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Robin Capacity and Extremal Length

PETER DUREN

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

AND

JOHN PFALTZGRAFF

Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599

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1. INTRODUCTION

Let $\Omega \subset \hat{\mathbb{C}}$ be a domain in the extended complex plane, containing the point at infinity and bounded by a finite number of analytic Jordan curves. Let the boundary be divided into two disjoint subsets A and B. The *Robin function* $R(z, \infty)$ is harmonic in Ω except at infinity, where $R(z, \infty) - \log |z|$ is harmonic. It vanishes on A and its normal derivative vanishes on B. If $A = \partial\Omega$, the Robin function reduces to Green's function of Ω. The *Robin capacity* of A with respect to Ω is defined by $\delta(A) = e^{-\rho(A)}$, where

$$\rho(A) = \lim_{z \to \infty} \{R(z, \infty) - \log |z|\}.$$

By way of comparison, the ordinary logarithmic capacity of A is $d(A) = e^{-\gamma(A)}$, where

$$\gamma(A) = \lim_{z \to \infty} \{g(z, \infty) - \log |z|\}$$

and $g(z, \infty)$ is Green's function of the unbounded component of $\hat{\mathbb{C}} \setminus A$.

It is easily seen that the Robin capacity is invariant under *admissible* conformal mappings of Ω. More precisely, if Ω is mapped conformally onto a domain $\tilde{\Omega}$ by a function f of the form

$$f(z) = z + b_0 + b_1 z^{-1} + \cdots \tag{1}$$

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near infinity, then $\delta(f(A)) = \delta(A)$; that is, the Robin capacity of $\tilde{A} = f(A)$ with respect to $\tilde{\Omega}$ is equal to the Robin capacity of A with respect to Ω. Duren and Schiffer [6] recently found by a variational method that $d(f(A)) \geq \delta(A)$ for every conformal mapping f of Ω with the normalization (1), and that the inequality is sharp for each given domain Ω and for each subset $A \subset \partial\Omega$. In other words, the Robin capacity provides a sharp lower bound for the distortion of capacity of boundary sets under normalized conformal mappings. Thus it seems worthwhile to make a further study of Robin capacity. Our purpose in the present paper is to investigate it from a more geometric viewpoint.

Ahlfors and Beurling [2, 1] introduced the notion of "reduced extremal distance" and related it to Robin capacity. Let C_r be a large circle $|z| = r$ in Ω, surrounding all of the boundary, and let Ω_r be the part of Ω lying inside C_r. The *extremal distance* $\lambda(A, C_r)$ is the extremal length of the family of curves in Ω_r connecting A to C_r. It can be shown (see also Ohtsuka [7, pp. 241–245]) that

$$\lim_{r \to \infty} \{2\pi\lambda(A, C_r) - \log r\} = \rho(A), \tag{2}$$

the same quantity defined above in terms of the Robin function.

After looking at some examples where the Robin capacity can be computed explicitly, we use the extremal distance approach to derive some monotonicity properties that seem less apparent from the standpoint of potential theory. We also point out that the inequality $d(f(A)) \geq \delta(A)$ follows directly from the extremal distance form of the definition, but it is not clear how to show by this method that the inequality is sharp. The paper concludes with a connection between Robin capacity and the exterior modulus of a quadrilateral.

2. EXAMPLES

Let us begin by calculating the Robin capacity for some specific configurations. Some corresponding calculations of ordinary capacity may be found for instance in a paper of Pólya and Szegö [10].

(i) *Arcs of a Circle.* First let Ω be the domain $|z| > 1$ and let A be an arc of the circle $|z| = 1$ that subtends an angle α at the origin. It can be shown by conformal mapping that the ordinary capacity of A is $d(A) = \sin(\alpha/4)$. One strategy is to use a Moebius transformation to map the complement of the arc onto the complement of a linear segment, then to apply the inverse of a Joukowski mapping to send the resulting domain onto the exterior of a disk. A final Moebius transformation will send the image of infinity back to infinity. (See Pommerenke [11, pp. 14–15].)

