1\alpha_2\cdots\alpha_n)_1=(a_1a_2\cdots a_n\mid\alpha_1\alpha_2\cdots\alpha_n)_r. \quad ∎$$

To simplify the notations, we define

$$(u)=[n_1]_{q^{-2}}!\,[n_2]_{q^{-2}}!\cdots[n_s]_{q^{-2}}!$$

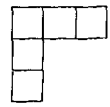
for any word u in $\mathrm{Mon}(L)$ of content $a_1^{n_1}a_2^{n_2}\cdots a_s^{n_s}$. Then Theorem 6 can be stated as

$$\frac{1}{(u)}(u\mid\mu)_1=\frac{1}{(\mu)}(u\mid\mu)_r \tag{12}$$

for all non-decreasing words $u\in\mathrm{Mon}(L)$ and $\mu\in\mathrm{Mon}(P)$. This is the form to be used in the next section.

4. Quantum Straightening Formula

In this section we prove the straightening formula for the supersymmetric quantum letterplace algebra Super$[L|P]_q$. First recall some basic concepts. A *shape* $\lambda=(\lambda_1, \lambda_2, ..., \lambda_k)$ is a finite sequence of positive integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$. We usually visualize a shape by an array of squares. For example,

$(3,1,1)=$ 

A *tableau* of shape λ on L (resp. P) is obtained by filling the squares of λ with letters of L (resp. places of P). The *content* of a tableau T, denoted by *cont*(T), is the multiset of the elements appeared in T. The *row sequence* of a tableau is the word obtained by lining up its rows one after another from top to bottom. A tableau on L or on P is called *standard* if

(i) all the rows and columns are non-decreasing,

(ii) no negative letter or place occurs more than once in any row,

(iii) no positive letter or place occurs more than once in any column.

Let

$$T=\begin{matrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{matrix} \quad \text{and} \quad T^{\dagger}=\begin{matrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{matrix}$$

be tableaux on L and on P of the same shape, where u_i and μ_i are words in Mon(L) and Mon(P), respectively. We define the *left quantum bitableau* $(T|T^{\dagger})_l$ and the *right quantum bitableau* $(T|T^{\dagger})_r$,

$$(T|T^{\dagger})_l=\left(\begin{array}{c|c} u_1 & \mu_1 \\ u_2 & \mu_2 \\ \vdots & \vdots \\ u_k & \mu_k \end{array}\right)_l := (-1)^{\Sigma_{i>j}|u_i|\,|\mu_j|}\,(u_1 \mid \mu_1)_l\,(u_2 \mid \mu_2)_l \cdots (u_k \mid \mu_k)_l,$$

$$T|T^{\dagger})_r=\left(\begin{array}{c|c} u_1 & \mu_1 \\ u_2 & \mu_2 \\ \vdots & \vdots \\ u_k & \mu_k \end{array}\right)_r := (-1)^{\Sigma_{i>j}|u_i|\,|\mu_j|}\,(u_1 \mid \mu_1)_r\,(u_2 \mid \mu_2)_r \cdots (u_k \mid \mu_k)_r.$$

In these notations, the Laplace expansions simply become

$$(u \mid \mu v)_{\mathrm{r}} = \sum_{u} \begin{pmatrix} u_{(1)} & \mu \\ u_{(2)} & v \end{pmatrix}_{\mathrm{r}}$$

and

$$(uv \mid \mu)_{\mathrm{l}} = \sum_{\mu} \begin{pmatrix} u & \mu_{(1)} \\ v & \mu_{(2)} \end{pmatrix}_{\mathrm{l}},$$

where the signs in the original formulas (7) and (8) are hidden.

Given a quantum bitableau $(T \mid T^{\dagger})_{\mathrm{r}}$ or $(T \mid T^{\dagger})_{\mathrm{l}}$, its *shape* is the common shape of T and $T^{\dagger}$; its *content*, denoted by *cont*$(T \mid T^{\dagger})$, is the union of *cont*(T) and *cont*$(T^{\dagger})$; it is called *standard* if both T and $T^{\dagger}$ are standard tableaux. We are now ready to state the main theorem of this paper, which is called the quantum straightening formula or the standard basis theorem for $\mathrm{Super}[L \mid P]_q$. Again we assume that $[n]_q \neq 0$ for all $n > 0$.

