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A Connection between Alternating Sign Matrices and Totally Symmetric Self-Complementary Plane Partitions

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We give a lattice path interpretation for totally symmetric self-complementary plane partitions. This is a first step in solving the long standing problem of enumerating such plane partitions. Another outstanding problem in enumerative combinatorics is the search for a bijection between alternating sign matrices and totally symmetric self-complementary plane partitions. From the lattice path interpretation, we discover a new statistic on totally symmetric self-complementary plane partitions which should correspond to the position of the 1 in the top row of an alternating sign matrix under such a bijection.

1. INTRODUCTORY DEFINITIONS

This paper deals with three different classes of combinatorial objects. We define them in this section. The remainder of the paper gives connections between these apparently unrelated things.

DEFINITION. A *plane partition* π is an array $\pi = [\pi_{i,j}]$, $i, j \geqslant 1$ of nonnegative integers $\pi_{i,j}$ with finite sum $|\pi| = \sum \pi_{i,j}$, which is weakly decreasing in both its rows and columns.

EXAMPLE 1.

$$\begin{matrix} 5 & 4 & 4 & 3 & 2 \\ 5 & 4 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & \\ 2 & 1 & 1 & & \end{matrix}$$

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The zero entries are not usually written down. The following definitions give some special classes of plane partitions.

DEFINITION. (a) A plane partition is *symmetric* if $\pi_{i,j} = \pi_{j,i}$ for all i, j.

(b) A plane partition is *cyclically symmetric* if the ith row of π, when regarded as an integer partition, is the conjugate of the ith column, for all i.

(c) A plane partition is *totally symmetric* if it is symmetric and cyclically symmetric.

(d) A plane partition is *(r, s, t)-self-complementary* if π has no more than r rows, no more than s columns, $\pi_{i,j} \leqslant t$, and

$$\pi_{i,j} + \pi_{r-i+1,\, s-j+1} = t$$

for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$.

EXAMPLE 2. The following four matrices illustrate the above definitions.

$$(1)\quad \begin{matrix} 4 & 4 & 3 & 2 \\ 4 & 4 & 1 & \\ 3 & 1 & 1 & \\ 2 & & & \end{matrix} \qquad (2)\quad \begin{matrix} 5 & 3 & 2 & 2 & 1 \\ 4 & 3 & 2 & & \\ 2 & 2 & & & \\ 1 & 1 & & & \\ 1 & & & & \end{matrix}$$

$$(3)\quad \begin{matrix} 4 & 3 & 3 & 1 \\ 3 & 3 & 2 & \\ 3 & 2 & 1 & \\ 1 & & & \end{matrix} \qquad (4)\quad \begin{matrix} 4 & 3 & 2 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 3 & 2 & 2 & 1 & 0 \end{matrix}$$

(1) is symmetric but does not have any of the other properties. (2) is cyclically symmetric but not totally symmetric. (3) is totally symmetric. Note that in a totally symmetric matrix, each row is in fact self-conjugate. (4) is (3, 5, 4)-self-conjugate but does not have any of the other properties. In self-complementary plane partitions, we write down zero entries that occur within the $r \times s$ rectangle.

The first class of combinatorial objects that we are interested in is totally symmetric $(2n, 2n, 2n)$-self-complementary plane partitions. To save writing the $2n$ time and time again, we call these totally symmetric self-complementary plane partitions of order n.

EXAMPLE 3. There are two totally symmetric self-complementary plane partitions of order 2:

$$\begin{array}{cccc} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{array} \qquad \begin{array}{cccc} 4 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}$$

The seven totally symmetric self-complementary plane partitions of order 3 are listed in the Appendix.

THEOREM 1.1. *The number of totally symmetric self-complementary plane partitions of order n is equal to A_n, where*

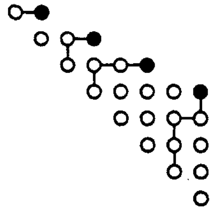
$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

This was first conjuctured by W. Mills, D. Robbins, and H. Rumsey [MRR2]. The proof of this required the work of three different people. The first portion is done in this paper. Theorem 1.2 converts counting totally symmetric self-complementary plane partitions into counting a certain class of non-intersecting lattice paths. The second step, which is due to J. Stembridge [Ste], converts counting the lattice paths into evaluating a determinant. The last step is due G. Andrews [A], who evaluated the determinant.

The second class of combinatorial objects is the above mentioned configuration of non-intersecting lattice paths. A *lattice path* is a sequence $\{(u_j, v_j) \in Z^2\colon 1 \leqslant j \leqslant t\}$ such that $(u_{j+1}, v_{j+1}) - (u_j, v_j)$ equals $(0, 1)$ or $(-1, 0)$. Assigning Z^2 matrix-style coordinates, a lattice path is a sequence of steps down 1 or left 1. The lattice path is said to *start* at (u_1, v_1) and *end* at (u_t, v_t). We refer to $((u_j, v_j), (u_{j+1}, v_{j+1}))$ as a *move*. If the difference is $(0, 1)$, it is a *downward* move. If the difference is $(-1, 0)$, it is a *leftward* move.

DEFINITION. A *lattice pattern* on n points is a set of n non-intersecting lattice paths, such that the ith path starts at $(2i, i)$ and ends somewhere on the diagonal $\{(j, j)\colon j \in Z\}$.

EXAMPLE 4. This is a lattice pattern on 4 points.



Since they play no role, the coordinate axes are not shown. The main result of this paper is the following theorem. It is proven in Sections 2–4.

THEOREM 1.2. *The number of totally symmetric self-complementary plane partitions of order n equals the number of lattice patterns on $n-1$ points.*

The final combinatorial object of interest is defined next.

