JOURNAL OF FUNCTIONAL ANALYSIS 114, 106–231 (1993)

Resonant One Dimensional Nonlinear Geometric Optics

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Communicated by R. B. Melrose

Received July 1, 1992

In this paper, we study the existence and resonant interaction of oscillatory wave trains in one space dimension, giving a rigorous proof of the validity of the corresponding expansions of weakly nonlinear optics. We consider both semilinear and quasilinear systems, the latter before shock formation. Some important features of the study are the following.

(1) We prove the existence of families of exact solutions which have asymptotic expansions governed by weakly nonlinear optics. Equations with variable coefficients, nonconstant background fields and nonlinear phases are permitted. Our weak transversality hypotheses allow us to justify expansions where even a formal theory did not exist before.

(2) We make a detailed study of resonances. The geometry associated with such resonances is related to the theory of planar webs.

(3) We study the smoothness of the profiles. Their regularity is ruled by a sum law analogous to that describing the propagation of singularities in one dimension.

(4) The expansions are justified up to the breakdown of the profiles which coincides with a suitably defined breakdown for exact solutions.

* Partially supported by NATO Grant CRG 890904.

† Partially supported by NSF Grant DMS 8601783.

0022-1236/93 $5.00

1. Introduction

The aim of this paper is to give a rigorous description of solutions to 1-d nonlinear hyperbolic equations with highly oscillatory initial data. The results are analogues of the expansions of linear geometric optics. If the initial data oscillate with phase $\varphi^0(x)$ then the solution takes the form

$$\sum_{k=1}^{N} U_k(t, x, \varphi_k(t, x)/\varepsilon),$$

where the $U_k(t, x, \theta_k)$ are periodic in θ_k and the $\varphi_k(t, x)$ are solutions of the eikonal equation with initial data $\varphi^0(x)$. The form of the amplitudes or profiles U_k reveals two distinct nonlinear effects. The periodicity takes into account generation of harmonics but, more interestingly, profiles contain information about resonant interaction between the different wave trains. It is precisely to study such nonlinear effects that the expansions of weakly nonlinear geometrics were developed and it is our goal to put a class of these multiphase expansions on a solid mathematical footing. We consider both semilinear and quasilinear systems, the latter before shock formation. Some important features of the study are the following.

(1) We show that there are families u_ε of exact solutions which have asymptotic expansions of the form

$$u_\varepsilon(t, x) = \sum_{k=1}^{N} U_k(t, x, \varphi_k(t, x)/\varepsilon) + o(1)$$

in the semilinear case and

$$u_\varepsilon(t, x) = u_0(t, x) + \varepsilon \sum_{k=1}^{N} U_k(t, x, \varphi_k(t, x)/\varepsilon) + o(\varepsilon)$$

in the quasilinear case. The profiles $U_k(t, x, \cdot)$ are almost periodic. Of particular interest is that equations with variable coefficients, nonconstant background fields $u_0(t, x)$, and nonlinear phases are permitted because the transversality or coherence hypotheses which we impose are very weak. The coherence assumption of J. Hunter, A. Majda, and R. Rosales [HMR] hold only for problems which, after a change of dependent variables, have constant coefficients and linear phases. With out weak hypotheses we rigorously justify expansions where even a formal theory did not exist before.

(2) The integrodifferential equations determining the profiles U_k involve averaging operators determined by the resonance relations which exist between the phases. The geometry associated with such resonances is

related to the theory of planar webs, that is of pairwise transverse families of foliations by curves in the plane. The notion of equivalence natural for the study of resonance is weaker than that of equivalence of webs under diffeomorphisms. An upper bound, proved by Poincaré [P], on the maximal dimension of the space of all resonances is extended from the C^ω to the C^∞ category with an independent proof.

(3) The smoothness of the profiles varies from point to point and is different for different groups of the fast variables. This regularity measures the rate of decay of the Fourier coefficients and thereby the manner that energy is distributed among the overtones. This regularity is ruled by a sum law analogous to that describing the propagation of singularities in one dimension.

(4) The expansions are justified up to the breakdown of the profiles which coincides with a suitably defined breakdown for exact solutions. As a consequence, the examples of R. Pego [Pe] show rigorously that shock formation can be indefinitely postponed by resonant interaction.

There is a rich literature devoted to the construction and application of weakly nonlinear asymptotic expansions of the type we discuss. Most of that literature is restricted to constant background states and phase functions which are linear. See the articles of J. Hunter, J. Keller, A. Majda, R. Rosales, and Kalyakin [HK, MR, HMR, Kal] for recent results.

The first rigorous justifications that we know of are due to L. Tartar [T1, T2] and concern semilinear equations from the kinetic theory of gases. These results inspired J-L. Joly [J] who treated general semilinear constant coefficients hyperbolic systems with linear phases and introduced the technique of simultaneous Picard iteration which we adopt. In this method, the Picard iterates u_ε^ν converging to the solution u_ε and the iterates U^ν converging to the solution of the profile equations are considered simultaneously. One estimates $u_\varepsilon^\nu(t, x) - U^\nu(t, x, \varphi(t, x)/\varepsilon)$ for each ν and $0 \leqslant \varepsilon \leqslant 1$. This is different from the standard method of finding enough terms in an asymptotic expansion so that the residual is very small and then appealing to continuity results with respect to data. The occurrence of small divisor problems suggests that the latter approach is not promising in the present setting.

Further rigorous results for semilinear problems with linear phases are found in [Ho, McLPT]. Similar linear phases results including quasilinear problems have been the object of much study in the former USSR, see [Kal]. For nonconstant background and nonlinear phases there has been no rigorous justification. The essential difficulty here is the occurrence of oscillatory integrals with phases which may be stationary on small but otherwise very complicated sets. An infinite number of such estimates may occur in the justification of a single expansion.

Finally we mention the important work of R. Diperna and A. Majda [DM] which contains the only rigorous results valid beyond the formation of shocks. They discuss two different sorts of weakly nonlinear expansions: those which correspond to rapid oscillations and those which correspond to bump of height and width of order ε. Moreover, the oscillatory results are restricted to 2×2 systems where resonance is not possible. The study of the validity of oscillatory weakly nonlinear optics beyond shock formation remain an outstanding open problem.

For multidimensional problems with just one phase, nonresonant formal expansions were constructed by Choquet–Bruhat [CB] and justified by J.-L. Joly and J. Rauch [JR3, JR4] and O. Gues [G].

1.1. *Statement of the Problem*

Let (t, x) denote the variable in $\mathbb{R}^2$ and consider a first order system of N equations

$$\partial_t u + A(t, x, u)\,\partial_x u = b(t, x, u) \tag{1.1.1}$$

for $u(t, x)$ which takes its values in $\mathbb{R}^N$; A and b are smooth functions of (t, x, u). As usual, the system (1.1) is called semilinear if A does not depend on u.

We assume strict hyperbolicity on $\mathbb{R}^2$ with t as time, that is that all the eigenvalues of $A(t, x, u)$ are real and simple. We denote them $\lambda_1(t, x, u) < \cdots < \lambda_N(t, x, u)$ and they depend smoothly on $(t, x, u)\in\mathbb{R}^2\times\mathbb{R}^N$.

In this paper we first consider the Cauchy problem

$$u|_{t=0} = h_\varepsilon \tag{1.1.2}$$

with families of Cauchy data of the form

$$h_\varepsilon(x) = H(x, \boldsymbol{\varphi}^0(x)/\varepsilon) + o(1) \tag{1.1.3}$$

in the semilinear case, and

$$h_\varepsilon(x) = h_0(x) + \varepsilon H(x, \boldsymbol{\varphi}^0(x)/\varepsilon) + o(\varepsilon) \tag{1.1.4}$$

in the quasilinear case. The precise meaning of the $o(1)$ and $o(\varepsilon)$ will be given later on, but one may think of them as taken in L^∞. The profile $H(x, \theta^0)$ is a given function on $\mathbb{R}\times\Theta^0$, almost periodic in θ^0, Θ^0 being some finite dimensional real space. The phase $\boldsymbol{\varphi}^0$ is a smooth function on $\mathbb{R}$ valued in Θ^0.

Our goal is to prove that the exact solution u_ε of (1.1.1) (1.1.2) exists on a domain independant of ε and has the form

$$u_\varepsilon(t, x) = \sum_{k=1}^{N} U_k(t, x, \varphi_k(t, x)/\varepsilon) + o(1) \tag{1.1.5}$$

in the semilinear case, and

$$u_\varepsilon(t, x) = u_0(t, x) + \varepsilon \sum_{k=1}^{N} U_k(t, x, \varphi_k(t, x)/\varepsilon) + o(\varepsilon) \tag{1.1.6}$$

in the quasilinear case, where $u_0(t, x)$ is a smooth solution to (1.1.1), with $h_0(x)$ as initial data at $t=0$.

Another problem we want to look at is as follows. Suppose that u_ε is a family of exact solutions of (1.1.1) in the past $\{t<0\}$, of the form (1.1.5) or (1.1.6). Then we want to extend u_ε as a solution of (1.1.1) to positive times, on a domain independent of ε, so that (1.1.5) or (1.1.6) is still valid. For instance, this covers the case where u_ε is in the past the superposition of two wave trains, $U_1 + U_2$ with disjoint supports that meet at times $t=0$, and in the future u_ε will be the superposition of U_1, U_2 and of all the other waves produced by interaction, called here resonances.

The profiles $U_k(t, x, \theta_k)$ are defined for θ_k running in some finite dimensional space Θ_k, and the φ_k are suitable "phase" functions valued in Θ_k. We will find that $U = (U_k)$ must solve an integrodifferential system (see [T1, T2, McLPT, MR, HMR, J, Ho]).

Beside existence properties for the Cauchy problem or continuation results, we also study some qualitative properties of nonlinear oscillating waves:

(1) Propagation of the smoothness in the Θ-variables, smoothness which is related to the decay of the Fourier coefficients for high frequencies. We show in Section 8 that the strength of the new oscillations created by interaction is ruled by a sum law. This phenomenon, though analogous to its well-known parallel for singularities [RR], is actually different and relies on smoothing properties of the averaging operators (Sections 1.5 and 4.2).

(2) Life span of solutions. We will show in Section 7 that the life span of the profiles is equal to a suitably defined life span of the family u_ε which takes some uniform boundedness into account and is of course in general different from the individual life span of each u_ε. This equality of the life spans amounts to saying that the approximate solution contains information about blow-up.

(3) The former property relies on an L^∞ estimate for the restriction of almost-periodic functions on some cones. This as a by product provides the uniqueness of the profile associated to oscillating solutions by (1.1.3)–(1.1.6).

Remark 1.1. Instead of considering oscillating Cauchy data, one could create the oscillations by inserting an oscillating source term in the right hand side of (1.1.1). For example, one could replace $b(t, x, u)$ by $b(t, x, c_\varepsilon, u)$, where c_ε is a given control of the form (1.1.5) or (1.1.6). Moreover, one could also allow oscillations in the coefficients of the left hand side and consider a matrix A of the form $A(t, x, c_\varepsilon, u)$ with c_ε of the form (1.1.6), even in the semilinear case. The proofs given below in Sections 5 and 6 would cover these extensions.

1.2. *An Example*

Next, we would like to explain briefly what resonances are. Consider the following example already used by J. Rauch–M. Reed [RR] in their study of the singularities of semilinear waves,

$$\begin{aligned} X_1 u_1 &= 0 \\ X_2 u_2 &= 0 \\ X_3 u_3 &= u_1 u_2, \end{aligned} \tag{1.2.1}$$

where X_1, X_2, X_3 denote three vector fields on $\mathbb{R}^2$ such that the associated operator is strictly hyperbolic, the first coordinate t of $\mathbb{R}^2$ being timelike.

Suppose that u_1 and u_2 define two regular incoming oscillating waves

$$u_k^\varepsilon(t, x) = a_k(t, x)\, e^{i\varphi_k(t,x)/\varepsilon}, \qquad k = 1, 2 \text{ and } \varepsilon > 0 \tag{1.2.2}$$

such that the amplitudes a_k are supported in characteristic tubes Γ_k of X_k, which do not intersect in the past $t < 0$ and cross each other on a compact set K that lies in the future $\{t > 0\}$.

The third wave, which we assume to vanish in the past, can be explicitely computed, using the third equation. It is supported on the forward characteristic tube $\Gamma_3^+(K)$ issued from K and is given by the oscillatory integral

$$u_3^\varepsilon = X_3^{-1}(a_1 a_2 e^{i(\varphi_1 + \varphi_2)/\varepsilon}) \tag{1.2.3}$$

the symbol X_3^{-1} meaning integration along the characteristics of X_3. The behaviour of u_3^ε depends on the size of the set of critical points of the phase function $\varphi_1 + \varphi_2$

$$C := \{(t, x) \in \mathbb{R}^2;\ X_3(\varphi_1 + \varphi_2) = 0\}$$

in a neighbourhood of K. If C is small (the precise meaning of small will be defined later on) then

$$u_3^\varepsilon = o(1), \qquad \varepsilon \to 0$$

in some L^p space. In that case, the two incoming oscillations with phases φ_1 and φ_2 do not interact. They just cross each other as in the linear case.

On the other hand, if C is a neighbourhood of K, then on $\Gamma_3^+(K)$

$$u_3^\varepsilon = e^{i(\varphi_1+\varphi_2)/\varepsilon} X_3^{-1}(a_1 a_2)$$

describes a new oscillation created by the nonlinear interaction of the two incoming waves: this is *resonance.*

Consider next a more general interaction $b(u_1, u_2)$ in the right hand side of the third equation of (1.2.1). Approximating b by polynomials suggests that one investigates not only $\varphi_1 + \varphi_2$ but also the set of phases $n\varphi_1 + m\varphi_2$, $n \in \mathbb{Z}$, $m \in \mathbb{Z}$. Moreover, if a resonance exists, that is if $\varphi_3 = n_0\varphi_1 + m_0\varphi_2$ is solution to $X_3\varphi_3 = 0$, then in general u_3 will not remain a pure exponential of $i\varphi_3/\varepsilon$. It will also contain all the harmonics so that u_3 will appear as a periodic function of φ_3/ε.

Consider now the case where u_3 does not vanish in the past, but also oscillates with its own phase φ_3 which satisfies $X_3\varphi_3 = 0$. Then one can imagine several possibilities.

(1) There are no resonances and nothing happens.

(2) There is a resonance of the form $n_0\varphi_1 + m_0\varphi_2 = \alpha\varphi_3$ with $\alpha \notin \mathbb{Q}$. Then u_3 becomes quasi-periodic, with the two $\mathbb{Q}$-independent phases φ_3 and $\alpha\varphi_3$.

(3) There is a resonance of the form $n_0\varphi_1 + m_0\varphi_2 = \varphi_3^\#$ with $X_3\varphi_3^\# \equiv 0$, while $\varphi_3^\#$ and φ_3 are linearly independent. Then u_3 oscillates with the two phases $\varphi_3^\#$ and φ_3.

(4) There is a resonance of the form $n_0\varphi_1 + m_0\varphi_2 = \varphi_3 + c$ with $c \in \mathbb{R}\setminus\{0\}$. Then u_3 oscillates with the two phases φ_3 and $\varphi_3 + c$.

These remarks are important to understand the setting of the problem that we will adopt below. Because of the nonlinearity, profiles do not remain pure exponentials and we must consider periodic functions of φ_k/ε. Moreover, because of (2), it is natural to enlarge the class of profiles $U_k(t, x, \varphi_k(t, x)/\varepsilon)$ to quasi, and even to almost-periodic functions of $\varphi_k(t, x)/\varepsilon$. In its turn (suppose, for instance, in the example above, that u_k is an almost-periodic function of φ_k/ε for $k = 1, 2$) this extension leads us to investigate the resonances among all the linear combinations $\alpha_1\varphi_1 + \alpha_2\varphi_2$ wih $\alpha_i \in \mathbb{R}$, and to introduce the spaces Φ_k generated by the φ_k. Now, because of remark (3), it is natural to allow vector valued $\boldsymbol{\varphi}_k$, say in some finite dimensional space Θ_k, or equivalently to consider finite dimensional spaces Φ_k of phases associated to each mode X_k. Note that these spaces are in duality by the mapping

$$\alpha \in \Theta_k^* \to s_\alpha \in \Phi_k \qquad \text{with} \quad s_\alpha(t, x) = \langle \alpha, \boldsymbol{\varphi}_k(t, x) \rangle_{\Theta_k^* \times \Theta_k}. \quad (1.2.4)$$

Finally condition (4) forces us to allow phase displacements. This will be taken into account by considering profiles U_k that are functions of $(t, x, \theta_k) \in \mathbb{R}^2 \times \Theta_k$ and of an extra variable called τ. The substitution in (1.1.5) or (1.1.6) is

$$U_k(t, x, \varphi_k(t, x)/\varepsilon, 1/\varepsilon). \tag{1.2.5}$$

Equivalently, one can view (1.2.5) as the introduction of the extra phase $\varphi_0 \equiv 1$.

1.3. *Conditions on the Phases*

Before stating precise results in the next section, consider a general system (1.1.1) and let us make a few comments on the conditions that must be imposed upon the set of phases (φ_k). This will serve as an introduction to the somewhat abstract presentation given in the beginning of Section 2. First, recall that in expansions like (1.1.5) or (1.1.6), the phases φ_k are solutions of the eikonal equation for the linearized operator

$$L_0 \equiv \partial_t + A_0(t, x)\, \partial_x, \tag{1.3.1}$$

where $A_0(t, x) = A(t, x, u_0(t, x))$.

Let $\lambda_{k,0}(t, x) = \lambda_k(t, x, u_0(t, x))$ denote the eigenvalues of $A_0(t, x)$. Because of the strict hyperbolicity, a function s satisfies the eikonal equation for L_0

$$\det\{(\partial_t s)\, \mathrm{Id} + (\partial_x s)\, A_0(t, x)\} = 0 \tag{1.3.2}$$

if and only if

$$X_k s \equiv \partial_t s + \lambda_{k,0}(t, x)\, \partial_x s = 0 \tag*{$(1.3.2)_k$}$$

for some $1 \leqslant k \leqslant N$.

The function φ_k that will enter in (1.1.5), (1.1.6), or in (1.2.5) is a solution to $(1.3.2)_k$ valued in some finite dimensional space Θ_k. With (1.2.4), this is equivalent to the data of finite dimensional spaces Φ_k of $\mathbb{R}$-valued solutions of $(1.3.2)_k$.

As explained in the example above, nonlinear interaction, for instance, in the right hand side of (1.1.1), immediately yields terms of the form $B(t, x, \varphi(t, x)/\varepsilon)$, where

$$\varphi(t, x) = \{\varphi_k(t, x)\}_{1 \leqslant k \leqslant N} \tag{1.3.3}$$

is valued in $\Theta = \Theta_1 \times \cdots \times \Theta_N$ and B is almost-periodic in the variables $\theta = (\theta_1, ..., \theta_N)$. Expanding B in a series of exponentials, a typical problem (see (1.2.3)) is to solve equations like

$$X_k u(t, x) = a(t, x)\, e^{is(t,x)/\varepsilon} \tag{1.3.4}$$

with, for instance, $u_{|t=0} = 0$, and phases s of the form

$$s(t, x) = s_\alpha(t, x) = \langle \alpha, \varphi(t, x) \rangle = \sum \langle \alpha_k, \varphi_k(t, x) \rangle, \tag{1.3.5}$$

where $\alpha = (\alpha_1, ..., \alpha_N) \in \Theta_1^* \times \cdots \times \Theta_N^*$.

It is well known that the magnitude of the solution depends on whether s is a phase for X_k, or not. If it is, that is if

$$X_k s(t, x) \equiv 0 \tag{1.3.6}$$

then the solution of (1.3.4) looks like $u_\varepsilon(t, x) = \tilde{a}(t, x, \varepsilon)\, e^{is(t,x)/\varepsilon}$ with a symbol $\tilde{a} = O(1)$. In that case, the conclusion is that the phase s must appear in the principal term of (1.1.5) or (1.1.6) so we are led to require the following condition, which is called *closedness* in [HMR],

> for any s of the form (1.3.5), the condition $X_k s(\lambda) \equiv 0$ implies that s belongs to Φ_k, up to a constant. (1.3.7)

On the other hand, if

$$X_k s \neq 0 \qquad \text{almost everywhere,} \tag{1.3.8}$$

then the solution of (1.3.4) is $o(1)$, say in L^p (in x) for all $p < \infty$ (see Section 4 below). If (1.3.8) is strengthened in

$$X_k s \neq 0 \qquad \text{almost everywhere on each integral curve of } X_k \tag{1.3.9}$$

then the solution of (1.3.4) is $o(1)$ in L^∞. When everything is real analytic, then (1.3.6) and (1.3.8) are the only two possibilities. Otherwise, there is still a small gap and we will assume that for each $\alpha \in \Theta^*$, either (1.3.6) or (1.3.8) [resp. (1.3.9)] is valid. This will be called the weak [resp. strong] *transversality condition*. Note that this assumption is much weaker than the *coherence* assumption of [HMR], which implies that either $X_k s \equiv 0$, or $X_k s \neq 0$ everywhere. Examples of phase spaces, natural in the context of resonance, that satisfy either weak or strong condition are given in Section 3.

Remark 1.2. The magnitude in ε of the solution of (1.15) depends strongly on the order of vanishing of $X_k s$ on the characteristics of X_k. If $X_k s \neq 0$ everywhere, then the solution is $O(\varepsilon)$ (indeed it is of the form $\varepsilon \tilde{a}(t, x, \varepsilon)\, e^{is(t,x)/\varepsilon}$). If $X_k s$ vanishes at the first order transversally on each characteristic, then, by the standard stationary phase theorem, the solution is $O(\sqrt{\varepsilon})$. In fact, it is not difficult in the example (1.2.1) to produce for any p, interactions u_3 that are of order $\varepsilon^{1/p}$. Moreover, one has to keep in mind that the parameter α varies in a large set (the almost-periodic spectrum of the function B) so that this behaviour as ε tends to 0, also depends on α and there is a problem of summability in α. The conclusion is that, unless strong assumptions are added, there will be no theory of "the second [resp. third] term "in an expansion like (1.1.5) [resp. (1.1.6)]. See [JMR].

1.4. *Existence of Resonances*

In our setting, in particular under the closedness assumption (1.3.7), resonances occur when there are functions $s_k \in \Phi_k$ such that $\sum s_k =$ constant. More generally, given vector fields X_k, one can ask whether there exist $\mathbb{R}$-valued functions s_k in $C^\infty(\Omega)$ (modulo the constants) such that

$$X_k s_k = 0 \qquad \text{for} \quad k = 1, ..., N \qquad \text{and} \qquad \sum s_k = \text{constant}, \tag{1.4.1}$$

in some open set $\Omega \subset \mathbb{R}^2$. This question, which is related to the theory of webs in differential geometry, will be investigated in Section 3. In particular, let us mention that the space $S(\Omega) \subset \{C^\infty(\Omega)\}$ of all the solutions $s = (s_1, ..., s_N)$ of (1.4.1) has a finite dimension. Indeed, for "generic" vector fields X_k, this space is $\{0\}$. The conclusion is that, in order to have resonances, first the vectors fields X_k must be suitably chosen and second, the phases are also to be chosen carefully. The most famous resonance condition concerns triplets of vector fields or rather the foliations associated to their integral curves and is called the *hexagonal closure property*. See Fig. 1.

When the space $\Phi = \Phi_1 \times \cdots \times \Phi_N$ is specified, the resonances that will enter in our problem are associated to the space $\Phi \cap S(\Omega)$.

In Section 3.4, we will have a special interest in the existence of resonances for systems of conservation laws. If the unperturbed state u_0 is constant, then the linearized operator L_0 has constant coefficients, as well as the vector fields X_k. In that case it is well known that resonances do occur with linear phases $x - \lambda_{k,0} t$, but we will also point out that other resonances appear with polynomial phases. We will study the case where u_0 is a simple wave, and special attention will be paid to Euler's system of gas dynamics. In this case, we will show that if u_0 is a simple wave associated to the linearly degenerate eigenvalue there are always resonances (i.e., nontrivial solutions to (1.4.1)), which form a space of dimension 1. On the other hand, if u_0 is associated to one of the two genuinely nonlinear eigenvalues, then no resonances can occur, except if u_0 is centered.

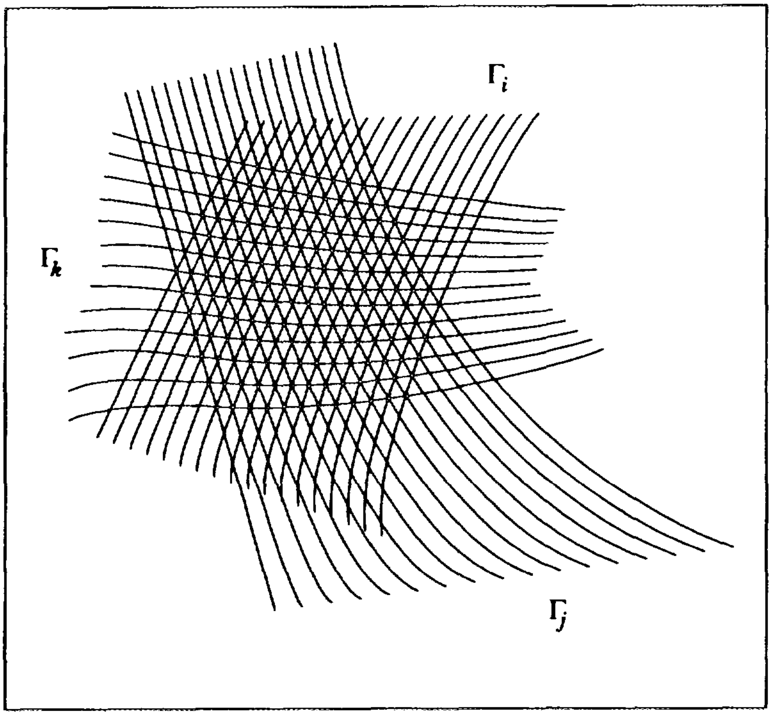

FIG. 1. The hexagonal closure property.

1.5. *The Averaging Operators*

We next describe briefly the averaging operators that enter in the integro-differential equations which determine the profile U_k. For simplicity we assume here that for all k, $\Theta_k \equiv \mathbb{R}$, and that there are no phase displacement, i.e., that

$$\text{for any } s = s_\alpha \text{ of the form (1.3.5), the condition } X_k s_\alpha \equiv 0 \text{ implies that } s_\alpha \in \Phi_k \tag{1.5.1}$$

Note that this condition is stronger than (1.3.7), but in practice it is often satisfied (see Section 2), and indeed the closedness condition is stated that way in [HMR].

As above we call φ_k the Θ_k-valued solution of $(1.3.2)_k$, which is now a basis of Φ_k. The analysis above (together with (1.5.1) leads us to introduce the space of resonances that is the set of those $s = (s_1, ..., s_N) \in \Phi = \Phi_1 \times \cdots \times \Phi_N$ such that $s_1 + \cdots + s_N \equiv 0$, or equivalently the space

$$R = \left\{ \alpha \in \mathbb{R}^N;\ \sum_{j=1}^{N} \alpha_j \varphi_j \equiv 0 \right\}. \tag{1.5.2}$$

R describes the linear relations among the φ_k and it is clear that the function $\boldsymbol{\varphi}$, as defined in (1.3.3), takes its values in the space

$$\Psi = R^{\perp} = \{\theta \in \mathbb{R}^N; \forall \alpha \in R, \langle \alpha, \theta \rangle = 0\}. \tag{1.5.3}$$

Therefore, it is natural when considering functions of the form $B(\boldsymbol{\varphi}(t, x)/\varepsilon)$, to think of B as a function defined on Ψ.

For $\alpha \in \mathbb{R}^N$, take $B(\theta) = e^{i\langle \alpha, \theta \rangle}$ and consider again Eq. (1.3.4). Then, with Definition (1.5.2) and condition (1.5.1), we see that $X_k s_\alpha = 0$, if and only if $\alpha \in R \oplus \tilde{\Phi}_k$, where $\tilde{\Phi}_k = \{\alpha \in \mathbb{R}^N;\ \alpha_j = 0 \text{ for all } j \neq k\}$. Therefore, assuming tranversality as in the discussion sketched above, we see that the solution u of (1.3.4) only depends, up to $o(1)$ terms, on the quantity

$$E_k(e^{i\langle \alpha, \theta \rangle}) = \begin{cases} 0 & \text{if } \alpha \notin R \oplus \tilde{\Phi}_k \\ e^{i\langle \alpha, \theta \rangle} & \text{if } \alpha \in R \oplus \tilde{\Phi}_k. \end{cases} \tag{1.5.4}$$

To solve (1.3.4) with a general almost periodic function $B(\boldsymbol{\varphi}(t, x)/\varepsilon)$ in the right hand side, we expand $B(\theta)$ in a series of exponentials, and the solution depends only, up to $o(1)$ terms, on $E_k(B)$, where E_k is the natural extension of (1.5.4) to almost periodic functions on Ψ. This extension is defined as follows. Let

$$\Psi_k := \{\theta = (\theta_1, ..., \theta_N) \in \Psi; \theta_k = 0\} \subset \Psi. \tag{1.5.5}$$

Then the operator E_k, which extends (1.5.4), is the projection mapping on the set of functions which are invariant by translations parallel to Ψ_k. This operator is obtained by just averaging in the directions parallel to Ψ_k

$$\{E_k(B)\}(\theta) = \lim_{T \to +\infty} T^{-q} \int_{TQ} B(\theta + \psi)\, d\psi, \tag{1.5.6}$$

where q is the dimension of Ψ_k, $d\psi$ a Lebesgue measure on Ψ_k and Q a cube in Ψ_k of measure one. In fact, $\{E_k(B)\}(\theta)$ is clearly invariant under translations parallel to Ψ_k, and, as we shall show in the next section, $\{E_k(B)\}(\theta)$ only depends on the variable θ_k (see, for instance, (1.5.4) and remember that $\theta \in \Psi$).

Finally, we shall see in Section 5 that the solution u_ε of

$$X_k u_\varepsilon = B(\boldsymbol{\varphi}(t, x)/\varepsilon), \qquad u_{\varepsilon|_{t=0}} = 0 \tag{1.5.7}$$

is $u_\varepsilon(t, x) = U(t, x, \varphi_k(t, x)/\varepsilon) + o(1)$, where $U(t, x, \theta_k)$ is the almost-periodic solution of

$$X_k U \equiv \partial_t U + \lambda_{k,0} \partial_x U = E_k(B),\ U_{|t=0} = 0. \tag{1.5.8}$$

This is the key point in the understanding of the problem, even if the things are technically complicated in the quasilinear case. With formulas like (1.5.8), it is clear why the averaging operators E_k are present in the integro-differential equations that determine the profiles. We shall make a more detailed study of these operators in Sections 2 and 4, and in particular we will make them more explicit in the case of a quadratic interaction which is the only one which shows up in the quasilinear case. With this approach we will of course recover the same equations as in [HMR] by quite different considerations. Another way of introducing averaging operators and equations for the profiles which is formal but more in the spirit of geometric optics may be found in [JMR].

2. Main Results

2.1. *Notations*

Consider a strictly hyperbolic system (1.1.1). In the quasilinear case, u_0 is assumed to be a solution of (1.1.1). We assume that the coefficients and u_0 are defined and smooth on an open set $\mathcal{O} \subset \mathbb{R}^2$, whose intersection with the x-axis is an interval I. As in Section 1.3, $L_0 := \partial_t + A_0(t, x)\,\partial_x$ is the linearized operator (1.3.1) and $\lambda_{k,0}(t, x) := \lambda_k(t, x, u_0(t, x))$ are the eigenvalues of $A_0(t, x) := A(t, x, u_0(t, x))$. We assume strict hyperbolicity, that is

$$\forall (t, x) \in \mathcal{O} : \lambda_{1,0}(t, x) < \lambda_{2,0}(t, x) < \cdots < \lambda_{N,0}(t, x). \tag{2.1.1}$$

As in $(1.3.2)_k$, introduce the propagation fields

$$X_k := \partial_t + \lambda_{k,0}(t, x)\,\partial_x. \tag{2.1.2}$$

Their integral curves play a fundamental role. Denote by $t \to \Gamma_k(t; t', y) := (t, \gamma_k(t; t', y))$ the integral curve of X_k passing through the point (t', y). To begin with, fix a possibly small interval $[y_-, y_+] \subset I$ and $T_0 > 0$ so that the characteristic curves $t \to \Gamma_k(t; 0, y)$ are defined for $t \in [0, T_0]$ and $y \in [y_-, y_+]$ and remain in $\mathcal{O}$. Choose T_0 sufficiently small, so that $\gamma_N(T_0; 0, y) \leqslant \gamma_1(T_0; 0, y_+)$. Let

$$\Omega_0 = \{((t, x) \in \mathbb{R}^2 / 0 \leqslant t \leqslant T_0, \gamma_N(t; 0, y_-) \leqslant x \leqslant \gamma_1(t; 0, y_+)\}. \tag{2.1.3}$$

Then Ω_0 is contained in the domain of determinacy of $[y_-, y_+]$, that is

> For any point $(t, x) \in \Omega_0$, and any $k \in \{1, ..., N\}$, the backward characteristic $t' \to \Gamma_k(t'; t, x)$ is defined for $t' \in [0, t]$ and remains in Ω_0. In particular it intersects the line $\{t' = 0\}$ at some point $y \in [y_-, y_+]$. (2.1.4)

Forward characteristics may leave Ω_0 through its sides so, for $y \in [y_-, y_+]$, the characteristic curve $t \to \Gamma_k(t; 0, y)$ remains in Ω_0 for t in a maximal interval $[0, T_k(y)] \subset [0, T_0]$.

In contrast to being a solution of $X_k s \equiv 0$, we say that a function $s \in C^1(\Omega_0)$ is *transverse* to X_k, [resp. *weakly transverse*] when

$$\forall y \in [y_-, y_+], X_k s(\cdot, \gamma_k(\cdot; 0, y)) \neq 0 \qquad \text{a.e.} \quad \text{on } [0, T_k(y)] \tag{2.1.5}$$

[resp. when

$$X_k s(t, x) \neq 0 \qquad \text{a.e.} \quad \text{on } \Omega_0]. \tag{2.1.6}$$

Choose smooth dual bases $r_k(t, x, u)$ [resp. $\ell_k(t, x, u)$] of right [resp. left] eigenvectors of $A(t, x, u)$, normalized by the condition

$$\ell_j \cdot r_k = \delta_{j,k}. \tag{2.1.7}$$

Let $r_{k,0}(t, x) := r_k(t, x, u_0(t, x))$ [resp. $\ell_{k,0}(t, x) := \ell_k(t, x, u_0(t, x))$].

2.2. *Resonances*

As explained in Section 1, the existence of resonances is a phenomenon associated to the system of vector fields X_k. The map taking $c \in \mathbb{R}$ to the constant function on Ω_0 with value c maps $\mathbb{R}$ into $C^\infty(\Omega_0)$. Denote by $C^\infty(\Omega_0)/\mathbb{R}$ the space of functions on Ω_0 modulo the constants.

DEFINITION 2.2.1. A resonance for the system of vector fields X_k, $k = 1, ..., N$ is an N-tuple $\dot{s} = (\dot{s}_1, ..., \dot{s}_N) \in \{C^\infty(\Omega_0)/\mathbb{R}\}^N$ such that

$$\forall k \in \{1, ..., N\} : X_k \dot{s}_k = 0 \qquad \text{on } \Omega_0 \tag{2.2.1}$$

and

$$\sum_{k=1}^{N} d\dot{s}_k = 0 \qquad \text{on } \Omega_0. \tag{2.2.2}$$

The set of all resonances on Ω_0 is denoted $S(\Omega_0)$. The support of a resonance $\dot{s}$ is the set of indices $k \in \{1, ..., N\}$ such that $\dot{s}_k \neq 0$. The number of elements in the support of a nontrivial resonance is called the order of the resonance. A resonance $\dot{s}$ with support contained in $J \subset \{1, ..., N\}$, is called a J-resonance.

Section 3 is devoted to a detailed study of the space $S(\Omega_0)$. An immediate consequence of the independence of the vector fields X_k is that X_i and X_j, for $i \neq j$, have no common solutions, except the constants. Thus we have

LEMMA 2.2.2. *The order of a nontrivial resonance is at least* 3.

2.3. *Conditions on the Phases*

As indicated in Section 1, for each $k = 1, ..., N$, we consider a *finite dimensional vector space* $\Phi_k \subset C^\infty(\Omega_0)$ of solutions to $(1.3.2)_k$

$$\forall s \in \Phi_k : X_k s = 0 \qquad \text{on } \Omega_0. \tag{2.3.1}$$

Because constant phases do not create oscillations, we eliminate from Φ_k the nonzero constant solutions. Keeping in mind that we act locally, we will *assume* that

$$\forall s \in \Phi_k,\ s \not\equiv 0 \Rightarrow ds(t, x) \neq 0 \qquad \text{a.e.} \quad \text{in } \Omega_0. \tag{2.3.2}$$

Define Φ to be the cartesian product $\Phi_1 \times \cdots \times \Phi_N$ and $\tilde{\Phi}_k$ the subspace $\{0\}^{k-1} \times \Phi_k \times \{0\}^{N-k} \subset \Phi$. We use two different notations because it is important to distinguish between the Φ_k's as subspaces of $C^\infty(\Omega_0)$ which may have a nondirect sum, and the $\tilde{\Phi}_k$'s whose direct sum is the space Φ.

