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Decomposable three-way layouts

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Abstract: In this paper we introduce an apparently new, and we believe important, subclass of three-way layouts or two-way elimination of heterogeneity designs specified by the full information matrix $S_{3.12}$ being decomposable in the following way, $S_{3.12}=\xi_1 S_{3.1}+\xi_2 S_{3.2}-\xi_0 S_{3.0}$, ξ_1, ξ_2 and $\xi_0>0$; the information matrices $S_{3.1}$, $S_{3.2}$, and $S_{3.0}$ correspond, respectively, to the treatment-row subdesign, the treatment-column subdesign, and the model in which both rows and columns are ignored. The special case when $\xi_1=\xi_2=\xi_0=1$ was introduced in Baksalary and Shah (1990). Our subclass comprises designs for which the study of relationships between properties of the three-way design itself, and corresponding properties of its treatment-row and treatment-column subdesigns, is simplified. As the study of block designs is more straightforward than that of three-way designs, we see that the level of difficulty is here reduced from one three-way level to two two-way level designs. We believe that our decomposability property identifies the most general subclass of three-way layouts or two-way elimination of heterogeneity designs for which certain results on connectedness, orthogonality and balance hold.

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1. Preliminaries

We consider the usual three-way layout of experimental design

$$\mathscr{E}(y)=X_1\boldsymbol{\alpha}+X_2\boldsymbol{\beta}+X_3\boldsymbol{\tau}=(X_1:X_2:X_3)\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\\ \boldsymbol{\tau}\end{pmatrix}=X\boldsymbol{\gamma}, \tag{1}$$

where the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\tau}$ consist of the row, column and treatment effects,

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respectively. The matrices X_1, X_2, X_3 are $n\times r$, $n\times c$ and $n\times v$ 'design matrices' identifying the correspondence between the elements of y and, respectively, the rows, columns and treatments, of the three-way layout; the partitioned matrix $X=(X_1:X_2:X_3)$, therefore, is the $n\times(r+c+v)$ design matrix for the full three-way layout. Since exactly one treatment is applied to each observation which appears in precisely one row and one column, we have $X_1e^{(r)}=X_2e^{(c)}=X_3e^{(v)}=e^{(n)}$, where $e^{(a)}$ is the $a\times 1$ vector of ones.

We write $N_{12}=X_1'X_2$ for the incidence matrix whose (i,j)th element is the number of units treated in the ith row and jth column. We then denote its transpose by interchanging the two subscripts, i.e., $N_{12}'=N_{21}=X_2'X_1$. Similarly we let $N_{13}=X_1'X_3$ and $N_{23}=X_2'X_3$. Their transposes are, respectively, $N_{31}=X_3'X_1$ and $N_{32}=X_3'X_2$.

We let $k_1=X_1'e^{(n)}$ denote the vector of row sizes, $k_2=X_2'e^{(n)}$ the vector of column sizes and $k_3=X_3'e^{(n)}$ the vector of treatment sizes or replications. The three matrices $D_1=X_1'X_1$, $D_2=X_2'X_2$ and $D_3=X_3'X_3$ are all diagonal and positive definite, with the successive elements of k_1, k_2 and k_3, respectively, as their diagonal elements.

If we consider rows and columns as sets of nuisance parameters in the model we call the three-way layout a 'two-way elimination of heterogeneity design' (cf., e.g., Agrawal, 1966a). The matrix $S_{3.12}$ is often called the 'information matrix' (John, 1987, pp. 8, 95), the 'C-matrix' (Raghavarao and Federer, 1975) or the 'coefficient matrix' (Pearce, 1983, p. 59) from which both row and column effects have been eliminated. We define this matrix as

$$S_{3.12}=X_3'X_3-X_3'H_{12}X_3=X_3'M_{12}X_3,$$

where the 'residual matrix' M may be defined by $M=I-XX^+$, with X^+ the Moore–Penrose inverse of X, and so M is the orthogonal projector on the null space $\mathscr{N}(X')$. The matrix $H=XX^+=I-M$ is the 'hat matrix' associated with the design matrix X and is the orthogonal projector on the column space, or range, $\mathscr{C}(X)$. Here X is the augmented matrix $(X_1:X_2)$.