DEFINITION. An $n \times n$ *alternating sign matrix* is an $n \times n$ $(1, -1, 0)$-matrix such that

(i) all the row and column sums are 1, and
(ii) the non-zero entries in each row and column alternate in sign.

These have been studied by several people [MRR1, MR, Sta1].

EXAMPLE 5. The 7 3×3 alternating sign matrices are

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

Mills, Robbins, and Rumsey [MRR1] have made the following

Conjecture. The number of $n \times n$ alternating sign matrices equals A_n.

There is no known bijection between $n \times n$ alternating sign matrices and totally symmetric self-complementary plane partitions of order n or lattice patterns on $n-1$ points. In Section 7, we give a numerical connection between alternating sign matrices and lattice patterns.

2. BASIC RESULTS

The self-complementary and totally symmetric properties, when combined, force some of the entries in the plane partition to have certain values. In this section, we describe these necessary conditions.

PROPOSITION 2.1. *Let $\pi = [\pi_{i,j}]$ be a totally symmetric self-complementary plane partition of order n. Then*

(i) $\pi_{i,\,2n+1-i} = n$ *for all* $i = 1, 2, \ldots, 2n$, *and*

(ii) $\pi_{1,j} = \pi_{j,1} = 2n$ *for all* $j = 1, \ldots, n$, *and*

(iii) $\pi_{2n,j} = \pi_{j,\,2n} = 0$ *for all* $j = n+1, \ldots, 2n$.

Proof. (i) Since π is symmetric, $\pi_{i,\,2n+1-i} = \pi_{2n+1-i,\,i}$. Since π is $(2n, 2n, 2n)$-self-complementary, $\pi_{i,2n+1-i} + \pi_{2n+1-i,i} = 2n$. Hence $\pi_{i,2n+1-i} = n$.

(ii) A special case of (i) is $\pi_{1,\,2n} = n$. As was noted in Example 2, the first row of π, i.e., $(\pi_{1,1}, \pi_{1,2}, \ldots, \pi_{1,\,2n})$, is a self-conjugate partition. Since the $2n$th value of the partition is n, the first n values of the partition must be greater than or equal to $2n$. Since the $(2n+1)$th value of the partition is 0, all the parts must be less than or equal to $2n$. Thus, $\pi_{1,1} = \pi_{1,2} = \cdots = \pi_{1,n} = 2n$. By symmetry, $\pi_{1,1} = \pi_{2,1} = \cdots = \pi_{n,1} = 2n$.

(iii) From self-complementariness, $\pi_{2n,j} + \pi_{1,\,2n+1-j} = 2n$. As j ranges over $n+1$ to $2n$, $2n+1-j$ ranges over n to 1. So by (ii), we get $\pi_{2n,j} = 2n - 2n = 0$. Symmetry gives the rest. ∎

The necessary values of a totally symmetric self-complementary plane partition are as follows:

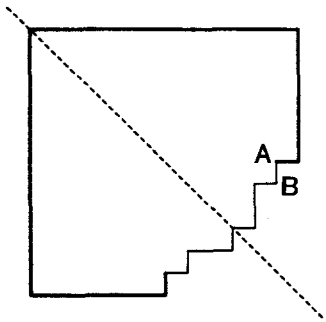
$$\begin{array}{cccccc}
2n & \cdots & 2n & & & n \\
\vdots & & & & n & \\
2n & & & \iddots & & \\
 & & \iddots & & & 0 \\
 & n & & & & \vdots \\
n & & & 0 & \cdots & 0
\end{array}$$

DEFINITION. An *allowable* partition of length $2n$ is a self-conjugate partition with $2n$ parts such that the first n parts equal $2n$ and the last part equals n.

The obvious reason for this definition is that the first row of a totally symmetric self-complementary plane partition of order n must be an allowable partition of length $2n$. Since the last part in an allowable partition is n and it is self-conjugate, the $(n+1)$th thru $(2n-1)$th parts must be strictly less than $2n$.

PROPOSITION 2.2. *There are 2^{n-1} allowable partitions of length 2n.*

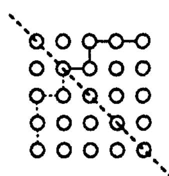
Proof. The Ferrers diagram of an allowable partition looks like



The thick part is fixed by the definition. At point A, you have two choices as how to continue the diagram. You may move down one step or left one step. At the next point, B, you have the same two choices. Continue making these choices until you hit the diagonal line. By self-conjugacy, the rest of the partition is gotten by taking the mirror image across the diagonal. There are $n-1$ independent choices to make. Thus there are 2^{n-1} allowable partitions. ∎

The proof of this proposition sets up a correspondence between allowable partitions of length $2n$ and lattice paths with $n-1$ steps.

EXAMPLE 6. The lattice path



corresponds to the partition (10, 10, 10, 10, 10, 7, 6, 5, 5, 5).

This correspondence is used frequently in the paper.

3. AN INTERMEDIATE RESULT

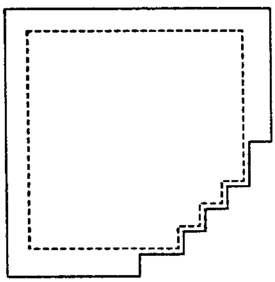
In order to prove the bijection between totally symmetric self-complementary plane partitions of order n and lattice patterns on $n-1$ points, we need the following technical result.

THEOREM 3.1. *The number of ordered pairs (A, B), where $A=[a_i]$ is an allowable partition of length $2n$ and $B=[b_{i,j}]$ is a totally symmetric self-complementary plane partition of order $n-1$ such that*

$$b_{i,1}+1 \leqslant a_{i+1} \qquad (*)$$

equals the number of totally symmetric self-complementary plane partitions of order n.

