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On the complexity of a special basis problem in LP

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Abstract

In a linear program (LP) in standard form, we show that the problem of finding a cheapest feasible basic vector among those containing a specified variable as a basic variable, is an NP-hard problem.

Keywords. Linear program, feasible basic vector, NP-hard problem.

1. Introduction

We consider the linear program (LP)

$$\begin{aligned} \text{minimize} \quad & z(x) = cx, \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0 \end{aligned} \tag{1}$$

where all the data are integer and A is an $m \times n$ matrix of rank m. A feasible basic vector for (1) is a vector of m variables among $x_1, \ldots, x_n$, called *basic variables*, associated with a linearly independent set of columns in (1), the basic solution corresponding to which is feasible. See [3] for definitions of LP terminology. Define the cost of a feasible basic vector to be the cost of the associated basic feasible solution. The problem of finding the cheapest feasible basic vector for (1) containing a specified subset of variables as basic variables has been studied in [2] for its relationship with the travelling salesman problem, and thereby shown to be NP-hard. Here we will investigate the following special case of this problem.

Problem 1.1. Find a cheapest feasible basic vector for (1) among those containing one specified variable as a basic variable.

In this paper we show that Problem 1.1 is NP-hard.

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2. The equal sums partition problem

The input to this problem is n positive integers $d_1, \dots, d_n$, whose sum is w, even. It is required to find a subset $S_1 \subset \{1, \dots, n\}$ such that S_1 and its complement S_2 satisfy $\sum_{j \in S_1} d_j = \sum_{j \in S_2} d_j = w/2$. We will consider the following slightly modified version.

Problem 2.1. Find a subset $S_1 \subset \{1, \dots, n\}$ satisfying $-2 + (w/2) \le \sum_{j \in S_1} d_j \le 2 + (w/2)$.

Thus, if S_1 is a solution to Problem 2.1 and S_2 its complement, then $|\sum_{j \in S_1} d_j - \sum_{j \in S_2} d_j| \le 4$ (it is actually 0 or 2, or 4). It is well known that Problem 2.1 is NP-hard [1]. We will now show that Problem 2.1 can be formulated as a special case of Problem 1.1 through a balanced transportation model. We will assume that all of $d_1, \dots, d_n$ are ≥ 2.

3. The transportation model

We will construct a $3 \times (n+1)$ balanced transportation model corresponding to Problem 2.1, with the following data

	Cell cost coefficients					
Col	1	2	...	n	$n+1$	Supply
Row 1					$-\alpha$	$w/2 + 1$
Row 2						$w/2 + 1$
Row 3						1
Demand	d_1	d_2	...	d_n	3	

(2)

A similar model has been used in [4] by Partovi to show the NP-hardness of another transportation related problem. Here we also have an objective function which is required to be minimized. The cost coefficient is zero in all the cells except cell $(1, n+1)$ where it is $-\alpha$, α being a positive integer. Clearly this is a special case of (1). The variable x_{ij} in this model is associated with cell (i, j), $i = 1$ to 3, $j = 1$ to $n+1$, so here, instead of basic vectors of variables, we will talk about the corresponding basic sets of cells. We have the following facts on (2) from well-known results on the transportation problem [3].

(i) Every basic set for (2) consists of $n+3$ basic cells.

(ii) If β is a feasible basic set for (2) in which all the cells in column $n+1$ are basic cells, then β contains exactly one cell among $(1, j)$, $(2, j)$ as a basic cell for each

$j = 1$ to n [4]. In this case let $S_r(\beta) = \{j: 1 \le j \le n$, and (r, j) is a basic cell in $\beta\}$, $r = 1, 2$. Then $(S_1(\beta), S_2(\beta))$ is a partition of $\{1, \ldots, n\}$. Let $(\bar{x}) = (\bar{x}_{ij})$ be the basic feasible solution (BFS) of (2) associated with β. There are three subcases to consider here.

Subcase 1: $\bar{x}_{r,n+1} = 1$ *for* $r = 1, 2, 3$. In this subcase the cost of β is $-\alpha < 0$. $\sum_{j \in S_r(\beta)} d_j = w/2$ for $r = 1, 2$, and hence $S_1(\beta)$ is a solution for Problem 2.1.

