Outer Automorphisms of Upper Triangular Matrices

J. S. Maginnis

University of Michigan, Ann Arbor, Michigan 48109

Communicated by Walter Feit

Received January 3, 1990

The outer automorphism group of the upper triangular matrices over the field of two elements is calculated. A. J. Weir (Proc. Amer. Math. Soc. 6 (1955), 454-464) performed a similar calculation for fields of odd characteristic, and we borrow the term extremal automorphism from his work. The results have implications in the study of stable splittings: the classifying space of $U_n$ has three dominant summands when $n = 4$ and only one dominant summand when $n \geq 5$, in the sense of G. Nishida (Stable homotopy type of classifying spaces of finite groups, preprint (1986)). © 1993 Academic Press, Inc.

Let $U_n$ denote the subgroup of upper triangular matrices in $GL_n(\mathbb{F}_2)$. The outer automorphism group of $U_n$ is generated by the obvious symmetry, perhaps called a flip or an anti-transpose, the central automorphisms and the extremal automorphisms. The central automorphisms lie in the kernel of the map $\text{Out}(U_n) \to \text{Out}(U_n/\text{center})$. The extremal automorphisms are described later. The term is borrowed from the work of A. J. Weir [3].

G. Nishida [2] has shown that the idempotents of the semisimple quotient of the group ring $\mathbb{F}_2[\text{Out}(U_n)]$ lift to idempotents in the ring of 2-local stable self-maps of the classifying space $BU_n$, and correspond to dominant summands, which are those not detected on any proper subgroups. This yields summands of the cohomology ring $H^*(U_n; \mathbb{F}_2)$ as a module over the Steenrod algebra. The results of this paper imply that $BU_4$ has three dominant summands (although two are isomorphic) and $BU_n$ has only one dominant summand for $n \geq 5$.

Explicitly, we have:

**Theorem.** The outer automorphism groups of $U_n$ are

1. $\text{Out}(U_3) = \text{Out}(D_8) \cong \mathbb{Z}/2$
2. $\text{Out}(U_4) \cong S_3 \times \mathbb{Z}/2$
3. $\text{Out}(U_n) \cong (\mathbb{Z}/2)^{n-1} \times \mathbb{Z}/2$ for $n \geq 5$.

**Remark.** In part (3), the notation signifies a semi-direct product.

267

0021-8693/93 $5.00

Copyright © 1993 by Academic Press, Inc.
All rights of reproduction in any form reserved.
The split $\mathbb{Z}/2$ is the anti-transpose, and the normal subgroup $(\mathbb{Z}/2)^{n-1}$ is generated by $n-3$ central automorphisms and two extremal automorphisms.

**Corollary.** The semisimple quotient of $\mathbb{F}_2[\text{Out}(U_n)]$ is trivial unless $n = 4$, in which case the quotient is $\mathbb{Z}/2 \times M_2(\mathbb{F}_2)$. Thus $BU_4$ has three dominant summands, and the two corresponding to the Steinberg idempotents of $M_2(\mathbb{F}_2)$ are isomorphic.

**Proof of the Theorem.** $U_n$ has $n-1$ generators, the "off diagonal" matrices $I_n + e_{i,i+1}$. The automorphisms are determined by the action on these generators. There is an important automorphism $\sigma: U_n \to U_n$ which is a flip or anti-transpose $I_n + e_{i,i+1} \to I_n + e_{n-i,n-1-i}$. This is the only non-trivial element of $\text{Out}(D_8)$, and is a split quotient of every $\text{Out}(U_n)$.