We prove this with the use of several lemmas. The restriction ($*$) can be thought of as saying that the Ferrers diagram for the first row of B must fit inside the Ferrers diagram of A with all the boxes whose coordinates contain either a 1 or a $2n$ eliminated. Consulting the figure below, the Ferrers diagram for A is represented by the solid line. The Ferrers diagram for the top row of B must fit inside the dotted line.



We now construct a plane partition $C = [c_{i,j}]$ from the pair (A, B). The construction proceeds in three steps. The first is to let the top row of C be the partition A. Once this has been done, other values of C are gotten by symmetry and self-complementariness:

$$\begin{aligned}
&\text{For } i = 1, ..., 2n,\\
&\quad c_{1,i} \leftarrow a_i\\
&\quad c_{i,1} \leftarrow a_i\\
&\quad c_{2n,i} \leftarrow 2n - a_{2n-i+1}\\
&\quad c_{i,2n} \leftarrow 2n - a_{2n-i+1}.
\end{aligned} \qquad \text{(Step 1)}$$

Since we want each row of C to self-conjugate, some of the values of C are now forced to be either 0 or $2n$. This is taken care of by step 2:

$$\begin{aligned}
&\text{For } i = 2, ..., n\\
&\quad \text{If } c_{i,2n} > 1 \text{ then}\\
&\quad\quad \text{For } j = 2, ..., c_{i,2n}\\
&\quad\quad\quad c_{i,j} \leftarrow 2n\\
&\quad\quad\quad c_{2n-i+1,2n-j+1} \leftarrow 0.
\end{aligned} \qquad \text{(Step 2)}$$

Step 3 fills the middle of C with the values from B incremented by 1.

$$\begin{aligned}
&\text{For } i = 2, ..., 2n-1\\
&\quad \text{For } j = 2, ..., 2n-1\\
&\quad\quad \text{If } c_{i,j} \text{ has not yet been assigned a value}\\
&\quad\quad\quad c_{i,j} \leftarrow b_{i-1,j-1} + 1.
\end{aligned} \qquad \text{(Step 3)}$$

The reason for the restriction $b_{i,1}+1 \leqslant a_{i+1}$ becomes clear. Without this restriction C would not necessarily be weakly decreasing in its columns. We denote this process by f: $(A, B) \mapsto C$.

EXAMPLE 7. If A is (10, 10, 10, 10, 10, 9, 8, 7, 6, 5) and B is

$$\begin{matrix} 8 & 8 & 8 & 8 & 7 & 5 & 5 & 4 \\ 8 & 8 & 8 & 7 & 5 & 4 & 4 & 3 \\ 8 & 8 & 8 & 7 & 4 & 4 & 4 & 3 \\ 8 & 7 & 7 & 6 & 4 & 4 & 3 & 1 \\ 7 & 5 & 4 & 4 & 2 & 1 & 1 & 0 \\ 5 & 4 & 4 & 4 & 1 & 0 & 0 & 0 \\ 5 & 4 & 4 & 3 & 1 & 0 & 0 & 0 \\ 4 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \end{matrix}$$

Then $C=f(A, B)$ is

$$\begin{matrix} 10 & 10 & 10 & 10 & 10 & 9 & 8 & 7 & 6 & 5 \\ 10 & 10 & 10 & 10 & 9 & 8 & 6 & 6 & 5 & 4 \\ 10 & 10 & 10 & 9 & 8 & 6 & 5 & 5 & 4 & 3 \\ 10 & 10 & 9 & 9 & 8 & 5 & 5 & 5 & 4 & 2 \\ 10 & 9 & 8 & 8 & 7 & 5 & 5 & 4 & 2 & 1 \\ 9 & 8 & 6 & 5 & 5 & 3 & 2 & 2 & 1 & 0 \\ 8 & 6 & 5 & 5 & 5 & 2 & 1 & 1 & 0 & 0 \\ 7 & 6 & 5 & 5 & 4 & 2 & 1 & 0 & 0 & 0 \\ 6 & 5 & 4 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Now we define the inverse map to f. Given $C=[c_{i,j}]$ a totally symmetric self-complementary plane partition of order n, we construct A and B as in Theorem 3.1. The partition A is simply the first row of C.

For $i=1, ..., 2n$
  $a_i \leftarrow c_{1,i}$ (Step 4)

The matrix B comes from the inner values of C decreased by 1.

For $i=1, ..., 2n-2$
  For $j=1, ..., 2n-2$
    If $c_{i+1,j+1}=2n$, then $b_{i,j} \leftarrow 2n-2$ (Step 5)
    If $c_{i+1,j+1}=0$, then $b_{i,j} \leftarrow 0$
    Otherwise, $b_{i,j} \leftarrow c_{i+1,j+1}-1$

We denote steps 4 and 5 by g: $C \mapsto (A, B)$.

In order to prove Theorem 3.1, we need to show that f and g are indeed the desired bijections. In particular, we must show that

(a) $f(A, B)$ is a totally symmetric self-complementary plane partition of order n, and

(b) $g(C)$ is a pair with the properties listed in Theorem 3.1, and

(c) g is the inverse of f.

These are proven in the next three lemmas.

LEMMA 3.2. *$f(A, B)$ is a totally symmetric self-complementary plane partition of order n.*

Proof. Let $C = f(A, B)$. Considering the comments made after the description of step 3, it is clear that C is a plane partition. We must show C has each of the properties needed to be a totally symmetric self-complementary plane partition.