On the other hand, it is not hard to see that the Robin capacity of A with respect to Ω is $\delta(A)=\sin^2(\alpha/4)$. Suppose for convenience that A has its center at the point 1 and apply the Joukowski mapping $\zeta=z+1/z$. This maps Ω onto the complement of the segment $[-2,2]$, sending A onto the doubly covered segment from $2\cos(\alpha/2)$ to 2. The inverse of an appropriate Joukowski mapping can now be applied to obtain a normalized conformal mapping f of Ω onto the complement of a disk of radius $r=\sin^2(\alpha/4)$ with a radial slit. The set A is mapped onto the circle $\tilde{A}$ of radius r, while the set B is mapped onto the slit. The Robin function of $\tilde{\Omega}=f(\Omega)$ is obviously $\tilde{R}(w,\infty)=\log|w|-\log r$, and so the Robin capacity is $\delta(A)=\delta(\tilde{A})=r=\sin^2(\alpha/4)$.

Suppose now that $\alpha=\pi$ and let f be the normalized mapping just constructed. Let A now be the union of n arcs on the unit circle of length π/n, centered at the nth roots of unity. The normalized mapping $w=g(z)=\{f(z^n)\}^{1/n}=z+\cdots$ carries Ω onto the domain $|w|>2^{-1/n}$ with n radial slits, sending A onto the circle $|w|=2^{-1/n}$ and B onto the slits. The Robin capacity is therefore $\delta(A)=2^{-1/n}$. It follows that the ordinary capacity is $d(A)=2^{-1/2n}$, since it is known [6] that $\delta(A)=d(A)^2$ for subsets of the unit circle.

It should be observed that if A is an arc of the unit circle with central angle α, then its complementary arc B subtends and angle $\beta=2\pi-\alpha$, and has Robin capacity $\delta(B)=\sin^2(\beta/4)=\cos^2(\alpha/4)$. Thus $\delta(A)+\delta(B)=1=d(A\cup B)$. It follows by conformal invariance that if A and B are complementary arcs of a smooth Jordan curve C and Ω is the domain outside C, then $\delta(A)+\delta(B)=d(C)$. This is no longer true for arbitrary subsets A and $B=C\backslash A$, but at least the inequalities $\delta(A)\leqslant d(A)\leqslant d(C)$ and $\delta(B)\leqslant d(C)$ show that $\delta(A)+\delta(B)\leqslant 2d(C)$. That the constant 2 is best possible is shown by the example above, where A is the union of n arcs of length π/n centered at the nth roots of unity. Then

$$\delta(A)=\delta(B)=2^{-1/n}\to 1 \qquad \text{as} \quad n\to\infty.$$

(ii) *The Half-Disk.* Let Ω be the domain exterior to the half-disk bounded by the semicircle

$$A=\{z\colon |z|=1, \operatorname{Im}\{z\}\geqslant 0\}$$

and the linear segment $B=(-1,1)$ of the real axis. The capacity of the semidisk is $d(\partial\Omega)=4/(3\sqrt{3})$, as is seen by constructing an admissible conformal mapping of Ω onto the outside of a disk (cf. [10, p. 10]). Under this mapping it may be observed that the image of A is a circular arc with central angle $\alpha=4\pi/3$, so that $\delta(A)=(4/3\sqrt{3})\sin^2(\alpha/4)=1/\sqrt{3}$. Similarly, $\delta(B)=(4/3\sqrt{3})\cos^2(\alpha/4)=1/(3\sqrt{3})$. It may also be noted that $d(A)=\sin(\pi/4)=1/\sqrt{2}$ and $d(B)=L/4=1/2$.

(iii) *Arcs of an Ellipse.* Let Ω be the domain outside an ellipse, defined by the inequality $x^2/a^2 + y^2/b^2 > 1$, where $a = c + 1/c$ and $b = c - 1/c$ for some number $c > 1$. Then $c = \frac{1}{2}(a+b)$. Observe first that the function $z = f(w) = cw + 1/cw$ maps the domain $\{|w| > 1\}$ onto Ω, so the ellipse has capacity $d(\partial\Omega) = c$. Let A be the arc of the ellipse which is the image under f of the circular arc from 1 to $e^{i\varphi}$, where $0 < \varphi < 2\pi$. Then the Robin capacity of A with respect to Ω is $\delta(A) = ((a+b)/2)\sin^2(\varphi/4)$. The arc A extends from a to the point $a\cos\varphi + ib\sin\varphi$, whose central angle ψ is defined by the equation $\tan\psi = (b/a)\tan\varphi$. Using the half-angle formulas, one may express $\delta(A)$ in terms of the angle ψ, but the formula is complicated and will not be given here.