The center of $U_n$ is a single copy of $\mathbb{Z}/2$, which must be fixed by all automorphisms. Thus we have a map $\text{Out}(U_n) \to \text{Out}(U_n/\text{center})$. Then the center of $U_n/\text{center}$ must be preserved as a subgroup, and so on, and we obtain a map

$$\text{Out}(U_n) \to \text{Out}((\mathbb{Z}/2)^{n-1}) \cong GL_{n-1}(\mathbb{F}_2).$$

For $n \geq 5$, the image of this map is just a $\mathbb{Z}/2$ generated by the flip $\sigma$, but for $n = 4$ the map $\text{Out}(U_4) \to GL_4(\mathbb{F}_2)$ has image isomorphic to $GL_2(\mathbb{F}_2) \cong \Sigma_3$. In $U_4$, the normalizer of the $\mathbb{Z}/2$ subgroup generated by either $I_4 + e_{1,2}$ or $I_4 + e_{3,4}$ is isomorphic to $D_8 \times \mathbb{Z}/2$, but the normalizer of the subgroup $\langle I_4 + e_{2,3} \rangle$ is $(\mathbb{Z}/2)^4$. Perhaps the best interpretation of this image $GL_2(\mathbb{F}_2)$ is as linear maps of the $\mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup of $H^2((\mathbb{Z}/2)^3; \mathbb{F}_2)$ generated by the two $K$-invariants for the central extension:

$$1 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to U_4/\text{center} \to \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1.$$

The elements in the kernel of the map $\text{Out}(U_n) \to \text{Out}(U_n/\text{center})$ are referred to as central automorphisms, which constitute an elementary abelian subgroup $(\mathbb{Z}/2)^{n-3}$. The generators are automorphisms $\varphi_i$ that twist with the central element

$$\varphi_i(I_n + e_{i,i+1}) = I_n + e_{i,i+1} + e_{n-1,n}$$

and

$$\varphi_i(I_n + e_{j,j+1}) = I_n + e_{j,j+1} \quad \text{for} \quad i \neq j.$$ 

Note that for $i = 1$ or $i = n - 1$ the automorphisms $\varphi_1$ and $\varphi_{n-1}$ are inner; for example $\varphi_1$ is conjugation by $I_n + e_{2,n}$.

Let $\varphi: U_n \to U_n$ be any automorphism, and let us consider the image
of the element \( I_n + e_{1,2} \). Note that the commutator \([I_n + e_{1,2}, I_n + e_{2,n}] = I_n + e_{1,n}\) is nontrivial, and also the \((\mathbb{Z}/2)^3\) subgroup \(\langle I_n + e_{1,n-1}, I_n + e_{2,n}, I_n + e_{1,n} \rangle\) is preserved. This implies that either \(\varphi(I_n + e_{1,2}) = (I_n + e_{1,2}) \cdot e\) or that \(\varphi(I_n + e_{1,2}) = (I_n + e_{n-1,n}) \cdot e'\). In the latter situation, \(\sigma \circ \varphi(I_n + e_{1,2}) = (I_n + e_{1,2}) \cdot e\). By composing with an inner automorphism we can assume that \(e \in U_{n-1}\), that is, the expression or “word” \(e\) contains no elements or “letters” of the first row of \(U_n\). In fact, since \(I_n + e_{1,2}\) is of order 2, it is clear that \(e\) contains no elements of the second row, so \(e \in U_{n-2}\).

Assume momentarily that \(e \in U_{n-3}\) (no elements from the third row). Note that the subgroup consisting of the first row of \(U_n\) has several properties: it is elementary abelian, normal, and equals the closure under inner automorphisms of the first element \(I_n + e_{1,2}\). These properties will be preserved by any automorphism \(\varphi\). Consider the conjugation of \((I_n + e_{1,2}) \cdot e\) given by

\[
(I_n + e_{2,3})(I_n + e_{1,2})(e)(I_n + e_{2,3}) = (I_n + e_{1,3})(I_n + e_{1,2}) \cdot e.
\]

Then the image of the first row subgroup contains the elements \(I_n + e_{1,3}\), and thus by continued conjugation, the rest of the row \(I_n + e_{1,k}\) with \(k > 2\). This image should be an abelian group, so \(e\) must commute with \(I_n + e_{1,k}\) for \(k > 2\). But then \(e = 1\) and so \(\varphi\) (or perhaps \(f_{\text{inner}} \circ \sigma \circ \varphi\)) fixes the matrix \(I_n + e_{1,2}\).