(1) Symmetry. Choose $i, j \in \{1, ..., 2n\}$. There are three cases based on how $c_{i,j}$ is defined. (a) If either i or j equals 1 or $2n$, then $c_{i,j}$ is defined in step 1. By inspection of this procedure, it is clear that $c_{i,j} = c_{j,i}$.

(b) Note that the values in B range from 0 to $2n-2$. So the values of C defined in step 3 range from 1 to $2n-1$. Thus, if $c_{i,j} = 0$ or $2n$ but is not covered in case (a), then $c_{i,j}$ is defined in step 2. Assume $c_{i,j} = 2n$. The argument for $c_{i,j} = 0$ is similar. This means that $j < c_{i,2n} + 1$. We need to show that $i < c_{j,2n} + 1$. From step 1, $c_{i,2n} = 2n - a_{2n-i+1}$. Combining this with $j < c_{i,2n} + 1$ gives

$$\begin{aligned} j &< 2n - a_{2n-i+1}, \\ j - 2n - 1 &< -a_{2n-i+1}, \\ 2n - j + 1 &> a_{2n-i+1}. \end{aligned}$$

Thus the box $(2n-j+1, 2n-i+1)$ is not inside the Ferrers diagram for the partition A. Since A is self-conjugate, $(2n-i+1, 2n-j+1)$ is not inside the Ferrers diagram for the partition A. Then

$$\begin{aligned} 2n - i + 1 &> a_{2n-j+1}, \\ i - 2n - 1 &< -a_{2n-j+1}, \\ i &< 2n - a_{2n-j+1} + 1, \\ i &< c_{j,2n} + 1. \end{aligned}$$

So step 2 assigns $c_{j,i}$ the value $2n$.

(c) If i, j falls in neither case (a) nor case (b), then $c_{i,j}$ is defined in step 3. Since B is symmetric,

$$c_{i,j} = b_{i-1,j-1} + 1 = b_{j-1,i-1} + 1 = c_{j,i}.$$

Therefore, C is symmetric.

(2) Self-Complementary. Choose $i, j \in \{1, ..., 2n\}$ and show $c_{i,j} + c_{2n-i+1,2n-j+1} = 2n$. Again we break the problem into three cases based on how $c_{i,j}$ is defined. (a) If either i or j equals 1 or $2n$, then $c_{i,j}$ and $c_{2n-i+1,2n-j+1}$ are defined in step 1. By inspection of the procedure, $c_{i,j} + c_{2n-i+1,2n-j+1} = 2n$.

(b) If $c_{i,j}$ is defined in step 2, then by inspection of that procedure, $c_{2n-i+1,2n-j+1}$ is also defined in step 2 and $c_{i,j} + c_{2n-i+1,2n-j+1} = 2n$.

(c) If $c_{i,j}$ and $c_{2n-i+1,2n-j+1}$ are defined in step 3, then because B is $(2n-2, 2n-2, 2n-2)$-self-complementary

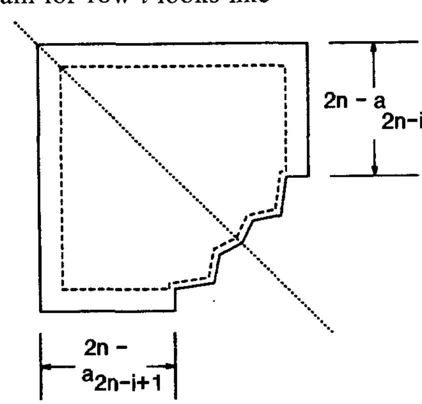
$$\begin{aligned} c_{i,j} + c_{2n-i+1,2n-j+1} &= (b_{i-1,j-1} + 1) + (b_{2n-i,2n-1} + 1) \\ &= (b_{i-1,j-1} + b_{2n-i,2n-j}) + 2 \\ &= (2n-2) + 2 \\ &= 2n. \end{aligned}$$

Therefore C is self-complementary.

(3) Total symmetry. We have already shown that C is symmetric. All we need to show is that each row when considered as a partition is self-conjugate. The first row is the partition A which is assumed to be self-conjugate. Pick $i \in \{2, ..., n\}$. Row i looks like

$$\underbrace{2n, ..., 2n}_{\substack{2n - a_{2n-i+1} \\ \text{times}}}, b_{i-1,2n-a_{2n-i+1}} + 1, ..., b_{i-1,2n-2} + 1, 2n - a_{2n-i+1}$$

The Ferrers diagram for row i looks like



The dotted line is the Ferrers diagram for the $(i-1)$th row of B. Since B is totally symmetric, its $(i-1)$th row is self-conjugate. Thus the ith row of C is self-conjugate. The self-conjugacy of rows $n+1$ thru $2n$ follows from the self-conjugacy of rows 1 to n and the self-complementariness of the plane partition. Therefore, C is totally symmetric. ▌

LEMMA 3.3. *If $g(C)=(A, B)$, then A is an allowable partition of length $2n$, B is a totally symmetric self-complementary plane partition of order n, and $a_{i+1} \geqslant b_{1,i}+1$.*

Proof. By step 4, A is the first row of C and thus A is an allowable partition of length $2n$. To show that B is a totally symmetric self-complementary plane partition of order $n-1$, we go through the properties one at a time.

(1) Symmetry. Choose $i, j \in \{1, ..., 2n-2\}$. If $c_{i+1,j+1}=2n$, then $b_{i,j}=2n-2$. Since C is symmetric, $c_{j+1,i+1}=2n$ and $b_{j,i}$ gets assigned the value $2n-2$. A similar argument works in the case $c_{i+1,j+1}=0$. If $c_{i+1,j+1}$ equals neither 0 nor $2n$, then because C is symmetric, $b_{i,j}=c_{i+1,j+1}-1=c_{j+1,i+1}-1=b_{j,i}$.