Other information matrices of importance are those obtained by ignoring one of the two sets of nuisance parameters. When we ignore the column effects, we call the resulting two-way layout the 'treatment-row subdesign' and the information matrix is given by

$$S_{3.1}=X_3'M_1X_3.$$

For the treatment-column subdesign (where the row effects are ignored) the information matrix is given by

$$S_{3.2}=X_3'M_2X_3.$$

The information matrix for the model in which both rows and columns are ignored will be denoted by $S_{3.0}$, where

$$S_{3.0}=X_3'X_3-\frac{k_3k_3'}{n}=X_3'C_nX_3,$$

and $C_n=I_n-J_n=I_n-(1/n)e^{(n)}e^{(n)\prime}$ is the $n\times n$ centering matrix.

We define efficiency matrices for the treatment-row and treatment-column subdesigns in the two-way elimination of heterogeneity as,

$$A_{3.1}=D_3^{-1/2}S_{3.1}D_3^{-1/2} \quad \text{and} \quad A_{3.2}=D_3^{-1/2}S_{3.2}D_3^{-1/2},$$

respectively. The efficiency matrix for the full design, after eliminating rows and columns, is given by

$$A_{3.12}=D_3^{-1/2}S_{3.12}D_3^{-1/2}.$$

Definition 1. A two-way elimination of heterogeneity design is connected for treatments whenever all elementary treatment contrasts $c'\tau$, for any $v\times 1$ vector satisfying $c'e^{(v)}=0$, are unbiasedly estimable in the design.

Definition 2. A two-way elimination of heterogeneity design is said to be variance balanced or to have variance balance whenever the ordinary least squares estimators of all normalized contrasts in the treatments have the same variance.

Definition 3. A two-way elimination of heterogeneity design is said to be efficiency balanced or to have efficiency balance whenever the ordinary least squares estimators of all normalized contrasts in the treatments have the same efficiency.

A well-known necessary and sufficient condition (cf. Kshirsagar, 1957; Singh, Dey and Nigam, 1979) for a two-way elimination of heterogeneity design, connected for treatments, to be variance balanced is that the information matrix $S_{3.12}$ be a scalar multiple of the centering matrix:

$$S_{3.12}=\lambda C_v=\lambda[I-(1/v)e^{(v)}e^{(v)\prime}]. \quad \lambda>0. \tag{2}$$

An also well-known characterization (cf. Jones, 1959) for an efficiency-balanced, treatment-connected two-way elimination of heterogeneity design is

$$S_{3.12}=\vartheta S_{3.0}=\vartheta[D_3-(1/n)k_3k_3'], \quad \vartheta\in(0,1], \tag{3}$$

where ϑ represents the efficiency with which each treatment contrast $c'\tau$ is estimated.

We define variance balance and efficiency balance for the treatment-row and treatment-column subdesigns in a similar fashion, i.e., the subdesigns are variance balanced whenever the ordinary least squares estimators of all the normalized contrasts in the treatments have the same variance, and efficiency balanced whenever the ordinary least squares estimators of all the normalized contrasts in the treatments have the same efficiency. Whenever the subdesigns are connected we can say that they have

$$\text{variance balance} \quad \Leftrightarrow \quad S_{3.h} = \vartheta_h C_v, \quad h = 1, 2, \tag{4}$$

$$\text{efficiency balance} \quad \Leftrightarrow \quad S_{3.h} = \vartheta_h S_{3.0}, \quad h = 1, 2. \tag{5}$$

Commutativity of the efficiency matrices is an important property for a design to possess. In the context of fixed effect two-way elimination of heterogeneity designs, Baksalary and Shah (1990) simply call this the 'commutativity property'. If the commutativity property holds then the efficiency matrices $A_{3.1}$, $A_{3.2}$ and $A_{3.0}$ are all spanned by the same set of eigenvectors, i.e., there exists an orthogonal matrix U such that $U'A_g U$, $g = 3.1, 3.2, 3.0$, all are diagonal matrices.

2. Decomposability

An apparently new, and we believe important, subclass of two-way elimination of heterogeneity designs is specified by the information matrix $S_{3.12}$ being decomposable in the following way,

$$S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0}, \qquad \xi_1, \xi_2, \xi_0 > 0, \tag{6}$$

cf. Bérubé (1991). This subclass comprises designs for which the study of relationships between properties of the three-way design itself, and corresponding properties of its treatment-row and treatment-column subdesigns, is simplified. As the study of block designs is more straightforward than that of three-way designs, we can see that when (6) is satisfied, the level of difficulty in analyzing the design would be reduced from one three-way level to two two-way level designs. As we will see later in this paper, our decomposability property (6) seems to be, up to now, probably the most general form of designs for which certain results on connectedness, orthogonality and balance hold.