(iv) *The Rectangle.* Our next objective is to compute the Robin capacity of opposite sides or a single side of a rectangle, with respect to its exterior domain Ω. To this end we now construct a conformal mapping of $\Delta = \{|z| > 1\}$ onto Ω, preserving certain symmetries. It is convenient to begin with a mapping of Δ onto the upper half-plane $H = \{\operatorname{Im}\{\zeta\} > 0\}$, followed by a mapping of H onto Ω. The side-lengths of the rectangle will then be determined by standard elliptic integrals with a parameter k, $0 < k < 1$.

Choose an angle α, $0 < \alpha < \pi/2$, and let $k = \tan^2(\alpha/2)$. Consider first the mapping

$$\zeta = h(z) = -\frac{i}{\sqrt{k}}\frac{1+z}{1-z}$$

of Δ onto H, with $h(-1) = 0$, $h(-e^{-i\alpha}) = 1$, $h(e^{i\alpha}) = 1/k$, $h(1) = \infty$, $h(-e^{i\alpha}) = -1$, $h(e^{-i\alpha}) = -1/k$, and $h(\infty) = i/\sqrt{k}$. Next consider the formula

$$w = g(\zeta) = \int_0^\zeta \frac{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}{(1+k\zeta^2)^2}\,d\zeta,$$

which defines a Schwarz–Christoffel mapping of H onto the domain Ω outside a rectangle R. Here the denominator of the integrand is dictated by the fact that infinity is an interior point of Ω. (See [8, p. 329, Th. 9.9.] or [9, p. 193].) Now $g(i/\sqrt{k}) = \infty$, $g(0) = 0$, $g(1) = J$, and $g(1/k) = J - iL$, where

$$J = J(k) = \int_0^1 \frac{\sqrt{(1-x^2)(1-k^2x^2)}}{(1+kx^2)^2}\,dx \tag{3}$$

and

$$L = L(k) = \int_1^{1/k} \frac{\sqrt{(x^2-1)(1-k^2x^2)}}{(1+kx^2)^2}\,dx. \tag{4}$$

Furthermore, $g(-1)=-J$ and $g(-1/k)=-J-iL$. Thus the rectangle R has vertices $\pm J$ and $\pm J-iL$.

Now let $f=g\circ h$ be the composition of the two mappings h and g. Note that $f(\infty)=\infty$. A straightforward calculation shows that

$$f'(z)=g'(h(z))\,h'(z)\to -i\,\frac{1+k}{8k} \quad \text{as} \quad z\to\infty,$$

and it follows that R has capacity $d(R)=(1+k)/8k$.

The two elliptic integrals (3) and (4) can be transformed by the identity

$$2\,\frac{\sqrt{(1-x^2)(1-k^2x^2)}}{(1+kx^2)^2}=\frac{1-kx^2}{\sqrt{(1-x^2)(1-k^2x^2)}}+\frac{d}{dx}\left\{\frac{x\sqrt{(1-x^2)(1-k^2x^2)}}{1+kx^2}\right\}.$$

Hence

$$2J=\int_0^1 \frac{1-kx^2}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx=\frac{1}{k}\,[E-(1-k)\,K], \tag{5}$$

where $K=K(k)$ and $E=E(k)$ are the standard complete elliptic integrals of first and second kinds, respectively (see [3]). Similarly,

$$2L=\int_1^{1/k} \frac{kx^2-1}{\sqrt{(x^2-1)(1-k^2x^2)}}\,dx=\frac{1}{k}\,(E'-kK'), \tag{6}$$

where $K'(k)=K(k')$, $E'(k)=E(k')$, and $k'=(1-k^2)^{1/2}$. The last integral is evaluated by the substitution $x=(1-k'^2t^2)^{-1/2}$ and the identity

$$kk'^2\,\frac{dK}{dk}=E-k'^2K.$$

It may be remarked that $2kJ$ increases from 0 to 1 and $2kL$ decreases from 1 to 0 as k increases from 0 to 1. Thus the ratio $2J/L$ increases from 0 to ∞.