(2) Self-complementariness. Choose $i, j \in \{1, ..., 2n-2\}$. If $c_{i+1,j+1}=2n$, then $c_{2n-i,2n-j}=0$ due to the fact that C is self-complementary. In this case, $b_{i,j}=2n-2$ and $b_{2n-2-i+1,2n-2-j+1}=0$. Hence B is self-complementary for this choice of i, j. A similar result occurs if $c_{i+1,j+1}=0$.
If $c_{i+1,j+1}$ equals neither $2n$ nor 0, then

$$\begin{aligned}
b_{i,j}+b_{2n-2-i+1,2n-2-j+1} &= b_{i,j}+b_{2n-1-i,2n-1-j} \\
&= (c_{i+1,j+1}-1)+(c_{2n-i,2n-j}-1) \\
&= (c_{i+1,j+1}+c_{2n-i,2n-j})-2 \\
&= 2n-2.
\end{aligned}$$

Hence, B is self-complementary.

(3) Total symmetry. We have to show that each row of B is self-conjugate. This follows directly from that fact each row of C is self-conjugate. By inspection of step 5, the Ferrers diagram for the ith row of B is the Ferrer diagram for the $(i+1)$th row of C with all the squares with either coordinate being 1 or $2n$ removed. Since the Ferrers diagram for each row of C is symmetric about the diagonal and the removal process is symmetric, the Ferrer diagram for each row of B is symmetric. Thus, B is totally symmetric. Therefore, B is a totally symmetric self-complementary plane partition of order $n-1$.

The last item to show is that A and B satisfy $a_{i+1} \geqslant b_{1,i}+1$. By step 4, $a_{i+1}=c_{1,i+1}$. By step 5, $b_{1,i} \leqslant c_{2,i+1}-1$. Since C is a plane partition, $c_{1,i+1} \geqslant c_{2,i+1}$. Putting all of this together,

$$a_{i+1}=c_{1,i+1} \geqslant c_{2,i+1} \geqslant b_{1,i+1}. \quad \blacksquare$$

LEMMA 3.4. $g(f(A, B))=(A, B)$.

Proof. Let $A=[a_i]$ and $B=[b_{i,j}]$ be of the proper type. Let $C=[c_{i,j}]=f(A, B)$. Let $(A', B')=([a_i'], [b_{i,j}'])=g(C)$. We need to show that $A=A'$ and $B=B'$.

Examination of steps 1 and 4 shows that $A=A'$. The B's are a little more interesting. If $c_{i+1,j+1}=2n$ where $i, j \in \{1, ..., 2n-2\}$, then $c_{i+1,j+1}$ was defined in step 2. By step 5, $b_{i,j}'=2n-2$. So we must show that $b_{i,j}=2n-2$. This is done in the following.

1.	$c_{i+1,j+1}=2n$	assumption
2.	$j+1 \leqslant c_{2n,i+1}$	from 1 and step 2
3.	$j<c_{2n,i+1}$	from 2 and since all the values are integers
4.	$c_{2n,i+1}=2n-a_{2n-i}$	from step 2
5.	$j<2n-a_{2n-i}$	from 3 and 4
6.	$2n-j>a_{2n-i}$	rewrite of 5
7.	$a_{2n-j} \geqslant b_{1,2n-i-1}+1$	assumption made about A and B
8.	$2n-j>b_{1,2n-i-1}+1$	from 6 and 7
9.	$2n-2=b_{1,2n-i-1}+b_{2n-2,i}$	since B is self-complementary
10.	$2n-j>2n-2-b_{2n-2,i}+1$	from 8 and 9
11.	$j<b_{2n-2,i}+1$	rewrite of 10
12.	$j \leqslant b_{2n-2,i}$	from 11 and since all the values are integers
13.	$b_{i,j}=2n-2$	since each row of B is self-conjugate

If $c_{i+1,j+1}=0$, then by a similar argument, $b_{i,j}=0=b_{i,j}'$. If $1 \leqslant c_{i+1,j+1} \leqslant 2n-1$, then from steps 3 and 5, $b_{i,j}=c_{i+1,j+1}-1=b_{i,j}'$. Therefore, $B=B'$. $\blacksquare$

Since the sets involved are finite, $f(g(C))=C$. Thus, f and g give the bijections needed to prove Theorem 3.1.

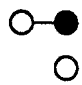
4. Totally Symmetric Self-Complementary Plane Partitions and Lattice Patterns

In this section, we prove the main theorem of the paper. In fact, we prove an even stronger result than the one stated in Theorem 1.2.

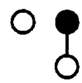
THEOREM 4.1. *There is bijection between totally symmetric self-complementary plane partitions of order n and lattice patterns on $n-1$ points. Moreover, the bijection sends a totally symmetric self-complementary plane partition to a lattice pattern such that the top row of the plane partition corresponds, as in Proposition* 2.2, *to the longest lattice path in the lattice pattern.*

Proof. By induction on n. When $n=2$, the bijection is

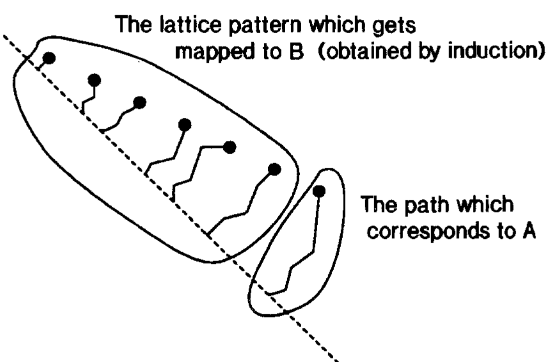
$$\begin{array}{cccc} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{array}$$