The special case of condition (6) when $\xi_1 = \xi_2 = \xi_0 = 1$ was introduced very recently in Baksalary and Shah (1990), where the two-way elimination of heterogeneity design is then said to satisfy the 'decomposability property', i.e.,

$$S_{3.12} = S_{3.1} + S_{3.2} - S_{3.0}. \tag{7}$$

We will say that the set of designs for which (6) holds, but for which (7) does not hold, satisfies the 'generalized decomposability property', while those for which (7) holds, and hence also (6), we will say satisfy the 'reduced decomposability property'.

Agrawal (1966b) constructed designs for which each of $S_{3.12}$, $S_{3.1}$, $S_{3.2}$ and $S_{3.0}$ has the form $aI + bJ$, i.e., all diagonal elements equal and all off-diagonal elements equal. Although this kind of design does not necessarily satisfy the reduced decomposability property (7), it very often satisfies our generalized decomposability property (6). Since in our generalized decomposability property, the matrices $S_{3.12}$, $S_{3.1}$, $S_{3.2}$ and $S_{3.0}$ need have no particular form, the class of designs satisfying our generalized decomposability property is more general than this special class of designs considered by Agrawal (1966b).

If the two-way elimination of heterogeneity design is ordinary (equal row sizes $\boldsymbol{k}_1 = k_1 \boldsymbol{e}^{(r)}$ and equal column sizes $\boldsymbol{k}_2 = k_2 \boldsymbol{e}^{(c)}$ for some positive integers k_1 and k_2 such that $k_1 r = k_2 c = n$), then the reduced decomposability property (7) is equivalent to

$$S_{3.12} = \boldsymbol{D}_3 - \frac{N_{31}N_{13}}{k_1} - \frac{N_{32}N_{23}}{k_2} + \frac{\boldsymbol{k}_3\boldsymbol{k}_3'}{n}. \tag{8}$$

Any row-column design, i.e., any three-way layout with incidence matrix $N_{12} = \boldsymbol{e}^{(r)}\boldsymbol{e}^{(c)\prime}$, provides a simple example of a design satisfying the reduced decomposability property. Since now the row sizes $k_1 = c$, the column sizes $k_2 = r$, and the total number of observations $n = rc$, the equation (8) becomes

$$S_{3.12} = \boldsymbol{D}_3 - \frac{N_{31}N_{13}}{c} - \frac{N_{32}N_{23}}{r} + \frac{\boldsymbol{k}_3\boldsymbol{k}_3'}{rc}.$$

A somewhat different decomposition of the information matrix $S_{3.12}$ was introduced in Baksalary and Siatkowski (1990) with designs for which the information matrix takes the form

$$S_{3.12} = \boldsymbol{D}_3 - \upsilon_1 N_{31}N_{13} - \upsilon_2 N_{32}N_{23} + \varrho \boldsymbol{k}_3\boldsymbol{k}_3', \quad \upsilon_1, \upsilon_2, \varrho > 0, \tag{9}$$

of which clearly (8) is the special case with $\upsilon_1 = 1/k_1$, $\upsilon_2 = 1/k_2$, and $\varrho = 1/n$. We will say that designs for which (9) holds satisfy the 'extended decomposability property'.

Our generalized decomposability property (6), the extended decomposability property (9) and the reduced decomposability property (7) are not equivalent, as we will show in the following two examples.

Example 1. As an example of a three-way layout that satisfies both (9) and (6) but not (7), we consider the following design with seven rows, seven columns and seven treatments, taken from Agrawal (1966b),

$$\begin{array}{ccccccc}
* & 3 & 5 & * & 2 & * & * \\
* & * & 4 & 6 & * & 3 & * \\
* & * & * & 5 & 7 & * & 4 \\
5 & * & * & * & 6 & 1 & * \\
* & 6 & * & * & * & 7 & 2 \\
3 & * & 7 & * & * & * & 1 \\
2 & 4 & * & 1 & * & * & *
\end{array} \tag{10}$$