Then the *closedness* property is

> If $(s_1, s_2, ..., s_N) \in \Phi$, $k \in \{1, ..., N\}$ and $X_k (\sum s_j) = 0$ on Ω_0 then the function $s := \sum s_j$ belongs to $\Phi_k \oplus \mathbb{R}$. $(\mathscr{C})$

When the phase displacements of (4) in Section 1.2 do not occur, this condition can be strengthened to the *restricted closedness* property which is

> If $(s_1, s_2, ..., S_N) \in \Phi$, $k \in \{1, ..., N\}$ and $X_k(\sum s_j) = 0$ on Ω_0 then, the function $s := \sum s_j$ belongs to Φ_k. $(r - \mathscr{C})$

As indicated in the introduction, this condition is supplemented by *transversality* requirements

> For any $\{s_j\} \in \Phi$ and for any $k \in \{1, ..., N\}$ the condition $X_k \sum s_j \not\equiv 0$ on Ω_0 implies that the function $s = \sum s_j$ is transverse to X_k, i.e., satisfies (2.1.5). $(\mathscr{T})$

Replacing (2.1.5) by (2.1.6) leads to the *weak transversality* condition $(w - \mathscr{T})$.

Note that condition (2.3.2) implies that $s_j \in \Phi_j$ is transverse to X_k when $j \neq k$. Indeed, because $X_j s_j = 0$, and because X_j and X_k are transverse to each other, if $X_k s_j = 0$ at some point $p \in \Omega_0$, then $ds_j = 0$ at p. Commuting ∂_x and X_j we see that $\partial_x s_j = 0$ and hence $ds_j = 0$ all along $\Gamma_j(p)$, the jth characteristic passing through p. If $X_k s_j$ vanished on a set E of positive measure on a kth characteristic, that would imply that $ds_j = 0$ on $\Gamma_j(E)$ which has positive measure in $\mathbb{R}^2$.

In the quasilinear case, only quadratic interactions occur (and this is also the case in semilinear problems with quadratic F.) We are naturally led to the following conditions of *closure under quadratic interaction.*

If $(i, j, k) \in \{1, ..., N\}^3$, $s' \in \Phi_i$, $s'' \in \Phi_j$, and $X_k(s' + s'') = 0$ then $s' + s'' \in \Phi_k \oplus \mathbb{R}$. $(\mathscr{C}q)$

Analogously, the condition of restricted closure under quadratic interactions is.

If $(i, j, k) \in \{1, ..., N\}^3$, $s' \in \Phi_i$, $s'' \in \Phi_j$, and $X_k(s' + s'') = 0$ then $s' + s'' \in \Phi_k$. $(r - \mathscr{C}q)$

Similarly, we have *transversality* under quadratic interaction.

If $(i, j, k) \in \{1, ..., N\}^3$, $s' \in \Phi_i$, $s'' \in \Phi_j$, and $X_k(s' + s'') \not\equiv 0$ then $s' + s''$ satisfies (2.1.5). $(\mathscr{T}q)$

Similarly, *weak transversality condition under quadratic interaction*, $(w\text{-}\mathscr{T}q)$, is defined, with (2.1.6) in place of (2.1.5).

Remark 2.3.1. The coherence assumption of [MR, HMR] requires that either $X_k s \equiv 0$, or $X_k s(t, x) \neq 0$ everywhere on Ω_0. This is what happens when the X_k's are constant and the phases linear in (t, x). This coherence is much stronger than the transversality requirements above. In fact coherence implies that for all k, $\dim \Phi_k \leqslant 1$ and if more than one Φ_k is nontrivial, then there exists a change of the independent variables such that, in the new variables, the X_k's are constant and the phases linear in (t, x). To show this, first note that, from $(\mathscr{C})$ and coherence, follows that $s \in \sum \Phi_k$ either is a constant or satisfies $ds \neq 0$ everywhere. A first consequence of this observation is that $\dim \Phi_k \leqslant 1$ since any two functions in Φ_k have colinear differentials. A second consequence is that $\dim(\sum \Phi_k/\mathbb{R}) \leqslant 2$ since above each (t, x) the differentials of three functions span a two dimensional vector space. Assume now that, for instance, $\dim \Phi_1 = \dim \Phi_2 = 1$ and choose $s_i \in \Phi_i$, $i = 1, 2$, as new coordinates, which is possible since, ds_i never vanishes. From the second consequence, any nontrivial Φ_k is spanned, up to a constant, by a linear function of the new variables. It follows that X_k can be normalized so that it becomes a linear combination of ∂_{s_1} and ∂_{s_2} which completes the proof of the assertion.

Remark 2.3.2. Assuming the coefficients $A(t, x, u)$, the unperturbed solution u_0, and the phases in Φ_j are real analytic, the weak transversality conditions are always satisfied.

Remark 2.3.3. Some of the spaces Φ_k may reduce to $\{0\}$, which means that in the expansions (1.1.5) or (1.1.6) there will be no waves propagating at the corresponding speed. Nevertheless, condition $(\mathscr{C})$ and $(\mathscr{T})$ (or $(r-\mathscr{C})$, $(w-\mathscr{T})\cdots)$ must be checked for all $k \in \{1, ..., N\}$.

Remark 2.3.4. Suppose that $\Phi^{\#}$ is a finite dimensional subspace of $C^{\infty}(\Omega_0)$, and take Φ_k to be the set of the $s \in \Phi^{\#}$ such that $X_k s = 0$. Then the $(r-\mathscr{C})$ condition is satisfied. Indeed, if $s_j \in \Phi \subset \Phi^{\#}$, then the sum $s = \sum s_j$ belongs to $\Phi^{\#}$ and, by definition, the condition $X_k s = 0$ implies that $s \in \Phi_k$.

2.4. *Spaces for the Fast Variables*

Let Θ_k denote the dual space of Φ_k, and let $\varphi_k \in C^{\infty}(\Omega_0; \Theta_k)$ be defined by

$$t, x \in \Omega_0, \qquad s \in \Phi_k \to \langle \varphi_k(t, x), s \rangle := s(t, x). \tag{2.4.1}$$

Then, φ_k is a Θ_k-valued solution of $(1.3.2)_k$. We also introduce the dual space Θ of Φ, which we identify with $\Theta_1 \times \cdots \times \Theta_N$. We then have the Θ-valued mapping $\varphi = (\varphi_1, ..., \varphi_N)$.

The *space of resonances* in Φ, denoted S_Φ, is the set of those $s = (s_1, ..., s_N) \in \Phi$ such that $s_1 + \cdots + s_N$ is constant on Ω_0. If we call $\sigma \to \dot{\sigma}$ the mapping from $C^{\infty}(\Omega_0)$ onto $C^{\infty}(\Omega_0)/\mathbb{R}$, and $s \to \dot{s}$ the corresponding map from $\{C^{\infty}(\Omega_0)\}^N$ onto $\{C^{\infty}(\Omega_0)/\mathbb{R}\}^N$, then S_Φ is the space of those $s \in \Phi$ such that $\dot{s} \in S(\Omega_0)$.

In order to take into account the possible phase displacements, we introduce the space

$$R := \left\{(s, c) = (s_1, ..., s_N, c) \in \Phi \times \mathbb{R}; \sum s_j + c = 0 \text{ on } \Omega_0\right\} \subset S_\Phi \times \mathbb{R}. \tag{2.4.2}$$

Note that R is isomorphic to S_Φ, and, there is a unique linear map $s \to c(s)$ from S_Φ to $\mathbb{R}$ such that

$$R = \{(s, c(s)); s \in S_\Phi\}. \tag{2.4.3}$$

Remark 2.4.1. When the restricted closedness assumption holds, for $s = (s_1, ..., s_N) \in \Phi$, the condition $\sum s_k = \text{constant}$ is equivalent, because of (2.3.2), to $\sum s_k = 0$, so that $R = S_\Phi \times \{0\}$. In this case the last component c can be ignored.

Because of Definition (2.4.1), it is clear that if $(s, c) \in R$, then $\langle \varphi(t, x), s \rangle + c \equiv 0$, so the function $(\varphi, 1)$ takes values in

$$\Psi := R^{\perp} \subset \Theta \times \mathbb{R}. \tag{2.4.4}$$

EXAMPLE 2.4.2. Suppose that bases $\varphi_{k,p}$ $(p = 1, ..., m_k)$ of Φ_k are fixed, as well as a basis $\rho_\alpha = \sum_{k,p} \rho^\alpha_{k,p} \varphi_{k,p}$ ($\alpha = 1, ..., \mu$) of S_Φ. Let $c_\alpha \in \mathbb{R}$ be such that $(\rho_\alpha, c_\alpha) \in R$, i.e., such that

$$\forall (t, x) \in \Omega_0 : \sum_{k,p} \rho^\alpha_{k,p} \varphi_{k,p}(t, x) + c_\alpha = 0. \tag{2.4.5}$$

The dual bases identify $\Theta_k = \Phi_k^*$ with $\mathbb{R}^{m_k}$, $\Theta = \Phi^*$ with $\mathbb{R}^m := \prod_k \mathbb{R}^{m_k}$, and Ψ with the space of $\theta = (\theta_{k,p}, \tau) \in \mathbb{R}^{m+1}$ such that

$$\sum_{k,p} \rho^\alpha_{k,p} \theta_{k,p} + \tau c_\alpha = 0 \qquad \text{for} \quad \alpha = 1, ..., \mu. \tag{2.4.6}$$

Then, $\boldsymbol{\varphi}$ is the map

$$(t, x) \to \{\varphi_{k,p}(t, x)\}_{k,p}, \tag{2.4.7}$$

and, it is clear from (2.4.4), (2.4.6) that $(\boldsymbol{\varphi}, 1)$ is valued in Ψ.

EXAMPLE 2.4.3. If there are no resonances in Φ, that is $S_\Phi = \{0\}$, then $R = \{0\}$ and Ψ is simply the space $\Theta \times \mathbb{R}$.

EXAMPLE 2.4.4 The Cauchy Problem. Consider the Cauchy Problem for (1.1.1) with initial data (1.1.3) or (1.1.4). Suppose that a finite dimensional subspace $\Phi^0 \subset C^\infty([y_-, y_+])$ is given, with $ds^0 \neq 0$ a.e. for all $s^0 \in \Phi^0 \setminus \{0\}$. Define Φ_k as the space of the solutions of

$$X_k s = 0; \qquad s|_{t=0} = s^0 \in \Phi^0. \tag{2.4.8}$$

We can take $T_0 > 0$ sufficiently small so that the functions in Φ_k are defined and C^∞ on Ω_0. Note that the condition (2.3.2) is satisfied.

Let $\Phi^\#$ be the span in $C^\infty(\Omega_0)$ of $\bigcup \Phi_k$. Then Remark 2.3.4 applies, and the restricted closedness condition is satisfied. Only the transversality conditions need to be checked.

Note that (2.4.8) provides us with an isomorphism $\imath_k$ from Φ^0 onto Φ_k taking $s^0 \in \Phi^0$ to the solution of $X_k s = 0$ with Cauchy data s^0. Therefore, the transposed mapping $\rho_k = \imath_k^*$ allows us to identify $\Theta_k = \Phi_k^*$ and $\Theta^0 = \Phi^{0*}$. An equivalent point of view is to consider $\boldsymbol{\varphi}^0$ be the Θ^0-valued function defined by $\langle \varphi^0(t, x), s^0 \rangle := s^0(t, x)$, and to define $\boldsymbol{\varphi}_k$ to be the Θ^0-valued solution of

$$X_k \boldsymbol{\varphi}_k = 0; \qquad \boldsymbol{\varphi}_{k|_{t=0}} = \boldsymbol{\varphi}^0. \tag{2.4.9}$$

For the Cauchy problem, it is natural to consider all the Θ_k as copies of Θ^0, so that Ψ will be viewed as a subspace of $\{\Theta^0\}^N \times \mathbb{R}$.

EXAMPLE 2.4.5 Interaction of Waves. Suppose $p \neq q$ and that Φ_p and Φ_q are finite dimensional subspaces of $C^\infty(\Omega_0)$, solutions of $(1.3.2)_p$ and $(1.3.2)_q$ respectively, as well as satisfying (2.3.2). Let $\Phi^\# \subset C^\infty(\Omega_0)$ be the sum of Φ_p and Φ_q. The sum is direct and contains no constant function other than 0. For $k = 1, ..., N$, let Φ_k be the space of the $s \in \Phi^\#$ such that $X_k s = 0$, so that for $k = p$ or q, we recover the spaces Φ_p and Φ_q.

Remark 2.3.4 applies, and the restricted closedness condition is satisfied. It remains to check the transversality conditions.

Note that generically the construction above yields $\Phi_k = \{0\}$ for $k \neq p$ and q, which means that no resonances occur. On the other hand, if the vector fields X_k are constant and the phases are linear in (t, x), then resonances exist. We refer to Section 3 for further examples of resonances.

2.5. *Projections*

It is important to understand the position of Ψ in the cartesian product $\Theta_1 \times \cdots \times \Theta_N \times \mathbb{R}$. This is linked to the existence of J-resonances.

First we introduce some notation. If J is a subset of $\{1, ..., N\}$, we can restrict our attention to the set of vector fields $\{X_j\}_{j \in J}$ and repeat the construction of Section 2.4 for this set. This yields the space $S_J(\Omega_0)$ of J-resonances, the spaces

$$\Phi_J := \prod_{j \in J} \Phi_j, \qquad \Theta_J := \prod_{j \in J} \Theta_j,$$
$$S_{\Phi_J} := \left\{ s \in \Phi_J;\ \sum_{j \in J} s_j = \text{constant} \right\}, \tag{2.5.1}$$

and, the corresponding spaces $R_J \subset S_{\Phi_J} \times \mathbb{R}$ and $\Psi_J := R_J^\perp \subset \Theta_J \times \mathbb{R}$.

PROPOSITION 2.5.1. *Let π_J denote the projection of $\Theta \times \mathbb{R}$ onto $\Theta_J \times \mathbb{R}$. Then $\pi_J(\Psi) = \Psi_J$. In particular, $\pi_J(\Psi) = \Theta_J \times \mathbb{R}$ if and only if there are no J-resonances or equivalently if $S_{\Theta_J} = \{0\}$.*

Proof. We show that Ψ_J and $\pi_J(\Psi)$ have the same annihilator in $\Phi_J \times \mathbb{R}$, namely R_J. For Ψ_J, this is just the definition. On the other hand, suppose that $(s', c) \in \Phi_J \times \mathbb{R}$ and let $s \in \Phi$ be such that $s_j = s'_j$ if $j \in J$ and $s_j = 0$ if $j \notin J$. Then (s', c) is orthogonal to the image $\pi_J(\Psi)$ if and only if for all $(\theta, \tau) = (\theta_1, ..., \theta_N, \tau) \in R^\perp$ one has $\langle s, \theta \rangle + c\tau = \langle (s', c), \pi_J(\theta, \tau) \rangle = 0$, and hence, if and only if $(s, c) \in R^{\perp\perp} = R$. With the representation (2.4.3), we see that this is equivalent to saying that $c = c(s)$ and s is a J-resonance, and hence to saying that $(s', c) \in R_J$.

At last we note that $\Psi_J = \Theta_J \times \mathbb{R}$ if and only if $R_J = \{0\}$ or equivalently $S_{\Phi_J} = \{0\}$ (see Example (2.4.3)). The proposition is proved.

Because of Lemma 2.2.2, $S_{\Phi_J} = \{0\}$ when J has one or two elements, so we have

COROLLARY 2.5.2. *For any k and any $j \neq k$, the projections $\pi_k : (\theta, \tau) \to (\theta_k, \tau)$ and $\pi_{j,k} : (\theta, \tau) \to (\theta_j, \theta_k, \tau)$ map Ψ onto $\Theta_k \times \mathbb{R}$ and $\Theta_j \times \Theta_k \times \mathbb{R}$, respectively.*

We now introduce the spaces Ψ_k

$$\Psi_k := R^{\perp} \cap \ker(\pi_k) = \{(\theta, \tau) \in \Psi;\ \theta_k = 0 \text{ and } \tau = 0\} \tag{2.5.2}$$

which plays a crucial role, as indicated in Section 1.5. If $k \in J \subset \{1, ..., N\}$, one defines similarly the space $\Psi_{J,k} = \{(\theta, \tau) \in \Psi_J;\ \theta_k = 0 \text{ and } \tau = 0\}$. Proposition 2.5.1 implies the following corollary.

COROLLARY 2.5.3. (i) *For $J \subset \{1, ..., N\}$, $\pi_J(\Psi_k) = \Psi_{J,k}$.*

(ii) *If the set of J-resonances $S_{\Phi_J} = \{0\}$, then for $k \in J$ the projection*

$$(\theta, \tau) \to (\theta_j)_{j \in J, j \neq k} \tag{2.5.3}$$

maps Ψ_k onto $\prod_{j \in J, j \neq k} \Theta_j$.

Moreover, because π_k maps Ψ onto $\Theta_k \times \mathbb{R}$ by Corollary 2.5.2, we also have,

COROLLARY 2.5.4. *π_k induces an isomorphism between the quotient space Ψ/Ψ_k and $\Theta_k \times \mathbb{R}$.*

2.6. *Mean Value Operators*

Recall that the space of continuous almost periodic functions on a finite dimensional real vector space Ψ is the closure in $L^{\infty}(\Psi)$ of the linear span of the exponential functions $e^{i\langle \lambda, \theta \rangle}$ with $\lambda \in \Psi^*$ (see [Kat]). We restrict attention to real valued functions and denote by $C^0_{pp}(\Psi)$ the space of real almost periodic functions on Ψ. We recall in Section 4 the main properties of the space $C^0_{pp}(\Psi)$.

If V is a linear subspace of Ψ, then the averaging operator E_V is defined as

$$\{E_V u\}(\theta) = \lim_{T \to +\infty} T^{-q} \int_{TQ} u(\theta + \psi)\, d\psi, \tag{2.6.1}$$

where q is the dimension of V, $d\psi$ is a Lebesgue measure on V and Q a rectangle in V of measure 1. For $u(\theta) = e^{i\langle \lambda, \theta \rangle}$, the limit in (2.6.1) exists and

$$E_V u = \begin{cases} 0 & \text{if } \lambda \notin V^{\perp} \\ u & \text{if } \lambda \in V^{\perp}. \end{cases} \tag{2.6.2}$$

E_V extends to $C^0_{pp}(\Psi)$ by density. This also shows that Definition (2.6.1) does not depend on the choice of Q.

For $u \in C^0_{pp}(\Psi)$, $E_V u$, as defined by (2.6.1), is a function in $C^0_{pp}(V)$. Moreover, $E_V u$ is invariant by translations parallel to V so it is also a function in $C^0_{pp}(\Psi/V)$.

In Section 4, we prove several properties of these averaging operators, including the following result. If π is a linear map from Ψ into another space Ψ', then for $V \subset \Psi$ and $u \in C^0_{pp}(\Psi')$ one has

$$E_V(u \circ \pi) = \{E_{\pi V}(u)\} \circ \pi \tag{2.6.3}$$

Returning to the situation described in Sections 2.4 and 2.5, we are given the space Ψ and subspaces $\Psi_k \subset \Psi$. According to formula (2.6.1), to each Ψ_k, is associated an averaging operator E_k. Using Corollary 2.5.4, we identify $E_k u$ with a function on $\Theta_k \times \mathbb{R} \approx \Psi/\Psi_k$, and with this identification E_k maps $C^0_{pp}(\Psi)$ to $C^0_{pp}(\Theta_k \times \mathbb{R})$.

We next make this identification more explicit. If u is a function on $\Theta_k \times \mathbb{R}$, it can be lifted to Θ, and hence to Ψ, by the formula

$$u(\theta, \tau) := u(\pi_k(\theta, \tau)) = u(\theta_k, \tau). \tag{2.6.4}$$

It is an abuse of notation to denote by u the lifted function, but it is a convenient one. Any function u on Ψ which is invariant under translations parallel to Ψ_k can be factored as $u = \tilde{u} \circ \pi_k$ with $\tilde{u}$ defined on $\Theta_k \times \mathbb{R}$. As in (2.6.4), we drop the tilda and identify u and $\tilde{u}$.

In the same vein, we write $u(\theta_k, \tau)$ for a function on Ψ that only depends on $(\theta_k, \tau) = \pi_k(\theta, \tau)$, and more generally $u(\theta_J, \tau)$ a function on Ψ that can be factored by π_J, or equivalently, that only depends on $(\theta_J, \tau) = \pi_J(\theta, \tau) \in \Psi_J$.

Let $(\lambda, c) = (\lambda_1, ..., \lambda_N, c) \in \Phi \times \mathbb{R}$ ($\simeq \mathbb{R}^{m+1}$) and let u_λ be the restriction to Ψ of the function $e^{i\{\langle \lambda, \theta \rangle + c\tau\}}$. Then, as in (2.6.2), we have

$$E_k(u_\lambda) = \begin{cases} 0 & \text{if } (\lambda, c) \notin R \oplus (\tilde{\Phi}_k \times \mathbb{R}) \\ u_\lambda & \text{if } (\lambda, c) \in R \oplus (\tilde{\Phi}_k \times \mathbb{R}) \end{cases}. \tag{2.6.5}$$

Indeed Ψ_k is the intersection of Ψ with the kernel of π_k, so that the orthogonal of Ψ_k in $\Phi \times \mathbb{R}$ is $R + (\tilde{\Phi}_k \times \mathbb{R})$ and the sum is direct. This formula extends (1.5.4).

We end this subsection with several important examples and remarks that illustrate the definitions of the operators E_k.

EXAMPLE 2.6.1. If no resonances are present, that is $R = \{0\}$, then everything above is simple with $\Theta = \Phi^* = \Phi_1^* \times \cdots \times \Phi_N^*$, $\Psi = \Theta \times \mathbb{R}$ and $\Psi_k = \Phi_1^* \times \cdots \times \{0\} \times \cdots \times \Phi_N^* \times \{0\}$ with the first factor $\{0\}$ in kth slot.

E_k is then the operator which averages over all the variables different from θ_k

$$E_k u(\theta_k, \tau) = \lim_{T \to +\infty} T^{-\mu} \int_{TQ} u(\theta, \tau)\, d\theta', \tag{2.6.6}$$

where μ is the dimension of Ψ_k, Q is a product of rectangles of measure one in Θ_j, $j \neq k$, and $\theta' = (\theta_j)_{j \neq k}$. When resonances exist, the operators E_k describe the coupling between the different phases.

Remark 2.6.2. If $u = u(\theta_k, \tau)$, only depends on θ_k (see (2.6.4)), then $E_k u = u$. More generally, if u is a product of the form $v(\theta_k, \tau)\, w(\theta, \tau)$, then $E_k u = E_k(vw) = v E_k w$.

Remark 2.6.3. Assume that u is a function that only depends on (θ_J, τ) for some $J \subset \{1, ..., N\}$, that is, $u = \tilde{u} \circ \pi_J$ with $\tilde{u} \in C^0_{pp}(\Psi_J)$. Then, from Proposition 2.5.1, Corollary 2.5.3, and formula (2.6.3), we deduce that, if $k \in J$,

$$E_k u = E_{\Psi_J, k}(\tilde{u}) \tag{2.6.7}$$

these functions being considered as functions of (θ_k, τ). The meaning of (2.6.7) is that, if u only depends on (θ_J, τ), then to compute $E_k u$, one can forget all the vector fields X_j with $j \notin J$ and all the resonances that are not supported in J and act as if Φ were Φ_J. This fact is not as trivial as it may seem, and it motivates the identification of u and $\tilde{u}$ above.

In particular, if there are no J-resonances, then Example 2.6.1 can be applied to the computation of $E_{\Psi_J, k}(u)$ so if u is a function that only depends on (θ_J, τ), one has, for $k \in J$

$$E_k u(\theta_k, \tau) = \lim_{T \to +\infty} T^{-\mu} \int_{TQ} u(\theta_J, \tau)\, d\theta', \tag{2.6.8}$$

where μ is the dimension of $\prod_{j \in J, j \neq k} \Theta_j$, $d\theta'$ a Lebesgue measure on it and Q a rectangle of measure 1 in that space.

EXAMPLE 2.6.4. As a special case of (2.6.8), we see that if $u = u(\theta_j, \theta_k, \tau)$ is an almost-periodic function which depends only on $(\theta_j, \theta_k, \tau)$. Then

$$E_k u(\theta_k, \tau) = \lim_{T \to +\infty} T^{-m_j} \int_{TQ} u(\theta_j, \theta_k, \tau)\, d\theta_j, \tag{2.6.9}$$

where Q is a rectangle in Θ_j of measure 1 and m_j is the dimension of Θ_j.

EXAMPLE 2.6.5. Suppose next that the Φ_j have dimension 1, and that φ_j is a basis of Φ_j. Consider three different indices i, j, and k and a function

$u=u(\theta_i, \theta_j, \theta_k, \tau)$ which only depends on the indicated variables. Because of Remark 2.6.3, we can compute $E_k u$ as if $\Phi=\Phi_i\times\Phi_j\times\Phi_k$.

(a) If there are no resonances between φ_i, φ_j, and φ_k, then (2.6.8) applies so

$$E_k u(\theta_k, \tau)=\lim_{T\to+\infty} T^{-2}\int_0^T\int_0^T u(\theta_i, \theta_j, \theta_k, \tau)\, d\theta_i\, d\theta_j. \qquad (2.6.10)$$

(b) If there is a relation between φ_i, φ_j, and φ_k, Corollary 2.5.2 implies that it can be written as

$$\varphi_i=\alpha\varphi_j+\beta\varphi_k+c. \qquad (2.6.11)$$

Therefore,

$$\theta_i=\alpha\theta_j+\beta\theta_k+c\tau \qquad (2.6.12)$$

is the equation of Ψ in $\Theta_i\times\Theta_j\times\Theta_k\times\mathbb{R}\approx\mathbb{R}^4$, and the function u appears on Ψ as a function of the variables $(\theta_j, \theta_k, \tau)$, namely $v(\theta_j, \theta_k, \tau)=u(\alpha\theta_j+\beta\theta_k+c\tau, \theta_j, \theta_k)$. So, we are back to Example 2.6.4 and we have

$$E_k u(\theta_k, \tau)=\lim_{T\to+\infty} T^{-1}\int_0^T u(\alpha s+\beta\theta_k+c\tau, s, \theta_k, \tau)\, ds. \qquad (2.6.13)$$

Note here the effect of the phase shift c which is mixed with the θ variables in the integral. This example illustrates how the dependence of u on the variables i and j affects the θ_k dependence of the mean. This is an expression of the nonlinear interaction between modes in the presence of resonance.

EXAMPLE 2.6.6. Consider the general case where Φ_j has dimension $m_j\geqslant 1$. Examples of situations where $m_j>1$ will be given in the Section 3. We still assume that $u=u(\theta_i, \theta_j, \theta_k, \tau)$ only depends on the indicated variables. According to Remark 2.6.3, we again compute $E_k u$ as if $\Phi=\Phi_i\times\Phi_j\times\Phi_k$. Now, Theorem 3.1.5 asserts that, given the three vector fields (X_i, X_j, X_k), the space of resonances has dimension at most 1. Thus the discussion performed in Example 2.6.5 is, in fact, general.

(a) If there are no resonances in $\Phi_i\times\Phi_j\times\Phi_k$, then (2.6.8) gives

$$E_k u(\theta_k, \tau)=\lim_{T\to\infty} T^{-m_i-m_j}\int_{TQ} u(\theta_i, \theta_j, \theta_k, \tau)\, d\theta_i\, d\theta_j \qquad (2.6.10)'$$

with Q a rectangle of measure 1 in $\Theta_i\times\Theta_j$.

(b) There is one resonance, or more precisely the dimension of the space $R_{\{i,j,k\}}$ is 1, and hence $\Psi_{\{i,j,k\}}$ has codimension 1 in $\Theta_i \times \Theta_j \times \Theta_k \times \mathbb{R}$ because of Corollary 2.5.2. One can choose coordinates (θ_i^0, θ_i') in Θ_i, (θ_j^0, θ_j') in Θ_j and (θ_k^0, θ_k') in Θ_k, with scalar first components (and second component of length $\dim \Theta_m - 1$, for $m = i, j, k$) such that the Eq. (2.6.12) of $\Psi_{\{i,j,k\}}$ is

$$\theta_i^0 = \alpha\theta_j^0 + \beta\theta_k^0 + c\tau. \tag{2.6.12$'$}$$

Then formula (2.6.13) is replaced

$$E_k u(\theta_k, \tau)$$
$$= \lim_{T \to \infty} T^{1-m_i-m_j} \int_0^T \int_{TQ} u(\alpha s + \beta\theta_k^0 + c\tau, \theta_i', s, \theta_j', \theta_k^0, \theta_k')\, d\theta_i'\, d\theta_j'\, ds, \tag{2.6.13$'$}$$

where Q is a rectangle of measure one in the space of the (θ_i', θ_j') variables. The whole thing means that there may be extraneous variables θ_i' and θ_j', in which case what you have to do is just average over them.

Remark 2.6.7. If the restricted closedness condition is satisfied, then $R = S_\Phi \times \{0\}$, as in Remark 2.4.1, so that $\Psi = \Psi' \times \mathbb{R}$, where Ψ' is the orthogonal of S_Φ in Θ. On the other hand, $\Psi_k \subset \Psi' \times \{0\}$ and can be identified with a subspace of Ψ'. In that case, it is clear that if u does not depend on τ, then neither does $E_k u$.

Note the difference with (2.6.13). If $c \neq 0$, even if u does not depend on τ, $E_k u$ may.

2.7. *Function Spaces*

We now introduce the spaces of functions we need. If Ω is a closed domain contained in Ω_0, we denote by $\mathscr{C}^0(\Omega; \Psi) := C^0(\Omega; C^0_{pp}(\Psi))$ the space of continuous functions from Ω into $C^0_{pp}(\Psi)$. This is a Banach space equipped with the obvious norm. As usual we will consider $U \in \mathscr{C}^0(\Omega; \Psi)$ as a function on $\Omega \times \Psi$, and $\mathscr{C}^0(\Omega; \Psi)$ appears as a closed subspace of $L^\infty(\Omega \times \Psi)$. However, let us emphasise that the condition $U \in \mathscr{C}^0(\Omega; \Psi)$ is much stronger than just requiring that $U(t, x, \cdot) \in C^0_{pp}(\Psi)$ for each $(t, x) \in \Omega$. In particular, any $u \in \mathscr{C}^0(\Omega; \Psi)$ can be represented as

$$u(t, x, \theta, \tau) \sim \sum_{(\lambda, c) \in \Lambda} a_{\lambda, c}(t, x)\, e^{i\{\langle \lambda, \theta \rangle + c\tau\}}, \tag{2.7.1}$$

where the spectrum, Λ, is a denumberable in set in Ψ^* which is independent of (t, x) (see Proposition 4.1.3 below).

For $k \in \mathbb{N}$, define the space $\mathscr{C}^k(\Omega; \Psi)$ of those functions $U \in \mathscr{C}^0(\Omega; \Psi)$ whose derivatives in (t, x, θ, τ), $\partial^\alpha_{t,x,\theta,\tau} U$, of order $|\alpha|$ less or equal to k, belong to $\mathscr{C}^0(\Omega; \Psi)$.

We also need the usual spaces $C^0(\Omega)$ of continuous functions on Ω, and $C^k(\Omega)$ of functions whose derivatives of order $\leqslant k$ belong to $C^0(\Omega)$. These spaces will be equipped with a family of norms,

$$|u|_{\varepsilon,k,p,\Omega} := \sum_{|\alpha| \leqslant k} \varepsilon^{|\alpha|} \sup_t \|\partial^\alpha_{t,x} u(t, \cdot)\|_{L^p(\Omega_t)}, \tag{2.7.2}$$

where $\Omega_t = \{x : (t, x) \in \Omega\}$. When $p = +\infty$ and $k = 0$ this is just the L^∞ norm on Ω.

We shall use the following terminology.

DEFINITION 2.7.1. (i) A family $u_\varepsilon \in C^k(\Omega)$ is bounded in $C^k_\varepsilon(\Omega)$ if the norms $|u_\varepsilon|_{\varepsilon,k,\infty,\Omega}$ are bounded.

(ii) We say that $u_\varepsilon = o(1)$ in $C^k_\varepsilon(\Omega)$ [resp. in $L^\infty W^{k,p}_\varepsilon(\Omega)$] if $|u_\varepsilon|_{\varepsilon,k,\infty,\Omega} \to 0$ [resp. $|u_\varepsilon|_{\varepsilon,k,p,\Omega} \to 0$] as $\varepsilon \to 0$.

Similar definitions hold for functions on $[y_-, y_+]$.

2.8. *The Semilinear Cauchy Problem*

Here the matrix $A(t, x)$ in (1.1.1) does not depend on u and we consider the Cauchy problem

$$\begin{aligned} Lu \equiv \partial_t u_\varepsilon + A(t, x)\, \partial_x u_\varepsilon &= b(t, x, u_\varepsilon(t, x)) \\ u_{\varepsilon|t=0} &= h_\varepsilon. \end{aligned} \tag{2.8.1}$$

Dropping the subscript o, we denote by λ_k, r_k, l_k the eigenvalues , and eigenvectors of $A(t, x)$.

We fix spaces Φ_k satisfying (2.3.1) (2.3.2), and perform the constructions of Section 2.4.

THEOREM 2.8.1. *Assume that the condition* ($\mathscr{C}$) *and* ($\mathscr{T}$) [*resp.* ($w - \mathscr{T}$)] *are satisfied and that*

The family h_ε $(0 < \varepsilon \leqslant 1)$ *is bounded in* $L^\infty([y_-, y_+])$. (2.8.2)

There exist $H_k \in \mathscr{C}^0([y_-, y_+]; \Theta_k \times \mathbb{R})$ *such that* $\ell_k(0, x) \cdot h_\varepsilon(x) - H_k(x, \varphi_k(0, x)/\varepsilon, 1/\varepsilon) = o(1)$ *in* $L^\infty([y_-, y_+])$, [*resp. in* $L^p([y_-, y_+])$]. (2.8.3)

Then

(i) *There is* $T > 0$ *such that for all* $\varepsilon \in]0, 1]$ *the solution* u_ε *of the*

Cauchy problem (2.8.1) *exists on* $\Omega = \Omega_0 \cap \{t \leqslant T\}$ *and the family* u_ε *is bounded in* $L^\infty(\Omega)$.

(ii) *There are functions* $U_k \in \mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R})$ *such that the difference*

$$u_\varepsilon(t, x) - \sum_{k=1}^{N} U_k(t, x, \varphi_k(t, x)/\varepsilon, 1/\varepsilon)\, r_k(t, x)$$
$$= o(1) \text{ in } L^\infty(\Omega), [\text{resp. in } L^\infty L^p(\Omega)]. \tag{2.8.4}$$

(iii) *The* U_k*'s are the unique solutions of the system of* N *integro-differential equations*

$$\begin{aligned} X_k U_k(t, x, \theta_k, \tau) &= E_k\{\ell_k(t, x) \cdot b_k(t, x, U(t, x, \theta, \tau))\} \\ U_{k|t=0}(x, \theta_k, \tau) &= H_k(x, \theta_k, \tau), \end{aligned} \qquad (2.8.5)_k$$

where U *denotes the function*

$$U(t, x, \theta, \tau) := \sum_{k=1}^{N} U_k(t, x, \theta_k, \tau)\, r_k(t, x) \tag{2.8.6}$$

and

$$b_k(t, x, U) := b(t, x, U) - \sum_{j=1}^{N} (X_k r_j)\, U_k \tag{2.8.7}$$

Remark 2.8.2. In the right hand side of $(2.8.5)_k$, E_k is applied for each $(t, x) \in \Omega$ to the function $(\theta, \tau) \to B(t, x, \theta, \tau) = \ell_k(t, x) \cdot b_k(t, x, U(t, x, \theta, \tau))$, and therefore $E_k B$ is a function of (t, x, θ_k, τ) which belongs to $\mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R})$ (see Section 4). In the left hand side, X_k is applied to the function $U_k(t, x, \theta_k, \tau)$, so it makes sense to say that $U_k \in \mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R})$ is a (weak) solution of Eq. $(2.8.5)_k$.

Remark 2.8.3. It is part of the theorem to prove that the system (2.8.5) has a solution U with $U_k \in \mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R})$ for all k.

Remark 2.8.4. If one assumes $(r - \mathscr{C})$ instead of $(\mathscr{C})$ and if the H_k do not depend on τ, that is if

$$\ell_k(0, x) \cdot h_\varepsilon(x) - H_k(x, \varphi_k(0, x)/\varepsilon) = o(1), \tag{2.8.8}$$

then U does not depend on τ either, so that u_ε has the form

$$u_\varepsilon(t, x) = \sum_{k=1}^{N} U_k(t, x, \varphi_k(t, x)/\varepsilon)\, r_k(t, x) + o(1). \tag{2.8.9}$$

This is the classical form for the expansions of nonlinear geometric optics.

It is worth emphasizing that the weaker closure hypothesis yields the weaker conclusion (2.8.4) even if the Cauchy data satisfy (2.8.8).

Remark 2.8.5. Theorem 2.8.1 applies to the Cauchy problem for Section 1.1 which data of the form

$$h_\varepsilon(x) = H(x, \varphi^0(x)/\varepsilon) + o(1). \tag{2.8.10}$$

Indeed, one constructs the space Φ as indicated in Example 2.4.4 and condition $(r - \mathscr{C})$ is satisfied. The Eqs. (2.4.9), $\varphi_k(0, x) = \varphi^0(x)$, and (2.8.10) imply (2.8.8) with $H_k = \ell_k \cdot H$. Thus assuming $(\mathscr{T})$ or $(w - \mathscr{T})$ and combining Theorem 2.8.1 with Remark 2.8.4 above, we obtain the existence of u_ε on a domain Ω independent of ε with an expansion (2.8.9).

Remark 2.8.6. Theorem 2.8.1 also contains a continuation result. Let u_ε be a bounded family of continuous solutions of (2.8.1) on $\Omega_1 = \Omega_0 \cap \{t \leqslant T_1\}$, which satisfies (2.8.4) on Ω_1 with profiles $U_k \in \mathscr{C}^0(\Omega_1; \Theta_k \times \mathbb{R})$, for some $T_1 < T_0$. Then Theorem 2.8.1 applies to the Cauchy problem with initial data $u_\varepsilon(T_1, \cdot)$ on $t = T_1$, and there is $T > T_1$ such that the solutions u_ε and the profiles U_k can be continued on $\Omega = \Omega_0 \cap \{t \leqslant T\}$ so that (2.8.4) still holds on Ω. In particular, this remark together with Example 2.4.5, solves the problem of the interaction of two wave trains that was stated in Section 1.1 of the introduction.