If $\alpha=\pi/4$, then $k=\tan^2(\pi/8)=3-2\sqrt{2}$ and it can be shown that R is a square of sidelength $2J=1.446\ldots$. To show that $2J=L$, it is enough to invoke standard transformations of elliptic integrals (see [3, p. 319]) under the correspondence $k\mapsto(1-k)/(1+k)$, which carries $1/\sqrt{2}$ to $3-2\sqrt{2}$.

Now let A be the union of the two circular arcs from $e^{i\alpha}$ to $e^{-i\alpha}$ and from $-e^{i\alpha}$ to $-e^{-i\alpha}$. The Robin capacity of A with respect to Δ is easily computed by a square-root transformation as in Example (i) above. One finds $\delta(A)=\sin\alpha$ and $\delta(B)=\cos\alpha$. Thus the set $\tilde{A}=f(A)$, which consists of the two opposite sides of R of length $2J$, has Robin capacity

$$\delta(\tilde{A})=\frac{1+k}{8k}\sin\alpha, \qquad \text{while} \quad \delta(\tilde{B})=\frac{1+k}{8k}\cos\alpha. \tag{7}$$

Next let A consist only of the arc from $e^{i\alpha}$ to $e^{-i\alpha}$. Its image $\tilde{A} = f(A)$ is then a single side of R of length $2J$. The Robin capacity of A is now $\delta(A) = \sin^2(\alpha/2)$, so the Robin capacity of $\tilde{A}$ with respect to Ω is

$$\delta(\tilde{A}) = \frac{1+k}{8k}\sin^2(\alpha/2) = \frac{1}{8}, \qquad \text{while} \quad \delta(\tilde{B}) = \frac{1+k}{8k}\cos^2(\alpha/2) = \frac{1}{8k}.$$

3. Monotonicity Properties

Let us now turn to the alternate definition of the Robin capacity, as given by (2) in terms of extremal length. As indicated in the introduction, this more geometric description has the advantage of leading directly to some general inequalities for Robin capacity. Recall that the *extremal length* $\lambda(\Gamma)$ of *a* curve-family Γ in a domain D is defined by

$$\frac{1}{\lambda(\Gamma)} = \inf_{\rho} \iint_D \rho(z)^2\, dx\, dy,$$

where the infimum is taken over all metrics ρ for which

$$\int_\gamma \rho(z)\,|dz| \geqslant 1 \qquad \text{for all} \quad \gamma \in \Gamma.$$

Such metrics are called *admissible* for the family Γ.

As a first application of the definition via extremal length, we now show that when a domain is enlarged while a boundary set A is held fixed, the Robin capacity of A must increase.

Theorem 1. *Let Ω and $\tilde{\Omega}$ be smoothly bounded finitely connected domains with a common boundary set A, and suppose $\Omega \subset \tilde{\Omega}$. Let $\delta(A)$ be the Robin capacity of A with respect to Ω, and $\tilde{\delta}(A)$ its Robin capacity with respect to $\tilde{\Omega}$. Then $\delta(A) \leqslant \tilde{\delta}(A)$.*

Proof. Let r be large enough to make the circle C_r surround both $\partial\Omega$ and $\partial\tilde{\Omega}$. Let Γ and $\tilde{\Gamma}$ be the families of arcs in Ω_r and $\tilde{\Omega}_r$, respectively, joining A to C_r. Since $\Gamma \subset \tilde{\Gamma}$, it is clear that every metric admissible for $\tilde{\Gamma}$ is also admissible for Γ, and it is clear that the extremal length $\lambda(\Gamma) \geqslant \lambda(\tilde{\Gamma})$. In view of relation (2), this shows that $\rho(A) \geqslant \tilde{\rho}(A)$ and hence that $\delta(A) \leqslant \tilde{\delta}(A)$.

As an illustration, let Ω be the domain outside the unit circle, let A be the semicircular arc of $\partial\Omega$ in the upper half-plane, and let $\tilde{\Omega}$ be the domain outside the semidisk $\{z: |z| \leqslant 1, \operatorname{Im}\{z\} \geqslant 0\}$. Then the formulas in Section 2 give $\delta(A) = \sin^2(\pi/4) = 1/2$ and $\tilde{\delta}(A) = 1/\sqrt{3} > 1/2$.