THEOREM 9 (Quantum Straightening Formula). *Both standard right bitableaux and standard left bitableaux form a linear basis of* $\mathrm{Super}[L \mid P]_q$. *Moreover, if*

$$(T \mid T^{\dagger})_{\mathrm{l}} = \sum a_{SS^{\dagger}} (S \mid S^{\dagger})_{\mathrm{l}}, \qquad a_{SS^{\dagger}} \neq 0,$$

where each $(S \mid S^{\dagger})_{\mathrm{l}}$ *is standard, then* (i) $\mathrm{cont}(S \mid S^{\dagger})_{\mathrm{l}} = \mathrm{cont}(T \mid T^{\dagger})_{\mathrm{l}}$, (ii) *the shape of* $(S \mid S^{\dagger})_{\mathrm{l}}$ *is greater than or equal to the shape of* $(T \mid T^{\dagger})_{\mathrm{l}}$ *in lexicographic order*, (iii) *if* $(S \mid S^{\dagger})$ *and* $(T \mid T^{\dagger})$ *have the same shape, then the row sequence of S followed by the row sequence of* $S^{\dagger}$ *is less than the row sequence of T followed by the row sequence of* $T^{\dagger}$ *in lexicographic order. The same statement is true for right bitableaux.*

We prove the theorem for left bitableaux. The proof consists of two parts:

(I) The set of left standard bitableaux spans $\mathrm{Super}[L \mid P]_q$ as a vector space.

(II) Left standard bitableaux are linearly independent.

Part (I) needs four lemmas below.

The first one serves a similar role to the exchange identity in [GRS, Proposition 10], although it is in a weaker form.

LEMMA 10 (Exchange Identity). *Let u, v, w and μ, ν, ω be non-decreasing words. Then*

$$\sum_{v} \frac{1}{(u)(v_{(1)})(v_{(2)})(w)} \begin{pmatrix} uv_{(1)} & \mu \\ v_{(2)}w & \nu \end{pmatrix}_{\mathrm{l}} = \sum_{\mu,\nu} \frac{1}{(\mu_{(1)})(\mu_{(2)})(\nu_{(1)})(\nu_{(2)})} \begin{pmatrix} u & \mu_{(1)} \\ v & \mu_{(2)}\nu_{(1)} \\ w & \nu_{(2)} \end{pmatrix}_{\mathrm{r}} \tag{13}$$

and

$$\sum_{\nu} \frac{1}{(\mu)(\nu_{(1)})(\nu_{(2)})(\omega)} \begin{pmatrix} u & \mu\nu_{(1)} \\ v & \nu_{(2)}\omega \end{pmatrix}_{\mathrm{r}} = \sum_{u,v} \frac{1}{(u_{(1)})(u_{(2)})(v_{(1)})(v_{(2)})} \begin{pmatrix} u_{(1)} & \mu \\ u_{(2)}v_{(1)} & \nu \\ v_{(2)} & \omega \end{pmatrix}_{\mathrm{l}} \tag{14}$$

where $\Delta u = \sum_u u_{(1)} \otimes u_{(2)}$ and $\Delta\mu = \sum_\mu \mu_{(1)} \otimes \mu_{(2)}$ denote the coproducts defined on Super$[L]_{q^{-1}}$ *and* Super$[P]_{q^{-1}}$, *respectively, such that the components $u_{(1)}$, $u_{(2)}$, $\mu_{(1)}$, $\mu_{(2)}$ are non-decreasing words (with suitable coefficients).*

Remarks. (1) The exchange identity in [GRS, Proposition 10] for the supersymmetric letterplace algebra (where $q = 1$) has the form

$$\sum_{v} \begin{pmatrix} uv_{(1)} & \mu \\ v_{(2)}w & \nu \end{pmatrix} = (-1)^{|u|\,|v|} \sum_{u,\nu} (-1)^{\mathrm{length}(u_{(2)})} \begin{pmatrix} vu_{(1)} & \mu\nu_{(1)} \\ u_{(2)}w & \nu_{(2)} \end{pmatrix}.$$

This is a stronger identity than formula (13), and it is not true in general for the quantum case. This is the reason that in the quantum straightening formula stated in Theorem 9 one cannot use the dominance order for shapes; instead, its linear extension, the lexicographic order for shapes, has to be used.