2.9. *The Quasilinear Cauchy Problem*

We return to the notations introduced at the beginning of Section 2.1. In particular Ω_0 is defined in (2.1.3) and $u_0 \in C^\infty(\Omega_0)$ has Cauchy data $h_0 \in C^\infty([y_-, y_+])$.

For convenience, we write the Cauchy data (1.1.2) in the form $h_0 + \varepsilon h_\varepsilon$ and look for a solution of the form $u_0 + \varepsilon u_\varepsilon$. The equations for the new unknowns are

$$\begin{aligned} \partial_t u_\varepsilon + A^\#(t, x, \varepsilon u_\varepsilon)\, \partial_x u_\varepsilon &= b^\#(t, x, \varepsilon u_\varepsilon(t, x)) \cdot u_\varepsilon \\ u_{\varepsilon|t=0} &= h_\varepsilon, \end{aligned} \tag{2.9.1}$$

where $A^\#(t, x, v) := A(t, x, u_0(t, x) + v)$, and $b^\#$ is the sum of two smooth matrices b' and b''

$$b'(t, x, v) \cdot v := \{A(t, x, u_0(t, x)) - A(t, x, u_0(t, x) + v)\}\, \partial_x u_0$$

$$b''(t, x, v) := b(t, x, u_0(t, x) + v) - b(t, x, u_0(t, x)).$$

Because the characteristics of $A^\#$ now depend on v we must slightly decrease the domain Ω_0 so that it remains in the domain of determinacy of $[y_-, y_+]$, for all sufficiently small functions v. Fix $\rho > 0$ and introduce

$$\Omega^\rho := \{(t, x) \in \mathbb{R}^2 : 0 \leqslant t \leqslant T_0, \gamma_N(t; 0, y_-) + \rho t \leqslant x \leqslant \gamma_1(t; 0, y_+) - \rho t\}. \tag{2.9.2}$$

We assume that spaces $\Phi_k \subset C^\infty(\Omega_0)$ are given and satisfy (2.3.1), (2.3.2). The constructions of Section 2.4 yield the crucial averaging operators.

THEOREM 2.9.1. *Assume that the condtion* $(\mathscr{C}q)$ *and* $(\mathscr{T}q)$ *[resp.* $(w - \mathscr{T}q)$*] are satisfied. Assume in addition that*

the family h_ε *is bounded in* $C^1_\varepsilon([y_-, y_+])$, *and,* (2.9.3)

there exists $H_k \in \mathscr{C}^1([y_-, y_+]; \Theta_k \times \mathbb{R})$ *such that* $\ell_{k,0}(0, x) \cdot h_\varepsilon(x) - H_k(x, \varphi_k(0, x)/\varepsilon, 1/\varepsilon) = o(1)$ *in* $C^1_\varepsilon([y_-, y_+])$ *[resp. in* $L^\infty W^{1,p}_\varepsilon([y_-, y_+])$*].* (2.9.4)

Then,

(i) *There is* $T > 0$ *and* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon \leqslant \varepsilon_0$ *the solution* u_ε *of the Cauchy problem* (2.9.1) *exists on* $\Omega = \Omega^\rho \cap \{t \leqslant T\}$ *and the family* u_ε *is bounded in* $C^1_\varepsilon(\Omega)$.

(ii) *There are functions* $U_k \in \mathscr{C}^1(\Omega; \Theta_k \times \mathbb{R})$ *such that*

$$u_\varepsilon(t, x) - \sum_{k=1}^{N} U_k(t, x, \varphi_k(t, x)/\varepsilon, 1/\varepsilon)\, r_{k,0}(t, x) = o(1) \text{ in } C^1_\varepsilon(\Omega) \quad [\text{resp. in } L^\infty W^{1,p}_\varepsilon(\Omega)]. \tag{2.9.5}$$

(iii) *The* U_k*'s are the unique solutions of the following system of* N *equations*

$$\begin{gathered} X_k U_k + E_k\left(\sum_{i,j} \Gamma^k_{i,j} U_i (D_j U_j)\right) = E_k(\ell_{k,0} \cdot b_k U) \\ U_{k\,|t=0}(x, \theta_k, \tau) = H_k(x, \theta_k, \tau). \end{gathered} \tag{2.9.6$_k$}$$

In these equations,

$$U(t, x, \theta, \tau) := \sum_{k=1}^{N} U_k(t, x, \theta_k, \tau)\, r_{k,0}(t, x), \tag{2.9.7}$$

$$b_k U := \bar{b} U - \sum_{j=1}^{N} (x_k r_{j,0})\, U_k \tag{2.9.8}$$

the D_i are vector fields in θ_i

$$(D_i V)(t, x, \theta, \tau) := \partial_x \varphi_i(t, x) \cdot \partial_{\theta_i} V(t, x, \theta, \tau) \tag{2.9.9}$$

(*note that* $\partial_x \varphi_i(t, x) \in \Theta_i$, *while* $\partial_{\theta_i} V(t, x, \theta, \tau) \in \Theta_i^*$), *and*

$$\Gamma_{i,j}^k(t, x) := \ell_{k,0}(t, x) \cdot \left(\frac{\partial A^\#}{\partial v}(t, x, 0)\, r_{i,0}(t, x) \right) \cdot r_{j,0}(t, x) \tag{2.9.10}$$

$$\bar{b}(t, x) := b^\#(t, x, 0). \tag{2.9.11}$$

Remark 2.9.2. In (2.9.10), $((\partial/A^\#/\partial v)(t, x, 0) r_{i,0})$ is the derivative of $A^\#$ in the direction $r_{i,0}$ of the v-space. It is a matrix and the $\Gamma_{i,j}^k(t, x)$ are scalar.

$D_j U_j(t, x, \theta_j)$ is also a scalar valued function so $\Gamma_{i,j}^k U_i(D_j U_j)$ is a scalar function on $\Omega \times \Psi$. Since $E_k\{\Gamma_{i,j}^k U_i(D_j U_j)\}$ is a well defined scalar valued function on $\Omega \times \Theta_k \times \mathbb{R}$, Eq. (2.9.6) makes sense.

Again, it is part of the theorem that this equation has a solution.

The Remarks 2.8.4–2.8.6 which follow Theorem 2.8.1, about the Cauchy problem, the continuation problem, and the irrelevance of τ when $(r - \mathscr{C})$ holds, apply equally well in the present, quasilinear case. In addition one can use the results of Example 2.6.6 to compute explicitely the action of E_k on $U_i(D_j U_j)$. In particular when the Φ_j have dimension 1, $(2.9.6)_k$ together with (2.6.13) recover the equations of profiles that were given in [MR, HMR] under the more restrictive assumption of coherence. The equations for the profiles in the case of higher dimensional spaces Θ or with nonrestricted closureness are new.

2.10. *Life Span and Uniqueness*

Consider first, the semilinear problem (2.8.1). We call $T(\varepsilon)$ the life span of the solution u_ε. $T(\varepsilon)$ is the supremum of the $T \in]0, T_0]$ such that (2.8.1) has a unique solution in $L^\infty(\Omega_0 \cap \{t \leqslant T\})$. It is known that, if $T(\varepsilon) < T_0$, then

$$\liminf_{t \to T(\varepsilon)} \|u_\varepsilon(t, \cdot)\|_{L^\infty(\Omega_t)} = +\infty. \tag{2.10.1}$$

Similarly, we call T_* the life span of U, that is the supremum of the $T \in]0, T_0]$ such that (2.8.5) has a unique solution in $\mathscr{C}^0(\Omega_0 \cap \{t \leqslant T\}; \Psi)$. One can prove that if $T_* < T_0$, then

$$\liminf_{t \to T_*} \|U(t, \cdot)\|_{L^\infty(\Omega_t \times \Psi)} = +\infty. \tag{2.10.2}$$

We will show that under the strong transversality assumption $(\mathscr{T})$,

$$T_* \leqslant \liminf_{\varepsilon \to 0} T(\varepsilon). \tag{2.10.3}$$

First we give a simple example, involving only ordinary differential equations, showing that strict inequality can occur in (2.10.3).

EXAMPLE 2.10.1. Consider the system

$$\begin{aligned} &\partial_t u_\varepsilon = (u_\varepsilon)^2 - v_\varepsilon \cdot (u_\varepsilon)^3, \qquad && u_{\varepsilon|_{t=0}} = 1 + \cos^2(x/\varepsilon) \\ &(\partial_t + \partial_x)\, v_\varepsilon = 0, && v_{\varepsilon|_{t=0}} = \varepsilon. \end{aligned} \tag{2.10.4}$$

The life span of u_ε is the domain of existence of the solution of the equation $y' = y^2 - \varepsilon y^3$, $y(0) = a \in [1, 2]$, so for any $\varepsilon > 0$ we have $T(\varepsilon) = +\infty$.

On the other hand, the equations for the profiles are

$$\begin{aligned} &\partial_t U = U^2, \qquad && U_{|_{t=0}} = 1 + \cos^2(\theta) \\ &(\partial_t + \partial_x) V = 0, && V_{|_{t=0}} = 0. \end{aligned} \tag{2.10.5}$$

Comparing with the time of existence of the solution of the $y' = y^2$; $y(0) = a \in [1, 2]$, we see that $T_* = 1$.

What happens in this example is that at time $t = 1$, the solution u_ε develops a singularity as $\varepsilon \to 0$, and although the Cauchy data remain bounded, there is no uniform control of u_ε in L^∞ after time $T = 1$. In other words, there is a loss of continuity, and even of boundedness of the mapping $h_\varepsilon \to u_\varepsilon$. This motivates the introduction of the following more relevant quantity.

> Let $T_*(\delta)$ be the supremum of the $T \in \,]0, T_0]$, such that (2.8.1) has a uniformly bounded family of solutions u_ε, for $0 < \varepsilon \leqslant \delta$. (2.10.6)

Note that $T_*(\delta)$ is a decreasing function of $\delta > 0$. The following statement makes (2.10.3) more precise.

THEOREM 2.10.2. *Assume* $(\mathscr{C})$ *and* $(\mathscr{T})$ *together with* (2.8.2) *and* (2.8.3) *with* $p = +\infty$. *Then,*

$$T_* = \lim_{\delta \to 0} T_*(\delta) \tag{2.10.7}$$

and the approximation (2.8.4) *is valid for all* $T < T_*$.

Remark 2.10.3. We show by example that this result and even (2.10.3) fails if one only assumes the weak transversality condition $(w - \mathscr{T})$.

Consider the problem

$$\begin{aligned} &\partial_t u_\varepsilon = (u_\varepsilon)^2 + 2v_\varepsilon \cdot w_\varepsilon, && u_{\varepsilon|_{t=0}} = 0 \\ &(\partial_t + \partial_x)\, v_\varepsilon = 0, && v_{\varepsilon|_{t=0}} = \cos(x^2/\varepsilon) \\ &(\partial_t - \partial_x)\, w_\varepsilon = 0, && w_{\varepsilon|_{t=0}} = \cos(x^2/\varepsilon). \end{aligned} \tag{2.10.8}$$

The three phases in play given by the Cauchy problem with initial condition x^2 are: $\varphi_0 = x^2$, $\varphi_1 = (x+t)^2$, and $\varphi_2 = (x-t)^2$. There are no resonances, but we have $\partial_t(\varphi_1 - \varphi_2) \equiv 0$ on the entire characteristic $\{x=0\}$ of the vector field ∂_t. Thus strong transversality is violated. The equations for the profiles are

$$\begin{aligned} &\partial_t U = (U)^2 + 2E_0(VW), && U_{|_{t=0}} = 0 \\ &(\partial_t + \partial_x)\, V = 0, && V_{|_{t=0}} = \cos(\theta_1) \\ &(\partial_t - \partial_x)\, W = 0, && W_{|_{t=0}} = \cos(\theta_2). \end{aligned} \tag{2.10.9}$$

Then, $V(t, x, \theta_1) = \cos(\theta_1)$ and $W(t, x, \theta_2) = \cos(\theta_2)$. Since there are no resonances, θ_0, θ_1, and θ_2 are independent variables, and $E_0(VW) = 0$ because the mean values of V and W are 0. Thus $U \equiv 0$ and $T_* = +\infty$.

On the other hand, $u_\varepsilon(t, 0)$ is solution of the ordinary differential equation

$$y' = y^2 + 1 + \cos(2t^2/\varepsilon), \qquad y(0) = 0.$$

Let $a(t) := \int_0^t \cos(2\tau^2)\, d\tau$. This function is bounded (and in fact it has a limit as $t \to +\infty$). If y is defined on $[0, T]$, then because $y' \geqslant 1 + \cos(2t^2/\varepsilon)$, one has $y(t) \geqslant t + \sqrt{\varepsilon}\, a(t/\sqrt{\varepsilon})$ and if ε is small enough, $y(1) \geqslant \frac{1}{2}$. Then, because $y' \geqslant y^2$, it is clear that $y \to +\infty$ before time $t = 2$.

Summing up, we have shown that for this example we have $T_* = +\infty$, and that $T(\varepsilon) \leqslant 2$, if ε is small enough.

One has a similar result for quasilinear systems.

> Let $T_*(\delta)$ be the supremum of the $T \in\,]0, T_0]$, such that (2.9.1) has solutions u_ε, such that $\varepsilon u_\varepsilon$ remain bounded in $C^1(\Omega^\rho \cap \{t \leqslant T\})$ for $0 < \varepsilon \leqslant \delta$. (2.10.10)

Let T_* be the supremum of the $T \in\,]0, T_0]$, such that (2.9.6) has a solution in $\mathscr{C}^1(\Omega^\rho \cap \{t \leqslant T\}; \Psi)$.

THEOREM 2.10.4. *Assume* ($\mathscr{C}q$), ($\mathscr{T}q$), (2.9.3), *and*, (2.9.4) *with* $p = +\infty$. *Then,*

$$T_* = \lim_{\delta \to 0} T_*(\delta) \tag{2.10.11}$$

and the approximation (2.9.5) *is valid for all* $T < T_*$.

In this statement, the parameter ρ which enters in the Definition (2.9.2) of Ω^ρ, is fixed. In fact, as we will see in Section 6, the domain Ω_0 itself (or more precisely, $\Omega_0 \times \Psi$) is a domain of determinacy for the system (2.9.6). A refinement is possible with ρ tending to 0 in (2.10.11).

The proofs of Theorems 2.10.2 and 2.10.4 are based on continuation arguments and two inequalities. The first one is trivial and asserts that if

$$v_\varepsilon(t, x) = V(t, x, \varphi_k(t, x)/\varepsilon, 1/\varepsilon), \tag{2.10.12}$$

then,

$$\|v_\varepsilon(t, \cdot)\|_{L^\infty(\Omega_t)} \leqslant \|V(t, \cdot)\|_{L^\infty(\Omega_t \times \Theta_k \times \mathbb{R})}. \tag{2.10.13}$$

The second inequality is nontrivial. It relies on theorem of Kronecker and Baire and asserts that, asymptotically, the converse estimate is true.

THEOREM 2.10.5. *Let Φ_k be a finite dimensional space that satisfies* (2.3.2), *Θ_k its dual space and let φ_k be as in* (2.4.1). *Let $V \in \mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R})$, and, v_ε be given by* (2.10.12). *Then,*

$$\|V(t, \cdot)\|_{L^\infty(\Omega_t \times \Theta_k \times \mathbb{R})} \leqslant \limsup_{\varepsilon \to 0} \|v_\varepsilon(t, \cdot)\|_{L^\infty(\Omega_t)}. \tag{2.10.14}$$

This theorem has an immediate corollary which is the uniqueness of the principal symbol in the expansion of any family v_ε.

COROLLARY 2.10.6. *Suppose that φ_k are as above and that v_ε is a bounded family in $C^0(\Omega)$. Then there is at most one $V \in \mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R})$ such that*

$$v_\varepsilon(t, x) - V(t, x, \varphi_k(t, x)/\varepsilon, 1/\varepsilon) = o(1) \qquad \textit{in } L^\infty(\Omega). \tag{2.10.15}$$

When applied to the solutions u_ε given by Theorems 2.8.1 or 2.9.1, this uniqueness result is, in the nature of the case, different from the uniqueness of the solution to the equations of profiles (2.8.5) or (2.9.6).

2.11. *The Sum Law*

In this section we study the smoothness in the variables θ of the solutions to Eqs. (2.8.5) or (2.9.6). We treat first the semilinear case in detail. At the end of the section we sketch the corresponding result for quasilinear systems. For simplicity, we assume that the restricted closure condition $(r - \mathscr{C})$ is satisfied and that the profiles H_k and U_k do not depend on τ, so that (2.8.5) becomes

$$X_k U_k(t, x, \theta_k) = E_k\{\ell_k(t, x) \cdot b_k(t, x, U(t, x, \theta))\} \qquad (2.11.1)_k$$
$$U_{k|_{t=0}}(x, \theta_k) = H_k(x, \theta_k).$$

In what follows $\Omega = \Omega_0 \cap \{t \leq T\}$ is fixed. We first give some definitions.

DEFINITION. (i) For $\sigma \in \mathbb{N}$, $C^{\sigma}_{pp}(\Theta_k)$ is the space of functions $U \in C^0_{pp}(\Theta_k)$ whose derivatives of order less or equal to σ belong to $C^0_{pp}(\Theta_k)$

(ii) A function $U \in \mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R})$ is called of class σ in θ_k, at $(t_0, x_0) \in \Omega$, if there is a neighborhood ω of (t_0, x_0) such that the restriction of U to $\omega \cap \Omega$ is continuous in (t, x) with values in $C^{\sigma}_{pp}(\Theta_k)$.

These definitions are extended to $\sigma = +\infty$ by taking intersections for all $\sigma \in \mathbb{N}$.

Similarly, we say that $H \in \mathscr{C}^0([y_-, y_+]; \Theta)$ is of class σ at $x_0 \in [y_-, y_+]$ if there exists an open set ω in $[y_-, y_+]$ containing x_0 such that $H \in C^0(\omega; C^{\sigma}_{pp}(\Theta))$.

Suppose $H_k \in \mathscr{C}^0([y_-, y_+]; \Theta_k)$ is given. The smoothness in θ_k of H_k is measured by a function $x \to \sigma^0_k(x)$ from $[y_-, y_+]$ into $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ such that

$$H_k \text{ is of class } \sigma^0_k(x) \text{ at } x \in [y_-, y_+]. \tag{2.11.2}$$

Our goal is to predict the smoothness at a point $(t, x) \in \Omega$ of the solution U of (2.8.5). The result depends on the set of resonances and we are led to introduce the

DEFINITION. Given $k \in \{1, ..., N\}$, we call $\mathfrak{J}(k)$ the collection of subsets $J \subset \{1, ..., N\}$ such that $k \in J$ and there is a resonance $s = (s_j)_{1 \leq j \leq N} \in R$, whose support $\{j : s_j \neq 0\}$ is J.

Thus $\mathfrak{J}(k)$ is the set of the supports of those resonances that create oscillations in the kth mode.

Let $\{\sigma_k(t, x)\}_{1 \leq k \leq N}$ be the largest functions from Ω_0 into $\bar{\mathbb{N}}$ such that

$$\sigma_k(0, x) \leq \sigma^0_k(x) \tag{2.11.3}$$

$$\sigma_k \text{ is a nonincreasing function of } t \text{ along the characteristic curves of } X_k. \tag{2.11.4}$$

$$\text{For all } J \in \mathfrak{J}(k), \qquad \sigma_k(t, x) \leq \sum_{\substack{j \in J \\ j \neq k}} \sigma_j(t, x). \tag{2.11.5}$$

This function is very similar to the one introduced in [RR] to measure the H^{σ} smoothness of solutions. We note two differences. First the regularity here is restricted to integer values. We believe that this restriction is not essential. Our proof relies on Theorem 4.2.1 which in turn is proved only for integers. Second, note that (2.11.5) takes into account the detailed structure of resonances. In [RR], all interactions were treated as if they were binary which yields a worst case estimate.

As in [RR] the functions σ_k can be constructed by an iterative process which we pause to describe. Let $\sigma_k^0(t, x)$ be the extension of $\sigma_k^0(x)$ which is constant along the characteristics of X_k, and, let the σ_k^ν, $\nu \geqslant 1$, be defined inductively by

$$\sigma_k^{\nu+1}(p) = \inf_{q \in \Gamma_k^-(p)} \min \left\{ \sigma_k^\nu(q), \inf_{J \in \mathfrak{J}(k)} \sum_{\substack{j \in J \\ j \neq k}} \sigma_j^\nu(q) \right\}, \tag{2.11.6}$$

where $\Gamma_k^-(p)$ denotes the backward k-characteristic from p to $t = 0$. Then σ_k^ν is larger than any function that satisfies (2.11.3)–(2.11.5) and decreases with ν. Therefore σ_k^ν converges toward σ_k.

The functions σ_k can be understood in a different way. Consider the oriented trees in Fig. 2. The branches are arcs of backward characteristics, labelled by an index (α) which corresponds to the vector field. Such an arc represents an oscillatory wave train of the αth mode. The vertex r represents a nonlinear interaction, a resonance of order three resulting in an outgoing oscillation of the ith mode.

The vertices, other than p, with $0 < t \leqslant t(p)$, are labelled by pairs (a, J), where a belongs to Ω and J is the support of a resonance. This indicates the presence of card$(J) - 1$ incoming waves interacting at a. For example, the vertex r has label $(r, \{m, n, i\})$, and an m-wave and an n-wave may

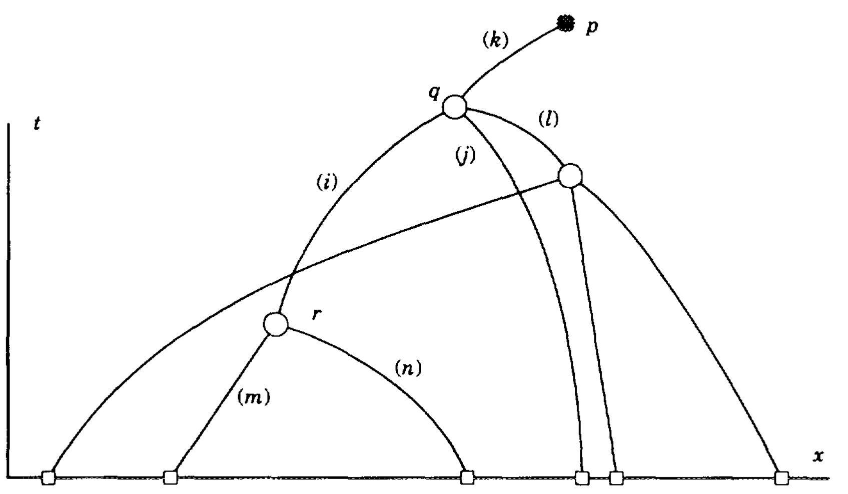

FIG. 2. The resonant tree in the sum law.

interact to produce an i-wave. The vertex q has label $(q, \{i, j, k, l\})$, and waves i, j, l may interact to create a k-wave.

Starting from p is a backward k-characteristic. Any other vertex is associated with the support of a resonance J. If the outgoing arc is an α-arc, then the incoming arcs are all the β-arcs with $\beta \in J \setminus \{\alpha\}$.

The intersection point, a, of an α-arc with $t(a) = 0$ has label (a, α).

We do not exclude the possibility that, for successive vertices (a, J) and (b, K), the points a and b coincide. In that case, $J \neq K$, and two distinct resonant interactions at a are possible.

For such a tree $\mathscr{A}$,

$$\sigma(\mathscr{A}) := \sum \sigma_\alpha^0(a) \tag{2.11.7}$$

the sum being taken over all the points (a, α) of $\mathscr{A}$ on $t = 0$. Then,

$$\sigma_k(p) := \operatorname{Inf} \sigma(\mathscr{A}), \tag{2.11.8}$$

where the infimun is taken over all the trees as above, that have p as a root and start with a k-arc.

THEOREM 2.11.1. *Suppose that* $H_k \in \mathscr{C}^0([y_-, y_+]; \Theta_k)$ *and* $U_k \in \mathscr{C}^0(\Omega; \Theta_k)$, $k = 1, ..., N$ *is a solution of* (2.11.1). *Assume that* (2.11.2) *holds, with strictly positive functions* σ_k^0, *and let* $\sigma_k(t, x)$ *be the function defined in* (2.11.8). *Then, at each point* $(t, x) \in \Omega$, U_k *is of class* $\sigma_k(t, x)$.

Remark 2.11.2. The restriction $\sigma_k^0 > 0$ is essential to our proof and also to the result. Assuming the σ_k^0 lower semicontinuous (l.s.c.), one can show that the σ_k are also l.s.c., provided $\sigma_k^0 > 0$. On the other hand, if one of the $\sigma_k^0 = 0$ for some x, the σ_k are no longer l.s.c. and are therefore not good candidates to define classes of regularity. This can be seen by adapting the example with 4 speeds of [RR]. In that case, if σ_k^0 vanishes somewhere, the index defined by (2.11.3)–(2.11.5) will not be l.s.c. and will not be given by (2.11.6) and (2.11.8). A new index is likely to be defined, which is l.c.s. but will not satisfy (2.11.6) and (2.11.8).

One could also work in Hölder spaces, keeping the same restrictions $\sigma_k^0 > 0$ as above.

Note that a similar restriction, $\sigma > \frac{1}{2}$ is made in [RR].

EXAMPLE 2.11.3. If there are no resonances at all, then σ_k is just σ_k^0 propagated along the characteristics of X_k.

EXAMPLE 2.11.4. Consider the three vector fields $X_1 = \partial_t - \partial_x$, $X_2 = \partial_t$ and $X_3 = \partial_t + \partial_x$ with the linear phases $\varphi_1 = x + t$, $\varphi_2 = x$, and $\varphi_3 = x - t$ so that there is one resonance $\varphi_1 - 2\varphi_2 + \varphi_3 = 0$. Assume that $\sigma_2^0 = +\infty$ and

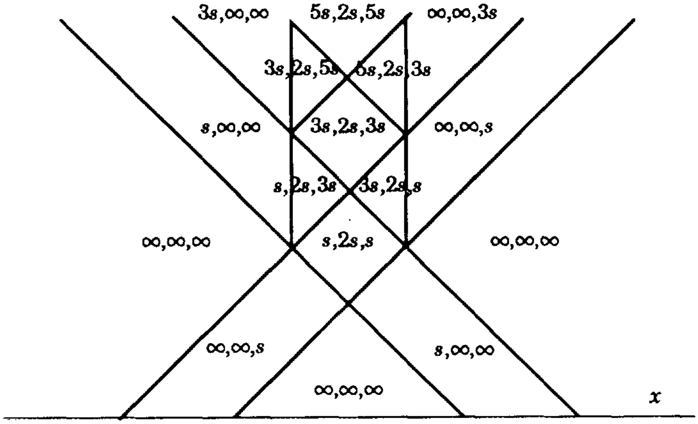

FIG. 3. The sum law.

$\sigma_1^0 = +\infty$ [resp. $\sigma_3^0 = +\infty$] except on the interval $[1, 2]$ [resp. $[-2, -1]$], where $\sigma_1^0 = s$ [resp. $\sigma_3 = s$]. See Fig. 3.

Finally we note that Theorem 2.11.1 extends to quasilinear systems without essential modification. Assume $(r - \mathscr{C})$, $H_k \in \mathscr{C}^1([y_-, y_+]; \Theta_k)$ and let $U_k \in \mathscr{C}^1(\Omega; \Theta_k)$, $k = 1, ..., N$ be the solution of (2.9.6). Assume that (2.11.2) holds, with functions $1 + \sigma_k^0$ that are strictly greater than one. Then, at each point $(t, x) \in \Omega$, U_k is of class $1 + \sigma_k(t, x)$, with the same function $\sigma_k(t, x)$ as above and $\mathfrak{J}(k)$ involving now only third order resonances.

3. Geometrical Aspects of Resonance

This section is devoted to the study of the properties of the solutions of (2.2.1), (2.2.2). For each operator L there is a finite dimensional space of resonances, denoted by S in Section 2.2. This space has additional structure when viewed as a subspace of the cartesian product $S_1 \times \cdots \times S_N$ of its projections on the N spaces of functions invariant by the flows of the X_n's. This structure determines the averaging operators and thereby the nature of the nonlinear interactions wich are possible. These considerations lead to a definition of resonantly equivalent operators. This equivalence is weaker than the equivalence of the N-web of characteristic curves under diffeomorphisms, at least when the space S is small.

Different operators have unequivalent interactions even when the corresponding spaces of resonances have the same dimension. The most studied case is that of constant coefficient operators and phases which are linear in t, x. For these operators we show that additional resonant phases exist which are polynomials of degree $d \in [2, N-2]$. This example is anything but typical, it has space S of maximal dimension.

Moreover, constant coefficient operators are only special examples in the class of the hexagonal operators for which all possible resonances of order 3 exist. In Section 3.3, we study this class in detail when $N=4$.

Finally, at the end of the section, we study the structure of resonances for the linearized operator of a quasilinear system, with special emphasis on the example of gas dynamics.

3.1. *The Spaces S of resonances*

Assume that we are given a finite number of two by two transverse C^∞ vector fields X_n, $n=1, ..., N$, in Ω, an open, connected subset of $\mathbb{R}^2$. We also suppose that the differential operator

$$L(t, x; \partial_t; \partial_x) = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_N \end{bmatrix}$$

is strictly hyperbolic, the first coordinate t of $\mathbb{R}^2$ being timelike.

DEFINITION 3.1.1. A resonance for L is any element $(s_1, s_2, ..., s_N) \in (C^\infty(\Omega)/\mathbb{R})^N$ solution in Ω of the system

$$\begin{aligned} X_n s_n &= 0, \qquad n = 1, ..., N \\ \sum_{n=1}^{N} ds_n &= 0. \end{aligned} \tag{3.1.1}$$

The order of a nontrivial resonance $(s_1, s_2, ..., s_N)$ is the number of nontrivial phase functions that it contains. This order is strict if the resonance is not a linear combination of resonances of smaller order. Recall that the order of a nontrivial resonance is at least 3. The space of all resonances for L is denoted by $S(\Omega)$.

Remarks. (1) As was shown in [JR1, JR3], the overdetermined system (3.1.1) is elliptic so distribution solutions to (3.1.1) are C^∞ or real analytic according to the regularity of the coefficients of L.

(2) The space S does not depend on the X_n but rather on the foliations defined by the characteristic curves of the X_n.

Let us introduce some notations. For short drop Ω and write S for $S(\Omega)$. Denote by S_n the image of S by π_n the nth projector: $s = (s_1, s_2, ..., s_N) \to s_n$. The spaces S and $\tilde{S}_n := \{0\}^{n-1} \times S_n \times \{0\}^{N-n}$ are linear subspaces of $S_1 \times \cdots \times S_N$.

Regarding the way L propagates oscillations, the choice of the phase space $\Phi = S_1 \times \cdots \times S_N$ is kind of canonical. The space Φ contains all possible resonance relations for L. Nonlinear interactions occur for phases only to the extent that the corresponding space intersect Φ. Thus it is natural to discuss the closure and the transversality properties of Section 2.3 for Φ. In that setting we observe that $(w\mathscr{T})$ is automatically satisfied if L is real analytic while the stronger property $(\mathscr{T})$ is not satisfied in general. When it holds, it relies on deeper properties of L (see Section 3.4). The closure property is automatic.

PROPOSITION 3.1.2. *The space $\Phi = S_1 \times \cdots \times S_N$ is closed in the sense that it satisfies property $(\mathscr{C})$ of Section 2.3.*

Proof. We must show that $\sum \tilde{S}_n \cap \{u; X_k u = 0\} \subset S_k$, the converse inclusion being clear. So let $(s_1, s_2, ..., s_N) \in S_1 \times \cdots \times S_N$ satisfy $X_k(\sum s_n) = 0$. We must show that $\sum s_n$ belong to S_k. Setting $\sigma = \sum_{n \neq k} s_n$ this reads $(s_1, ..., -\sigma, ..., s_N) \in S$ with σ in the kth slot. Thus σ and also $\sigma + s_k = \sum s_n$ belong to S_k.

Remark. In fact is still closed any product $\Phi_1 \times \cdots \times \Phi_N$ such that $\Phi_k \supset S_k$ for $k = 1, ..., N$.

Lemma 2.5.2 shows that the transversality of the fields X_n implies that $(S_m + S_n) \cap S = \{0\}$. Thus S is a linear subspace of $S_1 \times \cdots \times S_N$ which satisfies the two antagonistic conditions

$$\pi_n(S) = S_n \Leftrightarrow S^\perp \cap S_n = \{0\}, \qquad n = 1 \cdots N \tag{3.1.2}$$

$$S \cap (S_m + S_n) = \{0\}, \qquad m, n = 1 \cdots N. \tag{3.1.3}$$

Let r, r_n denote the dimensions of S, S_n, respectively, and $s^1 = (s_1^1, ..., s_N^1)$, ... $s^r = (s_1^r, ..., s_N^r)$ be a basis of S. Given a basis of each S_n, denote by $\mathfrak{Y}_n$ the matrix whose ith line describes the coordinates of s_n^i in S_n and by $\mathfrak{Y}$ the matrix

$$\mathfrak{Y} = [\mathfrak{Y}_1, ..., \mathfrak{Y}_N]. \tag{3.1.4}$$

The rank of $\mathfrak{Y}$ is r, the rank of $\mathfrak{Y}_n$ is r_n and (3.1.3) implies that the rank of $\hat{\mathfrak{Y}}_{i,j} = [\mathfrak{Y}_1, ..., \mathfrak{Y}_{i-1}, 0, \mathfrak{Y}_{i+1}, ..., \mathfrak{Y}_{j-1}, 0, \mathfrak{Y}_{j+1}, ..., \mathfrak{Y}_N]$ is also s for all i, j. This follows from

$$\operatorname{Im} \mathfrak{Y}_i + \operatorname{Im} \mathfrak{Y}_j \subset \sum_{k \neq i,j} \operatorname{Im} \mathfrak{Y}_k = \operatorname{Im} \hat{\mathfrak{Y}}_{i,j} \tag{3.1.5}$$

which is implied by Lemma 2.5.2. In fact, $\pi_{i,j}(S^\perp) = S_i + S_j$, implies that for every α_i, α_j there exist $\alpha_k k \neq i, j$, such that

$$\sum \mathfrak{Y}_n \alpha_n = 0.$$

Choose in Θ_n the dual basis of the basis of S_n and denote coordinates in Θ_n by $\boldsymbol{\theta}_n = (\theta_1, ..., \theta_{r_n})$. Fix the operator E_n as defined by (2.6.1) with $V = \Psi_n$ and

$$\Psi_n = \left\{ (\boldsymbol{\theta}_k)_{k=1,...,N}; \qquad \boldsymbol{\theta}_n = 0, \sum_{k \neq n} \mathfrak{Y}_k \boldsymbol{\theta}_k = 0 \right\}. \tag{3.1.6}$$

DEFINITION 3.1.3. Two systems L and L' are resonantly equivalent (r.e.) if there exists a linear isomorphism

$$T: S_1 \times \cdots \times S_N \to S'_1 \times \cdots \times S'_N \tag{3.1.7}$$

which, after relabelling if necessary, satisfies

$$\begin{aligned} T(S_n) &= S'_n, \qquad n = 1 \cdots N \\ T(S) &= S'. \end{aligned} \tag{3.1.8}$$

Remark 3.1.4. (1) It is not hard to show that r-equivalence is characterised by the usual equivalence of the N-tuple of matrices $\mathfrak{Y}_n$ in (3.1.4) under change of basis in the N domain spaces and the target space. Precisely, L and L' are r.e. if and only if there exist invertible matrices U, V_n such that

$$\mathfrak{Y}'_n = U \mathfrak{Y}_n V_n, \qquad 1 \leqslant n \leqslant N.$$

(2) Averaging operators E_n defined by (3.1.6) and $\Phi = S_1 \times \cdots \times S_N$ are T-intertwined transformations for equivalent L's. Just use the linear change of Θ-variables induced by T on the space of almost-periodic functions.

(3) Linear properties of strict resonances are preserved under r.e.

(4) A common equivalence of operators uses the following operators: linear change of dependent variables which may depend smoothly on t, x; change of independent variables; multiplication of L on the left by a smooth invertible matrix valued function; permutation of the rows. This relation corresponds to the equivalence of the unordered foliations generated by the X_n under diffeomorphisms in the t, x space. This is the standard equivalence in the theory of webs in differential geometry. Equivalence of webs implies r-equivalence and not conversely. For example, two 3-webs without resonances are r.e. However, generically, two such webs are not equivalent in the strong sense. Equivalence in the strong sense above will be called strong equivalence (s.e.).

(5) Property ($\mathscr{T}$) is invariant under s.e. but not under r.e.

(6) The two equivalence relations are not the same. For example consider two 3-webs without resonance, one satisfying ($\mathscr{T}$), the other not.

An example with resonance can be constructed as follows. Take a 3-web with one resonance. Add a fourth foliotion which creates no new resonance. The validity of $(\mathscr{T})$ depends on the choice of the fourth field. In summary, when there are few resonance, r.e. is a weak relation and the corresponding equivalence classes are large. On the other hand, when the resonance relations are very rich the opposite is true. In Section 3.3 we give examples of 4×4 systems for which r.e. implies s.e.

The next result shows that the space of all resonances is finite dimensional and gives an explicit upper bound for the dimension. In the case of real analytic vector fields, this result was proved by Poincaré [P, BB] as part of the theory of webs. Our proof is in the C^∞ category.