For another illustration, let Ω be the domain outside the rectangle R of Example (iv), Section 2. Choose $k=\tan^2(\pi/8)=3-2\sqrt{2}$, so that R is a square with vertices $\pm J$ and $\pm J-iL$, where $L=2J=1.446...$. Let $\tilde{\Omega}$ be the domain outside the corresponding semidisk of radius J in the lower half-plane. Then $\Omega\subset\tilde{\Omega}$ and the real segment $A=[-J, J]$ is part of both boundaries. Its Robin capacity with respect to $\tilde{\Omega}$ is $\tilde{\delta}(A)=J/(3\sqrt{3})=0.139...$, while $\delta(A)=1/8=0.125$.

Next fix the domain Ω while expanding the subset A of $\partial\Omega$.

THEOREM 2. *Let $\Omega\subset\hat{\mathbb{C}}$ be a finitely connected domain as above, and let $A\subset\tilde{A}\subset\partial\Omega$ be closed subsets of the boundary. Then $\delta(A)\leqslant\delta(\tilde{A})$.*

Proof. Again it is clear that the family Γ of arcs joining A to C_r is contained in the family $\tilde{\Gamma}$ joining $\tilde{A}$ to C_r, so $\lambda(\Gamma)\geqslant\lambda(\tilde{\Gamma})$ and $\rho(A)\geqslant\rho(\tilde{A})$. Thus $\delta(A)\leqslant\delta(\tilde{A})$.

Theorem 2 is implicit in a paper of Duren and Schiffer [5, pp. 268–269], where Hopf's lemma is invoked to show in a special case that the Robin function $R(z,\zeta)$ decreases as the set A is expanded. Some illustrations of Theorem 2 are provided by Examples (i) and (iii) in Section 2.

Finally, if the set B is held fixed while the domain expands through the set A, then the Robin capacity of A increases.

THEOREM 3. *Let Ω and $\tilde{\Omega}$ be smoothly bounded finitely connected domains with a common boundary set B, and suppose $\Omega\subset\tilde{\Omega}$. Let $A=\partial\Omega\setminus B$ and $\tilde{A}=\partial\tilde{\Omega}\setminus B$. Then $\delta(\tilde{A})\leqslant\delta(A)$.*

Proof. Let Γ be the family of arcs in Ω joining A to C_r, and let $\tilde{\Gamma}$ be the family of arcs in $\tilde{\Omega}$ joining $\tilde{A}$ to C_r in $\tilde{\Omega}$. It is clear that each arc $\gamma\in\Gamma$ can be extended to an arc $\tilde{\gamma}\in\tilde{\Gamma}$, so any metric admissible for Γ must also be admissible for $\tilde{\Gamma}$. This implies that $\lambda(\Gamma)\leqslant\lambda(\tilde{\Gamma})$ and $\rho(A)\leqslant\rho(\tilde{A})$, so $\delta(\tilde{A})\leqslant\delta(A)$.

4. COMPARISON WITH ORDINARY CAPACITY

It seems interesting to note that the inequality $d(A)\geqslant\delta(A)$ can be proved by the method of extremal length. Duren and Schiffer [6] used a variational method to show that the Robin capacity $\delta(A)$ is in fact the sharp lower bound for $d(f(A))$ under all admissible mappings f of Ω. Since the identity mapping is admissible, this shows in particular that $d(A)\geqslant\delta(A)$. More generally, Duren and Schiffer [5, p. 268] used Hopf's lemma to show in a special case that Robin's function majorizes Green's function. This again implies that $\rho(A)\geqslant\gamma(A)$ and so that $\delta(A)\leqslant d(A)$. We offer here a more geometric proof.