(2) We have to require in this lemma that $u_{(1)}$, $u_{(2)}$, etc. be non-decreasing words; otherwise, the expressions in (13) and (14) are not well defined.

Proof. We prove (13) here. Its left-hand side equals

$$\sum_{v,\mu,\nu} \frac{1}{(u)(v_{(1)})(v_{(2)})(w)} \begin{pmatrix} u & \mu_{(1)} \\ v_{(1)} & \mu_{(2)} \\ v_{(2)} & \nu_{(1)} \\ w & \nu_{(2)} \end{pmatrix}_{l}$$

$$= \sum_{v,\mu,\nu} \frac{1}{(\mu_{(1)})(\mu_{(2)})(\nu_{(1)})(\nu_{(2)})} \begin{pmatrix} u & \mu_{(1)} \\ v_{(1)} & \mu_{(2)} \\ v_{(2)} & \nu_{(1)} \\ w & \nu_{(2)} \end{pmatrix}_{r}$$

$$= \sum_{\mu,\nu} \frac{1}{(\mu_{(1)})(\mu_{(2)})(\nu_{(1)})(\nu_{(2)})} \begin{pmatrix} u & \mu_{(1)} \\ v & \mu_{(2)}\nu_{(1)} \\ w & \nu_{(2)} \end{pmatrix}_{r}.$$

Remarks. (1) If $L = L^{-}$ and $P = P^{-}$, then we may assume that all words in the exchange identity have no repetitions of negative variables. In this case, formulas (13) and (14) become

$$\sum_{v} \begin{pmatrix} uv_{(1)} & \mu \\ v_{(2)}w & \nu \end{pmatrix} = \sum_{\mu,\nu} \begin{pmatrix} u & \mu_{(1)} \\ v & \mu_{(2)}\nu_{(1)} \\ w & \nu_{(2)} \end{pmatrix}$$

and

$$\sum_{\nu} \begin{pmatrix} u & \mu\nu_{(1)} \\ v & \nu_{(2)}\omega \end{pmatrix} = \sum_{u,v} \begin{pmatrix} u_{(1)} & \mu \\ u_{(2)}v_{(1)} & \nu \\ v_{(2)} & \omega \end{pmatrix}.$$

(2) The identities in the definition of quantum flag scheme, namely, quantum shape-algebra (see [TT, Sect. 3]) can be verified immediately for $f_q(i_1, ..., i_s) := (1 \cdots s \mid i_1 \cdots i_s)$ by using the above two identities. For example, Young symmetry relations (the identity (3.2c) [TT, p. 20]) is equivalent to

$$0 = \sum_{i} \begin{pmatrix} 1 \cdots t & i_{(1)} \\ 1 \cdots s & i_{(2)}j \end{pmatrix},$$

where $1 \leqslant r \leqslant s \leqslant t$, $i = i_1 \cdots i_{t+r}$ and $j = j_1 \cdots j_{s-r}$ are increasing words; and the commutation relation (the identity (3.2d) [TT, p. 20]) is equivalent to

$$\begin{pmatrix} 1 \cdots r & j \\ 1 \cdots s & i \end{pmatrix} = (-q)^{-r(s-r)} \sum_{i} \begin{pmatrix} 1 \cdots s & ji_{(1)} \\ 1 \cdots r & i_{(2)} \end{pmatrix},$$

where $1 \leqslant r \leqslant s$, $i = i_1 \cdots i_s$ and $j = j_1 \cdots j_r$ are increasing words. Both displayed identities are special cases of the quantum exchange identity.

The next three lemmas deal with the commutativity of quantum minors. In general, it is far from true that $(u \mid \mu)(v \mid \nu) = (v \mid \nu)(u \mid \mu)$, where $(u \mid \mu)$ and $(v \mid \nu)$ are either left or right quantum minors. One of the major difficulty of the whole subject is that quantum minors, left or right, have very little commutative relation which can be witnessed in the following lemmas.