THEOREM 3.1.5. *The space S of resonances has dimension not greater than $(N-1)(N-2)/2$.*

Proof. Let us write $X_n = \partial_t + \lambda_n(t, x)\, \partial_x$, dividing, if necessary, the vector field by a nonzero C^∞ function. The N functions λ_n are real, C^∞ in Ω and satisfy $\lambda_m(t, x) \neq \lambda_n(t, x)$, $(t, x) \in \Omega$, $m \neq n$. The system defining $s = (s_n) \in (C^\infty)^N$ is a $N+2$ by N system, so we have to look at compatibility conditions. For a solution s of (3.1.1), $J_0 = \sum s_n$ is constant, thus $\partial_t J_0 = 0$. Using the first N equations, we obtain the new condition $J_1 = \sum \lambda_n \partial_x s_n = 0$. We continue the process, differentiating J_k, $k = 1, \ldots$ with respect to t and replacing each $\partial_t s_n$ by $-\lambda_n \partial_x s_n$ to obtain J_{k+1}. We thus obtain an infinite sequence of ordinary differential equations for $s = (s_n)$ with respect to the single variable x, the time t being viewed as a parameter. The kth equation $J_k = 0$ is of order k and its leading term has the form $\sum \lambda_n^k \partial_x^k s_n$. Thus the N equations $\partial_x^{N-1-k} J_k = 0$, $k = 0, \ldots, N-1$, are solvable in the highest order terms $\partial_x^{N-1} s_n$ the corresponding coefficient being the Vandermonde matrix

$$\begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_N \\ \lambda_1^{N-1} & \cdots & \lambda_N^{N-1} \end{bmatrix} \tag{3.1.9}$$

which is everywhere invertible because of the property of λ_n. This new system, which the resonance s should verify, has at most $N(N-1)$ linearly independent solutions, but in order to obtain it we differentiated J_0 $N-1$ times, ..., J_{N-2} 1 time. Since J_0 is not zero but a constant this shows that our initial system possesses at most

$$N(N-1) - (1 + 2 + \cdots + N - 1) + 1$$

linearly independant solutions s. Now, phases are considered only modulo the constants, we have to substract N to obtain the final result.

Remarks. (1) We found only an upper bound for the dimension of S because we have not taken into account an infinite number of compatibility conditions, namely $J_n = 0$, for $n \geqslant N$. Nevertheless, the upper bound is sharp. It is achieved for all N as the following proposition shows.

(2) For an approach of the theory of webs which uses systems of partial differential equations, see [H].

PROPOSITION 3.1.6. *The space S corresponding to an $N \times N$ constant coefficient operator has dimension $(N-1)(N-2)/2$.*

Proof. Since the speeds λ_k do not depend on t, x the computations in the proof of Theorem 3.1.5 gives

$$J_n = \sum_{k=1}^{N} \lambda_k^n \partial_x^n s_k = 0, \qquad n \geqslant 0. \tag{3.1.10}$$

Differentiating yields the homogeneous system of equations $\partial_x^{N-1-n} J_n = 0$ which has only the zero solution. We thus obtain $\partial_x^{N-1} s_k = 0$, $1 \leqslant k \leqslant N$ and since s_k is actually a function of $x - \lambda_k t$, this proves that this function is a polynomial of degree less than or equal to $N-2$,

$$s_k(t, x) = P_k(x - \lambda_k t). \tag{3.1.11}$$

Neglecting the constants, we write $P_k(\xi) = \sum_{n=1}^{N-2} a_{n,k} \xi^n$. For a resonance one has, $\sum s_k = \sum_{k,n} a_{n,k}(x - \lambda_k t)^n = 0$. Setting the coefficients of this polynomial equal to zero yields a system of equations for $a_{n,k}$

$$\begin{aligned} &\sum a_{1,k} = 0 \\ &\sum \lambda_k a_{1,k} = 0 \end{aligned} \tag{3.1.12}$$

$$\begin{aligned} &\sum a_{2,k} = 0 \\ &\sum \lambda_k a_{2,k} = 0 \\ &\sum \lambda_k^2 a_{2,k} = 0 \end{aligned} \tag{3.1.13}$$

$$\vdots$$

$$\begin{aligned} &\sum a_{N-2,k} = 0 \\ &\sum \lambda_k a_{N-2,k} = 0 \\ &\qquad \vdots \\ &\sum \lambda_k^{N-2} a_{N-2,k} = 0. \end{aligned} \tag{3.1.14}$$

The set of the solutions of the first system (3.1.12) describes the third-order resonances which span a $N-2$ dimensional linear space. The succeeding systems yield resonance space $\mathfrak{S}^n$ of decreasing dimension spanned by resonances of increasing strict order up to the last system (3.1.14) which gives a strict Nth-order resonance. We obtain for S the direct decomposition

$$S = \mathfrak{S}^3 \oplus \mathfrak{S}^4 \oplus \cdots \oplus \mathfrak{S}^N. \tag{3.1.15}$$

It follows that S has dimension $1+2+\cdots+N-2=(N-1)(N-2)/2$, which is maximum according to Theorem 3.1.5.

Remark. Note that in the above example strict nth-order resonances involve oscillations with polynomial phases of degree $n-2$ and that these resonances occur for any subset of n modes among the N.

Next we examine in more detail the space of phases and averaging operators for this example of constant coefficient.

Denote by $\Phi^p_{n,\mathrm{hom}}$, $1 \leqslant p \leqslant N-2$, $1 \leqslant n \leqslant N$, the subspace of homogeneous polynomials of degree p in the variable $x-\lambda_n t$ and by Φ^p_n, $1 \leqslant p \leqslant N-2$, $1 \leqslant n \leqslant N$, the space $\sum_{k=1}^p \Phi^k_{n,\mathrm{hom}}$.

The space $\Phi^p_{n,\mathrm{hom}}$ has dimension 1, while Φ^p_n has dimension p.

Define

$$\Phi^p = \Phi^p_1 \times \cdots \times \Phi^p_N, \qquad [\text{resp. } \Phi^p_{\mathrm{hom}} = \Phi^p_{1,\mathrm{hom}} \times \cdots \times \Phi^p_{N,\mathrm{hom}}].$$

Then Φ^p and Φ^p_{hom} clearly satisfy the closure property ($\mathscr{C}$) of Section 2.3. Note that

$$\Phi^p_{\mathrm{hom}} \cap S = \mathfrak{S}^{p+2},$$
$$\Phi^p \cap S = \mathfrak{S}^3 \oplus \mathfrak{S}^4 \oplus \cdots \oplus \mathfrak{S}^{p+2},$$

and that in Φ^p_{hom} all resonances have order greater than $p+2$.

Choose the basis in Φ^p_n and $\Phi^p_{n,\mathrm{hom}}$ formed by the monomials in $x-\lambda_n t$ and the dual basis for the corresponding Θ_n. With the notations of Section 2, we have

PROPOSITION 3.1.7. *The space Ψ_k corresponding to Φ^p_{hom} is spanned by the p independent vectors*

$$(\lambda_k - \lambda_1, \ldots, \lambda_k - \lambda_N) \tag*{(3.1.16)$_1$}$$

$$\vdots$$

$$((\lambda_k - \lambda_1)^p, \ldots, (\lambda_k - \lambda_N)^p). \tag*{(3.1.16)$_p$}$$

The associated averaging operator is

$$(E_k F)(\theta_k) = 1/T^p \int_0^T \cdots \int_0^T F(\theta_k + (\lambda_k - \lambda_1)\tau_1 + \cdots$$
$$+ (\lambda_k - \lambda_1)^p \tau_p, \ldots, \theta_k + (\lambda_k - \lambda_N)\tau_1 + \cdots$$
$$+ (\lambda_k - \lambda_N)^p \tau_p)\, d\tau_1 \cdots d\tau_p. \tag{3.1.17}$$

The space Ψ_k corresponding to Φ^p is the direct sum of the Ψ_k's corresponding to the homogeneous components $\Phi^p_{n,\mathrm{hom}}$ of Φ^p_n and E_k the product of the E_k's corresponding to the same homogeneous components.

3.2. *Third Order Resonances*

For $N = 3$, Theorem 3.1.5 gives dim $S \leqslant 1$. The following examples show that dimensions zero and one both occur.

EXAMPLE 3.2.1. $X_1 = \partial_t + \partial_x$, $X_2 = \partial_t - \partial_x$, $X_3 = \partial_t$ have the resonances

$$\mathbb{R}(x - t,\ x + t,\ -2x).$$

EXAMPLE 3.2.2. $X_1 = \partial_t + \partial_x$, $X_2 = \partial_t + x\partial_x$, $X_3 = \partial_t$ have outside $\{x = 0\}$, the resonances

$$\mathbb{R}(x - t,\ \mathrm{Log}\,|x| - t,\ x - \mathrm{Log}\,|x|).$$

More generally, $X_1 = \partial_t + \lambda(x)\,\partial_x$, $X_2 = \partial_t + \mu(x)\,\partial_x$, $X_3 = \partial_t$ have outside the zeros of λ, μ the resonances

$$\mathbb{R}\left(-\int_0^x (1/\lambda)\,dy + t,\ \int_0^x (1/\mu)\,dy - t,\ \int_0^x ((\lambda - \mu)/\lambda\mu)\,dy\right).$$

EXAMPLE 3.2.3. $X_1 = \partial_t + x\partial_x$, $X_2 = \partial_t - x\partial_x$, $X_3 = \partial_t + a\partial_x$ is a case in which for $a \neq 0$ no nontrivial resonance exists. For this system with arbitrary nonlinear coupling terms, high frequency oscillating waves propagate as in the linear case.

These examples show that

(1) Oscillations do not propagate as singularities. Anomalous singularities are produced for all L. Analogous oscillations are produced only when resonances exist.

(2) The difference depends only on the principal part of the operator which is linear.

The presence or absence of resonance is a property which is far from obvious. It is equivalent to the existence of so-called abelian relations on the web, defined by the three foliations Γ_n given by the integral curves of the fields X_n [BB].

We next describe the geometric property of the three fields or rather the associated foliations which corresponds to the existence of a resonance.

Suppose that, as in Example 3.2.1, there is a resonance which is defined by the relation

$$s_3 = s_1 + s_2,$$

where the modification of the sign of s_3 is introduced for the sake of simplicity.

We first show that s_1, s_2 define new local coordinates in Ω. Since $ds_i \in \mathrm{Char}(X_i)$ it suffices to show that ds_i never vanishes in Ω. Suppose for instance that ds_1 vanishes at some point in Ω. Applying X_3 to the resonance relation leads to $X_3 s_2 = 0$ at the same point, from which follows that the three differentials always vanish together. In the proof of Theorem 3.1.5 we showed that $v := \partial_x(s_1, s_2, s_3)$ satisfy a first order linear system of ordinary differential equations. This implies that v is identically zero and thus that the s_i are constant, a contradiction.

In the new coordinates, the three foliations are straightened, becoming three pencils of parallel straight lines $s_1 = \text{const}$, $s_2 = \text{const.}$ and $s_1 + s_2 = \text{const}$.

To the hexagonal figure drawn in the s_1, s_2 coordinates corresponds a curvilinear hexagon in the original coordinates (t, x), see Fig. 4. The lines leaving O as well as the sides of the hexagon are integral curves of the vector fields. Draw a curve as follows: Begin with the three characteristics through O and a point A on one of them; starting at A, turn around O along six arcs of orbits of the three vector fields. If a resonance exists, the curve sketched will return to A, a property which is clear in the

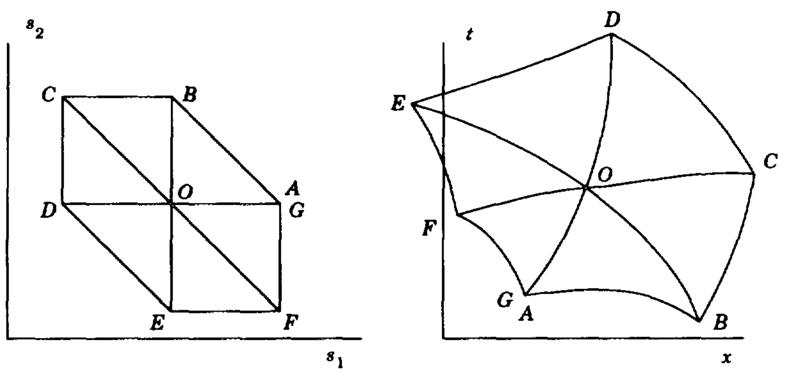

FIG. 4. The hexagonal closure property.

s-coordinates. Conversely, it is a famous result of Thomsen [BB] that this property, called the *hexagonal closure property* when it is satisfied for any couple of points A, B such that the above hexagonal picture lies in Ω, insures there exists a resonance.

We will call *hexagonal* a 3×3 operator whose characteristic web satisfy the hexagonal closure property. Thomsen's result may be settled as follows.

PROPOSITION 3.2.4. *For three vector fields to be resonant in an open set Ω, it is necessary and sufficient that they satisfy the hexagonal closure property in Ω.*

Regarding resonance, the 3×3 operators are easy to classify.

PROPOSITION 3.2.5. *There are two classes of 3×3 resonantly equivalent operators: the hexagonal class and the nonhexagonal class. Moreover all hexagonal operators are strongly equivalent.*

Proof. Let $s_n \in S_n$, $n = 1, 2, 3$ define the third-order resonance $s_1 + s_2 + s_3 = 0$ of an hexagonal operator. Choosing s_1, s_2 as new coordinates shows that this operator is *strongly equivalent* to the constant coefficient operator defined by ∂_{s_2}, ∂_{s_1}, $\partial_{s_2} + \partial_{s_1}$. On the other hand, two nonhexagonal operators are obviously resonantly equivalent, since their resonant spaces are both trivial.

COROLLARY 3.2.6. *L is a 3×3 hexagonal operator if and only if there exist never vanishing C^∞ functions $a_i(t, x)$ such that the vector fields $a_i X_i$, $i = 1, 2, 3$ commute.*

Proof. The result is a consequence of Proposition 3.2.5. It follows from the strong equivalence of a 3×3 hexagonal operator with a constant coefficient operator, which clearly satisfies the property of the corollary.

Remark. Two nonhexagonal operators are in general not strongly equivalent. The properties of the curvature of a 3-web will make this clear in Section 3.4. An operator with positive curvature cannot be s.e. to an operator with negative curvature.

Before ending this section let us examine the transversality properties of the phase space $\Phi = S_1 \times S_2 \times S_3$ for a 3×3 hexagonal operator L.

PROPERTIES 3.2.7. *The space $\Phi = S_1 \times S_2 \times S_3$ associated to a 3×3 hexagonal operator L satisfies the coherence assumption, hence the strong transversality property $(\mathcal{T})$.*

Proof. Properties of coherence and transversality are invariant by s.e. By Proposition 3.2.5, the result follows from the coherence of linear phases as noted in Remark 2.3.1.

3.3. *Hexagonal Operators for $N = 4$*

There exist N-webs with nontrivial resonance space and without third order resonances. Here is an example with $N = 4$. Let y, z be coordinates in the plane. Then, the four foliations defined by the level curves of the functions y, z, $y^2 + z^2 + yz$, $y^2 + z^2 + yz + y + z$ have this property. The sum of the first three is equal to the fourth, which gives a fourth order resonance. There are no third order resonance in this example. For example one can compute that the curvature, defined in the next section, is nonzero for each 3-web.

On the other hand, the proof of Proposition 3.1.6 shows that for systems with constant coefficients all possible third order resonances occur. More generally we now consider those operators such that every triplet of characteristic foliations possesses the hexagonal closure property. Following Blaschke-Bol, such systems are called *hexagonal.* The following result due to Mayrhofer, Reidemeister, Bol, and Blaschke (see [BB, B]) describes all such webs.

THEOREM 3.3.1. *When $N \neq 5$, an $N \times N$ operator is hexagonal if and only if it is equivalent, in the strong sense, to an operator whose characteristic web consists of N pencils of lines. If $N = 5$ there is one more possibility discovered by Bol, four pencils of lines and one pencil of hyperbolas.*

In view of Proposition 3.2.5, we already know what happens for $N = 3$. All 3×3 hexagonal operators form one class for both r.e., and s.e.

Consider next the case of 4×4 hexagonal operators. We show that there exists several different classes of resonantly equivalent 4×4 hexagonal operators and that the corresponding classification coincide with the projective classification.

Theorem 3.3.1 shows that after a change of coordinates, the characteristic web of a 4×4 hexagonal operator is four pencils of straight lines in Ω. Taking this first reduction into account, the classes of hexagonal operators will be shown to correspond to three different possibilities for the location of the four vertices of the pencils, A, B, C, D, as sketched in the figure. We also use A, B, C, D to denote the four different modes of the operator instead of 1, 2, 3, 4. For instance S_A replaces S_1, etc. Lower case a, b, c, ... denote the straight lines sketched in Fig. 5. We also denote by a, b, c, ... the polynomials of degree one which define the straight lines by the equations $a = 0$, $b = 0$, $c = 0$, ... With this convention, the equations for the pencils are easy to describe. For example, in Case 3 the pencil Γ_A with

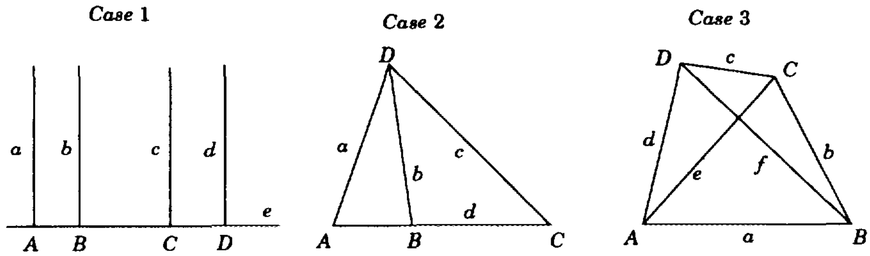

FIG. 5. Hexagonal 4 × 4 operators.

vertex A, which is the characteristic foliation of the mode A, is given by $a/d = \text{const}$ or $a/e = \text{const.}$ or more generally $F(a/d) = \text{const.}$ Hence a phase belonging to S_A is a function of the form $f(a/d)$.

In Case 1, the four vertices lie on a line. In Case 2, one vertex lies outside the line joining the three others. In Case 3, any line in the plane contains at most two vertices.

The open set Ω is chosen connected and included in the complement of the union of the lines connecting the vertices.

THEOREM 3.3.2. *Two* 4×4 *hexagonal operators are r.e. if and only if they are s.e. which here means projectively equivalent. Cases* 2 *and* 3 *both consist of one s.e. class. In Case* 1, *there is an infinite number of classes and each class is defined by the absolute value of the cross ratio of the vertices.*

Proof. The proof follows from the description of the spaces S, S_A, S_B, S_C, S_D in the different cases. In each case we will find three linearly independent resonances. Thus in each case S is of maximum dimension equal to $(4-1)(4-2)/2 = 3$.

Case 1. The straight lines a, b, c, d are *parallel.* The polynomials a, b, c, d can be chosen as polynomials of degree 1 in *one variable.* They span a two dimensional space and their squares span a three dimensional space. Thus there exist nontrivial α_n, β_n, γ_n such that

$$\begin{aligned} \alpha_1 a/e + \alpha_2 b/e + \alpha_3 c/e &= 0 \\ \beta_2 b/e + \beta_3 c/e + \beta_4 d/e &= 0 \\ \gamma_1 a^2/e^2 + \gamma_2 b^2/e^2 + \gamma_3 c^2/e^2 + \gamma_4 d^2/e^2 &= 0. \end{aligned} \tag{3.3.1}$$

The first two relations in (3.3.1) define two relations in (3.3.1) define two third-order resonances between the modes A, B, C, and B, C, D, respectively. They are therefore linearly independent. Two other third-order resonances, involving the mode C, D, A and D, A, B, C, are combinations of the first two.

The last relation in (3.3.1) is a strict resonance of order 4 between the modes A, B, C, D. It is independent of the first two.

In summary, the three resonances in (3.3.1) form a basis of the 3-dimensional space S. Furthermore, S admits the decomposition (3.1.15). In fact, in Case 1, the operator is strongly equivalent to a constant coefficient operator. One just has to send the line d to the line at infinity by a projective linear transformation.

Case 2. Since the lines a, b, c intersect in a single point, the corresponding polynomials are linearly dependent. Suppose that a, b, c are normalized so that $a+b+c=0$. Then we obtain a first resonance

$$a/d+b/d+c/d=0. \tag{3.3.2}$$

Two others are derived by using the two relations $a/d \cdot d/b \cdot b/a=1$ and $b/d \cdot d/c \cdot c/b=1$, which are linearized by taking the Log. This yields

$$\begin{aligned} \operatorname{Log}|a/d| + \operatorname{Log}|d/b| + \operatorname{Log}|b/a| &= 0 \\ \operatorname{Log}|b/d| + \operatorname{Log}|d/c| + \operatorname{Log}|c/b| &= 0. \end{aligned} \tag{3.3.3}$$

The three relations in (3.3.2) (3.3.3) are independent, thus forming a basis of S.

Since this basis consits of only third-order resonances, we have proven that no strict fourth-order resonance is present in Case 2, in contrast to Case 1. We note that among the 4 third-order resonances, 3 of them, the log-one relating the mode D to any two of the three others A, B, C, span a two-dimensional space. The last one, relating A, B, C, does not lie in that space.

In other words, in Case 2, the four third-order resonance are not in general position. This does not seem unreasonable given the location of the vertices A, B, C, D in the plane.

Case 3. As above we obtain a Log resonance for each triangle. In this case there are enough triangles so that these relations span the resonance space S. We may choose the basis

$$\begin{aligned} \operatorname{Log}|e/a| + \operatorname{Log}|a/b| + \operatorname{Log}|b/e| &= 0 \\ \operatorname{Log}|f/b| + \operatorname{Log}|b/c| + \operatorname{Log}|c/f| &= 0 \\ \operatorname{Log}|e/c| + \operatorname{Log}|c/d| + \operatorname{Log}|d/e| &= 0, \end{aligned} \tag{3.3.4}$$

thus proving the absence of strict fourth-order resonance and the general position of the four third-order resonances, one for each triplet of foliations.

In summary, the different cases correspond to different r.e. classes. (see Remark 3.1.2).

Moreover, Cases 2 and 3 both consist of one s.e. class since in each case any set of four vertices is mapped onto any other set of four vertices by a projective linear transformation which respects the pencils.

It remains to examine Case 1. By s.e. we may suppose $X_n = \partial_t + \lambda_n \partial_x$, with constant λ_n. In view of the description of the constant coefficient case in Section 3.1, the r.e. class is determined by the equivalence class of the matrices as defined in part 1 of Remark 3.1.4. The matrices are

$$\mathfrak{Y}_1 = \begin{bmatrix} \lambda_2 - \lambda_3 & 0 \\ 0 & 0 \\ 0 & \alpha_1 \end{bmatrix} \tag{3.3.5}$$

$$\mathfrak{Y}_2 = \begin{bmatrix} \lambda_3 - \lambda_1 & 0 \\ \lambda_3 - \lambda_4 & 0 \\ 0 & \alpha_2 \end{bmatrix} \tag{3.3.6}$$

$$\mathfrak{Y}_3 = \begin{bmatrix} \lambda_1 - \lambda_2 & 0 \\ \lambda_4 - \lambda_2 & 0 \\ 0 & \alpha_3 \end{bmatrix} \tag{3.3.7}$$

$$\mathfrak{Y}_4 = \begin{bmatrix} 0 & 0 \\ \lambda_2 - \lambda_3 & 0 \\ 0 & \alpha_4 \end{bmatrix}. \tag{3.3.8}$$

By right-multiplying $\mathfrak{Y}_1 \mathfrak{Y}_4$, we obtain any pairs of matrices of the same form. For $\mathfrak{Y}_2$, $\mathfrak{Y}_3$, the equivalence leads to matrices $\mathfrak{Y}'_2$, $\mathfrak{Y}'_3$ of the same form, but such that the cross ratios of their coefficients satisfy $|(\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4)| = |(\lambda_1, \lambda_2, \lambda_3, \lambda_4)|$. This last property is the condition saying the corresponding two webs of four pencils of straight lines are projectively equivalent.

We end this section by checking the transversality properties of the space $\Phi = S_A \times S_B \times S_C \times S_D$ for 4×4 hexagonal operators.

THEOREM 3.3.3. *The space Φ satisfies $(w - \mathcal{T})$. It satisfies $(\mathcal{T})$ if and only if the hexagonal operator is equivalent to Cases* 2 *or* 3.

Proof. As $(\mathcal{T})$ is preserved under s.e., it is enough to consider operators of Cases 1–3 of Theorem 3.3.2. In those case weak transversality is a consequence of real analyticity. We next consider strong transversality.

Case 1 *Does Not Satisfy* $(\mathcal{T})$. Take the three vector fields of Example 3.2.1 for three of the four constant coefficient vector fields. The combination of phases $(x+t)^2 - (x-t)^2 = 4xt$ does not belong to any S_n

but is constant on $\{x=0\}$ which is a characteristic curve of ∂_t. On the other hand, $(w\mathscr{T})$ follows by analiticity.

Case 2 *Satisfies* $(\mathscr{T})$. Set $s(\lambda)=\sum \lambda_n s_n$ with $s_n \in S_n n=A, B, C, D$. It suffices to check the transversality properties relative to the points A, D, the arguments for B, C being the same as for A. We must show that if $s(\lambda)$ is not in S_A (resp. in S_D) then its restrictions to lines of the A (resp. D) pencil are never constant. We will do this in two steps. First, using the resonances, we will eliminate terms in $s(\lambda)$, and this modulo Φ_A since these functions are constant on the A-foliation. This will reduce the problem to considering combinations of three instead of six independent functions. Then a simple study of the singularities of this reduced function will end the proof.

A-transversality. Using (3.3.7) (3.3.8) we can reduce $s(\lambda)$ to a combination of the three functions b/d, $\mathrm{Log}\,|b/d|$, $\mathrm{Log}\,|c/d|$. The restriction of this new $s(\lambda)$ to any line m of the A-pencil is smooth at all point of $m\backslash(b\cup c\cup d)$ the singular set consisting of at most three points. If $s(\lambda)$ is constant in a nonvoid connected open set of $m\backslash(b\cup c\cup d)$, by uniqueness of the analytic continuation of $s(\lambda)_{|m}$, $s(\lambda)_{|m}$ is constant on each connected component of its domain. As m always crosses either b or c and $s(\lambda)$ has only the corresponding Log as singular component near the intersection point, this term is necessarily zero. The result follows since $s(\lambda)$ is now reduced to b/d and the remaining $\mathrm{Log}(\cdot/d)$ term.

D-transversality. $s(\lambda)$ can be reduced to a combination of b/d, c/d, $\mathrm{Log}\,|b/d|$ modulo Φ_D. Since any m of the D-pencil crosses b, the $\mathrm{Log}\,|b/d|$ must not appear in $s(\lambda)$. Since b/c is constant on m, $s(\lambda)$ is proportional to c/d on m. As $s(\lambda)$ is also constant on m. The result follows.

Case 3 *Satisfies* $(\mathscr{T})$. The proof resembles the above case. It is even quicker. It is enough to check the X_A-tranversality. First, the elimination step yields a reduced $s(\lambda)$, combination of $\mathrm{Log}\,|b/c|$, $\mathrm{Log}\,|c/f|$, $\mathrm{Log}\,|c/d|$. Since any A-line crosses at least two of the lines b, c, f, this shows two out of the three terms are not present and thus ends the proof.

3.4. *Resonance for Systems of Conservation Laws*

If u_0 is a smooth solution of the quasilinear system,

$$\partial_t u_0 + A(u_0)\,\partial_x u_0 = 0, \tag{3.4.1}$$

ε-small perturbations are governed by the equation, whose principal part is the operator,

$$\partial_t + A(u_0)\,\partial_x. \tag{3.4.2}$$

The resonances for ε-small oscillations are those determined by this linear operator.

What does (3.4.1) imply regarding the structure of the resonances for (3.4.2)? If u_0 is a general wave, that is, if the values of u_0 describe locally an open set of a two-dimensional surface in $\mathbb{R}^N$, then one expects for (3.4.2) nothing simpler than a general operator with smooth variable coefficients. On the other hand, if u_0 is a simple wave which means that the range of u_0 is a curve, (3.4.2) inherits strong properties from (3.4.1). We will see that resonance depends heavily on the properties of the eigenvalue associated with the simple wave. If the simple wave is *linearly degenerate* then every third-order resonance whose support contains the mode of the simple wave is possible. If, on the other hand, the simple wave is *genuinely nonlinear*, resonance does not occur unless an extra condition is satisfied. For the case of gas dynamics, the simple wave must be *centered.*

Before giving precise statements, we need another characterisation of third-order resonance due to Blaschke–Bol [BB]. Let σ_n, $n = 1, 2, 3$ denote never vanishing differential forms of degree one, each of them annihilating the corresponding vector field X_n, i.e. $\langle \sigma_n(t, x), X_n(t, x)\rangle = 0$, for $(t, x) \in \Omega$. Since any two of the forms is a basis we may assume that the forms are normalized so that

$$\sum \sigma_n = 0. \tag{3.4.3}$$

For convenience, we assume that summations are taken over $\mathbb{Z}/3\mathbb{Z}$. Define the two-form τ by

$$\tau = \sigma_n \wedge \sigma_{n+1}. \tag{3.4.4}$$

Then $\tau \neq 0$ and (3.4.3) implies that it is independent of n. Define $h_n \in C^\infty$ and the one-form γ by

$$d\sigma_n = h_n \tau \tag{3.4.5}$$

$$\gamma = h_{n+1}\sigma_n - h_n \sigma_{n+1}, \tag{3.4.6}$$

where γ is independent of n by (3.4.3). Finally $k \in C^\infty$ is defined by

$$d\gamma = k\tau. \tag{3.4.7}$$

Blaschke calls k the *curvature* of the 3-web $(\Gamma_n,\ n = 1, 2, 3)$.

To understand the geometrical meaning of the curvature, first note that, once the σ_n are chosen, k is invariant under change of independent variables and permutation of the indices. However multiplying the σ_n's by a nonzero C^∞ factor does change k. In fact, if $\sigma'_n = \alpha \cdot \sigma_n$, $\alpha \in C^\infty \backslash \{0\}$ then

$\gamma' = \gamma - d(1/\alpha)$ and therefore $k' = k/\alpha^2$. This is an easy computation if one introduces the new vector fields $D_n \in C^\infty \cdot X_n$ defined by

$$df = D_{n+1} f \cdot \sigma_n - D_n f \cdot \sigma_{n+1}, \qquad f \in C^\infty. \tag{3.4.8}$$

Thus the sign and the zeros of the curvature are geometric properties of a 3-web. The D_n verify the interesting commutation relation

$$[D_i, D_j] = h_j D_i - h_i D_j. \tag{3.4.9}$$

With these notations, Blaschke–Bol prove

PROPOSITION 3.4.1. *The 3-web* $(\Gamma_n, n = 1, 2, 3)$ *has the hexagonal closure property if and only if its curvature is zero.*

Proof. Suppose first that there exist (s_n) solutions of (3.1.1). We may choose $\sigma_n = ds_n$ because we have shown in Section 3.2 that these differentials never vanish. With this choice, $h_n = 0$, $\gamma = 0$, $k = 0$, and the D_n commute. Conversely, if $k = 0$, there exists $g \in C^\infty$ with $\gamma = dg$. Now, $d(e^{-g}\sigma_n) = e^{-g}(d\sigma_n - \gamma \wedge \sigma_n) = 0$ so that the s_n, defined up to a constant by $ds_n = \bar{e}g) \cdot \sigma_n$, are solutions of (3.1.1).

We write this result in another form, using the notation

$$X_n = \partial_t + \lambda_n \partial_x. \tag{3.4.10}$$

As $\Delta^3 = \prod (\lambda_n - \lambda_{n+1})$ is never zero, we may define $K_n = (\lambda_{n+1} - \lambda_{n+2})/\Delta$. Then, $\sum K_n = \sum \lambda_n K_n = 0$ so

$$\sigma_n = K_n(dx - \lambda_n\, dt) \tag{3.4.11}$$

satisfies (3.4.3). With this choice, we have

$$\tau = \Delta\, dx \wedge dt, \tag{3.4.12}$$

and

$$h_n = -Z_n(K_n)/\Delta, \tag{3.4.13}$$

where $Z_n(u) := \partial_t u + \partial_x(\lambda_n u) = X_n u + (\partial_x \lambda_n) \cdot u$. Thus the curvature vanishes if and only if the following identity holds.

COROLLARY 3.4.2. *The hexagonal closure property is equivalent to*

$$Z_{n+1}(Z_n(K_n)\, K_{n+1}/\Delta) = Z_n(Z_{n+1}(K_{n+1})\, K_n/\Delta) \tag{3.4.14}$$

for some (and therefore all) $n \in \{1, 2, 3\}$.

As a first example, consider the linearization of a general quasilinear system at a k-simple wave when the mode k is linearly degenerate.

THEOREM 3.4.3. *Suppose u is a k-simple wave solution of the $N \times N$ hyperbolic quasilinear system*

$$\partial_t u + A(u)\, \partial_x u = 0 \tag{3.4.15}$$

and assume the mode k is linearly degenerate. Let $T_J(u)$ denote the web associated with the characteristic fields $\partial_t + \lambda_n(u)\, \partial_x$ of the modes $n \in J$. Then

(1) *$T_{i,j,k}(u)$ is hexagonal for every i, j.*

(2) *$\partial_t + A(u)\, \partial_x$ is resonantly equivalent to a constant coefficient operator if and only if for every m, n, p different from k there exist constants α, β with $\alpha + \beta = 1$, such that*

$$1/(\lambda_m - \lambda_k) = \alpha/(\lambda_n - \lambda_k) + \beta/(\lambda_p - \lambda_k). \tag{3.4.16}$$

Proof. (1) Since the mode k is linearly degenerate, $\lambda_k(u)$ is constant, $\lambda_k(u) \equiv \lambda_k$. The wave u satisfies $u = U(x - \lambda_k t)$, thus λ_n, K_n, Δ, as well as their derivatives with respect to t, x, which enter in formula (3.4.14), are functions of $x - \lambda_k t$. Since $Z_k = X_k$ and therefore $Z_k(K_k) = 0$, property (3.4.14) is equivalent to $X_k(Z_n(K_n)\, K_k/\Delta) = 0$, which satisfied since the functions inside the parentheses depend only on $x - \lambda_k t$. The result follows from Corollary 3.4.2.

The hexagonal closure property provides another short proof of the above fact. After a linear change of independent variables, we may suppose that $\lambda_k = 0$. Denoting by (t, x) the new independent variables, U depends only on x, $U = U(x)$. The result now follows from the fact that the vector field ∂_t together with any two autonomous (that is independent of t) vector fields, satisfy the hexagonal closure property. Indeed, drawing the picture as in Example 1, it is readily seen that the two curvilinear quadrilateral A, B, C, D and C, E, G, F coincide up to the time shift BE. We can also refer to Example 3.2.2.

(2) Assume $\lambda_k = 0$. Consider $J = \{k, 1, 2\}$. We know by 1 a resonance exists. The corresponding phases are (see Example 3.2.2)

$$\varphi_i = \left[\int (1/\lambda_i(y))\, dy\right] - t, \qquad i = 1, 2 \tag{3.4.17}$$

$$\varphi_k = \int (1/\lambda_1 - 1/\lambda_2)\, dy. \tag{3.4.18}$$

They yield the resonance $\varphi_k = \varphi_1 - \varphi_2$. Now, for (3.4.15) to be equivalent to a constant coefficient system it is necessary that all the phases φ_k given by (3.4.18) for different pair of indices are equal up to a multiplicative constant. This follows from the structure of the space of resonances as described in the proof of Proposition 3.1.4. Thus for every $n \neq k$ there must exist constants α_n, β_n such that for every $y \in \mathbb{R}$

$$1/\lambda_n(y) = \alpha_n/\lambda_1(y) + \beta_n/\lambda_2(y), \qquad \alpha_n + \beta_n = 1, \qquad n \neq k. \quad (3.4.19)$$

That this condition is also sufficient is readily seen taking any pair of phases as new coordinates in $\mathbb{R}^2$. Choosing for instance φ_1, φ_2, (3.4.19) provides the relations

$$\varphi_n = \alpha_n \varphi_1 + \beta_n \varphi_2, \qquad n \neq k \quad (3.4.20)$$

and using (3.4.18),

$$\varphi_k = \varphi_1 - \varphi_2. \quad (3.4.21)$$

Thus in the new coordinates the characteristic web is N pencils of parallel straight lines, and, the phases are homogeneous polynomials of degree one.

Returning to the original coordinates the condition in (2) is just (3.4.19), where each λ_n is replaced by $\lambda_n - \lambda_k$.

Remark. It would be interesting to have a condition guaranteeing that $T_{i,j,l}(u)$ is hexagonal for every i, j, l. This is less stringent than being equivalent to a constant coefficient operator.

We examine next the resonance properties of the linearization at a genuinely nonlinear simple wave. We will not consider the general case but restrict ourselves to the example of the Euler equations of gas dynamics. These are the conservation laws for mass, momentum, and energy. When viscosity, body forces, and heat conduction are neglected and the gas is assumed in thermodynamic equilibrium, this system may be written as

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) &= 0 \\ \partial_t(\rho e + 1/2\rho u^2) + \partial_x(\rho e u + 1/2\rho u^3 + pu) &= 0. \end{aligned} \quad (3.4.22)$$

In these equations, ρ is the mass density, u the flow velocity, p the pressure, and $e = e(\rho, p)$ the internal energy density.

Introducing the entropy S by

$$T\,dS = de + p\,d(1/\rho), \quad (3.4.23)$$

where T denote the temperature, the system (3.4.22) is equivalent, for smooth solutions, to the system

$$\begin{aligned} X_u\rho + \rho\partial_x u &= 0 \\ X_u u + (1/\rho)\,\partial_x p &= 0 \\ X_u S &= 0. \end{aligned} \tag{3.4.24}$$

Here $X_u = \partial_t + u(t, x)\,\partial_x$ denotes the time derivative following individual particles. Expressing $p = p(\rho, S)$, the sound speed c is given by $c^2 = (\partial_\rho p)_{S = cst}$. Note that the three vector fields, associated with the operator (3.4.1) (with u_0 replaced by (u, ρ, S) solution of (3.4.22) or (3.4.24)), may be defined, with obvious notations, as X_u, X_{u+c}, X_{u-c} the three characteristic speeds being equal to u, $u+c$, $u-c$. The corresponding foliations are denoted by Γ_u, Γ_{u+c}, $\Gamma+_{u-c}$.