The argument is again based on a simple comparison principle for extremal length. It is known (see Ahlfors [1, p. 54] or Ohtsuka [7, pp. 241–245]) that

$$\gamma(A) = \lim_{r\to\infty} \{2\pi\tilde{\lambda}(A, C_r) - \log r\}, \tag{8}$$

where the extremal distance $\tilde{\lambda}(A, C_r)$ is now the extremal length of the family $\tilde{\Gamma}$ of arcs in

$$\tilde{\Omega}_r = \{z : |z| < r\} \setminus A$$

joining A to C_r. Since $\Omega_r \subset \tilde{\Omega}_r$, it is clear that $\Gamma \subset \tilde{\Gamma}$ and so that $\lambda(\tilde{\Gamma}) \leqslant \lambda(\Gamma)$, or $\tilde{\lambda}(A, C_r) \leqslant \lambda(A, C_r)$. Hence a comparison of (2) and (8) shows that $\gamma(A) \leqslant \rho(A)$, or $d(A) \geqslant \delta(A)$.

It follows by conformal invariance that $d(f(A)) \geqslant \delta(A)$ for all admissible mappings f of Ω, but the extremal length approach does not seem capable of showing that the lower bound $\delta(A)$ is sharp for each set $A \subset \partial\Omega$.

5. Quadrilaterals

A *quadrilateral* Q is a Jordan domain $\Omega \subset \hat{\mathbb{C}}$ together with four distinct boundary points z_1, z_2, z_3, z_4 (called the *vertices* of Q) arranged in the positive sense with respect to Ω. The *exterior modulus* of Q is $m(Q) = a/b$ if there is a conformal mapping of Ω onto the exterior of a rectangle with side-lengths a and b, carrying vertices to vertices and making the boundary arcs (z_1, z_2) and (z_3, z_4) correspond to the sides of length a.

If Ω contains the point at infinity, we shall define the *Robin modulus* $\mu(Q) = \delta(A)/\delta(B)$, where A is the union of the two arcs (z_1, z_2) and (z_3, z_4) of $\partial\Omega$ and $B = \partial\Omega \setminus A$ is the union of the two complementary arcs. As usual, $\delta(A)$ and $\delta(B)$ denote the Robin capacities of A and B with respect to Ω. Observe that $m(Q)$ is a conformal invariant, while $\mu(Q)$ is invariant under conformal mappings that preserve infinity.

Now let us refer to Example (iv) in Section 2, where a conformal mapping f was constructed from the outside of the unit disk to the outside of a rectangle R with vertices $\pm J$ and $\pm J - iL$. By construction, f preserves infinity, it carries the points $\pm e^{\pm i\alpha}$ onto the vertices of R, and the side-lengths $2J$ and L are expressed by (5) and (6) in terms of complete elliptic integrals with parameter $k = \tan^2(\alpha/2)$. If A is the union of the two circular arcs $(e^{i\alpha}, e^{-i\alpha})$ and $(-e^{i\alpha}, -e^{-i\alpha})$, and if B is the union of the two complementary arcs, then their images $\tilde{A} = f(A)$ and $\tilde{B} = f(B)$ are the unions of the two opposite sides of R of lengths $2J$ and L, respectively. Their Robin capacities $\delta(\tilde{A})$ and $\delta(\tilde{B})$ are given by the formulas (7).

Observe that the ratio

$$\mu(Q)=\delta(A)/\delta(B)=\delta(\tilde{A})/\delta(\tilde{B})=\tan\alpha=\frac{2\sqrt{k}}{1-k}$$

is the Robin modulus of the quadrilateral Q consisting of the domain $|z|>1$ with vertices $e^{i\alpha}$, $e^{-i\alpha}$, $-e^{i\alpha}$, and $-e^{-i\alpha}$. The exterior modulus of Q is

$$m(Q)=\frac{2J}{L}=\frac{2[E-(1-k)K]}{E'-kK'}=\psi(k), \tag{9}$$

as given by the formulas (5) and (6). Standard expansion formulas for the elliptic integrals (see [4, p. 297]) may now be applied to show that $m(Q)\sim\pi k$ as $k\to 0$, while $m(Q)\sim(16/\pi)(1-k)^{-2}$ as $k\to 1$. Since $\mu(Q)^2$ has similar asymptotic properties, we have arrived at the following theorem.

THEOREM 4. *Let Q be a quadrilateral over a Jordan domain containing the point at infinity. Suppose that Q is conformally equivalent to a quadrilateral over the outside of a rectangle, under a mapping that preserves infinity. Let $m(Q)$ and $\mu(Q)$ denote the exterior modulus and the Robin modulus of Q. Then*

$$c_1 m(Q)\leqslant\mu(Q)^2\leqslant c_2 m(Q)$$

for some positive absolute constants c_1 and c_2.