where $*$ denotes an empty cell. For this design (10), $S_{3.1} = S_{3.2} = (\frac{7}{3})C_7$, $S_{3.0} = 3C_7$ and $S_{3.12} = C_7$, and so (9) holds with $\varrho = \frac{2}{21}$ and any υ_1 and υ_2 such that $\upsilon_1 + \upsilon_2 = 1$, $\upsilon_1, \upsilon_2 > 0$. Baksalary and Siatkowski (1990) use (10) as an example of a design satisfying (9) but not (7), since obviously here $S_{3.12} \neq S_{3.1} 3 S_{3.2} - S_{3.0}$. We can, however, express $S_{3.12}$ as in (6), i.e., this design satisfies our generalized decomposability property (6) with, for example, $\xi_1 + \xi_2 = 1$, $\xi_1, \xi_2 > 0$ and $\xi_0 = \frac{4}{9}$.

The following example exhibits a design which is not ordinary, and which satisfies our generalized decomposability property (6) but not the extended decomposability property (9). We have, however, not yet found a design which satisfies the extended decomposability property (9) but not our generalized decomposability property (6), nor have we been able to show whether or not there exists such a design.

Example 2. Consider the following design with three rows, three columns, and three treatments,

$$\begin{matrix} 1 & 2 & 3 \\ * & 1 & 2 \\ 3 & * & 1 \end{matrix} \tag{11}$$

where again * denotes an empty cell. It is straightforward to show that the associated information matrix for the full design

$$S_{3.12} = \tfrac{1}{15}\begin{pmatrix} 24 & -12 & -12 \\ -12 & 16 & -4 \\ -12 & -4 & 16 \end{pmatrix},$$

while the information matrices for the treatment-row and treatment-column subdesigns are equal and are given by

$$S_{3.1} = S_{3.2} = \tfrac{1}{6}\begin{pmatrix} 10 & -5 & -5 \\ -5 & 7 & -2 \\ -5 & -2 & 7 \end{pmatrix}, \tag{12}$$

and the information matrix ignoring both rows and columns is given by

$$S_{3.0} = \tfrac{1}{7}\begin{pmatrix} 12 & -6 & -6 \\ -6 & 10 & -4 \\ -6 & -4 & 10 \end{pmatrix}. \tag{13}$$

Hence,

$$S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \tfrac{7}{30} S_{3.0},$$

for any positive ξ_1, ξ_2 such that $\xi_1 + \xi_2 = \frac{6}{5}$.

However, there exist no $\upsilon_1, \upsilon_2, \varrho > 0$ such that $S_{3.12}$ could satisfy the extended decomposability property.

There are special cases when our generalized decomposability property (6) and the extended decomposability property (9) are equivalent. For example, the case of ordinary two-way elimination of heterogeneity designs, i.e., designs which have row sizes all equal to k_1 and column sizes all equal to k_2. For such designs, as Baksalary and Siatkowski (1990) point out, if we postmultiply (9) by $e^{(v)}$, we obtain the equality

$$\mathbf{0} = (1 - \upsilon_1 k_1 - \upsilon_2 k_2 + \varrho n) k_3, \tag{14}$$

implying that

$$1 = \upsilon_1 k_1 + \upsilon_2 k_2 - \varrho n. \tag{15}$$

The extended decomposability property can then be rewritten as

$$S_{3.12}=\upsilon_1 k_1\left(D_3-\frac{N_{31}N_{13}}{k_1}\right)+\upsilon_2 k_2\left(D_3-\frac{N_{32}N_{23}}{k_2}\right)-\varrho n\left(D_3-\frac{k_3k_3'}{n}\right), \tag{16}$$

which is equivalent to (6) with $\xi_1=\upsilon_1 k_1$, $\xi_2=\upsilon_2 k_2$ and $\xi_0=\varrho n$. Substituting into (15), yields

$$\xi_0=\xi_1+\xi_2-1. \tag{17}$$

Our generalized decomposability property (6) can then be rewritten as

$$S_{3.12}=\xi_1 S_{3.1}+\xi_2 S_{3.2}-(\xi_1+\xi_2-1)S_{3.0}. \tag{18}$$

For example, if we look again at the design (10) in Example 1, where $k_1=k_2=3$ and $n=21$, we see that the extended decomposability property is satisfied with $\upsilon_1+\upsilon_2=1$, $\upsilon_1,\upsilon_2>0$ and $\varrho=\frac{2}{21}$. This implies that we can have $\xi_1+\xi_2=3$, $\xi_1,\xi_2>0$ and $\xi_0=2$; in this case, it is obvious that our generalized decomposability property is equivalent to (18), i.e., $-2C_7=-\frac{2}{3}\xi_1 C_7-\frac{2}{3}\xi_2 C_7$, with $\xi_1+\xi_2=3$, $\xi_1,\xi_2>0$.