We next examine whether a solution (u, ρ, S), of (3.4.24), gives rise to a resonance. Using (3.4.14), it is readily seen that the web $(\Gamma_u, \Gamma_{u+c}, \Gamma_{u-c})$ is hexagonal if and only if the following condition is fulfilled

$$Z_{u+c}(Z_u(1)\,1/c) = Z_u(Z_{u+c}(1)\,1/c) \tag{3.4.25}$$

which, after elimination reads

$$\partial_x^2 u = \partial_t\partial_x \operatorname{Log}(c) + \partial_x(u\partial_x \operatorname{Log}(c)). \tag{3.4.26}$$

Summarizing, a solution gives rise to a resonance if and only if it satisfies (3.4.26). Together with the Euler equations, this forms an overdetermined system, reflecting the fact that generically resonances do not occur.

After integration with respect to x (3.4.26) yields,

$$\partial_x u = (X_u c)/c + f(t), \tag{3.4.27}$$

where f is an arbitrary function of t.

Let us consider first the particular case $f = 0$ in (3.4.27). It corresponds to the commutation property of the three fields.

PROPOSITION 3.4.4. *Let (u, ρ, S) be a solution of* (3.4.24) *and assume the gas law satisfies $2\partial_\rho p + \rho\partial_\rho^2 p > 0$. Then the three vector fields X_u, X_{u+c}, X_{u-c} pairwise commute if and only if the flow u is volume preserving i.e. satisfies $\partial_x u = 0$. In that case, the operator* (3.4.2) *is hexagonal.*

Proof. Straightforward computations show that the three characteristic fields commute if and only if $X_u c - c\partial_x u = 0$, that is if the solution of (3.4.24) satisfies (3.4.27) with $f = 0$.

Using (3.4.24) and the gas law, we obtain

$$X_u c - c\partial_x u = -(2c)^{-1}(2\partial_\rho p + \rho\partial_\rho^2 p)\,\partial_x u. \tag{3.4.28}$$

from which the result follows.

Flows with $\partial_x u = 0$ are very special. They satisfy $X_u p = X_u c = 0$, $X_u(\partial_x p) = \partial_x X_u p = 0$. Thus applying X_u to the second equation in (3.4.24): $\partial_t u + (1/\rho)\,\partial_x p = 0$, we obtain $X_u \partial_t u = \partial_t^2 u = 0$. Hence the volume preserving flows are those which satisfy

$$u(t, x) = \alpha t + \beta, \qquad \alpha, \beta \in \mathbb{R}. \tag{3.4.29}$$

Remark. A general volume-preserving flow is not a simple wave. Note however that $\alpha = 0$ corresponds to a solution which is a simple wave propagating with the constant speed $u = \beta$, the speed of the mode which is linearly degenerate. This is consistent with the previous result about linearly degenerate simple waves.

We next examine if threre exist, for an ideal gas, genuinely nonlinear simple waves that give rise to resonance. Consider waves with speed $u + c$. Then the simple waves satisfy (see [S])

$$S(t, x) = \alpha \tag{3.4.30}$$

$$u(t, x) - \frac{2}{\gamma - 1}\, c(t, x) = \beta \tag{3.4.31}$$

with constants α, β, since (3.4.33) and (3.4.34) are the two Riemann invariants associated with the mode $u + c$. To determine u there is one more equation

$$\partial_t u + \left(u\frac{\gamma + 1}{2} - \beta\frac{\gamma - 1}{2}\right)\partial_x u = 0 \tag{3.4.32}$$

which says that the values of u are constant along the characteristic curves with speed $u + c = u(\gamma + 1)/2 - \beta(\gamma - 1)/2$. These characteristic curves are therefore straight lines.

The resonance condition (3.4.27) and Eqs. (3.4.31), (3.4.32), together imply that the characteristic lines form a pencil. In fact, setting $Y := \partial_t + \beta\partial_x$, (3.4.27), (3.4.31) leads to

$$(Yu)/(u - \beta) + f(t) = 0 \tag{3.4.33}$$

so

$$\partial_x(Y(\mathrm{Log}(u - \beta))) = 0. \tag{3.4.34}$$

Since $[Y, \partial_x]=0$, (3.4.34) gives

$$u-\beta=A(x-\beta t)/B(t). \tag{3.4.35}$$

LEMMA 3.4.5. *If the level sets of a function $u(t, x)=A(x)/B(t)$ are locally straight lines in a simply connected open set, they necessarily belong to one of the three pencils*

$$\{x=\text{const.}\} \text{ or } \{t=\text{const.}\} \text{ or } \{(x-a)/(t-b)=\text{const.}\}. \tag{3.4.36}$$

Proof. At least one of the functions A or B is not constant, since otherwise u would be constant.

First assume that A' is nonzero in an open set crossed by level lines given by $A(x)=hB(t)$ for h a constant. The constant h varies in a neighbourhood of a fixed value, which, normalizing A if necessary may be taken equal to 1. Solving these equations for x, yields straight lines by assumption. The lines have equations $x=\alpha_h t+\beta_h$, with $\alpha_h \neq 0$ if B' is nonzero. Denoting by α and β the values of α_h and β_h for $h=1$, we may eliminate B to obtain a functional equation for A which reads $A(\alpha_h t+\beta_h)=hA(\alpha t+\beta)$. It follows that

$$\begin{aligned}(\alpha_h t+\beta_h-\alpha t-\beta)^{-1}&(A(\alpha_h t+\beta_h)-A(\alpha t+\beta))\\ &=A(\alpha t+\beta)(h-1)/((\alpha_h-\alpha)t+\beta_h-\beta)\end{aligned} \tag{3.4.37}$$

and, after taking the limit $h\to 1$

$$A'(\alpha t+\beta)=A(\alpha t+\beta)/(\alpha' t+\beta'). \tag{3.4.38}$$

We have thus proved that $A(x)=\gamma\,|x-a|^p$, $B(t)=\delta\,|t-b|^p$, with real constants a, b, p, γ, δ. Therefore the level lines belong to a pencil of straight lines with vertex (a, b). The other cases are left to the reader.

Return to the analysis of (3.4.35). First, if $du=0$, that is $dA=dB=0$, the wave is a constant state and the corresponding characteristic web consists of pencils of parallel straight lines as in Example 3.2.1. The resonant phases are polynomials of degree 1. Now assume $du\neq 0$. Applying Lemma 3.4.5 to (3.4.35 with a linear change of variables that respect pencils, we see that the characteristic curves of the simple wave satisfy either

$$x-\beta t=\text{const.} \tag{3.4.39}$$

or

$$(x-a)/(t-b)=\text{const.} \tag{3.4.40}$$

because finite speed of propagation excludes the possibility $t=\text{const.}$

In case (3.4.39) holds, we obtain $u+c=\beta$, which combined with (3.4.31), yields $c=0$. This case is excluded since we assumed $du \neq 0$. Only the case (3.4.40) remains. In particular the wave is centered. After a translation, we may assume $a=b=0$ and choose $y=x/t$, $s=\operatorname{Log}|t|$ as new coordinates to obtain for the three characteristic foliations

$$\Gamma_{u+c}: dy/ds = 0 \tag{3.4.41}$$

$$\Gamma_u: dy/ds = (\gamma+1)^{-1}(1-\gamma)(y-\beta) \tag{3.4.42}$$

$$\Gamma_{u-c}: dy/ds = 2(\gamma+1)^{-1}(1-\gamma)(y-\beta). \tag{3.4.43}$$

The change of variables $z=\operatorname{Log}|y-\beta|$ straightens the three foliations into pencils of parallel lines which satisfy the hexagonal closure property. Thus we have shown there exist an infinite family of genuinely nonlinear $(u+c)$ – simple waves leading to a resonance. They depend upon the four real parameters a, b, α, β. The corresponding resonant phases are easily computed using the preceding changes of variables. A similar family exists for the mode $u-c$. We thus have proved

THEOREM 3.4.6. *The $(u+c)$-simple waves that lead to resonance are centered. They satisfy*

$$S=\alpha, \qquad u-2c/(\gamma-1)=\beta, \qquad u+c=(x-a)/(t-b) \tag{3.4.44}$$

for some constants a, b, α, β.

The corresponding resonant phases are

$$\varphi_{u+c} = \operatorname{Log}\left(\frac{|x-a|}{|t-b|}-\beta\right),$$
$$\varphi_u = \operatorname{Log}\left\{\left(\frac{|x-a|}{|t-b|}-\beta\right)|t-b|^{(\gamma-1)/(\gamma+1)}\right\} \tag{3.4.45}$$

$$\varphi_{u-c} = \operatorname{Log}\left\{\left(\frac{|x-a|}{|t-b|}-\beta\right)|t-b|^{2(\gamma-1)/(\gamma+1)}\right\}. \tag{3.4.46}$$

Remark. On the line $|x-a|/|t-b|=\beta$ we obtain $c=0$, because of (3.4.31) and (3.4.44). The singularity that appears in (3.4.45), (3.4.46), is due to the presence of a vacuum.

4. Almost-Periodic Functions

We present in this paragraph some properties of almost-periodic functions which are needed in the proofs of the main theorems. After some elementary facts in 4.1, we state in Section 4.2 an important regularity result called the sum law for the averaging operators. Section 4.3 contains some lemmas peculiar to third-order resonance. The nonstationary phase lemma which is the keystone of the averaging method, is presented in Section 4.4. Section 4.5 contains the proof of Theorem 2.10.5.

4.1. *Algebras of Almost-Periodic Functions*

Let Ψ be a finite dimensional real vector space and let $\tilde{C}^0_{pp}(\Psi)$ denote the space of almost-periodic functions, that is the Banach subspace of $L^\infty(\Psi)$ generated by the exponentials $e^{i\langle\lambda,\theta\rangle}$ for $\theta\in\Psi$, $\lambda\in\Psi^*$. Recall that we denote by $C^0_{pp}(\Psi)$ the subspace of real valued functions.

To each m-dimensional subspace $V\subset\Psi$ we associate the V-averaging operator $E_V\in\mathscr{L}(\tilde{C}^0_{pp}(\Psi))$ defined by (2.6.1). If $E_{V,T}$ denotes the operator on the right hand side of (2.6.1), then $E_{V,T}$ has norm 1 in $\mathscr{L}(L^\infty(\Psi))$ and it converges pointwise in $L^\infty(\Psi)$ (see (2.6.2)) on the dense subset of trigonometric polynomials. Thus (2.6.1) defines E_V as the limit of $E_{V,T}$ in the strong operator topology. Furthermore E_V does not depend on Q, since by (2.6.2), the formula is independent of Q for trigonometric polynomials. We now prove formula (2.6.3).

LEMMA 4.1.1. *If π is a linear mapping from Ψ to another space Ψ', then*

$$E_V(u\circ\pi)=\{E_{\pi V}(u)\}\circ\pi. \tag{4.1.1}$$

Proof. One can choose coordinates (θ',θ'') in Ψ and (θ',τ) in Ψ' such that $\pi(\theta',\theta'')=(\theta',0)$. Let $H=\{\theta'':(0,\theta'')\in V\}$ and let $W=\{\theta':(\theta',0)\in V\}$; then $V\cap\operatorname{Ker}\pi=\{0\}\times H$ and $\pi V=W\times\{0\}$. Therefore there exists a linear map A such that $V=\{(\theta',\theta''+A\theta'):\theta'\in W$ and $\theta''\in H\}$. Taking $(\theta',\theta'')\in W\times H$ as coordinates on V, we see that the left hand side of (4.1.1), evaluated at $(\theta',\theta'')\in\Psi$, is equal to

$$\lim_{T\to+\infty} T^{q-r}\int_{TQ\times TR} u\circ\pi(\theta'+\sigma,\theta''+\rho+A\sigma)\,d\sigma\,d\rho$$

$$=\lim_{T\to+\infty} T^{-q}\int_{TQ} u(\theta'+\sigma,0)\,d\sigma,$$

where Q [resp. R] is a unit rectangle in W [resp. H] and q [resp. r] is the dimension of W [resp. H]. Now, the latter expression is also the right hand side of (4.1.1) and the lemma is proved.

To each $u\in C^0_{pp}(\Psi)$, one associates its Fourier series

$$u\sim\sum_{\lambda\in\Lambda}\hat{u}(\lambda)\,e^{i\langle\lambda,\theta\rangle}, \tag{4.1.2}$$

where Λ, the spectrum of u, is the set of those λ in Ψ^* such that the Fourier coefficient

$$\hat{u}(\lambda) = E_{\Psi}(u \cdot e^{-i\langle \lambda, \theta \rangle}) \tag{4.1.3}$$

does not vanish. The operator E_{Ψ} in (4.1.3) averages on the whole space and thus yields a scalar. Bessel's identity is (see [Kat])

$$E_{\Psi}(|u|^2) = \sum_{\lambda \in \Lambda} |\hat{u}(\lambda)|^2, \tag{4.1.4}$$

from which it follows that Λ is a countable set which is empty if and only if $u = 0$.

It is readily seen from (2.6.2) that the Fourier series of $E_V(u)$ is the sum of the Fourier modes of u with $\lambda \in V^{\perp}$. If $\Lambda \cap V^{\perp} = \{0\}$ (where as above Λ denotes the spectrum of u), then $E_V(u) = E_{\Psi}(u)$ which is a well-known result in ergodic theory.

In order to describe regularity properties of almost-periodic functions, introduce $C^k_{pp}(\Psi) = \{u \in L^{\infty}(\Psi) : \partial^{\alpha} u \in C^0_{pp}(\Psi)$ whenever $|\alpha| \leqslant k\}$. The members of this space are just uniformly differentiable almost-periodic functions. Recall the following classical result.

PROPOSITION 4.1.2. *If $C^k_u(\Psi)$ denotes the space of functions whose derivatives, up to the order k, are uniformly continuous and bounded, then $C^k_{pp}(\Psi) = C^k_u(\Psi) \cap C^0_{pp}(\Psi)$.*

Proof. Indeed, for $k = 1$, let $\partial_a u$ be the derivative of u in the direction a. Because $\partial_a u \in C^0_u(\Psi)$, applying the mean value theorem, we see that $\partial_a u$ is the uniform limit of $t^{-1}\{u(\cdot + ta) - u(\cdot)\}$ and hence belongs to the closure of $C^0_{pp}(\Psi)$ for the $L^{\infty}(\Psi)$ norm. The proposition follows by induction.

Given Ω, the closure of an open bounded subset of $\mathbb{R}^2$, we also defined in Section 2.7 the spaces $\mathscr{C}^k(\Omega; \Psi)$ whose elements are functions on $\Omega \times \Psi$ whose derivatives of order less than or equal to k belong to $\mathscr{C}^0(\Omega; \Psi) = C^0(\Omega; C^0_{pp}(\Psi))$. These spaces are equipped with the natural norms of uniform convergence on Ω with value in the Banach space $C^0_{pp}(\Psi)$.

We call trigonometric polynomial a function of the form

$$A(t, x; \theta) = \sum_{\text{finite}} \hat{A}(t, x; \lambda)\, e^{i\langle \lambda, \theta \rangle}. \tag{4.1.5}$$

Given $V \subset \Psi$, an operator on $\mathscr{C}^0(\Omega; \Psi)$ is defined by letting E_V act pointwise in (t, x). Abusing notation, this operator is also denoted E_V.

PROPOSITION 4.1.3. (i) *For every* k, $\mathscr{C}^k(\Omega; \Psi)$ *is a Banach algebra on which, by substitution,* C^∞ *functions operate continuously.*

(ii) *For every* $U \in \mathscr{C}^0(\Omega; \Psi)$ *there exists a countable subset* Λ *of* Ψ^* *which contains the spectrum of all* $U(t, x, \cdot)$, $t, x \in \Omega$

$$U(t, x; \theta) \sim \sum_{\lambda \in \Lambda} \hat{U}(t, x; \lambda)\, e^{i\langle \lambda, \theta \rangle}. \tag{4.1.6}$$

(iii) *Every* $U \in \mathscr{C}^k(\Omega; \Psi)$ *may be approximated in* $\mathscr{C}^k(\Omega; \Psi)$ *by a sequence of trigonometric polynomials whose spectra are included in the set* Λ *defined above.*

(iv) *For every* $V \subset \Psi$ *and* $k \geqslant 1$, *the linear operator* E_V *maps* $\mathscr{C}^k$, *into itself and for any vector field* $D(t, x, \partial_t, \partial_x, \partial_\theta)$ *on* $\Omega \times \Psi$ *whose coefficients are independent of* $\theta \in \Psi$, *one has* $D\{E_V(u)\} = E_V(Du)$.

Proof. (1) Completeness and closure under polynomial mappings are immediate. We turn to composition with smooth functions. Given a function F of N variables and $U = (U_1, U_2, ..., U_N)$ N functions in $\mathscr{C}^k$, we first prove that $F(U) = (F(U_1, ..., U_N)$ belongs to $\mathscr{C}^k$. Let K denote any compact set in $\mathbb{C}^N$ which contains the range of U. Approximate F by a sequence of polynomials P_n, uniformly on K. Since

$$\|F(U) - P_n(U)\|_{L^\infty(Q \times \Psi)} \leqslant \|F - P_n\|_{L^\infty(K)}$$

the first part of the lemma ends the proof for $k = 0$. The result for $k > 1$ is obtained by differentiation and using the $k = 0$ case together with the closure property under products.

(2) Let Δ be a countable dense subset of Ω and $\Lambda(t, x)$ denote the spectrum of $U(t, x, \cdot)$. Then

$$\Lambda := \bigcup_{t, x \in \Delta} \Lambda(t, x), \tag{4.1.7}$$

has the desired properties. This follows from the continuity of the Fourier coefficients with respect to the norm in $\mathscr{C}^0$.

(3) The usual Bohr summability process is easily adapted to our case. Enumerate Λ defined by (4.1.7), $\Lambda = \{\lambda_j\}_{j \in \mathbb{N}}$. Let $B_n(\theta)$ be a Bohr polynomial which satisfies (see [Kat], for instance, and take tensor product if there is more than one variable)

$$\begin{aligned} &B_n \geqslant 0 \\ &E_\Psi(B_n) = 1 \\ &\hat{B}_n(\lambda_j) \geqslant 1 - 1/n, \qquad j = 1, ..., n. \end{aligned} \tag{4.1.8}$$

The sequence U_n defined by the convolutions

$$U_n(t, x; \theta) = E_\Psi(U(t, x; \theta - \psi) B_n(\psi)) \tag{4.1.9}$$

satisfies $\|U_n(z') - U_n(z)\|_{L^\infty(\Psi)} \leqslant \|U(z') - U(z)\|_{L^\infty(\Psi)}$, hence is equicontinuous from Ω to $C^0_{pp}(\Psi)$. Now, the theorem of Bohr asserts that for every z, the trigonometric polynomials $U_n(z)$ tend to $U(z)$ as n tends to ∞. The result for $k = 0$ follows from Ascoli's theorem applied on compact subsets of Ω. For $k > 0$, we just differentiate U in the Bohr process.

(4) It is clear that averaging decreases the seminorms of $\mathscr{C}^0$. To show that averaging operates on $\mathscr{C}^k$ and commutes with D, just differentiate under integral.

LEMMA 4.1.4. *If $F \in \mathscr{C}^k(\Omega; \Psi)$ and $U \in \mathscr{C}^k(\Omega; \Psi)$, the shifted function $G(t, x; \theta) = F(t, x; \theta + U(t, x; \theta))$ also belongs to $\mathscr{C}^k(\Omega; \Psi)$.*

Moreover, if $V \subset \Psi$ and if U is invariant under translations parallel to V, then

$$E_V(G)(t, x; \theta) = E_V(F)(t, x; \theta + U(t, x; \theta)). \tag{4.1.10}$$

Proof. This lemma is not an immediate consequence of Proposition 4.1.3 because $\theta \to \theta$ is not almost-periodic.

(a) First we check that $G(t, x; \cdot) \in C^0_{pp}(\Psi)$ for fixed (t, x). If $F(t, x; \cdot)$ is a finite sum of exponentials, $G(t, x; \cdot)$ a sum of terms of the form $f_\lambda(t, x)\, e^{i\langle\lambda, \theta\rangle}\; e^{i\langle\lambda, U(t,x;\theta)\rangle}$ which all belong to $C^0_{pp}(\Psi)$. For a general $F(t, x; \cdot) \in C^0_{pp}$ a uniform approximation of $F(t, x; \cdot)$ by trigonometric polynomials yields a uniform approximation of $G(t, x, \cdot)$ by functions in $C^0_{pp}(\Psi)$.

(b) It remains to show that the mapping $(t, x) \to G(t, x, \cdot)$ is continuous from Ω into $C^0_{pp}(\Psi)$. We write $|G(t', x'; \theta) - G(t, x; \theta)| \leqslant A + B$, where

$$A := |F(t', x'; \theta + U(t', x'; \theta)) - F(t, x; \theta + U(t', x'; \theta))|$$

$$B := |F(t, x; \theta + U(t', x'; \theta)) - F(t, x; \theta + U(t, x; \theta))|.$$

Since $F \in \mathscr{C}^0$, A tends to 0, uniformly in θ, as t', x' tends to t, x. So does B, because $F(t, x; \cdot)$ being almost periodic is uniformly continuous with respect to $\theta \in \Psi$ and U being in $\mathscr{C}^0$, $U(t', x'; \theta)$ tends to $U(t, x; \theta)$ as t', x' tends to t, x, uniformly in θ.

(c) The result for $k > 0$ is obtained by differentiation.

(d) If U is invariant under translations parallel to V, then for $\sigma \in V$ one has

$$G(t, x; \theta + \sigma) = F(t, x; \theta + U(t, x; \theta) + \sigma)$$

and formula (4.1.10) follows by integration on V.

LEMMA 4.1.5. *Assume $\Omega = \{(t, x) : x \in [a, b]$ and $0 \leqslant t \leqslant T(x)\}$ for some Lipschitz function T. If $F \in \mathscr{C}^k(\Omega; \Psi)$, then*

$$(s, t, x, \theta) \to \int_s^t F(\sigma, x; \theta)\, d\sigma \in \mathscr{C}^k(\tilde{\Omega}; \Psi), \tag{4.1.11}$$

where $\tilde{\Omega} = \{(s, t, x) : x \in [a, b],\ 0 \leqslant t \leqslant T(x)$ and $0 \leqslant s \leqslant T(x)\}$.

Proof. When F is a trigonometric polynomial all we have to do is to integrate the coefficients of the polynomial, which clearly yields another polynomial. For general F we apply Proposition 4.1.3(iii). The result for $k > 0$ follows upon differentiating and applying the result for $k = 0$.

4.2. *The Sum Law for the Averaging Operators*

Assume $U \in \mathscr{C}^k(\Omega; \Psi)$ and that V is a linear subspace of Ψ. Proposition 4.1.3 asserts that $E_V U \in \mathscr{C}^k(\Omega; \Psi)$. On the other hand E_V averages in some directions, and behaves like convolution in some other directions (see Examples 2.6.5 and 2.6.6). It has smoothing properties in the latter directions. The aim of this section is to make this remark precise in the situation described in Section 2.6. This result is the key point in the proof of the sum law, Theorem 2.11.1.

Return to the situation described in Section 2. We are given the spaces Φ_j, $R \subset \Phi \times \mathbb{R}$ and $\Psi = R^\perp \subset \Theta \times \mathbb{R}$. The subspaces Ψ_k are defined in (2.5.2), and $E_k = E_{\Psi_k}$. As in Section 2.6, $E_k u$ is invariant by translations parallel to Ψ_k and so is viewed as a function on $\Theta_k \times \mathbb{R}$.

If $U \in \mathscr{C}^0(\omega; \Theta_k \times \mathbb{R})$ for some compact set $\omega \subset \Omega_0$, then as in Section 2.11, we say that U is of class $\sigma \in \mathbb{N}$ in θ_k, if the derivatives $\partial^\alpha_{\theta_k} U \in \mathscr{C}^0(\omega; \Theta_k \times \mathbb{R})$ for $|\alpha| \leqslant \sigma$.

THEOREM 4.2.1. *Suppose that Ψ and E_k are as above. Suppose in addition that $U_j \in \mathscr{C}^0(\omega; \Theta_j \times \mathbb{R})$ is of class σ_j in θ_j for $j = 1, ..., N$, and that $\mathscr{F}(t, x, u)$ is a continuous function on $\omega \times \mathbb{R}^N$, C^∞ with respect to the variables $u \in \mathbb{R}^N$. Let $\tilde{F} := \mathscr{F}(t, x, U_1, ..., U_N) \in \mathscr{C}^0(\omega; \Theta \times \mathbb{R})$ and $F := \tilde{F}_{|\Psi}$. Then $E_k(F)$ is of class μ in θ_k, where*

$$\mu = \operatorname{Min}\left\{\sigma_k,\ \inf_{J \in \mathfrak{J}(k)} \sum_{j \in J \setminus \{k\}} \sigma_j\right\}. \tag{4.2.1}$$

Recall that $\mathfrak{J}(k)$ is the set of the supports J of resonances in S_Φ, such that $k \in J$. The key point in the proof is the following lemma, which is a variation of Lemma 2.5.1.

LEMMA 4.2.2. *Suppose that* $k \in \{1, ..., N\}$ *and* $J' \subset \{1, ..., N\} \setminus \{k\}$ *have the property that for any subset* $K \subset J'$, $K \cup \{k\}$ *does not belong to* $\mathfrak{J}(k)$. *Let* π' *denote the projection from* $\Theta \times \mathbb{R}$ *onto* $\prod_{j \in J'} \Theta_j$. *Then* $\pi'(\Psi_k) = \pi'(\Psi_0)$ *if* $\Psi_0 := \{(\theta, \tau) \in \Psi : \tau = 0\}$.

Proof. Relabeling the variables, we may assume that $k = 1$, $J' = \{2, ..., \ell\}$ and call $(\theta_1, \theta', \theta'')$ the variables in Θ, so that $\pi'(\theta) = \theta'$.

Because π_1 maps Ψ onto $\Theta_1 \times \mathbb{R}$ (Corollary 2.5.2), Ψ can be parametrized by $\Theta_1 \times \mathbb{R} \times \Psi_1$,

$$\Psi \text{ is the set of points } (\theta_1,\ A'\theta_1 + \tau b' + \sigma',\ A''\theta_1 + \tau b'' + \sigma'',\ \tau) \text{ with } \theta_1 \in \Theta_1,\ \tau \in \mathbb{R} \text{ and } (0, \sigma', \sigma'', 0) \in \Psi_1. \tag{4.2.2}$$

with A' [resp. A''] some linear map from Θ_1 into Θ' [resp. Θ''] and b' [resp. b''] some vector in Θ' [resp. Θ''].

Since $\pi'(\Psi_1) \subset \pi'(\Psi_0)$, it suffices to show that $\{\pi'(\Psi_1)\}^\perp \subset \{\pi'(\Psi_0)\}^\perp$.

Let $s' = (s_2, ..., s_\ell) \in \Theta'^* = \prod_{j=2}^{\ell} \Theta_j$ be orthogonal to $\pi'(\Psi_1)$. Then,

$$(\forall (0, \sigma', \sigma'', 0) \in \Psi_1) \qquad \langle s', \sigma' \rangle = 0. \tag{4.2.3}$$

Let $s_1 \in \Theta_1^* = \Phi_1$ be defined by

$$\langle s_1, \theta_1 \rangle := -\langle s', A'\theta_1 \rangle \tag{4.2.4}$$

and let

$$c := -\langle s', b' \rangle \in \mathbb{R}. \tag{4.2.5}$$

If $s = (s_1, s', 0) \in \Phi_1 \times \Phi' \times \Phi''$, then formulas (4.2.3, 4, 5) together with the description (4.2.2) of Ψ, imply that

$$(\forall (\theta, \tau) \in \Psi) \qquad \langle s, \theta \rangle + c\tau = \langle s_1, \theta_1 \rangle + \langle s', \theta' \rangle + c\tau = 0. \tag{4.2.6}$$

Thus (s, c) is orthogonal to $\Psi = R^\perp$, so $(s, c) \in R$ and $s \in S_\Phi$. Let $K \subset J'$ be the support of s' (i.e., the set of indices $j \in J'$ such that $s_j \neq 0$). By assumption, $\{1\} \cup K$ cannot be the support of a resonance in S_Φ so necessarily $s_1 = 0$. Therefore, (4.2.6) implies that $\langle s', \theta' \rangle = 0$ for all $(\theta, 0) \in \Psi_0$ so that $s' \in \{\pi'(\Psi_0)\}^\perp$ and the lemma is proved.

Proof of Theorem 4.2.1. We show by induction on $p \leqslant \mu$ that for $|\alpha| \leqslant p$,

$$\partial_k^\alpha E_k(F) = \sum E_k(F^{(\beta)}), \tag{4.2.7}$$

where $F^{(\beta)}$ is the restriction to Ψ of a derivative $\tilde{F}^{(\beta)} = \partial_{\theta_1}^{\beta_1} \cdots \partial_{\theta_N}^{\beta_N} \tilde{F}$ and in the summation the indices β_j satisfy $|\beta_j| \leqslant \sigma_j$ for all j and $\sum |\beta_j| = |\alpha|$.

Equation (4.2.7) is trivial for $\alpha = 0$. Assume that it is proved for $|\alpha| \leqslant p < \mu$. We will prove it for order $p+1$. Applying (4.2.7) for $|\alpha| = p$, what we want is to differentiate (4.2.7) once more.

Consider a term $E_k(F^{(\beta)})$. First note that

$$|\beta_k| \leqslant p < \mu \leqslant \sigma_k \tag{4.2.8}$$

so that $\tilde{F}^{(\beta)}$ can be differentiated once more with respect to θ_k.

Next let $J' := \{j \neq k : |\beta_j| = \sigma_j\}$. Then for any $K \subset J'$ one has

$$\sum_K \sigma_j = \sum_K |\beta_j| \leqslant p < \mu \tag{4.2.9}$$

and by Definition (4.2.1) of μ we see that $K \cup \{k\} \notin \mathfrak{J}(k)$.

There is no restriction in assuming that $k = 1$ and $J' = \{2, ..., \ell\}$. Let $J'' := \{\ell + 1, ..., N\}$, then by definition of J'

$$(\forall j \in J'') \qquad |\beta_j| < \sigma_j. \tag{4.2.10}$$

Thus $\tilde{F}^{(\beta)}$ is a sum of terms of the form $\mathscr{G}(t, x, V_1, ..., V_N)$ with $V_j(t, x, \theta_j, \tau)$ of class $\sigma'_j = \sigma_j - |\beta_j|$ in θ_j. Moreover $\sigma'_j \geqslant 1$ for $j \notin J'$ and J' satisfies the assumption of Lemma 4.2.2. Call $\tilde{G}$ such a term, and G its restriction to Ψ.

As in the proof of that lemma, we write Θ as $\Theta_1 \times \Theta' \times \Theta''$ and use the parametrization (4.2.2) of the space Ψ. Let $H' = \pi'(\Psi_1)$ and let H'' be the projection on Θ'' of $\Psi_1 \cap \operatorname{Ker} \pi'$ (H'' is isomorphic to $\Psi_1 \cap \operatorname{Ker} \pi'$). Then one can parametrize Ψ_1 by $H' \times H''$, that is Ψ_1 is the space of points $(0, h', Bh' + h'', 0) \in \Theta \times \mathbb{R}$, where B is some linear map from H' into H''. Taking unit rectangles Q' and Q'' in H' and H'', respectively, we find that

$$E_1(G)(t, x, \theta_1, \tau) = \lim_{T \to +\infty} T^{-q} \int_{TQ' \times TQ''} \tilde{G}(t, x, \theta_1, A'\theta_1 + h' + \tau b', \\ h'' + A''\theta_1 + Bh' + \tau b'', \tau)\, dh'\, dh''. \tag{4.2.11}$$

By Lemma 4.2.2 (and Corollary 2.5.2), the range of A' is contained in H' so that one can perform a change of variables in H' to find that $E_1(G)$ is equal to

$$\lim_{T \to +\infty} T^{-q} \int_{TQ' \times TQ''} \tilde{G}(t, x, \theta_1, h' + \tau b', h'' \\ + \{A'' + BA'\}\, \theta_1 + Bh' + \tau b'', \tau)\, dh'\, dh''.$$

This expression can be differentiated with respect to θ_1 because $\tilde{G}$ is of class $\geqslant 1$ in the variables (θ_1, θ''). Moreover a θ_1 derivative of $E_1(G)$ is a linear combination for $j \notin J'$ of terms $E_1(G)^{(j)})$, where $G^{(j)}$ the restriction to Ψ of a θ_j derivative of $\tilde{G}$. Formula (4.2.7) at the order $p+1$ follows, and Theorem 4.2.1 is proved.

4.3. *Further Smoothing Properties*

In this section, we collect several lemmas that will be needed in Section 6. They are based upon the techniques of integration by parts and change of variables used in the last section.

Let φ_k be the Θ_k-valued solution of $X_k \varphi_k = 0$ defined in (2.4.1). Introduce the vector fields

$$D_k = \partial_x \varphi_k(t, x) \cdot \partial_{\theta_k}. \tag{4.3.1}$$

If $(\theta_{k,p})_{1 \leqslant p \leqslant m_k}$ are coordinates in Θ_k and $\varphi_k(t, x) = (\varphi_{k,p}(t, x))_{1 \leqslant p \leqslant m_k}$, then

$$D_k F = \sum_{p=1}^{m_k} \partial_x \varphi_{k,p} \cdot \frac{\partial F}{\partial \theta_{k,p}}.$$

We also introduce

$$D = \sum_{j=1}^{N} D_j \tag{4.3.2}$$

so that D is a vector field on Θ, or on $\Theta \times \mathbb{R}$, whose coefficients depend only on $(t, x) \in \Omega$.

LEMMA 4.3.1. *For each $(t, x) \in \Omega$, D is tangent to the space Ψ, so that for $F \in \mathscr{C}^1(\Omega; \Psi)$ and $k \in \{1, ..., N\}$ one has*

$$D_k\{E_k(F)\} = E_k(DF). \tag{4.3.3}$$

Proof. As in Section 2.4, $(\varphi, 1)$ is Ψ-valued and so is its derivative $(\partial_x \varphi, 0)$. This means that the vector field D is tangent to Ψ so that D acts on functions $F \in \mathscr{C}^1(\Omega; \Psi)$. According to Proposition 4.1.3 (iv), D and E_k commute. Moreover, if U is a function that depends only on (t, x, θ_k, t), then $DU = D_k U$, which means that if $U = U \circ \pi_k$ then $DU = (D_k \tilde{U}) \circ \pi_k$, and (4.3.3) follows.

PROPOSITION 4.3.2. *If $\lambda_{k,0}(t, x)$ denote the eigenvalues of $A(t, x, 0)$, the following integration by parts holds. If $\tilde{F} \in \mathscr{C}^1(\Omega; \Theta \times \mathbb{R})$ and $F := \tilde{F}|_\Psi \in \mathscr{C}^1(\Omega; \Psi)$, then*

$$\sum_{i=1}^{N} (\lambda_{k,0} - \lambda_{i,0})\, E_k\{(D_i \tilde{F})|_\Psi\} = 0. \tag{4.3.4}$$

Proof. $(\varphi, 1)$ and therefore $(\partial_t\varphi, 0)$ take their values in Ψ, which implies that the vector field

$$D' = \sum_{j=1}^{N} \partial_t \varphi_j(t, x) \cdot \partial_{\theta_j}$$

is tangent to Ψ. Because $X_k \varphi_k = 0$, we conclude that

$$Z := \lambda_{k,0} D - D' = \sum_{j=1}^{N} (\lambda_{k,0} - \lambda_{j,0}) D_j$$

is tangent to Ψ. Moreover, the kth component of Z is 0, as well as the τ-component, so that Z is in fact tangent to Ψ_k. Hence for $F \in \mathscr{C}^1(\Omega; \Psi)$, $E_k(ZF) = 0$. Formula (4.3.4) follows, once it is noted that $Z(\tilde{F}|_{\Psi}) = (Z\tilde{F})|_{\Psi}$.

As a direct application of this proposition we obtain

COROLLARY 4.3.3. *Consider three distinct indices i, j, and k, profiles $U(t, x; \theta_i, \tau) \in \mathscr{C}^1(\Omega; \Theta_i \times \mathbb{R})$, and $V(t, x; \theta_j, \tau) \in \mathscr{C}^1(\Omega; \Theta_j \times \mathbb{R})$. Then*

$$(\lambda_{k,0} - \lambda_{i,0}) E_k\{(D_i U) V\} + (\lambda_{k,0} - \lambda_{j,0}) E_k\{U(D_j V)\} = 0. \quad (4.3.5)$$

COROLLARY 4.3.4. *If i, j, k are three distinct indices, $U(t, x; \theta_i, \tau) \in \mathscr{C}^1(\Omega; \Theta_i \times \mathbb{R})$, and, $V(t, x; \theta_j, \tau) \in \mathscr{C}^0(\Omega; \Theta_j \times \mathbb{R})$, then*

$$D_k\{E_k(UV)\} = \frac{\lambda_{i,0} - \lambda_{j,0}}{\lambda_{k,0} - \lambda_{j,0}} E_k\{(D_i U) V\}. \quad (4.3.6)$$

Proof. If $V \in \mathscr{C}^1$, (4.3.6) follows directly from (4.3.3) and (4.3.5). The result for $V \in \mathscr{C}^0$ follows from a density argument using Proposition 4.1.3(iii).

Finally, we state a lemma which is a refinement of Theorem 4.2.1 in a special case.