LEMMA 11. *Let $a \in L$ and $\alpha \in P$. Suppose that u and μ are words on L and P such that* $\text{length}(u) - \text{length}(\mu) = s \geqslant 0$ *and that every place that occurs in μ is less than α. Then the element $(a \mid \alpha)(u \mid \mu\alpha^s)_l$ in* $\text{Super}[L \mid P]_q$ *is a linear combination of elements of the forms $(v \mid \nu)_l\,(b \mid \beta)$ or $(vb \mid \nu\beta)_l$ with the same content as $(a \mid \alpha)(u \mid \mu\alpha^s)_l$, where $b \in L$, $\beta \in P$, and v and ν are words on L and P with the same length as u.*

Proof. Assume u and μ are non-decreasing words.

Case (i). $s = 0$. Then $\text{length}(u) = \text{length}(\mu)$. We have

$$(au \mid \mu\alpha)_l = \sum_{\mu} \begin{pmatrix} a & \mu_{(1)} \\ u & \mu_{(2)}\alpha \end{pmatrix}_l + (-1)^{|\alpha|\,|\mu|} q^{-\text{length}(\mu)} \begin{pmatrix} a & \alpha \\ u & \mu \end{pmatrix}_l$$

$$= \sum_{\mu} \frac{(u)}{(\mu_{(2)}\alpha)} \begin{pmatrix} a & \mu_{(1)} \\ u & \mu_{(2)}\alpha \end{pmatrix}_r + (-1)^{|\alpha|\,|\mu|} q^{-\text{length}(\mu)} \begin{pmatrix} a & \alpha \\ u & \mu \end{pmatrix}_l.$$

Since α does not occur in μ, $(\mu_{(2)}\alpha) = (\mu_{(2)})$. By formula (14), the first term above equals

$$\sum_{u} \frac{(u)}{(u_{(1)})} \begin{pmatrix} au_{(1)} & \mu \\ u_{(2)} & \alpha \end{pmatrix}_l,$$

which proves case (i).

Case (ii). $s = 1$ and $|\alpha| = 1$. By (13), we have

$$\sum_{u} \frac{1}{(u_{(1)})} \begin{pmatrix} au_{(1)} & \mu\alpha^- \\ u_{(2)} & \alpha^- \end{pmatrix}_l$$

$$= \sum_{\mu} \frac{1}{(\mu_{(2)}\alpha^-)} \begin{pmatrix} a & \mu_{(1)} \\ u & \mu_{(2)}\alpha^- \cdot \alpha^- \end{pmatrix}_r + \frac{(-1)^{|\mu|} q^{-\text{length}(\mu)}}{(\mu)} \begin{pmatrix} a & \alpha^- \\ u & \mu \cdot \alpha^- \end{pmatrix}_r.$$

Note that each term in the first sum above vanishes. Hence the result follows from Theorem 7.

Case (iii). $s \geqslant 1$ and $|\alpha| = 0$. We have

$$(au \mid \mu\alpha^{s+1})_1 = \sum_{\mu} \begin{pmatrix} a & \mu_{(1)} \\ u & \mu_{(2)}\alpha^{s+1} \end{pmatrix}_1 + q^{-\mathrm{length}(\mu)} \begin{pmatrix} a & \alpha \\ u & \mu\alpha^{s} \end{pmatrix}_1$$

$$= \sum_{\mu} \frac{(u)}{(\mu_{(2)}\alpha^{s+1})} \begin{pmatrix} a & \mu_{(1)} \\ u & \mu_{(2)}\alpha^{s+1} \end{pmatrix}_r + q^{-\mathrm{length}(\mu)} \begin{pmatrix} a & \alpha \\ u & \mu\alpha^{s} \end{pmatrix}_1. \tag{15}$$

