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A UNIFYING PROBABILITY DENSITY FUNCTION

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Abstract—This paper presents a new unifying continuous probability density function. It is shown that the continuous distributions Weibull-, gamma-, Erlang-, χ^2 and exponential distributions can be derived from the proposed density function.

1. INTRODUCTION

It is well documented in the literature (e.g., [1,2]) that the Erlang-, χ^2- and exponential distributions are special cases of the gamma distribution. This paper presents a new little result, which unifies the above distributions with the Weibull distribution. It is shown that using one additional parameter, called a unifying parameter, all the above distributions can be derived from one distribution. The practical usefulness of the proposed probability density function may possibly be limited to software and programming tasks enabling the programmer to use the single density function to create the five distributions. This paper assumes familiarity with the above mentioned continuous density functions.

2. THE UNIFYING PROBABILITY DENSITY FUNCTION

THEOREM. *The unifying probability density function $f(x)$ is the parent of the Weibull-, gamma-, Erlang-, χ^2- and exponential density functions:*

$$\begin{aligned} f(x) &= \frac{\alpha\,\beta^{-\alpha^\theta}}{\Gamma((\alpha^\theta+1)^\theta)}\,x^{\alpha-1}\,e^{-\beta^{-1^\theta}x^{-\alpha^\theta+\alpha+1}}, \qquad x \ge 0, \\ f(x) &= 0, \qquad x < 0, \end{aligned}$$

where $\alpha > 0$, $\beta > 0$, and $\theta = 0$ or $\theta = 1$. Here, α and β are the distribution shape parameters and θ is the unifying parameter.

PROOF.

1. Obtain the Weibull distribution by setting $\theta = 0$.

$$f(x) = \frac{\alpha\,\beta^{-\alpha^0}}{\Gamma((\alpha^0+1)^0)}\,x^{\alpha-1}\,e^{-\beta^{-1^0}x^{-\alpha^0+\alpha+1}} \Rightarrow f(x) = \frac{\alpha\,\beta^1}{\Gamma(1)}\,x^{\alpha-1}\,e^{-\beta^1 x^{-1+\alpha+1}}.$$

Using the definition of the gamma function $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$, the Weibull distribution results:

$$\begin{aligned} f(x) &= \alpha\,\beta\,x^{\alpha-1}\,e^{-\beta x^\alpha}, \qquad x \ge 0, \\ f(x) &= 0, \qquad x < 0, \end{aligned}$$

where $\alpha > 0$, $\beta > 0$.

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2. Obtain the gamma distribution by setting $\theta = 1$.

$$f(x) = \frac{\alpha\,\beta^{-\alpha^1}}{\Gamma((\alpha^1+1)^1)}\,x^{\alpha-1}\,e^{-\beta^{-1^1}\,x^{-\alpha^1+\alpha+1}} \Rightarrow f(x) = \frac{\alpha}{\beta^{\alpha}\,\Gamma\,(\alpha+1)}\,x^{\alpha-1}\,e^{-\beta^{-1}x}.$$

Using the following property of the gamma function

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma\,(\alpha-1) \Rightarrow (\alpha-1) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-1)} \Rightarrow \alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)},$$

the gamma distribution results

$$\begin{aligned} f(x) &= \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)}\,x^{\alpha-1}\,e^{-x/\beta}, \qquad x \geq 0, \\ f(x) &= 0, \qquad x < 0, \end{aligned}$$

where $\alpha > 0$, $\beta > 0$.

3. The Erlang distribution is a gamma distribution with an integer valued shape parameter. The χ^2 distribution is obtained from the gamma distribution by setting $\alpha = \nu/2$ and $\beta = 2$. The exponential distribution is obtained from the gamma distribution by setting $\alpha = 1$. ∎

The relationships between the gamma-, χ^2-, and exponential distributions, including their means and variances, are presented in many probability text books, e.g., in [1,2].

REFERENCES

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2. R.E. Walpole and R.H. Myers, *Probability and Statistics for Engineers and Scientists*, MacMillan, New York, pp. 1–580, (1978).