LEMMA 4.3.5. *If $U \in \mathscr{C}^1(\Omega; \Theta_i)$ and $V \in \mathscr{C}^1(\Omega; \Theta_j)$ are as in the Corollary above, then $E_k\{(D_i U) V\} \in \mathscr{C}^1(\Omega; \Theta_k)$.*

Proof. We can assume that $k = 1$, $i = 2$, and $j = 3$, and $\theta = (\theta_1, \theta_2, \theta'')$ are the variable in Θ. From Corollary 2.5.2, we know that $\pi_{1,2}$ maps Ψ onto $\Theta_1 \times \Theta_2 \times \mathbb{R}$, and hence the projection $\pi'(\theta, \tau) = \theta_2$ maps Ψ_1 onto Θ_2. So, as in the proof of Theorem 4.2.1, one can parametrize Ψ_1 by $\Theta_2 \times H''$ and conclude that $E_1\{(D_2 U) V\}$ is given by a formula of the form

$$\lim_{T \to +\infty} T^{-q} \int_{TQ' \times TQ''} (D_2 U)(t, x, h', \tau)$$
$$\times V(t, x, h'' + \{A'' + BA'\}\, \theta_1 + Bh' + \tau b'', \tau)\, dh'\, dh'', \qquad (4.3.7)$$

where Q' and Q'' are unit rectangles in Θ_2 and H'', respectively.

This formula shows that $E_1\{(D_2 U)\, V\}$ can be differentiated with respect to θ_1.

Together with (4.3.5), it also shows that (formally) $\partial_\tau E_1\{(D_2 U)\, V\}$ is the sum of three terms

$$E_1\{(D_2 U)(\partial_\tau V)\} + E_1\{(D_2 U)(b''\, \partial_{\theta''} V)\} - \frac{\lambda_{1,0} - \lambda_{3,0}}{\lambda_{1,0} - \lambda_{2,0}}\, E_1\{(\partial_\tau U)(D_3 V)\}.$$

Indeed, this computation is certainly correct if U is a trigonometric polynomial, and Proposition 4.1.3(iii) implies that the result is still true for $U \in \mathscr{C}^1$. Thus $\partial_\tau E_1\{(D_2 U)\, V\} \in \mathscr{C}^0(\Omega; \Theta_1 \times \mathbb{R})$.

Similarly, for $\partial = \partial_t$ or ∂_x direct computations and (4.3.5), together with the same density argument, showh that $\partial E_1\{(D_2 U)\, V\}$ is a sum of three terms

$$E_1\{(D_2 U)(\partial V)\} - \frac{\lambda_{1,0} - \lambda_{3,0}}{\lambda_{1,0} - \lambda_{2,0}}\, E_1\{(\partial U)(D_3 V)\} + E_1\{([\partial, D_2]\, U)\, V\}$$

and $\partial E_k\{(D_i U)\, V\} \in \mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R})$.

4.4. *The Fundamental Lemma of Nonstationary Phase*

We refer to Section 2.1 formulas (2.1.5) and (2.1.6) for the definition of strong and weak transversality. In this section, we assume $\Omega \in \mathbb{R}^2$ is of the form

$$\Omega = \{(t, x) : x \in [a, b] \text{ and } 0 \leqslant t \leqslant T(x)\} \qquad (4.4.1)$$

for some Lipschitzean function T. We denote by $\tilde{\Omega}$ the set of $(s, t, x) \in \mathbb{R}^3$ such that $x \in [a, b]$, $0 \leqslant s \leqslant T(x)$ and $0 \leqslant t \leqslant T(x)$.

LEMMA 4.4.1. *Suppose that $a \in C^0(\tilde{\Omega})$ and $\psi \in C^\infty(\Omega)$. If ψ is transverse to ∂_t [resp. weakly transverse to ∂_t], then the function*

$$u_\varepsilon(s, t, x) := \int_s^t a(\sigma, t, x)\, e^{i\psi(\sigma, x)/\varepsilon}\, d\sigma \qquad (4.4.2)$$

tends to 0 in $L^\infty(\tilde{\Omega})$ [resp. in $L^\infty L^p(\tilde{\Omega})$ for all $p < +\infty$] as ε tends to 0.

In accordance with Definition 2.7.1, u_ε is said to be $o(1)$ (or to converge to 0) in $L^\infty L^p(\tilde{\Omega})$ when $\sup_{s,t} \|u_\varepsilon(s,t,\cdot)\|_{L^p(\Omega_{s,t})} \to 0$ as $\varepsilon \to 0$. Here we have used the definition with $\Omega_{s,t} := \{x \in [a,b] : (s,t,x) \in \tilde{\Omega}\}$.

Proof. *Step* 0. We may assume that a is C^1.

Step 1. For any $\delta > 0$, we let $A_\delta := \{(\sigma, x) \in \Omega : |\partial_t \psi(\sigma, x)| \leq \delta\}$ and $B_\delta := \{(\sigma, x) : |\partial_t \psi(\sigma, x)| > \delta/2\}$. Then there exist C^∞ functions on $\mathbb{R}^2$, χ_1 and χ_2, such that

$$\operatorname{supp} \chi_1 \cap \Omega \subset A_\delta, \qquad \operatorname{supp} \chi_2 \cap \Omega \subset B_\delta, \qquad \text{and} \qquad \chi_1 + \chi_2 = 1 \qquad \text{on } \Omega. \tag{4.4.3}$$

Inserting this partition of unity under the integral in (4.4.2), yields $u_\varepsilon = u_\varepsilon^1 + u_\varepsilon^2$. An integration by parts shows that

$$|u_\varepsilon^2(s,t,x)| \leq \varepsilon C(\delta), \tag{4.4.4}$$

where $C(\delta)$ depends only on δ, and the estimate (4.4.4) is uniform in (s,t,x).

On the other hand, if $A_\delta(x) := \{\sigma \in [0, T(x)] : (\sigma, x) \in A_\delta\}$ and

$$h_\delta(x) := \operatorname{meas}\{A_\delta(x)\}, \tag{4.4.5}$$

then we clearly have $|u_\varepsilon^1(s,t,x)| \leq C h_\delta(x)$ with a constant C independent of $(s,t,x) \in \tilde{\Omega}$. Summing up, we have proved that

$$|u_\varepsilon(s,t,x)| \leq \varepsilon C(\delta) + C h_\delta(x). \tag{4.4.6}$$

Step 2. Integrating (4.4.6) over $\Omega_{s,t}$ yields

$$\|u_\varepsilon(s,t,\cdot)\|_{L^1(\Omega_{s,t})} \leq \varepsilon C(\delta) + C \operatorname{meas}(A_\delta). \tag{4.4.7}$$

Weak transversality means that $\partial_t \psi(\sigma, x) \neq 0$ a.e. in Ω so that $\operatorname{meas}(A_\delta) \to 0$ when $\delta \to 0$. So (4.4.7) implies that $u_\varepsilon = o(1)$ in $L^\infty L^1(\tilde{\Omega})$.

On the other hand, there is a trivial uniform estimate $|u_\varepsilon(s,t,x)| \leq C$ so that u_ε is also $o(1)$ in $L^\infty L^p(\tilde{\Omega})$ for all $p < +\infty$.

Step 3. Assuming strong transversality means that for every $x \in [a,b]$, $\partial_t \psi(\sigma, x) \neq 0$ a.e. in $[0, T(x)]$ and hence that $h_\delta(x) \to 0$ as $\delta \to 0$ pointwise for $x \in [a,b]$.

We begin by showing that h_δ is an upper semicontinuous function. So suppose that $x_0 \in [a,b]$, $\eta > 0$ and an open neighborhood U of the compact set $A_\delta(x_0)$ in $\mathbb{R}$ are given with $\operatorname{meas}(U) \leq h_\delta(x_0) + \eta$. By compactness and continuity of $\partial_t \psi$ it is easy to show that there is $\rho > 0$ small enough such that for $|x - x_0| \leq \rho$ one has $A_\delta(x) \subset U$ and hence $h_\delta(x) \leq h_\delta(x_0) + \eta$.

On the other hand, the family h_δ is clearly decreasing with δ. Hence the h_δ form a decreasing family of upper semicontinuous functions that converge pointwise to 0. By Dini's theorem, the convergence is uniform in $x \in [a, b]$. Plugging this in (4.4.6) immediately implies that $u_\varepsilon \to 0$ in $L^\infty(\tilde{\Omega})$ as $\varepsilon \to 0$.

The proof is complete.

As in Section 2.3, we now assume that spaces Φ_k are given on Ω_0 and we fix a domain $\Omega = \Omega_0 \cap \{t \leqslant T\}$ or $\Omega = \Omega_\rho \cap \{t \leqslant T\}$ as in Section 2.8 or 2.9. For simplicity, we suppose here that (after a change of the x variable) the vector field X_k is equal to ∂_t. Because of (2.1.4), Ω is contained in the domain of determinacy of X_k and hence is of the form (4.4.1). Define $\tilde{\Omega}$ as above.

THEOREM 4.4.2. *Assume ($\mathscr{C}$) and ($\mathscr{T}$) [resp. ($w - \mathscr{T}$)]. Suppose that F belongs to $\mathscr{C}^0(\tilde{\Omega}; \Psi)$, and define u_ε and U by*

$$u_\varepsilon(s, t, x) := \int_s^t F(\sigma, t, x, \varphi(\sigma, x)/\varepsilon, 1/\varepsilon)\, d\sigma \tag{4.4.8}$$

$$U(s, t, x; \theta_k, \tau) := \int_s^t (E_k F)(\sigma, t, x, \theta_k, \tau)\, d\sigma. \tag{4.4.9}$$

Then

$$u_\varepsilon(s, t, x) - U(s, t, x, \varphi_k(x)/\varepsilon, 1/\varepsilon) = o(1) \tag{4.4.10}$$

in $L^\infty(\tilde{\Omega})$ [resp. in $L^\infty L^p(\tilde{\Omega})$ for all $p < +\infty$].

If one assumes that F has the form $\sum_{i,j} F_{i,j}(\sigma, t, x, \theta_i, \theta_j)$ and ($\mathscr{C}_2$) and ($\mathscr{T}_2$) [resp. ($w - \mathscr{T}_2$)] then the same conclusion holds for u.

Note that φ_k is a function of x alone because $X_k \varphi_k = \partial_t \varphi_k = 0$.

Proof. According to Proposition 4.1.3(iii) it suffices to prove the estimate when F is a trigonometric polynomial. By linearity it suffices to consider

$$F(\sigma, t, x, \theta, \tau) = a(\sigma, t, x)\, e^{i\{\langle \lambda, \theta \rangle + c\tau\}} \tag{4.4.11}$$

and we are led to study integrals of the form (4.4.2) with $\psi = \langle \lambda, \varphi \rangle + c$, where $(\lambda, c) \in \Theta^* \times \mathbb{R} = \Phi \times \mathbb{R}$.

By properties ($\mathscr{C}$) and ($\mathscr{T}$) [resp. ($w - \mathscr{T}$)], either $\partial_t \psi \equiv 0$ and $\psi \in \Phi_k \oplus \mathbb{R}$, or ψ is transverse [resp. weakly transverse] to ∂_t. In the second case, Lemma 4.4.1 implies that the corresponding integral (4.4.2) is $o(1)$ while the condition $\partial_t \psi \neq 0$ implies that $(\lambda, c) \notin R \oplus (\tilde{\Phi}_k \times \mathbb{R})$ so that $E_k(F) = 0$ by (2.6.5). Therefore (4.4.10) is proved in that case.

In the first case, ψ does not depend on t so

$$u_\varepsilon(s, t, x) = e^{i\psi(x)/\varepsilon} \int_s^t a(\sigma, t, x)\, d\sigma. \tag{4.4.12}$$

On the other hand, because $\psi \in \Phi_k \oplus \mathbb{R}$, there are $\mu_k \in \Phi_k$ and $c' \in \mathbb{R}$ such that $\psi = \mu_k + c'$. Let $\mu := (0, ..., \mu_k, 0, ..., 0) \in \tilde{\Phi}_k \subset \Phi$, the μ_k being in the kth slot. Then $\psi = \langle \lambda, \varphi \rangle + c$ implies that $(\lambda - \mu, c - c') \in R$. Thus $(\lambda, c) \in R \oplus (\tilde{\Phi}_k \times \mathbb{R})$. Therefore F is invariant under translations parallel to Ψ_k, so considering F as a function of (θ_k, τ) one has

$$F(\sigma, t, x, \theta_k, \tau) = E_k(F)(\sigma, t, x, \theta_k, \tau) = a(\sigma, t, x)\, e^{i\{\langle \mu_k, \theta_k \rangle + c'\tau\}}. \tag{4.4.13}$$

Comparing (4.4.12) and (4.4.9) we have $u_\varepsilon(s, t, x) = U(s, t, x, \varphi_k(x)/\varepsilon, 1/\varepsilon)$ exactly.

When $F = \sum_{i,j} F_{i,j}(\theta_i, \theta_j)$ only exponentials of the form

$$a(\sigma, t, x)\, e^{i\langle \lambda_j, \theta_j \rangle + \langle \lambda_l, \theta_l \rangle}$$

appear in (4.4.11). In this cases ($\mathscr{C}q$) and ($\mathscr{T}q$) [resp. ($w - \mathscr{T}q$)] suffice to apply Lemma 4.4.1.

This finishes the proof of Theorem 4.4.2.

Remark 4.4.3. With more information on the transversality (the order of vanishing of $\partial_t \psi$) one can improve the $o(1)$ estimate of Lemma 4.4.1. But, as mentioned in Remark 1.2, it is much more difficult to improve estimate (4.4.10) of Theorem 4.4.2. Such an improvement would probably require a lot of additional structure on Φ, and also restrictions on the class of functions F.

4.5. *An L^∞ Estimate*

The aim of this section is to prove Theorem 2.10.5. In fact we cast the problem in a slightly more general context. Consider an interval $[a, b] \subset \mathbb{R}$, with $a < b$, and a continuous map ψ from $[a, b]$ into $\mathbb{R}^m$ such that

> For any nonvoid open subinterval $I \subset [a, b]$, the components of ψ are linearly independent functions on I. (4.5.1)

Given a function $U \in \mathscr{C}^0([a, b]; \mathbb{R}^m)$ introduce

$$u_\varepsilon(x) := U(x, \psi(x)/\varepsilon). \tag{4.5.2}$$

THEOREM 4.5.1. *With the above notation, one has the estimate*

$$\|U\|_{L^\infty([a,b]\times\mathbb{R}^m)} = \limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty([a,b])}. \tag{4.5.3}$$

Proof of Theorem 2.10.5 *Assuming Theorem* 4.5.1. It suffices to consider for t fixed, the map $\psi: x \to (\varphi_k(t,x), 1)$ from the interval Ω_t into $\Theta_k \times \mathbb{R}$. Condition (4.5.1) is satisfied because if there were a nontrivial linear relation between the components of ψ on $I \subset \Omega_t$, then Definition (2.4.1) of φ_k would imply that there would exist $s \in \Phi_k \setminus \{0\}$ and $c \in \mathbb{R}$ such that $s + c = 0$ on I. This is impossible thanks to conditions (2.3.1) and (2.3.2).

The proof of Theorem 4.5.1 is based upon the following lemma.

LEMMA 4.5.2. *Suppose $(\alpha_\nu)_{\nu\in A}$ is a finite subset of $\mathbb{R}^m$ and denote by $\alpha_{\nu,j}$ the components of α_ν. Fix $\theta \in \mathbb{R}^m$, a nonvoid open subinterval I, $\delta > 0$, and $\lambda_0 \in \mathbb{R}$. Then there exist a point $x \in I$, a real $\lambda \geqslant \lambda_0$ and integers $k_{\nu,j} \in \mathbb{Z}$ such that, for all ν and j*

$$|\alpha_{\nu,j}\{\theta_j - \lambda\psi_j(x)\} - k_{\nu,j}| \leqslant \delta. \tag{4.5.4}$$

Proof. Step 1. For each j, consider the $\mathbb{Q}$-vector space generated in $\mathbb{R}$ by the $\{\alpha_{\nu,j}\}_{\nu\in A}$. Choose a basis, $\{\beta_{\mu,j}\}_{\mu\in A_j}$ for this space so that one can write

$$\alpha_{\nu,j} = \sum_{\mu\in A_j} p_{\nu,\mu,j}\beta_{\mu,j} \tag{4.5.5}$$

with $p_{\nu,\mu,j} \in \mathbb{Q}$. Indeed, dividing the $\beta_{\mu,j}$ by a common multiple of the denominators of the $p_{\nu,\mu,j}$ we can assume that all the coefficients $p_{\nu,\mu,j}$ are integers.

Step 2. Consider the map $x \to \Psi(x) = \{\beta_{\mu,j}\psi_j(x)\}_{1\leqslant j\leqslant m,\, \mu\in A_j}$ from I into $\mathbb{R}^M$ with $M = \sum \#(A_j)$. We claim that for any open interval I, the image $\Psi(I)$ is not contained in any rational hyperplane of $\mathbb{R}^M$.

Indeed, let $\rho = (\rho_{\mu,j}) \in \mathbb{Q}^M$ and assume that $\Psi(I) \subset \rho^\perp$. That means that

$$\forall x \in I, \qquad \sum_j \left(\sum_{\mu\in A_j} \rho_{\mu,j}\beta_{\mu,j}\right)\psi_j(x) = 0.$$

Because the ψ_j are linearly independent, this requires that for all j, $\sum \rho_{\mu,j}\beta_{\mu,j} = 0$. Since the $\{\beta_{\mu,j}\}_{\mu\in A_j}$ are $\mathbb{Q}$-independent, all the coefficients $\rho_{\mu,j}$ must vanish.

Step 3. Let I be a given interval and let $D \subset I$ be the set of those $x \in I$ such that the coordinates $\Psi_{\mu,j}(x) = \beta_{\mu,j}\psi_j(x)$ of Ψ, for $1 \leqslant j \leqslant m$, $\mu \in A_j$, are $\mathbb{Q}$-linearly independent. We claim that D is dense in I.

Indeed, for $\rho \in \mathbb{Q}^M \setminus \{0\}$, let $F_\rho = \{x \in I: \langle \rho, \Psi(x) \rangle = 0\}$. Ψ being continuous, F_ρ is closed in I and it follows from Step 2 that F_ρ has empty interior. Therefore, by Baire's theorem, the interior of the countable union $F = \bigcup F_\rho$ is still void, which means that $D = I \setminus F$, is dense in I, as claimed.

Step 4. For any I, and any $\lambda_0 \in \mathbb{R}$, there is $x \in I$ such that the half line $\{\lambda \Psi(x), \lambda \geqslant \lambda_0\}$ in $\mathbb{R}^M$ has a dense image in the torus $(\mathbb{R}/\mathbb{Z})^M$. In fact, by Kronecker theorem, it suffices to pick an $x \in I$ such that the reals $\{\Psi_{\mu,j}(x)\}$ are $\mathbb{Q}$-independent (see [A]).

Step 5. Let $\theta \in \mathbb{R}^m$, I, $\delta > 0$, and λ_0 be given. Let p be the maximum of the moduli of the integers $p_{\nu,\mu,j}$ that appear in (4.5.5). According to Step 4 there are $x \in I$, $\lambda \geqslant \lambda_0$ and integers $k_{\mu,j}$ such that for all j and $\mu \in A_j$

$$|\beta_{\mu,j}\{\theta_j - \lambda\psi_j(x)\} - k_{\mu,j}| \leqslant \delta/pM. \tag{4.5.6}$$

With (4.5.5) we find

$$\left|\alpha_{\nu,j}\{\theta_j - \lambda\psi_j(x)\} - \sum_{\mu \in A_j} p_{\nu,\mu,j} k_{\mu,j}\right| \leqslant \delta \tag{4.5.7}$$

and (4.5.4) is proved.

Next we prove the theorem in the special case that U is a trigonometric polynomial.

LEMMA 4.5.3. *If* $U = \sum_{\nu \in A} a_\nu e^{2i\pi\langle \alpha_\nu, \theta \rangle}$ *is a trigonometric polynomial and* u_ε *is defined by* (4.5.2), *then for any open interval I and any* $\varepsilon_0 > 0$ *one has*

$$\|U\|_{L^\infty(\mathbb{R}^m)} \leqslant \sup_{0 < \varepsilon \leqslant \varepsilon_0} \|u_\varepsilon\|_{L^\infty(I)}. \tag{4.5.8}$$

Proof. Let $\eta > 0$ be given. Choose $\delta > 0$ such that

$$\left(\sum_{\nu \in A} |a_\nu|\right) \sup_{|y| \leqslant \delta} |e^{2i\pi y} - 1| \leqslant \eta. \tag{4.5.9}$$

Fix $\theta \in \mathbb{R}^m$. According to Lemma 4.5.2, there are $x \in I$, $\lambda \geqslant 1/\varepsilon_0$, and integers $k_{\nu,j}$ such that (4.5.4) holds. Thus

$$U(\theta) - U(\lambda\psi(x)) = \sum_{\nu \in A} a_\nu e^{2i\pi\lambda\langle \alpha_\nu, \psi(x) \rangle}\{e^{2i\pi y_\nu} - 1\}$$

with $y_\nu = \alpha_{\nu,j}\{\theta_j - \lambda\psi_j(x)\} - k_{\nu,j}$. Then (4.5.4) and (4.5.9) imply that

$$|U(\theta)| \leqslant |U(\lambda\psi(x))| + \eta$$

and (4.5.8) follows.

Proof of Theorem 4.5.1. Let $\eta > 0$, $\varepsilon_0 > 0$, $x_0 \in [a, b]$ and $\theta \in \mathbb{R}^m$ be given. Because $U \in \mathscr{C}^0([a, b]; \mathbb{R}^m)$ there is $\rho > 0$ such that

$$\forall x \in [a, b], \qquad |x - x_0| \leqslant \rho \Rightarrow \|U(x, \cdot) - U(x_0, \cdot)\|_{L^\infty(\mathbb{R}^m)} \leqslant \eta. \tag{4.5.10}$$

On the other hand, there is a trigonometric polynomial $V(\theta)$ such that

$$(\forall \theta' \in \mathbb{R}^m) \qquad |U(x_0, \theta') - V(\theta')| \leqslant \eta. \tag{4.5.11}$$

Lemma 4.5.3 implies that there exist $\varepsilon \leqslant \varepsilon_0$ and $x \in [a, b]$ with $|x - x_0| \leqslant \rho$ such that

$$|V(\theta)| \leqslant |V(\psi(x)/\varepsilon)| + \eta. \tag{4.5.12}$$

Adding these inequalities, yields

$$|U(x_0, \theta)| \leqslant |U(x, \psi(x)/\varepsilon)| + 4\eta$$

and the estimate

$$\|U\|_{L^\infty([a,b] \times \mathbb{R}^m)} \leqslant \sup_{0 < \varepsilon \leqslant \varepsilon_0} \|u_\varepsilon\|_{L^\infty([a,b])}$$

follows. The converse estimate being obvious, Theorem 4.5.1 is proved.

5. The Semilinear Cauchy Problem

This section is devoted to the proof of Theorem 2.8.1. The more technical proof of Theorem 2.9.1 follows the same general scheme.

Consider the Cauchy problem (2.8.1) with the assumptions of Section 2.8 in force. First, we make a technical simplification. It is easy to check that the statement of Theorem 2.8.1 is invariant under a change of dependent variables $u = V(t, x)\,\tilde{u}$ with a smooth $N \times N$ invertible matrix V. Therefore we can assume that the basis (r_j) and (ℓ_j) are just the canonical basis of $\mathbb{R}^N$, or equivalently that $A(t, x)$ is diagonal. In that case,

The operator L of (2.8.1) is diagonal with entries X_k. (5.0.1)

5.1. *Outline of Proof*

The starting point is the construction of the solution of the Cauchy problem by the standard Picard iteration

$$\begin{aligned} Lu_\varepsilon^{\nu+1} &= b(t, x, u_\varepsilon^\nu(t, x)) \\ u_\varepsilon^{\nu+1}|_{t=0} &= h_\varepsilon \end{aligned} \tag{5.1.1}$$

with $u_\varepsilon^0 = 0$. The following lemma is an immediate consequence of the fundamental theorem of calculus.

LEMMA 5.1.1. *For any $T>0$, any $f \in L^\infty([0, T] \times \mathbb{R})$ and any $h \in L^\infty(\mathbb{R})$, the problem*

$$Lu = f \qquad u|_{t=0} = h \tag{5.1.2}$$

has a unique solution $u \in L^\infty([0, T] \times \mathbb{R})$ and

$$\|u\|_{L^\infty([0,T]\times\mathbb{R})} \leqslant \|h\|_{L^\infty(\mathbb{R})} + T\,\|f\|_{L^\infty([0,T]\times\mathbb{R})}. \tag{5.1.3}$$

Because the Cauchy data h_ε are uniformly bounded in L^∞, one deduces from Lemma 5.1.1 the following consequence (HW).

PROPOSITION 5.1.2. *There is a $T>0$ such that for all $\varepsilon \in]0, 1]$ the iteration scheme (5.1.1) defines a sequence $u_\varepsilon^\nu \in L^\infty(\Omega)$, $(\Omega = [0, T] \times \mathbb{R})$, such that*

(i) *there is M such that for all ν and all ε: $\|u_\varepsilon^\nu\|_{L^\infty(\Omega)} \leqslant M$.*

(ii) *the sequence u_ε^ν converges in $L^\infty(\Omega)$ to the solution of (2.8.1) uniformly in $\varepsilon \in]0, 1]$.*

The next proposition is second and main step in the proof of Theorem 2.8.1.

PROPOSITION 5.1.3. *We assume that condition $(\mathscr{T})$ [resp. $(w-\mathscr{T})$] holds. Then, there are profiles $U_k^\nu \in \mathscr{C}^0(\Omega; \Theta_k)$ such that*

(i) *for all ν, $u_{\varepsilon,k}^\nu(t, x) - U_k^\nu(t, x, \varphi_k(t, x)/\varepsilon, 1/\varepsilon)$ is $o(1)$ in $L^\infty(\Omega)$ [resp. in $L^\infty L^p(\Omega)$] as $\varepsilon \to 0$.*

(ii) *the U_k^ν are determined as the solutions of*

$$X_k U_k^{\nu+1} = E_k(b_k(t, x, U^\nu(t, x, \theta, \tau))) \qquad U_k^{\nu+1}|_{t=0}(x, \theta_k, \tau) = H_k(x, \theta_k, \tau). \tag{5.1.4$_k$}$$

In this statement, the subscript k for u_ε^ν, H, U^ν or b, indicates the kth component of the corresponding vector in $\mathbb{R}^N$ (recall that (r_j) and (ℓ_j) have been taken to be the canonical basis). Note that we do not claim in (i) that the $o(1)$ is uniform with respect to ν.

This proposition is proved by induction on ν in Section 5.3.

The third step is a study of the iteration scheme (5.1.4) for the profiles.

PROPOSITION 5.1.4. *Decreasing T if necessary, the sequence $U^\nu = (U_k^\nu)_{1 \leqslant k \leqslant N}$ is bounded in $\mathscr{C}^0(\Omega; \Psi)$, and converges in $L^\infty(\Omega \times \Psi)$ towards the (unique) solution $U = (U_k)_{1 \leqslant k \leqslant N}$ of*

$$\begin{aligned} X_k U_k &= E_k(b_k(t, x, U(t, x, \theta, \tau))) \\ U_k|_{t=0}(x, \theta_k, \tau) &= H_k(x, \theta_k, \tau). \end{aligned} \tag{5.1.5$_k$}$$

Theorem 2.8.1 is a consequence of the three preceding propositions. Indeed, point (i) of Theorem 2.8.1 is an immediate consequence of Proposition 5.1.2. Equations (2.8.5) of point (iii), are just those stated in (5.1.5). Finally, Propositions 5.1.2 and 5.1.4 imply that $u_\varepsilon^\nu(t, x) - U^\nu(t, x, \varphi(t, x)/\varepsilon, 1/\varepsilon)$ converges in $L^\infty(\Omega)$ (and hence in $L^\infty L^p(\Omega)$) to $u_\varepsilon(t, x) - U(t, x, \varphi(t, x)/\varepsilon, 1/\varepsilon)$, uniformly with respect to $\varepsilon \in]0, 1]$. By Proposition 5.1.3, for each ν, $u_\varepsilon^\nu(t, x) - U^\nu(t, x, \varphi(t, x)/\varepsilon, 1/\varepsilon) = o(1)$ in $L^\infty(\Omega)$ or in $L^\infty L^p(\Omega)$. Since the convergence is uniform in ε, it follows that $u_\varepsilon(t, x) - U(t, x, \varphi(t, x)/\varepsilon, 1/\varepsilon) = o(1)$, which is just point (ii) of Theorem 2.8.1.

Proposition 5.1.2 is a straightforward consequence of Lemma 5.1.1, and its proof will be omitted. Proposition 5.1.3 will be proved in Section 5.2, and Proposition 5.1.4 in Section 5.3. It is the latter section which is the main step in the proof. Times of existence T given by Proposition 5.1.2 and 5.1.4 are compared in Section 7.

5.2. *Equations for the Profiles*

Because of the averaging operators E_k the system (5.1.5) is integro-differential. It is solved by the standard Picard iteration scheme with only superficial modifications. Consider the scheme (5.1.4), starting from $U^0 = 0$. We note $\|F\|$ the norm of F in $\mathscr{C}^0(\Omega; \Psi)$ and $\|F(t)\|$ the norm of $F(t, \cdot, \cdot, \cdot)$ in $\mathscr{C}^0(\Omega_t; \Psi)$. Assuming U^ν is defined, by Proposition 4.1.3, the function $F^\nu(t, x, \theta, \tau) = b(t, x, U^\nu(t, x, \theta, \tau))$ belongs to $\mathscr{C}^0(\Omega; \Psi)$ and

$$\|F^\nu(t)\| \leqslant b^\#(\|U^\nu(t)\|), \tag{5.2.1}$$

where $b^\#$ is an increasing function from $[0, +\infty[$ into itself. By Proposition 4.1.3, the functions $G_k^\nu = E_k(F_k^\nu)$ belong to $\mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R})$ and satisfy

$$\|G_k^\nu(t)\| \leqslant \|F_k^\nu(t)\| \leqslant b^\#(\|U^\nu(t)\|). \tag{5.2.2}$$

Now, Eq. $(5.1.4)_k$ takes the form

$$X_k U_k^{\nu+1} = G_k^\nu \tag{5.2.3}$$

and its solution is found by integrating along the characteristics. With notations as in Section 2.1, one has

$$U_k^{\nu+1}(t, x, \theta, \tau) = H_k(\gamma_k(0, t, x), \theta, \tau) + \int_0^t G_k^\nu(s, \gamma_k(s; t, x), \theta, \tau)\, ds. \quad (5.2.4)$$

It follows from Lemma 5.1.4 that $U_k^{\nu+1} \in \mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R})$ and that

$$\|U_k^{\nu+1}(t)\| \leqslant \|H_k\| + \int_0^t \|G_k^\nu(s)\|\, ds. \quad (5.2.5)$$

Let $M = \|H\| + 1$ and choose $T > 0$ such that $Tb^{\#}(M) \leqslant 1$. Then it is clear by induction, from (5.2.2) and (5.2.5), that $\|U^\nu\| \leqslant M$ for all ν. Knowing that, one can write

$$\|G_k^\nu(t) - G_k^{\nu-1}(t)\| \leqslant \|F_k^\nu(t) - F_k^{\nu-1}(t)\| \leqslant C\, \|U^\nu(t) - U^{\nu-1}(t)\|. \quad (5.2.6)$$

Integrating along the characteristics yields

$$\|U_k^{\nu+1}(t) - U_k^\nu(t)\| \leqslant C \int_0^t \|G_k^\nu(s) - G_k^{\nu-1}(s)\|\, ds \quad (5.2.7)$$

and therefore

$$\|U^{\nu+1}(t) - U^\nu(t)\| \leqslant M \frac{(Ct)^\nu}{\nu!} \quad (5.2.8)$$

which implies the uniform convergence of U_k^ν to a function $U_k \in \mathscr{C}^0(\Omega, \Theta_k \times \mathbb{R})$. Passing to the limit shows that U is solution to (5.1.5).

Uniqueness is clear, because for two bounded solutions U and V one has an estimate similar to (5.2.7)

$$\|U(t) - V(t)\| \leqslant C \int_0^t \|U(s) - V(s)\|\, ds, \quad (5.2.9)$$

with C depending only on the L^∞ norms of U and V. The proof of Proposition 5.1.4 is now complete.

5.3. *Linear Propagation of Oscillations*

The proof of Proposition 5.1.3 is by induction. The u_ε^ν and U^ν are given by Propositions 5.1.2 and 5.1.4, and they are known to be uniformly bounded respectively in $L^\infty(\Omega)$ and $L^\infty(\Omega \times \Psi)$.

The first step in the proof is an application of Theorem 4.1.3.

LEMMA 5.3.1. *Assume that* $u_\varepsilon^\nu(t, x) - U^\nu(t, x, \varphi(t, x)/\varepsilon, 1/\varepsilon) = o(1)$ *in*

$L^\infty L^p(\Omega)$ $(1 \leqslant p \leqslant \infty)$. Then $b(t, x, u_\varepsilon^\nu(t, x)) - F^\nu(t, x, \varphi(t, x)/\varepsilon, \varepsilon) = o(1)$ in $L^\infty L^p(\Omega)$, where $F^\nu(t, x, \theta, \tau) = b(t, x, U^\nu(t, x, \theta, \tau)) \in \mathscr{C}^0(\Omega; \Psi)$.

Then, in order to prove Proposition 5.1.3 it suffices to check the following result.

PROPOSITION 5.3.2. *Assume that condition $(\mathscr{T})$ [resp. $(w-\mathscr{T})$] holds. Let h_ε be as in Theorem 2.8.1, and let f_ε be a family of functions in $L^\infty(\Omega)$ such that*

there is $F \in \mathscr{C}^0(\Omega; \Psi)$ such that $f_\varepsilon(t, x) - F(t, x, \varphi(t, x)/\varepsilon, 1/\varepsilon) = o(1)$ in $L^\infty(\Omega)$ [resp. in $L^\infty L^p(\Omega)$]. (5.3.1)

Let u_ε be the (unique) solution of

$$Lu_\varepsilon = f_\varepsilon, \qquad u_\varepsilon|_{t=0} = h_\varepsilon \tag{5.3.2}$$

and let U_k be the (unique) solution in $\mathscr{C}^0(\Omega, \Theta_k \times \mathbb{R})$ of

$$X_k U_k = E_k(F), \qquad U_k|_{t=0} = H_k(x; \theta_k, \tau) \tag{5.3.3}$$

Then,

$$u_\varepsilon(t, x) - U(t, x, \varphi(t, x)/\varepsilon, 1/\varepsilon) = o(1)$$

in $L^\infty(\Omega)$ [resp. in $L^\infty L^p(\Omega)$].

Proof. It suffices to consider *one* scalar equation $X_k u_\varepsilon = f_\varepsilon$. We perform the change of variables $y \to x = \gamma_k(t; 0, y)$ and we call $v_\varepsilon(t, y)$, $g_\varepsilon(t, y)$, $\psi_j(t, y)$ the functions that correspond to $u_\varepsilon(t, x)$, $f_\varepsilon(t, x)$, $\varphi_j(t, x)$. In the new coordinates, X_k is $\partial_t \psi_k = 0$. Thus $\psi_k = \psi_k(y)$ is a function of y only. We have $\varphi_k|_{t=0} = \varphi^0$ (see Section 2.4).

The equation for v_ε is

$$\begin{aligned} &\partial_t v_\varepsilon = g_\varepsilon \\ &v_\varepsilon(0, y) = h_\varepsilon(y) = H(y, \psi_k(y)/\varepsilon, 1/\varepsilon) + o(1). \end{aligned} \tag{5.3.4}$$

Therefore

$$v_\varepsilon(t, y) = h_\varepsilon(y) + \int_0^t g_\varepsilon(s, y))\, ds. \tag{5.3.5}$$

From (5.3.4) and (5.3.1) we obtain

$$v_\varepsilon(t, y) = H(y, \psi_k(y)/\varepsilon, 1/\varepsilon) + \int_0^t G(s, y, \psi(s, y)/\varepsilon, 1/\varepsilon)\, ds) + o(1), \tag{5.3.6}$$

where

$$G(t, y, \theta, \tau) = F(t, \gamma_k(t; 0, y), \theta, \tau) \in \mathscr{C}^0(\Omega; \Psi), \tag{5.3.7}$$

where Ω still denotes the same open set in the new variables. We are now in position to apply Theorem 4.4.2 to the integral in (5.3.6), and conclude that

$$v_\varepsilon(t, y) = V(t, y, \psi_k(y)/\varepsilon, 1/\varepsilon) + o(1), \tag{5.3.8}$$

where

$$V(t, y, \theta_k, \tau) = H(y, \theta_k, \tau) + \int_0^t (E_k G)(s, y, \theta_k, \tau)\, ds \in \mathscr{C}^0(\Omega; \Theta_k \times \mathbb{R}). \tag{5.3.9}$$

This integral equation is equivalent to the initial value problem

$$\begin{aligned} \partial_t V(t, y, \theta_k, \tau) &= (E_k G)(t, y, \theta_k, \tau) \\ V(0, y, \theta_k, \tau) &= H(y, \theta_k, \tau). \end{aligned} \tag{5.3.10}$$

Now, an important point is to remark that the change of variables (5.3.7) does not affect θ, so

$$(E_k G)(t, y, \theta_k, \tau) = (E_k F)(t, \gamma_k(t; 0, y), \theta_k, \tau). \tag{5.3.11}$$

Therefore, if U is the solution of

$$\begin{aligned} X_k U(t, x, \theta_k, \tau) &= (E_k F)(t, x, \theta_k, \tau) \\ U(0, x, \theta_k, \tau) &= H(x, \theta_k, \tau), \end{aligned} \tag{5.3.12}$$

it follows that

$$V(t, y, \theta_k, \tau) = U(t, \gamma_k(t; 0, y), \theta_k, \tau) \tag{5.3.13}$$

and (5.3.8) shows that $u_\varepsilon(t, x) = U(t, x, \varphi(t, x)/\varepsilon, 1/\varepsilon) + o(1)$. Proposition 5.3.2 is proved.