Now assume that \(\varphi(I_n + e_{1,2}) = (I_n + e_{1,2}) \cdot e\) with \(e \in U_{n-2}\), and let \(I_n + e_{3,k}\) be the first element of the third row of \(U_n\) that appears in the expression for \(e\). Write \(e = (I_n + e_{3,k}) \cdot e'\), and again consider the conjugation by \(I_n + e_{2,3}\):

\[
(I_n + e_{2,3})(I_n + e_{1,2}) e(I_n + e_{2,3}) = (I_n + e_{1,2})(I_n + e_{1,3})(I_n + e_{2,k})(I_n + e_{3,k}) (I_n + e_{2,3}) e'(I_n + e_{2,3})
\]

\[
= (I_n + e_{1,2})(I_n + e_{3,k}) e'(I_n + e_{1,3})(I_n + e_{2,k})(I_n + e_{1,k}) \cdot \lambda,
\]

where \(\lambda\) is an element of the abelian subgroup generated by \(I_n + e_{i,j}\) and \(I_n + e_{2,3}\) for \(j > k\).

Now conjugate \(\varphi(I_n + e_{1,2})\) by \(I_n + e_{k,n}\), if \(k < n\)

\[
(I_n + e_{k,n})(I_n + e_{1,2})(I_n + e_{3,k}) e'(I_n + e_{k,n}) = (I_n + e_{1,2})(I_n + e_{3,k})(I_n + e_{3,n}) e' \lambda',
\]

where \(\lambda'\) is in the subgroup of the last column of \(U_n\) generated by \(I_n + e_{j,n}\) for \(4 \leq j \leq k - 1\). Thus the image under \(\varphi\) of the first row subgroup contains both \((I_n + e_{1,3})(I_n + e_{2,k})(I_n + e_{1,k}) \lambda\) and \((I_n + e_{3,n}) \lambda'\). But the
commutator of these two elements is \( I_n + e_{1,n} \), which contradicts the fact that this subgroup should be abelian.

Unless \( k = n \), we see that \( e = 1 \) and \( I_n + e_{1,2} \) is fixed by \( \varphi \). When \( k = n \), we find a type of automorphism referred to as an extremal automorphism (for a similar definition, see [3]):

\[
\varphi_e(I_n + e_{1,2}) = (I_n + e_{1,2})(I_n + e_{3,n})
\]

\[
\varphi_e(I_n + e_{1,3}) = (I_n + e_{1,3})(I_n + e_{2,n})(I_n + e_{1,n})
\]

\[
\varphi_e(I_n + e_{i,j}) = I_n + e_{i,j} \quad \text{otherwise},
\]

There are only two extremal automorphisms, the \( \varphi_e \) above and \( \sigma \varphi_e \sigma \). These generate a \( \mathbb{Z}/2 \times \mathbb{Z}/2 \) subgroup of \( \text{Out}(U_n) \).

So, if necessary by composing with an extremal automorphism, we can assume our transformed automorphism \( \varphi \) fixes \( I_n + e_{1,2} \). Then the first row is preserved as a group, and by composing with further inner automorphisms, we can obtain a map \( \varphi \) fixing the top row element-wise.

The set of those automorphisms which fix both the top row and the quotient \( U_{n-1} \) can be shown to be the cohomology group \( H^1(U_{n-1}; (\mathbb{Z}/2)^{n-1}) \) with twisted coefficients [1]. This yields \( (\mathbb{Z}/2)^{n-2} \subset \text{Out}(U_n) \) generated by the central automorphisms and the extremal automorphisms \( \sigma \varphi_e \sigma \).

Now I claim that any automorphism of \( U_n \) which acts as the identify on the first row will also act as the identify on the quotient \( U_{n-1} \). We may inductively assume that \( \text{Out}(U_{n-1}) \cong (\mathbb{Z}/2)^{n-2} \times_{T} \mathbb{Z}/2 \), generated by \( n-4 \) central automorphisms, two extremal automorphisms, and the flip \( \sigma \). Showing that none of these extends to an automorphism of \( U_n \) fixing the top row element-wise follows from simple commutativity relations with \( I_n + e_{1,2} \) and \( I_n + e_{1,n-2} \). This completes the proof.

REFERENCES