On the other hand, by (13),

$$\sum_{u} \frac{1}{(u_{(1)})} \begin{pmatrix} au_{(1)} & \mu\alpha^{s} \\ u_{(2)} & \alpha \end{pmatrix}_1$$

$$= \sum_{\mu} \frac{1}{(\mu_{(2)}\alpha^{s})} \begin{pmatrix} a & \mu_{(1)} \\ u & \mu_{(2)}\alpha^{s}\cdot\alpha \end{pmatrix}_r + \frac{q^{-\mathrm{length}(\mu)}}{(\mu\alpha^{s-1})} \begin{pmatrix} a & \alpha \\ u & \mu\alpha^{s-1}\cdot\alpha \end{pmatrix}_r. \tag{16}$$

By eliminating the element

$$\sum_{\mu} \frac{1}{(\mu_{(2)}\alpha^{s})} \begin{pmatrix} a & \mu_{(1)} \\ u & \mu_{(2)}\alpha^{s}\cdot\alpha \end{pmatrix}_r$$

from (15) and (16), the element $(a \mid \alpha)(u \mid \mu\alpha^{s})_1$ can be written as a linear combination of elements of the desired forms. ∎

LEMMA 12. *Let $a \in L$ and $\alpha \in P$. Suppose that u and μ are words on L and P of the same length. Then the element $(a \mid \alpha)(u \mid \mu)_1$ in* $\mathrm{Super}[L \mid P]_q$ *can be written as a linear combination of elements of the forms $(v \mid \nu)_1 (b \mid \beta)$ or $(vb \mid \nu\beta)$ with the same content as $(a \mid \alpha)(u \mid \mu)_1$, where $b \in L$, $\beta \in P$, and v and ν are words on L and P with the same length as u.*

Proof. Assume that u and μ are non-decreasing words. Write $\mu = \nu\alpha^{s}\omega$, $s \geqslant 0$, where each place in ν is less than α and each place in ω is greater than α. We prove the statement by using induction on length(ω). The case when length$(\omega) = 0$ is settled in Lemma 11. If length$(\omega) > 0$, we have three cases.

Case (i). $s = 0$. Then $\mu = \nu\omega$ and α does not appear in μ. We have

$$(au \mid \nu\alpha\omega)_1 = (-1)^{|\alpha|\,|\nu|} q^{-\mathrm{length}(\nu)} \begin{pmatrix} a & \alpha \\ u & \nu\omega \end{pmatrix}_1 + \sum_{\nu} \begin{pmatrix} a & \nu_{(1)} \\ u & \nu_{(2)}\alpha\omega \end{pmatrix}_1$$

$$+ q^{-\mathrm{length}(\nu)-1} \sum_{\omega} (-1)^{|\omega_{(1)}|\,|\nu\alpha|} \begin{pmatrix} a & \omega_{(1)} \\ u & \nu\alpha\omega_{(2)} \end{pmatrix}_1. \tag{17}$$

By induction, the third term on the right-hand side above is a linear combination of the desired elements. On the other hand, by (13),

$$\begin{aligned}
&\sum_u \frac{1}{(u_{(1)})}\begin{pmatrix} au_{(1)} & \vline & v\omega \\ u_{(2)} & \vline & \alpha \end{pmatrix}_1 \\
&\quad = \sum_v \frac{1}{(v_{(2)}\omega)}\begin{pmatrix} a & \vline & v_{(1)} \\ u & \vline & v_{(2)}\omega\cdot\alpha \end{pmatrix}_r \\
&\quad + q^{\operatorname{length}(v)} \sum_\omega \frac{(-1)^{|v|\,|\omega_{(1)}|}}{(v\omega_{(2)})}\begin{pmatrix} a & \vline & \omega_{(1)} \\ u & \vline & v\omega_{(2)}\cdot\alpha \end{pmatrix}_r. \qquad (18)
\end{aligned}$$

By induction, each term on the second sum of the right-hand side above is a linear combination of elements of the desired forms. Furthermore, by eliminating the sum

$$\sum_v \begin{pmatrix} a & \vline & v_{(1)} \\ u & \vline & v_{(2)}\alpha\omega \end{pmatrix}_1 = \sum_v \frac{(u)}{(v_{(2)}\alpha\omega)}\begin{pmatrix} a & \vline & v_{(1)} \\ u & \vline & v_{(2)}\alpha\omega \end{pmatrix}_r$$

from (17) and (18), case (i) is settled.