Subcase 2: $\bar{x}_{1,n+1} = 2$, $\bar{x}_{2,n+1} = 0$, *and* $\bar{x}_{3,n+1} = 1$. In this subcase the cost of β is $-2\alpha < 0$. $\sum_{j \in S_1(\beta)} d_j = w/2 - 1$, and hence $S_1(\beta)$ is a solution for Problem 2.1.

Subcase 3: $\bar{x}_{1,n+1} = 0$, $\bar{x}_{2,n+1} = 2$, *and* $\bar{x}_{3,n+1} = 1$. In this subcase construct a new feasible basic set of cells, β_1, for (2) from β by the following rules.

(a) For each $j = 1$ to n such that $(2, j)$ is a basic cell in β, make $(1, j)$ a basic cell in β_1.

(b) For each $j = 1$ to n such that $(1, j)$ is a basic cell in β, make $(2, j)$ a basic cell in β_1.

(c) Leave $(r, n + 1)$ as a basic cell in β_1 for $r = 1, 2, 3$.

By symmetry, verify that β_1 is a feasible basic set for (2) which is exactly of the type discussed in Subcase 2 above.

These subcases cover all possibilities in this case. From this we conclude that if there is a feasible basic set of cells containing all cells in column $n + 1$ of the array in (2) as basic cells, then there is a feasible basic set of negative cost with the same property, and an $S_1 \subset \{1, \ldots, n\}$ satisfying $\sum_{j \in S_1(\beta)} d_j =$ either $w/2$ or $w/2 - 1$ (hence S_1 is a solution for Problem 2.1).

Conversely if (S_1, S_2) is a partition of $\{1, \ldots, n\}$ satisfying $\sum_{j \in S_1} d_j =$ either $w/2$ or $w/2 - 1$, define $\beta = \{(1, j): j \in S_1\} \cup \{(2, j): j \in S_2\} \cup \{(1, n + 1), (2, n + 1), (3, n + 1)\}$. Then β is a feasible basic set for (2) with negative cost.

Now we define the special case of Problem 1.1 corresponding to (2).

Problem 3.1. Find the cheapest feasible basic set of cells for (2) among those containing $(2, n + 1)$ as a basic cell.

Theorem 3.2. *There exists a subset $S_1 \subset \{1, \ldots, n\}$ which is a solution for Problem* 2.1 *iff the objective value associated with an optimum feasible basic set for Problem* 3.1 *is* < 0.

Proof. From (ii) above, we already know that there exists an $S_1 \subset \{1, \ldots, n\}$ satisfying $\sum_{j \in S_1} d_j =$ either $w/2$ or $w/2 - 1$ iff there exists a feasible basic set for (2) containing all the cells in column $n + 1$ as basic cells, and the cost of such a basic set is < 0.

Let β_1 be a feasible basic set of cells for (2) containing $(2, n + 1)$ as a basic cell, with negative cost. Since $(1, n + 1)$ is the only cell in (2) associated with a nonzero cost coefficient, this implies that $(1, n + 1)$ must also be a basic cell in β_1. There must be at least one basic cell in row 3. If $(3, n + 1)$ is also a basic cell in β_1, this case is covered above, so we assume that $(3, n + 1)$ is not a basic cell in β_1. Let $(3, p)$ be a basic cell in β_1, where $1 \le p \le n$. By assumption, each of $d_j \ge 2$, $j = 1$ to n; so for feasibility there must be at least one basic cell among $(1, j)$ and $(2, j)$ for each $j = 1$ to n. $(1, n + 1)$, $(2, n + 1)$, $(3, p)$ are already basic cells in β_1 and there are a total of only $n + 3$ basic cells. These facts imply that for each $1 \le j \le n$, exactly one of $(1,j)$ or $(2,j)$ is a basic cell in β_1. Define for $r = 1, 2$,

$$S_r(\beta_1) = \{j\colon (r,j) \text{ is a basic cell in } \beta_1\}.$$

So $S_1(\beta_1)$ and $S_2(\beta_1)$ forms a partition of $\{1,\dots,n\}$. Let $\bar{x} = (\bar{x}_{ij})$ be the BFS of (2) associated with β_1. Since the cost of β_1 is < 0, $\bar{x}_{1,n+1} > 0$, so its value is either 1 or 2 or 3. So $\bar{x}_{2,n+1}$ is either 2 or 1 or 0. $\bar{x}_{3,p} = 1$. So, for $r = 1, 2$, $j \in S_r(\beta_1)\backslash\{p\}$, $\bar{x}_{r,j} = d_j$. And if (t, p) is the basic cell in β_1 among $(1, p)$, $(2, p)$, then $\bar{x}_{t,p} = d_p - 1$. Therefore for $t = 1, 2$, we have