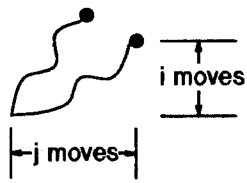
$$\begin{array}{cccc} 4 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}$$



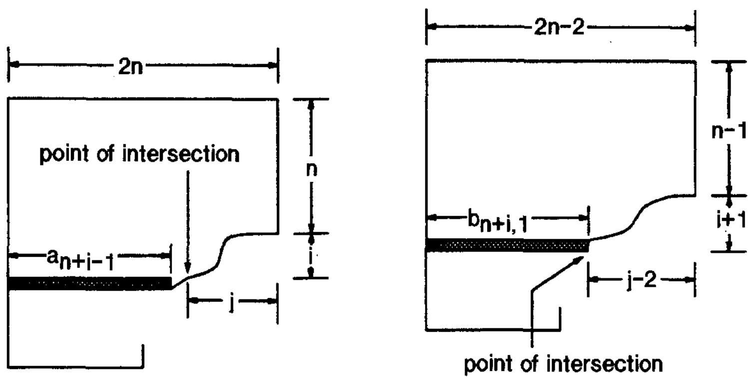
Suppose we have a bijection between totally symmetric self-complementary plane partitions of order $n-1$ and lattice patterns on $n-2$ points such that the top row of the plane partition corresponds to the longest lattice path. Given a totally symmetric self-complementary plane partition, C, of order n, let $(A, B)=g(C)$. Use the bijection to obtain a lattice pattern on $n-2$ points which corresponds to B. Add on the outside of this pattern the lattice path which corresponds to A.



We claim that this is the desired bijection. The only unclear point is whether the outer two lattice paths intersect or not. By the induction hypothesis, the second to last lattice path corresponds to the top row of B. Suppose they do intersect.



We need to determine bounds on a_{n+i-1} and $b_{n+1,1}$ in order to draw a contradiction. The Ferrers diagrams for A and the top row of B look like



From the pictures we can read off that $a_{n+i-1} \leqslant 2n-j-1$ and $b_{n+1,1} \geqslant 2n-j-1$. Thus, $b_{n+i,1} \geqslant a_{n+i-1}$ which contradicts condition (*) in Theorem 3.1. ∎

We could have started the induction at $n=1$ if we defined there to be one lattice pattern on 0 points, namely the empty lattice pattern. This proof of Theorem 1.2 is quite long and tedious. J. Stembridge gives a much shorter proof in [Ste]. The reason for the long proof is that in Section 6, we give a "lattice path" interpretation for cyclically symmetric self-complementary plane partitions, the proof of which is an easy adaptation of this proof. The other proofs do not give the results in Section 6.

EXAMPLE 8. For the case $n=3$, see the Appendix.

5. Counting Lattice Patterns

Counting totally symmetric self-complementary plane partitions directly has proven fruitless. However, the lattice pattern form of the problem has been solved. It is interesting that it required several people to do this, each working independently on a different part of the puzzle.

THEOREM 5.1. *The number of lattice patterns on $n-1$ points equals*

$$\begin{array}{ll} \det[a_{i,j}]_{0 \leqslant i,j \leqslant n-1} & \textit{if } n \textit{ is even} \\ \det[a_{i,j}]_{0 \leqslant i,j \leqslant n-1} & \textit{if } n \textit{ is odd,} \end{array}$$

where

$$a_{i,j} = \begin{cases} \sum_{r=2i-j+1}^{2j-i} \binom{i+j}{r}, & \textit{if} \quad i<j \\ -\sum_{r=2i-j+1}^{2j-i} \binom{i+j}{r}, & \textit{if} \quad i>j \\ 0, & \textit{if} \quad i=j. \end{cases}$$

Proof 1 (J. Stembridge). In [Ste], Stembridge developed a general method for counting the number of non-intersecting lattices paths with certain starting and ending regions. As a application of the method, he obtains the above result. ∎

Proof 2. It is a routine application of the Gessel–Viennot theorem [Sta3] on enumerating non-intersecting lattice paths with fixed starting and ending points that the number of lattice patterns on $n-1$ points equals the sum of all $(n-1)\times(n-1)$ minors of the matrix

$$\left[\binom{i}{j-i}\right]_{1 \leqslant i \leqslant n-1,\, 1 \leqslant j \leqslant 2n-1}.$$

D. Robbins noted [R1] that the methods of S. Okada [O] can be applied to this sum of minors to obtain the above result. ∎

The proof of Theorem 1.1 was completed by G. Andrews [A] who evaluated the determinants in Theorem 5.1.

6. CYCLICALLY SYMMETRIC SELF-COMPLEMENTARY PLANE PARTITIONS

In this section, we give a lattice path interpretation for cyclically symmetric self-complementary plane partitions of order n. We do not include most of the details, as this is a straight-forward generalization of the methods in Sections 2–4. Here is the result.

THEOREM 6.1. *The number of cyclically symmetric self-complementary plane partitions of order n equals the number of lattice path configurations with the following conditions.*

(i) *There are n lattice paths. The ith path starts at* $(2i, 0)$ *and ends at* $(0, 2i)$.

(ii) *Let* s_i *be the number of downward moves made by the ith path before its first leftward move. Let* t_i *be the number of leftward moves made after its last downward move. Then for all i,* $s_i + t_i = 2i$.