For designs where our generalized decomposability property (6) holds irrespective of the application of treatments, i.e., designs for which

$$H_{12}=\xi_1 H_1+\xi_2 H_2-\xi_0 J_n, \tag{19}$$

then (18) also holds since, again, if we postmultiply (19) by $e^{(n)}$, we obtain $1=\xi_1+\xi_2-\xi_0$ and hence $\xi_0=\xi_1+\xi_2-1$ as in (17).

3. Results

A problem which seems not yet to have been completely solved, concerns the relationship between connectedness for treatments in a two-way elimination of heterogeneity design and connectedness in its treatment-row and treatment-column subdesigns. Raghavarao and Federer (1975) showed that if a two-way elimination of heterogeneity design is connected for treatments, then the treatment-row and treatment-column subdesigns are also connected (the row-column subdesign need not, however, be connected). However, the converse of this statement is not generally true as was shown by Shah and Khatri (1973). Raghavarao and Federer (1975) show that for equireplicate row-column designs satisfying the condition $N_{13}N_{32}=ke^{(n)}e^{(n)\prime}$, connectedness of the treatment-row and treatment-column subdesigns does lead to treatment-connectedness. This result was first strengthened by Sia (1977) who showed that when $S_{3.1}$ and $S_{3.2}$ commute in an equireplicate row–column design (or equivalently when $N_{31}N_{13}$ and $N_{32}N_{23}$ commute and $N_{12}=e^{(r)}e^{(c)\prime}$), then connectedness of the treatment-row and treatment-column subdesigns implies treatment-connectedness if and only if the sums of the eigenvalues of $S_{3.1}$ and $S_{3.2}$ corresponding to the same eigenvectors are different from k, the number of replications of each treatment. The commutativity of $S_{3.1}$ and $S_{3.2}$ by itself is not sufficient for this result to still hold, however, as was again shown by the design in Shah and Khatri (1973) where $S_{3.1}$ and $S_{3.2}$ do commute.

The equireplicate condition was relaxed in Baksalary and Kala (1980), where the more general commutativity condition

$$A_{3.1}A_{3.2}=A_{3.2}A_{3.1}$$

was considered. In the following theorem, we give an extension for two-way elimination of heterogeneity designs with equal row sizes and equal column sizes, satisfying our generalized decomposability property (6). Our proof follows that of Baksalary and Kala (1980).

Theorem 1. *Consider a two-way elimination of heterogeneity design which is ordinary, i.e., with equal row and column sizes:* $\boldsymbol{k}_1=k_1\boldsymbol{e}^{(r)}$ *and* $\boldsymbol{k}_2=k_2\boldsymbol{e}^{(c)}$, *which satisfies both the generalized decomposability property*

$$S_{3.12}=\xi_1 S_{3.1}+\xi_2 S_{3.2}-\xi_0 S_{3.0},\quad \xi_1,\xi_2,\xi_0>0,$$

and the commutativity property

$$A_{3.1}A_{3.2}=A_{3.2}A_{3.1}.$$

If the treatment-row and treatment-column subdesigns are connected, then the design itself is connected for treatments if and only if

$$\xi_1\phi_s^{(3.1)}+\xi_2\phi_s^{(3.2)}\neq\xi_0,\quad s=1,\ldots,v-1, \tag{20}$$

where $\phi_s^{(3.1)}$ *and* $\phi_s^{(3.2)}$ *are eigenvalues of, respectively,* $A_{3.1}$ *and* $A_{3.2}$ *corresponding to the same eigenvector.*

Furthermore, (20) *is also equivalent to*

$$\xi_1 k_2\mu_s+\xi_2 k_1\omega_s\neq k_1k_2(\xi_1+\xi_2-\xi_0),\quad s=1,\ldots,v-1, \tag{21}$$

where μ_s *is an eigenvalue of* $N_{31}N_{13}\boldsymbol{D}_3^{-1}$ *not equal to* k_1, *and* ω_s *is that eigenvalue of* $N_{32}N_{23}\boldsymbol{D}_3^{-1}$ *not equal to* k_2 *and which corresponds to the same eigenvector as does the eigenvalue* μ_s.