The discussions for case (ii), $s=1$ and $|\alpha|=1$, and case (iii), $s\geqslant 1$ and $|\alpha|=0$, are similar to their counterparts in the proof of Lemma 11. ∎

Applying Lemma 12, we obtain

LEMMA 13. *Let u, v be words on L and μ, ν be words on P such that* $\operatorname{length}(u)=\operatorname{length}(\mu)$ *and* $\operatorname{length}(v)=\operatorname{length}(\nu)$. *Then the element* $(u\,|\,\mu)_1\,(v\,|\,\nu)_1$ *in* $\operatorname{Super}[L\,|\,P]_q$ *can be written as a linear combination of elements of the forms* $(w\,|\,\omega)_1\, m$, *where* $\operatorname{length}(w)=\operatorname{length}(\omega)\geqslant\operatorname{length}(v)=\operatorname{length}(\nu)$ *and m is a monomial in* $\operatorname{Super}[L\,|\,P]_q$.

We now go back to prove Theorem 9.

Proof of Part (I). Given two tableaux S and T of the same content, we say that S is less than T, denoted by $S<T$, if the shape of S is larger than or equal to the shape of T in lexicographic order, and in the latter case the row sequence of S is smaller than the row sequence of T in lexicographic order. For example,

$$\begin{matrix} 2\,1\,1 \\ 3\,2\,1 \end{matrix} > \begin{matrix} 1\,2\,1 \\ 3\,2\,1 \end{matrix} > \begin{matrix} 2\,1\,1\,3 \\ 2 \end{matrix}$$

Given bitableaux $(T\,|\,T^\dagger)_1$ and $(S\,|\,S^\dagger)_1$ of the same content, we say that $(T\,|\,T^\dagger)_1$ is less than $(S\,|\,S^\dagger)_1$, denoted $(T\,|\,T^\dagger)_1<(S\,|\,S^\dagger)_1$, if $T<S$, or $T=S$ and $T^\dagger<S^\dagger$. The smallest bitableau of a given content is of the form $(u\,|\,\mu)$,

where u and μ are non-decreasing words. Given an arbitrary bitableau $(T|T^\dagger)_l$, suppose any bitableau $(E|E^\dagger)_l < (T|T^\dagger)_l$ is a linear combination of the standard ones satisfying the conditions (i), (ii), (iii) in Theorem 9. We claim that so is $(T|T^\dagger)$. The proof proceeds as follows. First we may assume that the rows of T and $T^\dagger$ are non-decreasing and no negative letter or place appears more than once in a row. Suppose that T is not standard. (Similar discussions will work if $T^\dagger$ is not standard.) Consider a violation pair $b_i > a_i$, or $b_i = a_i$, $|b_i| = |a_i| = 0$, between two consecutive rows of T:

$$\begin{aligned} &b_1 b_2 \cdots b_i \cdots b_s \\ &a_1 a_2 \cdots a_i \cdots a_t, \qquad s \geqslant t. \end{aligned}$$

Let

$$(T|T^\dagger)_l = \pm (T_1 \mid T_1^\dagger)_l \begin{pmatrix} b_1 b_2 \cdots b_i \cdots b_s & \mu \\ a_1 a_2 \cdots a_i \cdots a_t & \nu \end{pmatrix}_l (T_2 \mid T_2^\dagger)_l.$$

Applying the exchange identity (13), by letting $u = b_1 b_2 \cdots b_{i-1}$, $v = a_1 a_2 \cdots a_i b_i \cdots b_s$, $w = a_{i+1} a_{i+2} \cdots a_t$, we get