6. The Quasilinear Cauchy Problem

This section contains the proof of Theorem 2.9.1. Consider an equation of the form (2.9.1), in which we drop the superscripts #

$$\partial_t u_\varepsilon + A(t, x, \varepsilon u_\varepsilon)\, \partial_x u_\varepsilon = b(t, x, \varepsilon u_\varepsilon)\, u_\varepsilon \tag{6.0.1}$$

$$u_\varepsilon|_{t=0} = h_\varepsilon \tag{6.0.2}$$

and we assume that the family h_ε satisfies assumptions (2.9.3), (2.9.4).

Recall (2.9.2) defines the domain $\Omega^\rho := \{(t, x) \in \mathbb{R}^2; 0 \leqslant t \leqslant T_0, \gamma_N(t; 0, y) + \rho t \leqslant x \leqslant \gamma_1(t; 0, y_+) - \rho t\}$ which remains in the domain of determinacy of $[y_-, y_+]$ for all $A(t, x, v)$ and v small enough.

After a linear change of dependent variables, we can assume that the matrix $A_0(t, x) = A(t, x, 0)$ is diagonal, with entries $\lambda_{k,0}(t, x)$, and that the eigenvectors $r_{k,0}(t, x) = r_k(t, x, 0)$ and $l_{k,0}(t, x) = l_k(t, x, 0)$ are the vectors of the canonical basis in $\mathbb{R}^N$. In these circumstances, the operator $L_0 \equiv \partial_t + A_0(t, x)\,\partial_x$ is diagonal with entries X_k.

6.1. *Outline of Proof*

The solution u_ε is constructed as the limit of the sequence u_ε^ν which is defined by $u_\varepsilon^0 = 0$ and

$$\partial_t u_\varepsilon^{\nu+1} + A(t, x, \varepsilon u_\varepsilon^\nu)\,\partial_x u_\varepsilon^{\nu+1} = b(t, x, \varepsilon u_\varepsilon^\nu(t, x))\, u_\varepsilon^\nu \tag{6.1.1}$$

$$u_\varepsilon^{\nu+1}|_{t=0} = h_\varepsilon. \tag{6.1.2}$$

The *first step* in the proof of Theorem 2.9.1 uses assumption (2.9.3) which says that the sequence h_ε is bounded in C_ε^1.

PROPOSITION 6.1.1. *There are $T > 0$ and $\varepsilon_0 > 0$, such that, for all $\varepsilon \in\,]0, \varepsilon_0]$ the iteration scheme* (6.1.1), (6.1.2) *defines a sequence $u_\varepsilon^\nu \in C^1(\Omega)$, $\Omega = \Omega^\rho \cap \{t \leqslant T\}$, such that*

(i) *there is M such that for all ν and $\varepsilon \leqslant \varepsilon_0$, one has $\|u_\varepsilon^\nu\|_{\varepsilon,1,\infty,\Omega} := \|u_\varepsilon^\nu\|_{L^\infty(\Omega)} + \varepsilon\, \|\nabla u_\varepsilon^\nu\|_{L^\infty(\Omega)} \leqslant M$*

(ii) *for each fixed $\varepsilon \in\,]0, \varepsilon_0]$, the sequence u_ε^ν converges in $C^1(\Omega)$ to the solution $u_\varepsilon \in C^1(\Omega)$ of* (6.0.1), (6.0.2).

(iii) *moreover $\|u_\varepsilon - u_\varepsilon^\nu\|_{L^\infty(\Omega)} \to 0$ as $\nu \to +\infty$, uniformly with respect to $\varepsilon \in\,]0, \varepsilon_0]$.*

Parallel to (6.1.1), (6.1.2), is the iteration scheme for the profiles U^ν defined by $U^0 = 0$ and

$$X_k U_k^{\nu+1} + E_k\Big(\sum_{i,j} \Gamma_{i,j}^k U_i^\nu (D_j U_j^{\nu+1})\Big) = E_k\Big(\sum_j \tilde{b}_{k,j} U_j^\nu\Big), \tag{6.1.3$_k$}$$

with the initial conditions

$$U_k^{\nu+1}|_{t=0}(x, \theta_k, \tau) = H_k(x, \theta_k, \tau). \tag{6.1.4$_k$}$$

As in (4.3.1), $D_j := (\partial_x \varphi_j) \cdot \partial_{\theta_j}$ and, $\Gamma_{i,j}^k(t, x) := \ell_{k,0}(t, x) \cdot ((\partial A/\partial v)(t, x, 0)\, r_{i,0}(t, x)) \cdot r_{j,0}(t, x)$, $\tilde{b}(t, x) := b(t, x, 0)$ as defined by (2.9.10), (2.9.11).

The *second step* in the proof of Theorem 2.9.1 consists in proving that this system has a unique solution.

PROPOSITION 6.1.2. (i) *Decreasing T if necessary, there is a sequence $U^\nu = (U_k^\nu)_{1 \leq k \leq N} \in \mathscr{C}^1(\Omega; \Psi)$, with U_k^ν depending only on θ_k, τ, of solutions of* (6.1.3), (6.1.4).

(ii) *This sequence converges in $\mathscr{C}^1(\Omega; \Psi)$.*

(iii) *The limit $U = (U_k)_{1 \leq k \leq N}$ belongs to $\mathscr{C}^1(\Omega; \Psi)$, U_k depends only on θ_k, τ and it is the unique solution of*

$$X_k U_k + E_k \left(\sum_{i,j} \Gamma_{i,j}^k U_i (D_j U_j) \right) = E_k \left(\sum_j \bar{b}_{k,j} U_j \right) \qquad (6.1.5)_k$$

$$U_k|_{t=0}(x, \theta_k, \tau) = H_k(x, \theta_k, \tau). \qquad (6.1.6)_k$$

The *third step* is

PROPOSITION 6.1.3. *Assume that condition $(\mathscr{T})$ [resp. $(w-\mathscr{T})$] holds. Then for all ν, $u_{\varepsilon,k}^\nu(t, x) - U_k^\nu(t, x, \varphi_k(t, x)/\varepsilon, 1/\varepsilon)$ is $o(1)$ in $C_\varepsilon^1(\Omega)$ [resp. in $L^\infty W_\varepsilon^{1,p}(\Omega)$] as $\varepsilon \to 0$.*

From the three propositions above one has that

$$u_{\varepsilon,k}(t, x) - U_k(t, x, \varphi_k(t, x)/\varepsilon, 1/\varepsilon) \text{ is } o(1) \text{ in } L^\infty(\Omega) \text{ [resp. in } L^\infty L^p(\Omega)]. \qquad (6.1.7)$$

In the *last step* we prove the approximation of the derivatives.

PROPOSITION 6.1.4. *Under the same assumptions, decreasing $T > 0$ if necessary, the following estimate holds*

$$u_{\varepsilon,k}(t, x) - U_k(t, x, \varphi_k(t, x)/\varepsilon, 1/\varepsilon) \text{ is } o(1) \text{ in } C_\varepsilon^1(\Omega) \text{ as } \varepsilon \to 0 \qquad (6.1.8)$$

$$[\text{resp. in } L^\infty W_\varepsilon^{1p}(\Omega)]. \qquad (6.1.9)$$

Theorem 2.9.1 is a combination of the four propositions above.

Remark 6.1.5. If one could prove that $u_\varepsilon^\nu \to u_\varepsilon$ in C_ε^1 uniformly in ε, estimate (6.1.8) would follow directly from Propositions 6.1.2 and 6.1.3. Unfortunately such a uniform convergence is probably false without further assumptions on h_ε (see Remark 6.2.8 below), so we have to provide a separate proof for the approximation of the derivatives.

Remark 6.1.6. In fact, it is not necessary to decrease T in order that (6.1.8) holds. However, for simplicity, we will do so in the proof given below. The study of the life span of solutions is deferred to Section 7.

Before proceeding to the detailed proofs let us show for the simple example of a single scalar equation how the equations for the profile U can be formally derived from the equations for u_ε. For simplicity we assume restricted closedness and that the profiles do not depend on τ (see Remark 2.8.4).

Consider the solution u_ε of

$$\partial_t u_\varepsilon + \varepsilon w_\varepsilon(t, x)\, \partial_x u_\varepsilon = f_\varepsilon \tag{6.1.10}$$

$$u_\varepsilon|_{t=0} = 0 \tag{6.1.11}$$

with oscillating coefficient and source term

$$w_\varepsilon(t, x) - W(t, x, \varphi(t, x)/\varepsilon) = o(1), \tag{6.1.12}$$

$$f_\varepsilon(t, x) - F(t, x, \varphi(t, x)/\varepsilon) = o(1). \tag{6.1.13}$$

We want to show that

$$u_\varepsilon(t, x) - U(t, x, \varphi_k(x)/\varepsilon) = o(1), \tag{6.1.14}$$

with U solution to

$$X_k U + E_k(W)\, D_k U = E_k(F), \tag{6.1.15}$$

$$U|_{t=0}(x, \theta_k) = 0. \tag{6.1.16}$$

We have $X_k = \partial_t$, $\varphi_k(t, x) = \varphi_k(x)$ and $D_k = \partial_x \varphi_k \cdot \partial_{\theta_k}$.

We introduce the characteristic curve of (6.1.10), $s \to (s, x + \varepsilon y_\varepsilon(s; t, x))$ defined by

$$\begin{aligned} &\frac{dy_\varepsilon(s; t, x)}{ds} = w_\varepsilon(s, x + \varepsilon y_\varepsilon(s; t, x)) \\ &y_\varepsilon(t; t, x) = 0. \end{aligned} \tag{6.1.17}$$

Taking (6.1.12) into account (6.1.17) yields (Section 6.4)

$$\frac{dy_\varepsilon(s; t\ x)}{ds} = W(s, x, \varphi(s, x)/\varepsilon + \partial_x \varphi(s, x)\ y_\varepsilon) + o(1), \tag{6.1.18}$$

or equivalently

$$y_\varepsilon(s; t, x) = \int_t^s W(\tau, x, \varphi(\tau, x)/\varepsilon + \partial_x \varphi(\tau, x)\ y_\varepsilon(\tau; t, x))\, d\tau + o(1). \tag{6.1.19}$$

Write $W(t, x, \theta) = \sum a_\alpha(t, x)\, e^{i\langle \alpha, \theta\rangle}$, so that

$$y_\varepsilon(s; t, x) = \sum \int_t^s a_\alpha(\tau, x)\, e^{i\langle \alpha, \varphi/\varepsilon\rangle} e^{i\langle \alpha, \partial_x \varphi\rangle\, y_\varepsilon}\, d\tau + o(1). \tag{6.1.20}$$

Apply to (6.1.20) the nonstationary phase result of Section 4. This gives

$$y_\varepsilon(s; t, x) = \int_t^s E_k W(\tau, x, \varphi_k(x)/\varepsilon + \partial_x \varphi_k(x)\, y_\varepsilon(\tau; t, x))\, d\tau + o(1), \tag{6.1.21}$$

from which follows that

$$y_\varepsilon(s; t, x) = Y(s; t, x, \varphi_k(x)/\varepsilon) + o(1), \tag{6.1.22}$$

where the scalar function Y satisfies

$$Y(s; t, x, \theta_k) = \int_t^s E_k W(\tau, x, \theta_k + Y(\tau; t, x, \theta_k)\, \partial_x \varphi_k(x))\, d\tau \tag{6.1.23}$$

which is equivalent to

$$\begin{aligned} \frac{dY(s; t, x, \theta_k)}{ds} &= (E_k W)(s, x, \theta_k + Y(s; t, x, \theta_k)\, \partial_x \varphi_k(x)) \\ Y(t; t, x, \theta_k) &= 0. \end{aligned} \tag{6.1.24}$$

Next compute $u_\varepsilon(t, x)$, solution to (6.1.10), by integrating along characteristics

$$u_\varepsilon(t, x) = \int_0^t F(\tau, x + \varepsilon y_\varepsilon(\tau; t, x), \varphi(\tau, x + \varepsilon y_\varepsilon(\tau; t, x))/\varepsilon)\, d\tau + o(1). \tag{6.1.25}$$

Use (6.1.22) to obtain

$$u_\varepsilon(t, x) = \int_0^t F(\tau, x, \varphi(\tau, x)/\varepsilon + Y(\tau; t, x, \varphi_k(x)/\varepsilon)\, \partial_x \varphi(\tau, x))\, d\tau + o(1). \tag{6.1.26}$$

Using again Section 4, leads to

$$u_\varepsilon(t, x) = \int_0^t E_k F(\tau, x, \varphi_k(x)/\varepsilon + Y(\tau; t, x, \varphi_k(x)/\varepsilon)\, \partial_x \varphi_k(x))\, d\tau + o(1). \tag{6.1.27}$$

This shows that u_ε is oscillating with profile U satisfying

$$U(t, x, \theta_k) = \int_0^t E_k F(\tau, x, \theta_k + Y(\tau; t, x, \theta_k)\, \partial_x \varphi_k(x))\, d\tau. \qquad (6.1.28)$$

The result (6.1.15) follows from (6.1.24), which says that the curves

$$s \to (s, x, \theta_k + Y(s; t, x, \theta_k)\, \partial_x \varphi_k(x)), \qquad (6.1.29)$$

are the characteristics of

$$\partial_t + E_k(W)\, D_k. \qquad (6.1.30)$$

The computations sketched above show how the analysis that worked in the semilinear case handles the quasilinear one. In (6.1.26), appears a bounded oscillating phase displacement with amplitude Y which gives rise to (6.1.28). This shift in the phase variables corresponds to the differential operator D_k in the dual space of the θ variables. This explains why, in the quasilinear case, the equation for the profiles is not only integral but also differential in the θ variables.

Section 6 is organized as follows. Section 6.2 contains the proof of Proposition 6.1.1 for u_ε. Except for minor changes due to the control in ε, this is a review of classical facts about C^1-waves. Section 6.3 deals with the profiles and the proof of Proposition 6.1.2. The proof combines the same classical lines with additional regularity properties of averaging operators, that were proved in Section 4.3, in order to transform the differential operator in the variables θ and obtain θ-smoothness and almost-periodicity. The last five sections concern asymptotics as ε tend to 0. As in the semilinear case, the result is established first for Picard iterates than in Sections 6.7 and 6.8, for the solution itself. In fact linear C^0-asymptotics is studied in Sections 6.4–6.6 with estimates that justify passage to the limit to obtain the nonlinear C^0-asymptotics in Section 6.7. The final C^1 result follows by differentiating the equations and is done in Section 6.8.

6.2. *Classical Results on Quasilinear Systems*

As the τ variable was already shown in Section 5 to act merely as a parameter in the equations, we assume henceforth for simplicity that the profiles are independent of τ and that the phase space satisfies restricted closedness under quadratic interactions. This section mostly reviews some classical facts about semilinear and quasilinear systems in one space dimension. See for instance [HW, and the Refs. therein]. Although they are well known, we recall them with some details through the different steps of the construction of solutions, for three reasons

(1) at several points, we must check uniformity in ε, (2) we need to carry out the asymptotics throughout the different steps and (3) we will follow the same scheme of resolution, for the less classical equations of profiles.

1. *C^0 and C^1 Solutions of a Scalar Equation.* Consider the scalar equation

$$\partial_t u_\varepsilon + \lambda_k(t, x, \varepsilon v_\varepsilon)\, \partial_x u_\varepsilon = f_\varepsilon \tag{6.2.1}$$

$$u_\varepsilon|_{t=0} = h_\varepsilon. \tag{6.2.2}$$

Let $\gamma_{\varepsilon,k}(s; t, x)$ be the characteristic curve through (t, x) that is the solution of

$$\begin{aligned} &\frac{d\gamma_{\varepsilon,k}(s; t, x)}{ds} = \lambda_k(s, \gamma_{\varepsilon,k}(s; t, x), \varepsilon v_\varepsilon(s, \gamma_{\varepsilon,k}(s; t, x))) \\ &\gamma_{\varepsilon,k}(t; t, x) = x. \end{aligned} \tag{6.2.3}$$

Let $\rho > 0$ be given. There exists $\eta > 0$ such that for all $t \in [0, T_0]$, all $v \in C^1(\Omega)$, $\Omega = \Omega^\rho \cap \{t \leqslant T\}$, such that

$$\|\varepsilon v_\varepsilon\|_{L^\infty(\Omega)} \leqslant \eta \tag{6.2.4}$$

$\gamma_{\varepsilon,k}(s; t, x)$ is defined, belongs to $C^1(\tilde{\Omega})$, where $\tilde{\Omega} = \{(s, t, x); 0 \leqslant s \leqslant t, (t, x) \in \Omega\}$, and satisfies $(s, \gamma_{\varepsilon,k}(s; t, x)) \in \Omega$. In the sequel, ρ and the corresponding η are fixed.

Then, for $f_\varepsilon \in L^\infty(\Omega)$ and $h_\varepsilon \in L^\infty([y_-, y_+])$, (6.2.1), (6.2.2) has a unique weak solution u_ε in $L^\infty(\Omega)$, given by

$$u_\varepsilon(t, x) = h_\varepsilon(\gamma_{\varepsilon,k}(0; t, x)) + \int_0^t f_\varepsilon(s, \gamma_{\varepsilon,k}(s; t, x))\, ds. \tag{6.2.5}$$

LEMMA 6.2.1. *Let T be given such that $0 < T \leqslant T_0$. For any $v_\varepsilon \in C^1(\Omega)$ satisfying* (6.2.4), *$f_\varepsilon \in C^0(\Omega)$ and $h_\varepsilon \in C^0(\mathbb{R})$, there is a unique weak solution $u_\varepsilon \in C^0(\Omega)$ of* (6.2.1), (6.2.2) *given by* (6.2.5). *Moreover*

$$\|u_\varepsilon(t)\| \leqslant \|h_\varepsilon\| + \int_0^t \|f_\varepsilon(s)\|\, ds, \tag{6.2.6}$$

where $\|\cdot\|$ denotes the norm in $L^\infty(\mathbb{R})$ and $u_\varepsilon(t)$ the function $u_\varepsilon(t, \cdot)$.

Introduce the modulus of continuity

$$\omega(\delta; t; u) = \sup |u(s, x) - u(s', x')|, \tag{6.2.7}$$

where the supremum is taken for (s, x) and (s', x') in Ω, such that $s, s' \leqslant t$, $|(s, x)-(s', x')| \leqslant \delta$. A similar definition holds for function of $x \in \mathbb{R}$. In the following statements ∇ means $\nabla_{t,x}$.

LEMMA 6.2.2. *There is C such that, if $\varepsilon \|v_\varepsilon\|_{L^\infty(\Omega)} \leqslant \eta$, and $1+\varepsilon \|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \leqslant M$, then*

$$\omega(\delta; t; u_\varepsilon) \leqslant Ce^{CMt}\omega(\delta; h_\varepsilon) + \int_0^t Ce^{CM|t-s|}\omega(\delta; s; f_\varepsilon)\, ds + \delta \|f_\varepsilon\|_{L^\infty([0,t]\times\mathbb{R})}.$$

Proof. This is a consequence of formula (6.2.5) and the estimates

$$|\partial_{t,x}\gamma_{\varepsilon,k}(s; t, x)| \leqslant Ce^{MC|t-s|} \tag{6.2.8}$$

if one keeps in mind that any modulus of continuity satisfies $\omega(C\delta) \leqslant (C+1)\,\omega(\delta)$.

It is clear that if h and f are C^1, then the solution u given by (6.2.5) is also C^1. But an important remark in [HW], who refer to K. O. Friedrichs [F], is the following fact.

LEMMA 6.2.3. *If f is only $C^0(\Omega)$ but of the form*

$$f_\varepsilon = \rho_\varepsilon\{\partial_t\sigma_\varepsilon + \lambda_k(t, x, \varepsilon v_\varepsilon)\,\partial_x\sigma_\varepsilon\} \tag{6.2.9}$$

with ρ_ε and σ_ε in $C^1(\Omega)$, then the solution u_ε of (6.2.1), (6.2.2) *belongs to $C^1(\Omega)$.*

Proof. If we call $\tilde\rho_\varepsilon(s; t, x) = \rho_\varepsilon(s, \gamma_{\varepsilon,k}(s; t, x))$, $\tilde\sigma_\varepsilon(s; t, x) = \sigma_\varepsilon(s, \gamma_{\varepsilon,k}(s; t, x))$, and $\tilde h_\varepsilon(t, x) = h_\varepsilon(\gamma_{\varepsilon,k}(0; t, x))$, then the solution u_ε is

$$u_\varepsilon(t, x) = \tilde h_\varepsilon(t, x) + \int_0^t \tilde\rho_\varepsilon(s; t, x)(\partial_s\tilde\sigma_\varepsilon)(s; t, x)\, ds \tag{6.2.10}$$

and integrating by parts, one can prove the existence of a continuous partial derivative $\partial_x u_\varepsilon \in C^0(\Omega)$, given by

$$\begin{aligned}\partial_x u_\varepsilon(t, x) = {} & \partial_x \tilde h_\varepsilon(t, x) + \int_0^t (\partial_x\tilde\rho_\varepsilon)(s; t, x)(\partial_s\tilde\sigma_\varepsilon)(s; t, x)\, ds \\ & + \tilde\rho_\varepsilon(\partial_x\tilde\sigma_\varepsilon)(t; t, x) - \tilde\rho_\varepsilon(\partial_x\tilde\sigma_\varepsilon)(0; t, x) \\ & - \int_0^t (\partial_s\tilde\rho_\varepsilon)(s; t, x)(\partial_x\tilde\sigma_\varepsilon)(s; t, x)\, ds.\end{aligned} \tag{6.2.11}$$

Going back to the equation, one can show that $\partial_t u \in C^0(\Omega)$ and the lemma is proved. As in [HW], estimates of ∇u_ε in L^∞ and of the modulus of continuity of ∇u_ε can be deduced from the proof above, but for the sake of brevity, we do not write them explicitely here.

2. *Diagonal Linear System.* Consider now a system whose principal part is diagonal

$$\partial_t u_\varepsilon + A(t, x, \varepsilon v_\varepsilon(t, x))\, \partial_x u_\varepsilon + m_\varepsilon u_\varepsilon = f_\varepsilon \tag{6.2.12}$$

$$u_\varepsilon|_{t=0} = h_\varepsilon, \tag{6.2.13}$$

where A is the diagonal matrix whose coefficients are λ_k, v_ε is a bounded family in $C^1_\varepsilon(\Omega)$ and m_ε is a bounded family of $N \times N$ matrices. The solution u_ε is constructed as the limit of the u^ν_ε that are defined inductively by $u^0_\varepsilon = 0$ and

$$\partial_t u^{\nu+1}_\varepsilon + A(t, x, \varepsilon v_\varepsilon(t, x))\, \partial_x u^{\nu+1}_\varepsilon = f_\varepsilon - m_\varepsilon u^\nu_\varepsilon \tag{6.2.14}$$

$$u^{\nu+1}_\varepsilon|_{t=0} = h_\varepsilon. \tag{6.2.15}$$

Indeed, if $\|m_\varepsilon\|_{L^\infty(\Omega)} \leqslant M$, Lemma 6.2.1 and induction on ν show that

$$\|u^\nu_\varepsilon(t)\| \leqslant e^{Mt}\, \|h_\varepsilon\| + \int_0^t e^{M(t-s)}\, \|f_\varepsilon(s)\|\, ds. \tag{6.2.16}$$

Moreover, writing the equation for $u^{\nu+1}_\varepsilon - u^\nu_\varepsilon$, one obtains by induction on ν that

$$\|(u^{\nu+1}_\varepsilon - u^\nu_\varepsilon)(t)\| \leqslant \frac{M^\nu t^\nu}{\nu!}\, \|u^1_\varepsilon\|_{L^\infty(\Omega)}. \tag{6.2.17}$$

Finally, let us recall that, for $h \in L^\infty$ and $f \in L^\infty$, uniqueness of L^∞ weak solutions of (6.2.12), (6.2.13) is a consequence of estimate (6.2.6). So, we can state

LEMMA 6.2.4. *Assume* (6.2.4) *and moreover that* $\|m_\varepsilon\|_{L^\infty(\Omega)} \leqslant M$ *for all* $\varepsilon \in]0, 1]$, *and that the families* h_ε *and* f_ε *are bounded respectively in* $L^\infty([y_-, y_+])$ *and* $L^\infty(\Omega)$. *Then the sequences* u^ν_ε *converge in* $L^\infty(\Omega)$ *uniformly with respect to* ε. *The limit* u_ε *is the unique* L^∞ *weak solution of* (6.2.12), (6.2.13) *and*

$$\|u_\varepsilon(t)\| \leqslant e^{Mt}\, \|h_\varepsilon\| + \int_0^t e^{M(t-s)}\, \|f_\varepsilon(s)\|\, ds. \tag{6.2.18}$$

We will also consider in Section 6.8, a semilinear system

$$\partial_t u_\varepsilon + A(t, x, \varepsilon v_\varepsilon(t, x))\, \partial_x u_\varepsilon + m_\varepsilon u_\varepsilon + Q_\varepsilon(u_\varepsilon) = f_\varepsilon \tag{6.2.19}$$

with initial condition (6.2.13). Here $Q_\varepsilon(u)$ is quadratic form in u valued in $\mathbb{R}^N$, with coefficients that are uniformly bounded in $L^\infty(\Omega)$. The corresponding iteration scheme is

$$\partial_t u_\varepsilon^{\nu+1} + A(t, x, \varepsilon v_\varepsilon(t, x))\, \partial_x u_\varepsilon^{\nu+1} = f_\varepsilon - m_\varepsilon u_\varepsilon^\nu - Q_\varepsilon(u_\varepsilon^\nu) \tag{6.2.20}$$

with the initial condition (6.2.15). Then, one has

LEMMA 6.2.5. *Assume* (6.2.4), *that* m_ε, f_ε *and the coefficients of* Q_ε *are uniformly bounded in* $L^\infty(\Omega)$, *and that* h_ε *is bounded in* $L^\infty([y_-, y_+])$. *Then there is* $T'>0$, *such that the sequences* $u_\varepsilon^\nu \in C^0(\Omega)$ *are uniformly bounded in* $L^\infty(\Omega')$, $\Omega' = \Omega \cap \{0 \leqslant T'\}$. *Furthermore they converge in* $L^\infty(\Omega')$ *uniformly with respect to* ε, *and the limit* $u_\varepsilon \in C^0(\Omega)$ *is the unique* $L^\infty(\Omega')$ *weak solution of* (6.2.19), (6.2.13).

3. C^0 *Estimates for a Linear System.* Consider now the linear system

$$\partial_t u_\varepsilon + A(t, x, \varepsilon v_\varepsilon)\, \partial_x u_\varepsilon + m_\varepsilon u_\varepsilon = f_\varepsilon \tag{6.2.21}$$

$$u_\varepsilon|_{t=0} = h_\varepsilon \tag{6.2.22}$$

with v_ε bounded in $C^1_\varepsilon(\Omega)$ and m_ε as above.

Strict hyperbolicity implies there is a smooth invertible matrix $P(t, x, v) \in C^\infty(\Omega_0 \times \mathbb{R}^N)$, constant for $|x|$ large, such that

$$P^{-1}(t, x, v)\, A(t, x, v)\, P(t, x, v) = \Lambda(t, x, v), \tag{6.2.23}$$

where $\Lambda(t, x, v)$ is, as above, the diagonal matrix with entries $\lambda_k(t, x, v)$. Because we have already assumed that $A(t, x, 0)$ is diagonal, we can choose P so that

$$P(t, x, 0) = \mathrm{Id}_N. \tag{6.2.24}$$

One performs the change of dependent variables

$$u_\varepsilon(t, x) = P(t, x, \varepsilon v_\varepsilon(t, x))\, \hat{u}_\varepsilon(t, x) \tag{6.2.25}$$

and similarly for $\hat{f}_\varepsilon$ and $\hat{h}_\varepsilon$. Then $\hat{u}_\varepsilon$ is solution to

$$\partial_t \hat{u}_\varepsilon + \Lambda(t, x, \varepsilon v_\varepsilon)\, \partial_x \hat{u}_\varepsilon = \hat{f}_\varepsilon - \hat{m}_\varepsilon \hat{u}_\varepsilon \tag{6.2.26}$$

$$\hat{u}_\varepsilon|_{t=0} = \hat{h}_\varepsilon, \tag{6.2.27}$$

where

$$\hat{m}_\varepsilon(t, x) = P_\varepsilon^{-1}\, \partial_t P_\varepsilon + \Lambda P_\varepsilon^{-1}\, \partial_x P_\varepsilon + P_\varepsilon^{-1} m_\varepsilon P_\varepsilon \tag{6.2.28}$$

with

$$P_\varepsilon(t, x) = P(t, x, \varepsilon v_\varepsilon(t, x)).$$

When $m_\varepsilon = 0$, the entries $\hat{m}_\varepsilon^{k,l}$ of $\hat{m}_\varepsilon$ have the following form

$$\hat{m}_\varepsilon^{k,l} = \sum_j \rho_\varepsilon^{k,j} \{ \partial_t \sigma_\varepsilon^{j,l} + \lambda_k(t, x, \varepsilon v_\varepsilon) \, \partial_x \sigma_\varepsilon^{j,l} \}, \tag{6.2.29}$$

with $\rho_\varepsilon^{k,j} = \rho_\varepsilon^{k,j}(t, x, \varepsilon v_\varepsilon(t, x))$ and $\sigma_\varepsilon^{j,l} = \sigma_\varepsilon^{j,l}(t, x, \varepsilon v_\varepsilon(t, x))$.

Lemma 6.2.4 can be applied to Eqs. (6.2.26), (6.2.27) to prove

LEMMA 6.2.6. *For any $v_\varepsilon \in C^1(\Omega)$ satisfying (6.2.4), $f_\varepsilon \in C^0(\Omega)$ and $h_\varepsilon \in C^0([y_-, y_+])$, there is a unique solution $u_\varepsilon \in C^0(\Omega)$ of (6.2.21), (6.2.22). Moreover there is a constant C such that if $1 + \varepsilon \|\nabla v_\varepsilon\|_{L^\infty(\Omega)} + \|m_\varepsilon\|_{L^\infty(\Omega)} \leqslant M$, then*

$$\|u_\varepsilon(t)\| \leqslant C \left\{ e^{MCt} \|h_\varepsilon\| + \int_0^t e^{MC(t-s)} \|f_\varepsilon(s)\| \, ds \right\}. \tag{6.2.30}$$

4. *C^1 Solutions of a Linear System.* Consider the problem

$$\partial_t u_\varepsilon + A(t, x, \varepsilon v_\varepsilon) \, \partial_x u_\varepsilon = f_\varepsilon \tag{6.2.31}$$

$$u_\varepsilon|_{t=0} = h_\varepsilon, \tag{6.2.32}$$

with v_ε bounded in $C_\varepsilon^1(\Omega)$, but where we now assume that $f_\varepsilon \in C^1(\Omega)$ and that $h_\varepsilon \in C^1([y_-, y_+])$. Then, because of the form (6.2.29) of the matrix $\hat{m}_\varepsilon$, we can apply Lemma 6.2.3 at each step of the iteration scheme (6.2.14), (6.2.15) associated to system (6.2.26), (6.2.27), and conclude that the corresponding $\hat{u}_\varepsilon^\nu$ are in $C^1(\Omega)$. Lemmas resembling Lemmas 6.2.2, 6.2.3, applied to ∇u (see [HW]) show that for ε fixed, the family $\{\nabla \hat{u}_\varepsilon^\nu, \nu \in \mathbb{N}\}$ is equicontinuous, and therefore $\hat{u}_\varepsilon^\nu \to \hat{u}_\varepsilon$ in $C^1(\Omega)$, so that the solution $\hat{u}_\varepsilon$ of (6.2.26), (6.2.27) is in $C^1(\Omega)$. The conclusion is that for each ε, the solution u_ε of (6.2.21), (6.2.22) also belongs to $C^1(\Omega)$. One could also obtain from [HW], estimates for ∇u_ε. However, *knowing that* u_ε is C^1 one can differentiate (6.2.31) to find that $z_\varepsilon = \varepsilon \, \partial_x u_\varepsilon$ is the $L^\infty(\Omega)$ weak solution of

$$\partial_t z_\varepsilon + A(t, x, \varepsilon v_\varepsilon) \, \partial_x z_\varepsilon = \varepsilon \, \partial_x f_\varepsilon - \left\{ \frac{\partial A}{\partial v}(t, x, \varepsilon v_\varepsilon) \cdot (\varepsilon \, \partial_x v_\varepsilon) \right\} z_\varepsilon - (\partial_x A)(t, x, \varepsilon v_\varepsilon) \, z_\varepsilon \tag{6.2.33}$$

$$z_\varepsilon|_{t=0} = \varepsilon \, \partial_x h_\varepsilon. \tag{6.2.34}$$

One can apply estimate (6.2.30) to this equation, and with Gronwall's lemma, we obtain

$$\|\varepsilon\,\partial_x u_\varepsilon(t)\| \leqslant C\left\{e^{MCt}\,\|\varepsilon\,\partial_x h_\varepsilon\| + \int_0^t e^{MC(t-s)}\,\|\varepsilon\,\partial_x f_\varepsilon(s)\|\,ds\right\} \tag{6.2.35}$$

with C depending on η, $\varepsilon\,\|v_\varepsilon\|_{L^\infty(\Omega)} \leqslant \eta$ and $1+\varepsilon\,\|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \leqslant M$.

Estimating next $\partial_t u_\varepsilon$ from the equation, yields

$$\|\varepsilon\,\nabla u_\varepsilon(t)\| \leqslant C\left\{e^{MCt}\,\|\varepsilon\,\partial_x h_\varepsilon\| + \|\varepsilon f_\varepsilon(0)\| + \int_0^t e^{MC(t-s)}\,\|\varepsilon\,\nabla f_\varepsilon(s)\|\,ds\right\}. \tag{6.2.36}$$

Summing up, we have

Lemma 6.2.7. *Assume that* $v_\varepsilon \in C^1(\Omega)$ *satisfies* (6.2.4), $f_\varepsilon \in C^1(\Omega)$ *and* $h_\varepsilon \in C^1([y_-, y_+])$. *Then the* $L^\infty(\Omega)$ *solution* u_ε *of* (6.2.21), (6.2.22) *belongs to* $C^1(\Omega)$. *Moreover there is a constant C such that, if* $1+\varepsilon\,\|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \leqslant M$, *then* (6.2.30) *and* (6.2.36) *hold.*

We can now proceed to the

5. *Proof of Proposition* 6.1.1. (i) Let R_0 and K_0 be such that

$$\forall \varepsilon \in]0, 1], \qquad \|h_\varepsilon\|_{L^\infty([y_-, y_+])} \leqslant R_0 \tag{6.2.37}$$

$$\forall \varepsilon \in]0, 1], \qquad \varepsilon\,\|\partial_x h_\varepsilon\|_{L^\infty([y_-, y_+])} \leqslant K_0. \tag{6.2.38}$$

Let $R = R_0 + 1$. Let C_1 be the constant defined in Lemma 6.2.7 and $K = C_1\{K_0 + R_0 + 1\}$. Assume that $u_\varepsilon^\nu \in C^1(\Omega)$ has been constructed such that

$$\forall \varepsilon \in]0, \varepsilon_0], \qquad \varepsilon\,\|u_\varepsilon^\nu\|_{L^\infty(\Omega)} \leqslant \eta \tag{6.2.39}$$

$$\forall \varepsilon \in]0, \varepsilon_0], \qquad \varepsilon\,\|\nabla u_\varepsilon^\nu\|_{L^\infty(\Omega)} + \|u_\varepsilon^\nu\|_{L^\infty(\Omega)} \leqslant K. \tag{6.2.40}$$

Then, $f_\varepsilon^\nu = b(t, x, \varepsilon u_\varepsilon^\nu)\, u_\varepsilon^\nu$ satisfies

$$\forall \varepsilon \in]0, \varepsilon_0], \qquad \varepsilon\,\|\nabla f_\varepsilon^\nu\|_{L^\infty(\Omega)} + \|f_\varepsilon^\nu\|_{L^\infty(\Omega)} \leqslant C_2 K, \tag{6.2.41}$$

where C_2 only depends on η and on the function b.

Lemma 6.2.7, shows that (6.1.1), (6.1.2) has a unique solution $u_\varepsilon^{\nu+1}$ which satisfies

$$\|u_\varepsilon^{\nu+1}\|_{L^\infty(\Omega)} \leqslant C_1 e^{MT}\{R_0 + TC_2 K\} \tag{6.2.42}$$

$$\varepsilon\,\|\nabla u_\varepsilon^{\nu+1}\|_{L^\infty(\Omega)} \leqslant C_1 e^{MT}\{K_0 + \varepsilon C_2 K + TC_2 K\} \tag{6.2.43}$$

with $M = C_1\{1+K\}$. Now it is clear that if $T>0$ and $\varepsilon_0>0$ are small enough, then (6.2.42) and (6.2.43) imply that the estimates (6.2.39), (6.2.40) are also satisfied by $u_\varepsilon^{\nu+1}$ and hence that they are satisfied for all ν and $\varepsilon \leqslant \varepsilon_0$. The point (i) of Proposition 6.1.1 is proved.