Proof. We have a two-way elimination of heterogeneity design with efficiency matrices satisfying the following relation:

$$A_{3.12}=\xi_1 A_{3.1}+\xi_2 A_{3.2}-\xi_0 A_{3.0},\quad \xi_1,\xi_2,\xi_0>0, \tag{22}$$

in view of our generalized decomposability property (6). Since we assume that the design satisfies the commutativity property, the three matrices $A_{3.1}$, $A_{3.2}$ and $A_{3.0}$ have a common set of eigenvectors. The zero eigenvalue for each matrix corresponds to the same eigenvector $\boldsymbol{D}_3^{1/2}\boldsymbol{e}^{(v)}$. The other $v-1$ eigenvalues of $A_{3.0}=\boldsymbol{I}-(1/n)\boldsymbol{D}_3^{-1/2}\boldsymbol{e}^{(v)}\boldsymbol{e}^{(v)\prime}\boldsymbol{D}_3^{-1/2}$ are all equal to 1. If the treatment-row and treatment-column subdesigns are connected, then the remaining eigenvalues of $A_{3.1}$ and $A_{3.2}$

are all nonzero, and equal, respectively, to

$$\phi_s^{(3.1)} = 1 - \frac{\mu_s}{k_1} \quad \text{and} \quad \phi_s^{(3.2)} = 1 - \frac{\omega_s}{k_2}; \qquad s = 1, \dots, v-1. \tag{23}$$

From (22) we find that the design itself is connected for treatments if and only if the $v-1$ eigenvalues of $A_{3.12}$

$$\xi_1 \phi_s^{(3.1)} + \xi_2 \phi_s^{(3.2)} - \xi_0 \neq 0, \quad s = 1, \dots, v-1, \tag{24}$$

or equivalently (20) holds. Furthermore, substituting (23) in (24) yields the inequality

$$\xi_1\left(1 - \frac{\mu_s}{k_1}\right) + \xi_2\left(1 - \frac{\omega_s}{k_2}\right) \neq \xi_0, \quad s = 1, \dots, v-1,$$

which implies (21). □

Baksalary and Kala (1980) obtained the special case of our Theorem 1 for row-column designs, i.e., with $\xi_1 = \xi_2 = \xi_0 = 1$, $k_1 = c$ and $k_2 = r$.

With our next theorem, we present a relationship between efficiency balance in a two-way elimination of heterogeneity design and efficiency balance in its sub-designs.

Theorem 2. *For a treatment-connected two-way elimination of heterogeneity design satisfying the generalized decomposability property* $S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0}$, ξ_1, ξ_2 *and* $\xi_0 > 0$, *any two of the following properties imply the third*:

(i) *the design is efficiency balanced*;

(ii) *the treatment-row subdesign is efficiency balanced*;

(iii) *the treatment-column subdesign is efficiency balanced.*

Proof. Follows at once from the characterizations in (3) and (5). □

A form of this theorem was first given in the first part of Theorem 2 in Singh, Dey and Nigam (1979). Our version is a slight extension of the version given in Baksalary, Shah and Siatkowski (1990), since we have replaced the more restrictive reduced decomposability property, by our less restrictive generalized decomposability property.

A theorem similar to our Theorem 2, but with one further condition, holds for designs that are variance balanced. We extend Theorem 3 in Baksalary, Shah and Siatkowski (1990) (which is a corrected version of the second part of Theorem 2 in Singh, Dey and Nigam, 1979) with our:

Theorem 3. *For a treatment-connected two-way elimination of heterogeneity design with the number of treatments* $v \geqslant 3$ *and satisfying the generalized decomposability property* $S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0}$, ξ_1, ξ_2 *and* $\xi_0 > 0$, *cf.* (6), *any three of the following properties imply the fourth*:

(i) *the design is variance balanced*;
(ii) *the treatment-row subdesign is variance balanced*;
(iii) *the treatment-column subdesign is variance balanced*;
(iv) *the design is equireplicated.*

Proof. Follows from the characterizations in (2) and (4). □

In our next theorem, we extend a result given by Baksalary and Shah (1990) for designs satisfying the reduced decomposability property, i.e., $S_{3.12}=S_{3.1}+S_{3.2}-S_{3.0}$, to designs satisfying our generalized decomposability property, i.e., $S_{3.12}=\xi_1 S_{3.1}+\xi_2 S_{3.2}-\xi_0 S_{3.0}$, ξ_1, ξ_2 and $\xi_0>0$.