$$\begin{aligned} &\pm (T_1 \mid T_1^\dagger)_l \sum_\sigma k_\sigma(q) \begin{pmatrix} b_1 \cdots b_{i-1} \sigma(b_i)\, \sigma(b_{i+1}) \cdots \sigma(b_s) & \mu \\ \sigma(a_1) \cdots \sigma(a_{i-1})\, \sigma(a_i)\, a_{i+1} \cdots a_t & \nu \end{pmatrix}_l (T_2 \mid T_2^\dagger)_l \\ &\quad = \pm (T_1 \mid T_1^\dagger)_l \sum_{\mu,\nu} \frac{1}{(\mu_{(1)})(\mu_{(2)})(\nu_{(1)})(\nu_{(2)})} \\ &\qquad \times \left(\begin{array}{c|c} b_1 b_2 \cdots b_{i-1} & \mu_{(1)} \\ a_1 \cdots a_i b_i \cdots b_s & \mu_{(2)} \nu_{(1)} \\ a_{i+1} \cdots a_t & \nu_{(2)} \end{array} \right)_r (T_2 \mid T_2^\dagger)_l, \end{aligned} \tag{19}$$

where each $k_\sigma(q)$ is a suitable coefficient and the sum of the left-hand side ranges over permutations σ of the multiset $\{a_1, ..., a_i, b_i, ..., b_s\}$ such that both $\sigma(a_1) \cdots \sigma(a_i)$ and $\sigma(b_{i+1}) \cdots \sigma(b_s)$ are non-decreasing words. Note that $(T|T^\dagger)$ appears (with the coefficient $k_{\mathrm{id}}(q)$) on the left-hand side once and ony once when $\sigma = \mathrm{id}$. Since $a_1 \leqslant \cdots \leqslant a_i \leqslant b_i \leqslant \cdots \leqslant b_s$, all other terms on the left-hand side have smaller row sequence in lexicographic order. Therefore, they can be expressed as linear combinations of standard bitableaux satisfying the conditions in Theorem 9. Now applying Lemma 13 to the terms on the right-hand side of (19), by letting $u = b_1 b_2 \cdots b_{i-1}$ and $v = a_1 \cdots a_i b_i \cdots b_s$, they can be written as a linear combination of elements of the form

$$(T_1 \mid T_1^\dagger)_l (w \mid \omega)_l\, m,$$

where $\text{length}(w) = \text{length}(\omega) \geqslant \text{length}(v) = s+1$ and m is a monomial in $\text{Super}[L \mid P]_q$. Therefore, by induction the right-hand side of (19) can be written as a linear combination of standard bitableaux satisfying the conditions in Theorem 9, since the shape of $(T_1 \mid T_1^\dagger)_l\,(w \mid \omega)_l\, m$ (when put in a bitableau form) is longer than the shape of $(T \mid T^\dagger)_l$. In conclusion, $(T \mid T^\dagger)_l$ is a linear combination of the standard bitableaux satisfying the conditions (i), (ii), (iii) in Theorem 9.

Proof of Part (II). By Theorem 3 in Section 2, $\text{Super}[L \mid P]_q = \bigoplus_{M,M'} V_{MM'}$ and $V_{MM'}$ is a K-subvector space spanned by the ordered monomials

$$(a_1 \mid \alpha_1)(a_2 \mid \alpha_2) \cdots (a_k \mid \alpha_k) \tag{20}$$

of letter content M and place content M'(M and M' are multisets on L and P of the same size), where (i) $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$, (ii) if $a_i = a_{i+1}$ for some i, then $\alpha_i \leqslant \alpha_{i+1}$, and (iii) if $a_i = a_{i+1}$ and $\alpha_i = \alpha_{i+1}$ for some i, then $|(a \mid \alpha)| = 0$. On the other hand, it was just proved that standard left (resp. right) quantum bitableaux $(S \mid S^\dagger)_l$ (resp. $(S \mid S^\dagger)_r$) with $\text{cont}(S) = M$ and $\text{cont}(S^\dagger) = M'$ form a spanning set of $V_{MM'}$. So in order to prove the linear independence, it is sufficient to know that the cardinality of the set of monomials in (20) is the same as the cardinality of the set of the pairs $(S, S^\dagger)$, where $S, S^\dagger$ are standard tableaux on L, P, respectively, with $\text{cont}(S) = M$ and $\text{cont}(S^\dagger) = M'$. This combinatorial result is proved directly in [BSV], where a supersymmetric Schensted correspondence was constructed. ∎

References

[AR] D. Anick and G.-C. Rota, Higher-order syzygies for the bracket algebra and for the ring of coordinates of the Grassmannian, *Proc. Nat. Acad. Sci. U.S.A* **88** (1991).