$$\sum_{j=1}^{n} \bar{x}_{tj} = \frac{w}{2} + 1 - \bar{x}_{t,n+1} = \begin{cases} \sum_{j \in S_t(\beta_1)} d_j, & \text{if } p \notin S_t(\beta_1), \\ -1 + \sum_{j \in S_t(\beta_1)} d_j, & \text{if } p \in S_t(\beta_1). \end{cases}$$

All the possibilities in this case are summarized in Table 1. Hence $|\sum_{j \in S_1(\beta_1)} d_j - \sum_{j \in S_2(\beta_1)} d_j| \le 4$ always, and therefore $S_1(\beta_1)$ is a solution for Problem 2.1.

To prove the converse, let $S_1 \subset \{1,\dots,n\}$ be a solution for Problem 2.1. If $\sum_{j \in S_1} d_j = w/2$, this case is covered by (ii) as discussed above.

If $\sum_{j \in S_1} d_j = w/2 - 1$, let β_2 be the basic set $\{(1,j)\colon j \in S_1\} \cup \{(2,j)\colon j \notin S_1\} \cup \{(2, p), (1, n + 1), (2, n + 1)\}$ where p is any element in the complement of S_1. If $\sum_{j \in S_1} d_j = w/2 + 1$, replace S_1 by its complement and construct β_2 exactly in the same way. It can be verified that β_2 is a feasible basic set for (2), and in the BFS, $\hat{x} = (\hat{x}_{ij})$ associated with it $\hat{x}_{1,n+1} = 2$ and hence the cost of β_2 is $-2\alpha < 0$.

If $\sum_{j \in S_1} d_j = w/2 - 2$, let β_3 be the basic set $\{(1,j)\colon j \in S_1\} \cup \{(2,j)\colon j \notin S_1\} \cup \{(2, p), (1, n + 1), (2, n + 1)\}$ where p is any element in the complement of S_1. If $\sum_{j \in S_1} d_j = w/2 + 2$, replace S_1 by its complement, and construct β_3 exactly the same way. It can be verified that β_3 is feasible to (2) and in the BFS, $\tilde{x} = (\tilde{x}_{ij})$ associated with it $\tilde{x}_{1,n+1} = 3$ and hence the cost of $\beta_3 = -3\alpha < 0$. □

Since Problem 2.1 is NP-hard, by Theorem 3.2 it follows that Problem 3.1 is NP-hard. Hence Problem 1.1, of which Problem 3.1 is a special case, is also NP-hard.

Table 1

Values of $(\bar{x}_{1,n+1}, \bar{x}_{2,n+1})$	Location of p	Summary of position
(3, 0)	$p \in S_1(\beta_1)$	$\sum_{j \in S_2(\beta_1)} d_j = w/2 + 1$, $\sum_{j \in S_1(\beta_1)} d_j = w/2 - 1$
	$p \in S_2(\beta_1)$	$\sum_{j \in S_1(\beta_1)} d_j = w/2 - 2$, $\sum_{j \in S_2(\beta_1)} d_j = w/2 + 2$
(2, 1)	$p \in S_1(\beta_1)$	$\sum_{j \in S_2(\beta_1)} d_j = w/2$, $\sum_{j \in S_1(\beta_1)} d_j = w/2$
	$p \in S_2(\beta_1)$	$\sum_{j \in S_1(\beta_1)} d_j = w/2 - 1$, $\sum_{j \in S_2(\beta_1)} d_j = w/2 + 1$
(1, 2)	$p \in S_1(\beta_1)$	$\sum_{j \in S_2(\beta_1)} d_j = w/2 - 1$, $\sum_{j \in S_1(\beta_1)} d_j = w/2 + 1$
	$p \in S_2(\beta_1)$	$\sum_{j \in S_1(\beta_1)} d_j = w/2$, $\sum_{j \in S_2(\beta_1)} d_j = w/2$

Clearly, the decision problem version of Problem 1.1 or 3.1 is also in NP, and hence it is NP-complete.

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