Theorem 4. *If a treatment-connected two-way elimination of heterogeneity design satisfying the generalized decomposability property, i.e.,* $S_{3.12}=\xi_1 S_{3.1}+\xi_2 S_{3.2}-\xi_0 S_{3.0}$, ξ_1, ξ_2 *and* $\xi_0>0$, *is efficiency balanced or if its treatment-row or treatment-column subdesign is efficiency-balanced, then the commutativity property holds.*

Proof. We first suppose that the treatment-row subdesign is efficiency-balanced. Then we can write

$$A_{3.1}=\vartheta A_{3.0},$$

and, postmultiplying by $A_{3.2}$, yields

$$A_{3.1}A_{3.2}=\vartheta A_{3.0}A_{3.2}=\vartheta A_{3.2},$$

which is symmetric and so the commutativity property holds. The proof for the column-treatment subdesign is similar.

When the row-column design itself is efficiency-balanced, then we have

$$A_{3.12}=\vartheta A_{3.0},$$

which is equivalent to

$$A_{3.2}=\frac{(\xi_0+\vartheta)A_{3.0}-\xi_1 A_{3.1}}{\xi_2}. \tag{25}$$

Therefore, if we now premultiply (25) by $A_{3.1}$, we obtain

$$A_{3.1}A_{3.2}=\frac{(\xi_0+\vartheta)A_{3.1}A_{3.0}-\xi_1 A_{3.1}^2}{\xi_2}=\frac{(\xi_0+\vartheta)A_{3.1}-\xi_1 A_{3.1}^2}{\xi_2},$$

which is symmetric and so the commutativity property holds. □

Another relationship between efficiency balance and the commutativity property is given in the following Theorem 5 which is a slight modification of Theorem 4.2 in Baksalary and Shah (1990) and the Lemma on page 7 in Baksalary and Siatkowski (1990). In their Theorem 4.2, Baksalary and Shah (1990) assume only the reduced decomposability property, and in the lemma on page 7, Baksalary and Siatkowski (1990) assume equal row and column sizes with the information matrix satisfying the extended decomposability property $S_{3.12} = D_3 - \upsilon_1 N_{31} N_{13} - \upsilon N_{32} N_{23} + \varrho k_3 k_3'$, $\upsilon_1, \upsilon_2, \varrho > 0$.

Theorem 5. *A treatment-connected two-way elimination of heterogeneity design, which satisfies* $S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0}$ *for some* ξ_1, ξ_2 *and* $\xi_0 > 0$, *is efficiency balanced if and only if it satisfies the commutativity property and* $\xi_1 \phi_s^{(3.1)} + \xi_2 \phi_s^{(3.2)}$ *is the same for all* $s = 1, \ldots, v-1$, *where the nonzero eigenvalues* $\phi_1^{(3.h)}, \ldots, \phi_{v-1}^{(3.h)}$, $h = 1, 2$, *are ordered correspondingly to a fixed set of common eigenvectors of* $A_{3.1}$ *and* $A_{3.2}$.

Proof. If $A_{3.12} = \vartheta A_{3.0}$ for some ϑ, then $S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0}$ implies that

$$\xi_1 A_{3.1} + \xi_2 A_{3.2} = (\xi_0 + \vartheta) A_{3.0},$$

and so $A_{3.1} A_{3.2} = A_{3.2} A_{3.1}$. This in turn implies that

$$\xi_1 \phi_s^{(3.1)} + \xi_2 \phi_s^{(3.2)} = \xi_0 + \vartheta \quad \text{for } s = 1, \ldots, v-1.$$

Conversely, if $A_{3.1} A_{3.2} = A_{3.2} A_{3.1}$ and $\xi_1 \phi_s^{(3.1)} + \xi_2 \phi_s^{(3.2)}$ is equal to a constant c, say, then $S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0}$ implies that

$$\phi_s^{(3.12)} = \xi_1 \phi_s^{(3.1)} + \xi_2 \phi_s^{(3.2)} - \xi_0 = c - \xi_0,$$

and so $S_{3.12}$ is a scalar multiple of $S_{3.0}$. □

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