(iii) *The lattice paths do not intersect.*

Proof (Sketch). If $C = [c_{i,j}]$ is a cyclically symmetric self-complementary plane partition of order n, then $c_{2n,1}$ equals the number of $2n$'s in the top row, since the first column is the conjugate of the first column. By self-complementariness, $c_{2n,1} + c_{1,2n} = 2n$. Hence an "allowable" partition for the top row of a cyclically symmetric self-complementary plane partition is one in which the largest part is $2n$ and the number of $2n$'s in the partition plus the value of the smallest part equals $2n$. This is why we have the strange restriction in (ii).

Next we need an result analogous to Theorem 3.1.

LEMMA 6.2. *The number of ordered pairs* (A, B), *where* $A = [a_i]$ *is an allowable (in just defined sense) partition,* $B = [b_{i,j}]$ *is a cyclically symmetric self-complementary plane partition of order* $n - 1$, *and*

$$b_{i,1} + 1 \leqslant a_{i+1} \tag{$**$}$$

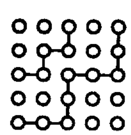
equals the number of cyclically symmetric self-complementary plane partitions of order n, $C = [c_{i,j}]$.

The construction of C proceeds as expected. Step 1: Let the top row C be the partition A, the first column of C be the conjugate of A, and fill in the bottom row and last column using self-complementary. Step 2: Since the ith of C row is the conjugate of the ith column, certain values of C must be either $2n$ or 0. Fill these in. Step 3: Fill in the all the missing values of C by $c_{i,j} = b_{i-1,j-1} + 1$.

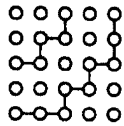
Finally, we complete the bijection between lattice path configurations and cyclically symmetric self-complementary plane partitions by recursively applying Lemma 6.2, just as was done in the proof of Theorem 4.1. The condition $(**)$ forces the lattice paths to be non-intersecting. ∎

EXAMPLE 9. For the case $n = 2$, the bijection is

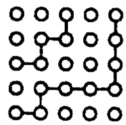
$$\begin{matrix} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{matrix}$$



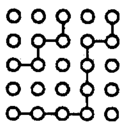
$$\begin{array}{cccc} 4 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}$$



$$\begin{array}{cccc} 4 & 4 & 4 & 1 \\ 3 & 3 & 2 & 1 \\ 3 & 3 & 1 & 1 \\ 3 & 0 & 0 & 0 \end{array}$$



$$\begin{array}{cccc} 4 & 3 & 3 & 3 \\ 4 & 3 & 3 & 0 \\ 4 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}$$



You should note that symmetric cyclically symmetric self-complementary plane partitions are in fact totally symmetric self-complementary plane partitions, and lattice configurations of the above type which are symmetric about the diagonal $\{(i, i): i \in \mathbf{Z}\}$ correspond to lattice patterns. In the symmetric case, $s_i = t_i = i$ and we do not draw the forced moves. So, this is a very natural generalization of the work done in the previous sections.

Using very different techniques, G. Kuperberg has enumerated cyclically symmetric self-complementary plane partitions [K].

THEOREM 6.2 (Kuperberg). *The number of cyclically symmetric self-complementary plane partitions of order n equals* A_n^2.

His method involves converting the plane partition problem into counting 1-factors in certain graphs. This leaves open the problem of finding a direct bijection between ordered pairs of totally symmetric self-complementary plane partitions and cyclically symmetric self-complementary plane partitions.

7. A Connection Between Alternating Sign Matrices and Lattice Patterns

Given Conjecture 1, the number of lattice patterns on $n-1$ points should be equal to the number of $n \times n$ alternating sign matrixs. This has been checked for $n \leqslant 20$. As a refinement of this conjecture, we define a statistic on lattice patterns which seems to correspond to the position of the 1 in the top row an alternating sign matrix.

DEFINITION. Let $A(n, k)$ denote the number of $n \times n$ alternating sign matrixs with a 1 in position $(1, k)$.

Two basic properties of $A(n, k)$ are given in the following proposition.

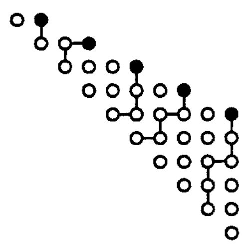
PROPOSITION 7.1. (i) $A(n, k) = A(n, n+1-k)$.

(ii) $A(n, 1) =$ *the number of* $(n-1) \times (n-1)$ *alternating sign matrices.*

A proof of this can be found in [MRR1]. The statistic on lattice patterns is defined next.

DEFINITION. The *descent value* of a lattice pattern is the number lattice paths in the pattern which make an odd number of downward moves.

EXAMPLE 10.



The two shortest paths make an odd number of downward moves. So they contribute to the descent value. The other three paths make an even number of downward moves. So they do not contribute. Hence, the value of this lattice pattern is 2.

DEFINITION. Let $L(n, k)$ be the number of lattice patterns on $n-1$ points with descent value $k-1$.

The connection between lattice patterns and alternating sign matrices is stated.

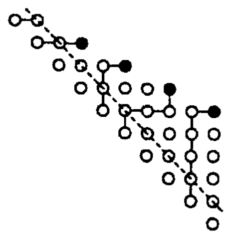
Conjecture. For all n and k,

$$A(n, k) = L(n, k)$$

This has been checked for $n \leqslant 10$. Adding more strength to this conjecture is that $L(n, k)$ also obeys the two conditions in Proposition 7.1. This is shown in the following the next two lemmas.