Finally, let us consider the extremal distance between opposite sides of a rectangle with respect to the exterior domain. Given a rectangle R with side-lengths a and b, let Ω be the domain outside R and let Γ be the family of curves in Ω that join the two opposite sides of lengths b. What is the extremal length of Γ?

Of course, it is well known that the extremal length of the corresponding curve-family *inside* R is a/b. To calculate $\lambda(\Gamma)$, we refer again to Example (iv) and choose the parameter k such that $a/b=\psi(k)=2J/L$. Since $\psi(k)$ is strictly increasing from 0 to ∞ as k increases from 0 to 1, this requirement determines k uniquely. By the conformal invariance of extremal length, we may as well assume that $a=2J$ and $b=L$. Now consider the conformal mapping g^{-1} of Ω onto the upper half-plane H. It carries the family Γ to a family $\tilde{\Gamma}$ of curves in H joining the real segments $(1, 1/k)$ and $(-1/k, -1)$. The extremal length is unchanged: $\lambda(\tilde{\Gamma})=\lambda(\Gamma)$. Now let H be mapped conformally onto the *interior* of a rectangle by the standard Schwarz–Christoffel function F (see [9, p. 280]) with $F(1)=K$,

$F(-1)=-K$, $F(1/k)=K+iK'$, and $F(-1/k)=-K+iK'$. Then $\tilde{\Gamma}$ is carried onto the curve-family Γ^* inside the rectangle, joining the opposite sides of length K'. Thus $\lambda(\tilde{\Gamma})=\lambda(\Gamma^*)=2K/K'$, and we have proved the following theorem.

THEOREM 5. *Let R be a rectangle with side-lengths a and b, and let Γ be the family of curves lying outside R and joining the opposite sides of length b. Then the extremal length of Γ is* $\lambda(\Gamma)=2K/K'$, *where* $k=\psi^{-1}(a/b)$ *and* $\psi(k)=2J/L$ *is defined by* (9).

We thank Fred Gehring for suggesting that a result of this type could be derived from our mapping function. One immediate corollary is the conformal invariance of the exterior modulus of a quadrilateral.

REFERENCES

1. L. V. AHLFORS, "Conformal Invariants: Topics in Geometric Function Theory," McGraw-Hill, New York, 1973.
2. L. V. AHLFORS AND A. BEURLING, Conformal invariants, *in* "Construction and Applications of Conformal Maps: Proceedings of a Symposium" (E. F. Beckenbach, Ed.), National Bureau of Standards, Appl. Math. Series, No. 18, pp. 243–245, U.S. Government Printing Office, Washington, DC, 1952.
3. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. TRICOMI, (Eds.), "Higher Transcendental Functions," Bateman Manuscript Project, Vol. II, McGraw-Hill, New York, 1953.
4. P. F. BYRD AND M. D. FRIEDMAN, "Handbook of Elliptic Integrals for Engineers and Scientists," 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York, 1971.
5. P. L. DUREN AND M. M. SCHIFFER, Robin functions and energy functionals of multiply connected domains, *Pacific J. Math.* **148** (1991), 251–273.
6. P. DUREN AND M. M. SCHIFFER, Robin functions and distortion of capacity under conformal mapping, *Complex Variables Theory Appl.* **21** (1993), 189–196.
7. M. OHTSUKA, "Dirichlet Problem, Extremal Length and Prime Ends," Van Nostrand Reinhold, New York, 1970.
8. A. I. MARKUSHEVICH, "Theory of Functions of a Complex Variable," Vol. III, Prentice-Hall, Englewood Cliffs, NJ, 1967.
9. Z. NEHARI, "Conformal Mapping," McGraw-Hill, New York, 1952; Dover reprint, 1975.
10. G. PÓLYA AND G. SZEGÖ, Über den transfiniten Durchmesser (Kapazitätskonstante) von Ebenen und räumlichen Punktmengen, *J. Reine Angew. Math.* **165** (1931), 4–49.
11. CH. POMMERENKE, "Univalent Functions," Vandenhoeck & Ruprecht, Göttingen, 1975.