[Be] G. Bergman, The Diamond Lemma for ring theory, *Adv. Math.* **29** (1978), 178–218.

[BPT] A. Brini, A. Palareti, and A. Teolis, Gordan–Capelli series in superalgebras, *Proc. Nat. Acad. Sci. U.S.A.* **85** (1988), 1330–1333.

[BR] D. A. Buchbaum and G.-C. Rota, Projective resolutions of Weyl modules, *Proc. Nat. Acad. Sci. U.S.A.*, to appear.

[BSV] F. Bonetti, D. Senato, and A. Venezia, La corrispondenza di Robinson–Schensted per la fourfold algebra, *Boll. Un. Mat. Ital. B (7)* **2** (1988), 541–554.

[BT] A. Brini and A. Teolis, Capelli bitableaux and Z-forms of general linear Lie superalgebras, *Proc. Nat. Acad. Sci. U.S.A.* **87** (1990), 56–60.

[DKR] J. Désarménien, J. P. S. Kung, and G.-C. Rota, Invariant theory, young bitableaux, and combinatorics, *Adv. Math.* **27** (1978), 63–92.

[DRS] P. Dubilet, G.-C. Rota, and J. A. Stein, On the foundations of combinatorial theory. IX. Combinatorial method in invariant theory, *Stud. Appl. Math.* **53** (1976), 185–216.

[GRS] F. D. GROSSHANS, G.-C. ROTA, AND J. A. STEIN, "Invariant Theory and Superalgebras," Regional Conference Series in Mathematics, No. 69, 1985.

[HRS] R. HUANG, G.-C. ROTA, AND J. A. STEIN, "Supersymmetric algebra, supersymmetric space, and invariant theory, *Ann. Scuola Norm. Sup.* (volume dedicated to L. Radicati) (1989).

[Hu] R. HUANG, Invariants of sets of linear varieties, *Proc. Nat. Acad. Sci. U.S.A.* **88** (1990), 4557–4560.

[KR] J. P. S. KUNG AND G.-C. ROTA, The invariant theory of binary forms, *Bull. Amer. Math. Soc.* **10** (1984), 27–85.

[Ma1] YU. MANIN, "Quantum Groups and Non-commutative geometry," Les publications du Centre de Recherches Mathematiques, Université de Montreal, 1988.

[Ma2] YU. MANIN, "Topics in Noncommutative Geometry," Princeton Univ. Press, Princeton, NJ, 1991.

[NYM] M. NOUMI, H. YAMADA, AND K. MIMACHI, Finite dimensional representations of the quantum group $GL_q(n;\mathbb{C})$ and the zonal spherical functions on $U_q(n-1)\backslash U_q(n)$, *Japanese J. of Math.* **19** (1993), 31–80.

[PW] B. PARSHALL, AND J. P. WANG, "Quantum general linear groups," Memoirs of the Amer. Math. Soc., Nomber 439, 1991.

[RS] G.-C. ROTA AND J. A. STEIN, Applications of Cayley algebras, *in* "Colloquio Internazionale sulle Teorie Comninatoriee," Tomo II, pp. 71–97, Accad. Naz. Lincei, Rome, (1976).

[St] R. STANLEY, "Enumerative Combinatorics," Vol. I, Wadsworth and Brooks/Cole Mathematics Series, 1986.

[TT] E. TAFT AND J. TOWBER, Quantum deformation of flag schemes and Grassmann schemes. I. A q-deformation of the shape-algebra for $GL(n)$, *J. Algebra* **142**, 1–36 (1991).

[Wh] N. L. WHITE, Multilinear Cayley factorization, *J. Symbolic. Comput.*, to appear.

[Wo] S. L. WORONOWICZ, Tannaka–Krein duality for compact matrix pseudogroups, Twisted $SU(N)$ groups, *Invent. Math.* **93** (1988), 35–76.

[WW] N. L. WHITE AND W. WHITELY, The algebraic geometry of stresses in frameworks, *SIAM J. Algebraic Discrete Methods* **4** (1983), 481–511.