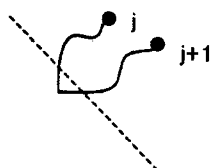
LEMMA 7.2. $L(n, 1) =$ *the number of lattice paths on* $n-2$ *points.*

Proof. Given any lattice pattern on $n-2$ points, we need to "extend" it to a lattice pattern on $n-1$ points in such a way that each path makes an even number of downward moves. To do this, we add one more move onto the end of each of the lattice paths. If a certain lattice path makes an odd number of downward moves, then add a downward move at the end. If a lattice path makes an even number of downward moves, then add a leftward move at the end:



After these additions are made, each lattice path makes an even number of downward moves. Finally, to make this a lattice pattern, add a lattice path with one leftward move at the upper left.

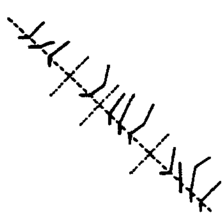
Does this extending process introduce any intersections? Suppose path j and $j+1$ intersect after the additions are made:



For this to be possible the paths must end in the original lattice pattern next to one another along the diagonal $\{(i, i): i \in Z\}$. Since path j starts one step higher than the $(j+1)$th path and end one step higher, the number of downward moves is the same in both. Thus the additional move must be in the same direction for both path, and an intersection is not possible. ∎

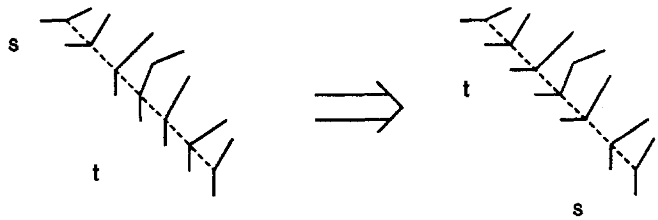
LEMMA 7.3. $L(n, k) = L(n, n+1-k)$.

Proof. Take a lattice path on $n-1$ points. Group together the paths whose second to last positions form a sequence $(i+1, i)$, $(i+2, i+1)$, ..., $(i+k+1, i+k)$:



What we want to do is switch the parity of downward moves in each path by changing the direction of its last move. The problem that can arise is that this might create an intersection. Note that the only possible intersections that could arise come from two paths of the same group.

Suppose a group has s members with an even number of downward moves and t with an odd number. Then make the switch as demonstrated in



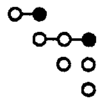
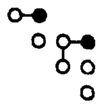
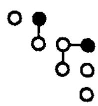
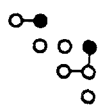
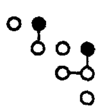
You leave everything above the dotted line alone. Now this group has t members with an even number of downward moves and s with an odd number. Do this for every group in the lattice pattern. This is an involution which takes lattice patterns with descent value k to a lattice pattern with descent value $n+1-k$. ∎

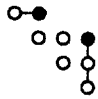
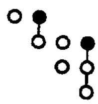
The proof of Proposition 7.1(i) essentially comes down to seeing that the vertical flip of an alternating sign matrix is also an alternating sign matrix. The switching described in the above proof does not correspond exactly to a vertical flip. The easiest way to see this is to compare the number of figures invariant under each operation. The number 5×5 alternating sign matrices fixed by a vertical flip is 3, but there are 8 lattice patterns on 4 points which are fixed by the above switching. So the process which corresponds to a vertical flip in alternating sign matrices must alter more than just the last move of the lattice paths.

It is known that $A(n, 2) = (n/2)\, A(n, 1)$ [MRR1]. It is unknown whether $L(n, 2) = (n/2)\, L(n, 1)$.

Finally, Mills, Robbins, and Rumsey have two other statistics on totally symmetric self-complementary plane partitions which have the same properties as the descent value [MRR2]. For the case $n = 3$ they are given in the Appendix. D. Robbins has noted [R2] that if you take any two of the three statistics, the number of totally symmetric self-complementary plane partitions with statistic 1 being i and statistic 2 being j equals the number of alternating sign matrices with the one of the top row in column $i+1$ and the one on the bottom row in column $j+1$. This means that not only should there be one bijection between totally symmetric self-complementary plane partitions and alternating sign matrices, but there should be three of them, one bijection for each of the three choices of statistics to maintain.

APPENDIX

TSSCPP	Lattice Pattern	DV	MRR #1	MRR #2
6 6 6 3 3 3 6 6 6 3 3 3 6 6 6 3 3 3 3 3 3 3 3 3 3 3 3		0	2	2
6 6 6 4 3 3 6 6 6 3 3 3 6 6 5 3 3 2 4 3 3 1 3 3 3 3 3 2		1	1	2
6 6 6 4 3 3 6 6 6 4 3 3 6 6 4 3 2 2 4 4 3 2 3 3 2 3 3 2		2	0	1
6 6 6 5 4 3 6 6 5 3 3 2 6 5 5 3 3 1 5 3 3 1 1 4 3 3 1 3 2 1		2	2	1
6 6 6 5 4 3 6 6 5 4 3 2 6 5 4 3 2 1 5 4 3 2 1 4 3 2 1 3 2 1		2	1	0

$\begin{matrix} 6 & 6 & 6 & 5 & 5 & 3 \\ 6 & 5 & 5 & 3 & 3 & 1 \\ 6 & 5 & 5 & 3 & 3 & 1 \\ 5 & 3 & 3 & 1 & 1 & \\ 5 & 3 & 3 & 1 & 1 & \\ 3 & 1 & 1 & & & \end{matrix}$		0	1	1
$\begin{matrix} 6 & 6 & 6 & 5 & 5 & 3 \\ 6 & 5 & 5 & 4 & 3 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 & \\ 5 & 3 & 2 & 1 & 1 & \\ 3 & 1 & 1 & & & \end{matrix}$		1	0	0

Note. DV: descent value. MRR #1 and #2: Mills, Robbins, Rumsey statistics.

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