(ii) Next, note that $u_\varepsilon^{\nu+1} - u_\varepsilon^\nu$ satisfies

$$L(u_\varepsilon^\nu)(u_\varepsilon^{\nu+1} - u_\varepsilon^\nu) = \{A(t, x, \varepsilon u_\varepsilon^{\nu-1}) - A(t, x, \varepsilon u_\varepsilon^\nu)\}\, \partial_x u_\varepsilon^\nu + b(t, x, \varepsilon u_\varepsilon^\nu)\, u_\varepsilon^\nu - b(t, x, \varepsilon u_\varepsilon^{\nu-1})\, u_\varepsilon^{\nu-1} \tag{6.2.44}$$

$$u_\varepsilon^{\nu+1} - u_\varepsilon^\nu|_{t=0} = 0. \tag{6.2.45}$$

Using the uniform bound (6.2.40) and estimate (6.2.30), a classical induction shows that

$$\|\{u_\varepsilon^{\nu+1} - u_\varepsilon^\nu\}(t)\| \leqslant C^\nu \frac{t^{\nu-1}}{\nu!}, \qquad \nu \geqslant 2 \tag{6.2.46}$$

with C independent of $t \in [0, T]$, $\varepsilon \in]0, \varepsilon_0]$ and ν. This shows that u_ε^ν converges in $L^\infty(\Omega)$, uniformly in $\varepsilon \in]0, \varepsilon_0]$. Call the limit u_ε.

(iii) On the other hand, it is shown in [HW], that *for ε fixed*, the derivatives of u_ε^ν are not only uniformly bounded as in (6.2.40), but also equicontinuous, so that the convergence $u_\varepsilon^\nu \to u_\varepsilon$ also holds in C^1. Hence, one can conclude that $u_\varepsilon \in C^1(\Omega)$ and that u_ε is solution to (6.0.1), (6.0.2). The proof of Proposition 6.1.1 is now complete.

Remark 6.2.8. We have not assume that the ∇h_ε are equicontinuous as ε tends to 0, and therefore the family $\{\nabla u_\varepsilon^\nu,\ \nu \geqslant 0,\ \varepsilon > 0\}$ is certainly not equicontinuous and we cannot conclude without further assumptions on the data that the convergence $\nabla u_\varepsilon^\nu \to \nabla u_\varepsilon$ is uniform in ε.

Given a C^1 solution u_ε and w_ε, we can differentiate (6.0.1) and obtain for $\varepsilon\, \partial_x u_\varepsilon$ an equation similar to (6.2.33), (6.2.34). In Section 6.8, we shall use the following result.

LEMMA 6.2.9. *Suppose that $u_\varepsilon \in C^1(\Omega)$ satisfies* (6.0.1), (6.0.2) *and let*

$$z_\varepsilon(t, x) = \varepsilon P^{-1}(t, x, \varepsilon u_\varepsilon(t, x))\, \partial_x u_\varepsilon(t, x). \tag{6.2.47}$$

Then z_ε is the unique $L^\infty(\Omega)$ weak solution of

$$\partial_t z_\varepsilon + A(t, x, \varepsilon u_\varepsilon)\, \partial_x z_\varepsilon + m(t, x, \varepsilon u_\varepsilon)\, z_\varepsilon + Q(t, x, \varepsilon u_\varepsilon, z_\varepsilon) = f_\varepsilon(t, x, \varepsilon u_\varepsilon) \tag{6.2.48}$$

$$z_\varepsilon|_{t=0} = \varepsilon P^{-1}(0, x, \varepsilon h_\varepsilon(x))\, \partial_x h_\varepsilon(x) \tag{6.2.49}$$

with

$$f_\varepsilon(t, x, v) := P^{-1}\, \partial_x b \cdot v \tag{6.2.50}$$

$$m(t, x, v)\, z := P^{-1}\{\partial_x(AP) + \partial_t P\}\, z + \sum_{\ell, m=1}^{N} b_{\ell,m} v_m P^{-1} \frac{\partial P}{\partial v_\ell} z$$

$$- \sum_{\ell, i=1}^{N} z_i P_{\ell,i} P^{-1} \frac{\partial b}{\partial v_\ell} \cdot v - P^{-1} b P z \tag{6.2.51}$$

and $Q(t, x, v, z) := \sum_{i,j=1}^{N} Q^{i,j}(t, x, v)\, z_i z_j$ *is a quadratic form in* z, *valued in* $\mathbb{R}^N$, *whose* kth *component has coefficients* $Q_k^{i,j}$ *given by*

$$Q_k^{i,j} := \delta_{k,j} \sum_{\ell=1}^{N} \frac{\partial \lambda_j}{\partial v_\ell} P_{\ell,i} + \sum_{m,\ell=1}^{N} P_{k,m}^{-1} \frac{\partial P_{m,j}}{\partial v_\ell} P_{\ell,i}(\lambda_j - \lambda_i), \tag{6.2.52}$$

where $\delta_{k,j}$ *is Kronecker's symbol. In these formulas, the matrices* P^{-1}, $\partial_x(AP)$, $\partial_t P$, $\partial P/\partial v_i$, b, *and* $\partial_x b$ *as well as the eigenvalues* λ_j *are evaluated at* (t, x, v).

6.3. *Equations for the Profiles*

In this section, which parallels the last section, we study the different equations of profiles which we will encounter in the sequel. Recall that we assume $(r - \mathscr{C}_q)$ and that the profiles do not depend on τ. Of course, $\Psi \subset \Theta^*$ and E_k maps $C^0_{pp}(\Psi)$ on $C^0_{pp}(\Theta_k)$. We consider the scheme (6.1.3), (6.1.4), and to begin with, make an important remark on the linear system

$$X_k U_k + E_k\left(\sum_{i,j} \Gamma^k_{i,j} V_i \cdot D_j U_j\right) = E_k(F_k) \tag{6.3.1$_k$}$$

$$U_k|_{t=0}(x, \theta_k) = H_k(x, \theta_k). \tag{6.3.2$_k$}$$

The data $V = (V_1, ..., V_N)$ and $F = (F_1, ..., F_N)$ are given functions of $(t, x, \theta) \in \Omega \times \Psi$ valued in $\mathbb{R}^N$. The vector fields D_j are those introduced in (4.3.1). The kth components U_k and V_k of $U = (U_1, ..., U_N)$ and $V = (V_1, ..., V_N)$ depend only on (t, x, θ_k).

Split the sum $\sum E_k\{\Gamma^k_{i,j} V_i(D_j U_j)\}$ into three parts.

The terms where $j = k$ yield an expression of the form $\gamma_k(V)(D_k U_k)$ with

$$\gamma_k(V) := E_k\left(\sum_i \Gamma^k_{i,k} V_i\right). \tag{6.3.3}$$

Next, for $j \neq k$ and $i = k$, Remark 2.6.2 shows that $E_k\{V_k(D_j U_j)\} = V_k E_k(D_j U_j)\} = 0$.

Eventually, when $j \neq k$ and $i \neq k$ Corollary 4.3.3 shows that, for $\mathscr{C}^1$ functions, the sum

$$\mathscr{E}_k(V, U) := E_k\Big(\sum_{i \neq k, j \neq k} \Gamma^k_{i,j} V_i (D_j U_j)\Big) \tag{6.3.4}$$

is equal to

$$\mathscr{E}'_k(V, U) := E_k\Big(\sum_{i \neq k, j \neq k} \Pi^k_{i,j} (D_i V_i)\, U_j\Big) \tag{6.3.5}$$

with

$$\Pi^k_{i,j} := \Gamma^k_{i,j} \frac{\lambda_{k,0} - \lambda_{i,0}}{\lambda_{j,0} - \lambda_{k,0}}. \tag{6.3.6}$$

With these notations, we have

PROPOSITION 6.3.1. *For $\mathscr{C}^1(\Omega)$ solutions, Eq. (6.3.1) is equivalent to*

$$X_k U_k + \gamma_k(V)\, D_k U_k + \mathscr{E}'_k(V, U) = E_k(F_k). \tag{6.3.7}$$

These computations achieve a new *diagonalization* of the principal part of the system (6.3.1) in the (t, x, θ) variables. Together with Lemma 4.3.5, Proposition 6.3.1 is crucial for the analysis which follows particularly in Lemma 6.3.7 and in the proof of Proposition 6.1.2.

Although the method of resolution for (6.3.7) is straightforward, we must pay attention to the almost-periodicity and so, we will go through the different steps of the construction.

1. *$\mathscr{C}^1$ Solutions of the Scalar Equation.* Consider the scalar equation

$$X_k U + \gamma_k D_k U = F_k \tag{6.3.8}$$

$$U|_{t=0} = H_k. \tag{6.3.9}$$

Below, as in Section 6.2 we denote by Ω the set $\Omega^\rho \cap \{t \leqslant T\}$ and by $\tilde{\Omega} = \{(s, t, x); 0 \leqslant s \leqslant t, (t, x) \in \Omega\}$.

LEMMA 6.3.2. *Assume that $\gamma_k \in \mathscr{C}^1(\Omega; \Theta_k)$, $F_k \in \mathscr{C}^0(\Omega; \Theta_k)$ and that $H_k \in \mathscr{C}^0([y_-, y_+], \Theta_k)$. Then (6.3.8), (6.3.9) has a unique weak solution U in $L^\infty(\Omega \times \Theta_k)$. This solution belongs to $\mathscr{C}^0(\Omega; \Theta_k)$, and denoting by $\|U(t)\|_0$ the L^∞ norm of $U(t, \cdot, \cdot)$, we have*

$$\|U(t)\|_0 \leqslant \|H_k\|_0 + \int_0^t \|F(s)\|_0\, ds. \tag{6.3.10}$$

Proof. By integration along the characteristics of $X_k + \gamma_k D_k$ it is clear that there is a unique L^∞ solution U on $\Omega \times \Theta_k$ which satisfies (6.3.10) (see Lemma 6.2.1). It remains to check that it is almost periodic.

Without loss of generality, we can assume that $X_k \equiv \partial_t$. Note that if $0 \leqslant s \leqslant t$ and (t, x) is in Ω, then (s, x) is also in Ω. The characteristics of $\partial_t + \gamma_k D_k$ are the curves $s \to (s, x, \theta + \mu(s; t, x, \theta))$, where μ is the solution of

$$\begin{aligned} &\frac{d\mu(s; t, x, \theta)}{ds} = \gamma_k(s, x, \theta + \mu(s; t, x, \theta))\, \partial_x \varphi_k(x) \\ &\mu(t; t, x, \theta) = 0. \end{aligned} \tag{6.3.11}$$

We claim that, because $\gamma_k \in \mathscr{C}^1(\Omega; \Theta_k)$, μ also belongs to $\mathscr{C}^1(\tilde{\Omega}; \Theta_k)$. Indeed, μ is the limit of μ^ν, where $\mu^0(s, x, \theta) \equiv 0$ and μ^ν is defined for $\nu \geqslant 1$, by the inductive formula

$$\mu^{\nu+1}(s; t, x, \theta) = \int_t^s \gamma_k(\tau, x, \theta + \mu^\nu(\tau; t, x, \theta))\, \partial_x \varphi_k(x)\, d\tau. \tag{6.3.12}$$

It follows from Lemma 4.1.4 that if $\mu^\nu \in \mathscr{C}^0(\tilde{\Omega}; \Theta_k)$, then $\gamma_k(\tau; t, x, \theta + \mu^\nu(\tau; t; x, \theta)) \in \mathscr{C}^0(\tilde{\Omega}; \Theta_k)$ and by Lemma 4.1.5 that $\mu^{\nu+1} \in \mathscr{C}^0(\tilde{\Omega}; \Theta_k)$.

Moreover, the scheme (6.3.12) is known to converge in $L^\infty(\tilde{\Omega} \times \Theta_k)$ and therefore, because $\mathscr{C}^0(\tilde{\Omega}; \Theta_k)$ is closed in $L^\infty(\tilde{\Omega} \times \Theta_k)$, the limit μ belongs to $\mathscr{C}^0(\tilde{\Omega}; \Theta_k)$. From the equation, it is clear that $\partial_s \mu$ also belongs to this space.

On the other hand, it is known that $\mu \in C^1(\tilde{\Omega} \times \Theta_k)$, and that any derivative μ' of μ with respect to t, x or θ is solution of an ODE of the form

$$\begin{aligned} &\frac{d\mu'}{ds} = \alpha(s, x, \theta + \mu) + \beta(s, x, \theta + \mu)\, \mu' \\ &\mu'(t; t, x, \theta) = \rho(t, x, \theta) \end{aligned}$$

with α and β and ρ in $\mathscr{C}^0(\Omega; \Theta_k)$. Again μ' is the limit in L^∞ of the μ'^ν that are defined by

$$\begin{aligned} \mu'^{\nu+1}(s; t, x, \theta) = \rho + \int_t^s \{&\alpha(\tau, x, \theta + \mu(\tau; t, x, \theta)) \\ &+ \beta(\tau, x, \theta + \mu(\tau; t, x, \theta)) \cdot \mu'^\nu(\tau; t, x, \theta)\}\, d\tau. \end{aligned}$$

Making use once more of Lemmas 4.1.4 and 4.1.5, shows that the μ'^ν are in $\mathscr{C}^0(\tilde{\Omega}; \Theta_k)$ as is their limit μ'. The conclusion is that $\mu \in \mathscr{C}^1(\tilde{\Omega}; \Theta_k)$ as claimed. We also have the following estimates

$$\sup_{(x,\theta)} |\nabla\mu(s; t, x, \theta)| \leqslant Ce^{CM|t-s|}, \tag{6.3.13}$$

where C depends only on the L^∞ norm of γ_k, and M on the L^∞ norm of $\nabla\gamma_k$, where ∇ means $\nabla_{s,t,x,\theta}$.

Now the end of the proof of Lemma 6.3.2 is easy, because the solution U of (6.3.8), (6.3.9) is given by

$$U(t, x, \theta) = H_k(x, \theta + \mu(0; t, x, \theta)) + \int_0^t F_k(s, x, \theta + \mu(s; t, x, \theta))\, ds \tag{6.3.14}$$

and the lemmas of Section 4.1 imply that $U \in \mathscr{C}^0(\tilde{\Omega}; \Theta_k)$. Note also that estimate (6.3.10) is trivial from (6.3.14).

For $U \in C^0(\Omega \times \Psi)$, denote by $\omega(\delta; t; U)$ the modulus of continuity of U on $\Omega_t \times \Psi = \{(s, x) \in \Omega; 0 \leqslant s \leqslant t\} \times \Psi$

$$\omega(\delta; t; U) = \sup |U(s, x, \theta) - U(s', x', \theta')| \tag{6.3.15}$$

the supremum being taken over the $(s, x, \theta) \in \Omega_t \times \Psi$ and $(s', x', \theta') \in \Omega_t \times \Psi$ such that $|(s, x, \theta) - (s', x', \theta')| \leqslant \delta$.

LEMMA 6.3.3. *With assumptions as in Lemma 6.3.2, we have*

$$\omega(\delta; t; U) \leqslant Ce^{CMt}\omega(\delta; H_k) + \int_0^t Ce^{CM(t-s)}\omega(\delta; s; F_k)\, ds + \delta\, \|F_k\|_{L^\infty(\Omega_t \times \Theta_k)}, \tag{6.3.16}$$

where C depends only on the L^∞ norm of γ_k, and M on the L^∞ norm of $\nabla\gamma_k$.

Proof. This is a consequence of (6.3.13), (6.3.14).

LEMMA 6.3.4. *Assume that $\gamma_k \in \mathscr{C}^1(\Omega; \Theta_k)$, $F_k \in \mathscr{C}^1(\Omega; \Theta_k)$, and that $H_k \in \mathscr{C}^1([y_-, y_+]; \Theta_k)$. Then the solution U to (6.3.8), (6.3.9) belongs to $\mathscr{C}^1(\Omega; \Theta_k)$ and*

$$\|U(t)\|_1 \leqslant Ce^{CMt}\, \|H_k\|_1 + \|F_k(0)\|_0 + \int_0^t Ce^{CM(t-s)}\, \|F_k(s)\|_1\, ds, \tag{6.3.17}$$

where C only depends on the $L^\infty(\Omega \times \Theta_k)$ norm of γ_k and $M = \|\nabla\gamma_k\|_{L^\infty}$, while ∇U denotes the gradient of U, and $\|U(t)\|_1 = \|U(t)\|_0 + \|\nabla U(t)\|_0$.

Proof. It is a consequence of the explicit formula (6.3.14) and of the estimates (6.3.13).

2. *$\mathscr{C}^0$ Estimates for a Diagonal System.* Consider the linear system

$$X_k U_k + \gamma_k(V) D_k U_k + \mathscr{M}_k(W, U) = E_k(F_k) \qquad (6.3.18)_k$$

$$U_k|_{t=0} = H_k(x, \theta_k), \qquad (6.3.19)_k$$

where $\mathscr{M}_k(W, U) = E_k(\sum_{i,j} \bar{m}^k_{i,j} W_i U_j)$ with coefficients $\bar{m}^k_{i,j} \in C^\infty(\Omega)$. Note that $\mathscr{M}_k$ is a continuous bilinear map from $\mathscr{C}^0(\Omega; \Psi) \times \mathscr{C}^0(\Omega; \Psi)$ into $\mathscr{C}^0(\Omega; \Theta_k)$, so that it makes sense for $U_k \in \mathscr{C}^0(\Omega; \Theta_k)$, to say that U is a weak solution of (6.3.18).

LEMMA 6.3.5. *Assume that $V \in \mathscr{C}^1(\Omega; \Psi)$, $F \in \mathscr{C}^0(\Omega; \Psi)$, $W_i \in \mathscr{C}^0(\Omega; \Theta_i)$, and that $H_k \in \mathscr{C}^0([y, y_+], \Theta_k)$. Then there is a unique solution $U \in \mathscr{C}^0(\Omega; \Psi)$ with U_k depending only on θ_k, to the problem* (6.3.18), (6.3.19). *Moreover, for all $t \in [0, T]$*

$$\|U(t)\|_0 \leqslant e^{CMt} \|H\|_0 + \int_0^t e^{CM(t-s)} \|F(s)\|_0 \, ds \qquad (6.3.20)$$

with $M = \|W\|_{L^\infty(\Omega)}$ and C a bound in $L^\infty(\Omega)$ for the $\bar{m}^k_{i,j}$.

Proof. As usual, the solution U is the limit of the U^ν that are defined by $U^0 = 0$ and

$$X_k U_k^{\nu+1} + \gamma_k(V) D_k U_k^{\nu+1} = E_k(F_k) - \mathscr{M}_k(W, U_k^\nu) \qquad (6.3.21)_k$$

$$U_k^{\nu+1}|_{t=0} = H_k(x, \theta_k). \qquad (6.3.22)_k$$

These equations are solved inductively, with the help of Lemma 6.3.2 and the estimate

$$\|\mathscr{M}_k(W, U)(t)\|_0 \leqslant C \, \|W(t)\|_0 \, \|U(t)\|_0. \qquad (6.3.23)$$

Lemma 6.3.2 shows, by induction, that $U^\nu \in \mathscr{C}^0(\Omega; \Psi)$, that $\|U^\nu(t)\|_0$ is bounded by the right hand side of (6.3.20), and that

$$\|U^{\nu+1}(t) - U^\nu(t)\|_0 \leqslant (CM)^\nu \frac{t^\nu}{\nu!} \|U^1\|_{L^\infty}, \qquad \nu \geqslant 1 \qquad (6.3.24)$$

Therefore U^ν converges in $C^0(\Omega \times \Psi)$ to $U \in \mathscr{C}^0(\Omega; \Psi)$ a solution of (6.3.18), (6.3.19) which satisfies (6.3.20).

For the moduli of continuity we have

LEMMA 6.3.6. *With assumptions as in Lemma 6.3.5, we have the following estimate*

$$\omega(\delta; t; U) \leqslant Ce^{CMt}\omega(\delta; H) + \int_0^t Ce^{CM(t-s)}\omega(\delta; s; F)\, ds$$

$$+ \int_0^t Ce^{CM(t-s)}\omega(\delta; s; W)\, ds + \delta C, \tag{6.3.25}$$

where C depends only on the $L^\infty(\Omega \times \Psi)$ norm of U, V, W, and F, and M on the $L^\infty(\Omega \times \Psi)$ norm of ∇V.

Proof. Pass $\mathscr{M}_k(W, U)$ to the right hand side of (6.3.18) and apply Lemma 6.3.3, noting that the modulus of continuity of $\mathscr{M}_k(W, U)$ satisfies

$$\omega(\delta; s) \leqslant C_0\{\delta \,\|W\|\, \|U\| + \|W\|\, \omega(\delta; s; U) + \|U\|\, \omega(\delta; s; W)\} \tag{6.3.26}$$

with C_0 depending only on the coefficients $\bar{m}$, and $\|\cdot\|$ denoting the norm in $L^\infty(\Omega \times \Psi)$. The modulus of U is absorbed from the right to the left thanks to Gronwall's lemma.

3. *$\mathscr{C}^0$ Solutions of a Semilinear System.* In Section 6.8, we will consider a semilinear problem of the following form

$$X_k U_k + \gamma_k(V)\, D_k U_k + \mathscr{M}_k(W, U) + \mathscr{Q}_k(U) = E_k(F_k) \qquad (6.3.27)_k$$

together with the initial condition (6.3.19), where $\mathscr{Q}_k$ is quadratic in U, $\mathscr{Q}_k(U) = E_k(\sum_{i,j} \bar{Q}_k^{i,j} U_i U_j)$ with coefficients $\bar{Q}_k^{i,j} \in C^\infty(\Omega)$. As above, we consider the associated iteration scheme, starting from $U^0 = 0$

$$X_k U_k^{\nu+1} + \gamma_k(V)\, D_k U_k^{\nu+1} = E_k(F_k) - \mathscr{M}_k(W, U_k^\nu) - \mathscr{Q}_k(U^\nu) \qquad (6.3.28)_k$$

with the initial condition (6.3.22). Making use of Lemma 6.3.2, it is not difficult to prove the following lemma.

LEMMA 6.3.7. *Assume that $V \in \mathscr{C}^1(\Omega; \Psi)$, $F \in \mathscr{C}^0(\Omega; \Psi)$, $W_i \in \mathscr{C}^0(\Omega; \Theta_i)$, and that $H \in \mathscr{C}^0([y, y_+]; \Theta^0)$. Then there is $T' \in]0, T]$ such that the sequence U^ν converges in $L^\infty(\Omega' \times \Psi)$ to the unique solution $U \in \mathscr{C}^0(\Omega'; \Psi)$ of* (6.3.27), (6.3.19), *where Ω' denotes the domain $\Omega \cap \{t \leqslant T'\}$.*

4. *Derivative Estimates for the Linear System of Proposition* 6.3.1. In this fourth step, we consider the system (6.3.7) with (6.3.2) as initial condition. This system is of the form (6.3.18) with $W = DV$ (i.e., $W_i = D_i V_i$) and $\mathscr{M}_k(W, U) = \mathscr{E}'_k(V, U)$. Thus for $V \in \mathscr{C}^1(\Omega; \Theta_k)$, $F \in \mathscr{C}^0(\Omega; \Psi)$, and $H_k \in \mathscr{C}^0([y_-, y_+]; \Theta_k)$ one can apply Lemma 6.3.5 and conclude that there is a unique solution $U \in \mathscr{C}^0(\Omega; \Psi)$, with U_k depending only on θ_k.

LEMMA 6.3.8. *Assume that* $V \in \mathscr{C}^1(\Omega; \Theta_k)$, $F \in \mathscr{C}^1(\Omega; \Psi)$, *and that* $H_k \in \mathscr{C}^1([y, y_+]; \Theta_k)$. *Then the solution* U *to* (6.3.7), (6.3.2) *belongs to* $\mathscr{C}^1(\Omega; \Psi)$ *and*

$$\|U(t)\|_1 \leqslant Ce^{CMt}(\|H\|_1 + \|F(0)\|_0) + C\int_0^t e^{CM(t-s)}\, \|F(s)\|_1\, ds, \qquad (6.3.29)$$

where C *only depends on the* $L^\infty(\Omega \times \Theta_k)$ *norm of* V *and* $M = 1 + \|\nabla V\|_{L^\infty}$, *while* $\|U(t)\|_1 := \|U(t)\|_0 + \|\nabla U(t)\|_0$ *as in Lemma* 6.3.4.

Proof. From Lemma 6.3.5, we know that U is the limit of the U^ν that are defined by

$$X_k U_k^{\nu+1} + \gamma_k(V)\, D_k U_k^{\nu+1} = E_k(F_k) - \mathscr{E}'_k(V, U_k^\nu) \qquad (6.3.30)_k$$

and the initial condition (6.3.22). From Lemma 4.3.5, we know that

$$\|\mathscr{E}'_k(V, U)(t)\|_1 \leqslant C_0\, \|V(t)\|_1\, \|U(t)\|_1 \qquad (6.3.31)$$

and we deduce from Lemma 6.3.4 that $U^\nu \in \mathscr{C}^1(\Omega; \Psi)$ for all ν. Moreover, with estimate (6.3.17), we see by induction on ν that $\|U^\nu(t)\|_1$ remains bounded by the right hand side of (6.3.29), if the constant C is suitably enlarged. Finally, (6.3.17) also implies the estimates

$$\|U^{\nu+1}(t) - U^\nu(t)\|_1 \leqslant (CM)^{\nu-1} \frac{t^{\nu-1}}{\nu!}\, \|U^1\|_{W^{1,\infty}}, \qquad \nu \geqslant 1 \quad (6.3.32)$$

and therefore $U^\nu \to U$ in $W^{1,\infty}(\Omega \times \Psi)$ so the limit $U \in \mathscr{C}^1(\Omega; \Psi)$ and satisfies (6.3.29).

Before leaving the problems with a linear principal part, we need to control the modulus of continuity of ∇U.

LEMMA 6.3.9. *With assumptions as in Lemma* 6.3.8, *the following estimate holds*

$$\omega(\delta; t; \nabla' U) \leqslant Ke^{Kt}\omega(\delta; \nabla' H) + \int_0^t Ke^{K(t-s)}\omega(\delta; s; \nabla' F)\, ds$$
$$+ \int_0^t Ke^{K(t-s)}\omega(\delta; s; \nabla' V)\, ds + \delta Ke^{Kt}, \qquad (6.3.33)$$

where K *depends only on the* $W^{1,\infty}(\Omega \times \Psi)$ *norm of* U, V *and* F, *and* ∇' *denotes the gradient in the* (x, θ)*-variables.*

Proof. Differentiate (6.3.7), taking Lemma 4.3.5 into account when dealing with the terms $\mathscr{E}'_k(V, U)$. We obtain an equation of the form

$$X_k \nabla' U_k + \gamma_k D_k \nabla' U_k = \mathscr{D}_k(\nabla' V, \nabla' U) + \mathscr{F}_k(\nabla' V, U) + \nabla' E_k(F_k) \tag{6.3.34}$$

with $\mathscr{D}_k(\nabla' V, \nabla' U)$ [resp. $\mathscr{F}_k(\nabla' V, U)$] being the image under E_k of a bilinear expression in $\nabla' V$ and $\nabla' U$ [resp. in $\nabla' V$ and U]. Furthermore, the initial value of $\nabla' U_k$ is clearly $\nabla' H_k$. We remark that the modulus of continuity of the right hand side of (6.3.34) satisfies

$$\omega(\delta; s) \leqslant K\omega(\delta; s; \nabla' U) + K\omega(\delta; s; \nabla' V) + K\delta + K\omega(\delta; s; F). \tag{6.3.35}$$

We are now in position to apply Lemma 6.3.6 and estimate (6.3.25). We obtain

$$\omega(\delta; t; \nabla' U) \leqslant Ke^{Kt}\omega(\delta; \nabla' H) + \int_0^t Ke^{K(t-s)}\omega(\delta; s; \nabla' F)\, ds + \int_0^t Ke^{K(t-s)} \times \{\omega(\delta; s; \nabla' U) + \omega(\delta; s; \nabla' V)\}\, ds + \delta\, Ke^{Kt} \tag{6.3.36}$$

and (6.3.33) follows.

5. *Proof of Proposition* 6.1.2. We now consider the iteration scheme (6.1.3), (6.1.4), and using Proposition 6.3.1 we write it

$$X_k U_k^{\nu+1} + \gamma_k(U^\nu) D_k U_k^{\nu+1} + \mathscr{E}'_k(U^\nu, U^{\nu+1}) = F_k(U^\nu) \qquad (6.3.37)_k$$

$$U_k^{\nu+1}|_{t=0}(x, \theta_k) = H_k(x, \theta_k) \qquad (6.3.38)_k$$

with $F_k(U) = E_k(\sum_j \bar{b}_{k,j} U_j)$. These equations are solved inductively thanks to Lemma 6.3.8 and to the trivial estimate

$$\|F(U)(t)\|_1 \leqslant C\, \|U(t)\|_1. \tag{6.3.39}$$

From estimates (6.3.29) and (6.3.39), one deduces the existence of a $T > 0$ such that the sequence U^ν is bounded in $\mathscr{C}^1(\Omega; \Psi)$, with $\Omega = \Omega^\rho \cap \{t \leqslant T\}$, (see for instance the proof of Proposition 6.1.1, point (i) at the end of Section 6.2).

Knowing that, one can write down the equation satisfied by $U^{\nu+1} - U^\nu$,

$$\begin{aligned} X_k\{U_k^{\nu+1} - U_k^\nu\} &+ \gamma_k(U^\nu) D_k\{U_k^{\nu+1} - U_k^\nu\} + \mathscr{E}'_k(U^\nu, U^{\nu+1} - U^\nu) \\ &= F_k(U^\nu) - F_k(U^{\nu-1}) + \{\gamma_k(U^{\nu-1}) - \gamma_k(U^\nu)\}\, D_k U_k^\nu \\ &\quad + \mathscr{E}'_k(U^{\nu-1} - U^\nu, U^\nu). \end{aligned} \qquad (6.3.40)_k$$

Because of Corollary 4.3.3, the last term $\mathscr{E}'_k(U^{\nu-1}-U^\nu, U^\nu)$ is equal to $\mathscr{E}_k(U^{\nu-1}-U^\nu, U^\nu)$ (see formulas (6.3.4) and (6.3.5)). Furthermore, F_k and γ_k are linear function of U, so the right hand side G_k^ν of (6.3.40) satisfies

$$\|G_k^\nu(t)\|_0 \leqslant K\,\|\{U^\nu - U^{\nu-1}\}(t)\|_0, \tag{6.3.41}$$

where K only depends on the uniform bound for $\|U^\nu\|_{W^{1,\infty}(\Omega\times\Psi)}$. Hence Lemma 6.3.5 shows that

$$\|U^{\nu+1}(t) - U^\nu(t)\|_0 \leqslant K^\nu \frac{t^{\nu-1}}{\nu!} \tag{6.3.42}$$

and U^ν converges in $L^\infty(\Omega\times\Psi)$ to a function $U\in\mathscr{C}^0(\Omega;\Psi)$ with components U_k depending only on θ_k.

Since the norms $\|U^\nu\|_{W^{1,\infty}(\Omega\times\Psi)}$ are uniformly bounded, we deduce from Lemma 6.3.9 applied to Eq. (6.3.37) that

$$\omega(\delta;t;\nabla' U^{\nu+1}) \leqslant Ke^{Kt}\omega(\delta;\nabla' H) + \int_0^t Ke^{K(t-s)}\omega(\delta;s;\nabla' U^\nu)\,ds + \delta\,Ke^{Kt}. \tag{6.3.43}$$

In fact, in the derivation of (6.3.43), we have used that $\omega(\delta;s;F_k(U)) \leqslant K\omega(\delta;s;U)$. Therefore, it is clear by induction on ν, that, with a new constant K, one has a uniform estimate

$$\omega(\delta;t;\nabla' U^{\nu+1}) \leqslant Ke^{Kt}\omega(\delta;\nabla' H) + \delta\,Ke^{Kt}. \tag{6.3.44}$$

Estimating $\omega(\delta;t;\partial_t U^{\nu+1})$ from Eq. (6.3.37), we find that the derivatives of the U^ν are uniformly equicontinuous on $\Omega\times\Psi$. Therefore, the convergence $U^\nu\to U$ holds in C^1 on any compact subset of $\Omega\times\Psi$, and U is C^1. Moreover the derivatives of U are uniformly continuous on $\Omega\times\Psi$. Thus almost-periodicity of the derivatives of U follows from Proposition 4.1.2 showing that $U\in\mathscr{C}^1(\Omega;\Psi)$.

It remains to show that U is solution to (6.1.5) by passing to the limit in (6.3.37). Because U^ν converge to U in $\mathscr{C}^0$, it is clear from Definition (6.3.3) that $\gamma_k(U^\nu)\to\gamma_k(U)$ in $L^\infty(\Omega\times\Theta_k)$. On the other hand, $\nabla U_k^\nu \to \nabla U_k$ in $L^\infty_{\text{loc}}(\Omega\times\Theta_k)$, so there is no difficulty in taking the limit in $L^\infty_{\text{loc}}(\Omega\times\Theta_k)$ of $X_k U_k^{\nu+1} + \gamma_k(U^\nu)\,D_k U_k^{\nu+1}$. Similarly, $F_k(U^\nu)\to F_k(U)$ in $L^\infty(\Omega\times\Psi)$. Finally, to pass to the limit in the $\mathscr{E}'$ terms we write

$$\mathscr{E}'_k(U^\nu, U^{\nu+1}) - \mathscr{E}'_k(U,U) = \mathscr{E}'_k(U^\nu, U^{\nu+1}-U) + \mathscr{E}'_k(U^\nu - U, U). \tag{6.3.45}$$

Since DU^ν is uniformly bounded in $L^\infty(\Omega\times\Psi)$ and $\{U^{\nu+1}-U\}\to 0$ in $L^\infty(\Omega\times\Psi)$, formula (6.3.5) clearly shows that $\mathscr{E}'_k(U^\nu, U^{\nu+1}-U)\to 0$ in $L^\infty(\Omega\times\Psi)$. Since we know that all the functions are in $\mathscr{C}^1$, we can transform $\mathscr{E}'_k(U^\nu - U, U)$ to $\mathscr{E}_k(U^\nu - U, U)$ and with Definition (6.3.4) it is now clear that $\mathscr{E}'_k(U^\nu - U, U) = \mathscr{E}_k(U^\nu - U, U) \to 0$ in $L^\infty(\Omega \times \Psi)$. The conclusion is that $\mathscr{E}'_k(U^\nu, U^{\nu+1}) \to \mathscr{E}'_k(U, U)$ in $L^\infty(\Omega \times \Theta_k)$.

Thus the limit in $L^\infty_{\text{loc}}(\Omega \times \Theta_k)$ of Eq. $(6.3.37)_k$ is just $(6.3.7)_k$, and by Proposition 6.3.1, U is solution to (6.1.5), (6.1.6).

6.4. $\mathscr{C}^0$ *Asymptotics*: *The Scalar Case*

In this section we consider the scalar problem and make precise the arguments sketched at the end of Section 6.1. The main tool will be Theorem 4.4.2, under the quadratic assumptions $(\mathscr{T}_q)$ and $(\mathscr{C}_q)$. Let u_ε be the solution of

$$\partial_t u_\varepsilon + \lambda_k(t, x, \varepsilon v_\varepsilon)\, \partial_x u_\varepsilon = f_\varepsilon \tag{6.4.1}$$

$$u_\varepsilon|_{t=0} = h_\varepsilon, \tag{6.4.2}$$

where

$$v_\varepsilon,\ f_\varepsilon \text{ and } h_\varepsilon \text{ are bounded in } C^1_\varepsilon(\Omega),\ C^0(\Omega) \text{ and } C^0_\varepsilon([y_-, y_+]). \tag{6.4.3}$$

We also assume that there are $\mathbb{R}^N$-valued $V \in \mathscr{C}^1(\Omega; \Psi)$, $\mathbb{R}$-valued $F \in \mathscr{C}^0(\Omega; \Psi)$, and $\mathbb{R}$-valued $H \in \mathscr{C}^0([y_-, y_+]; \Theta_k)$, such that V_k depends only on θ_k, F is of the form $\sum F^{i,j}(t, x, \theta_i, \theta_j)$, and

$$v_\varepsilon(t, x) - V(t, x, \varphi(t, x)/\varepsilon) = o(1) \qquad \text{in } L^\infty L^p \tag{6.4.4}$$

$$f_\varepsilon(t, x) - F(t, x, \varphi(t, x)/\varepsilon) = o(1) \qquad \text{in } L^\infty L^p \tag{6.4.5}$$

$$h_\varepsilon(x) - H(x, \varphi^0(x)/\varepsilon) = o(1) \qquad \text{in } L^p, \tag{6.4.6}$$

where $p = +\infty$ [resp. $p < +\infty$] if condition $(\mathscr{T}q)$ [resp. $(w - \mathscr{T}q)$] of Section 2.2 holds. Recall we assume that $(r - \mathscr{C}_q)$ holds.

Because of (6.4.3), we know from Lemma 6.2.1 that the solution u_ε of (6.4.1), (6.4.2) remains in a bounded set of $C^0(\Omega)$. In addition, we know from Lemma 6.3.2 that there is a unique solution $U \in \mathscr{C}^0(\Omega; \Theta_k)$ to the equation of the profiles

$$X_k U + \gamma_k(V)\, D_k U = E_k(F) \tag{6.4.7}$$

$$U|_{t=0}(x, \theta_k) = H(x, \theta_k). \tag{6.4.8}$$

The aim of this section is to prove the following result.

PROPOSITION 6.4.1. *With the above assumptions*

$$u_\varepsilon(t, x) - U(t, x, \varphi_k(t, x)/\varepsilon) = o(1) \qquad \textit{in } L^\infty L^p. \tag{6.4.9}$$

First we make some simplifications. We can change the coordinate x in Ω in such a way that X_k becomes ∂_t, and $\varphi_k = \varphi_k(x)$ only depends on x. In these coordinates, $\lambda_k(t, x, 0) = 0$. Moreover, there is a C^∞ function $\lambda'(t, x, v)$ such that $\lambda_k(t, x, v) = \lambda_k(t, x, 0) + v \cdot \lambda'(t, x, v)$, and with Lemma 5.3.1 we see that

$$\lambda_k(t, x, \varepsilon v_\varepsilon(t, x)) = \lambda_k(t, x, 0) + \varepsilon w_\varepsilon(t, x) = \varepsilon w_\varepsilon(t, x), \qquad (6.4.10)$$

where the family w_ε is bounded in $C^1_\varepsilon(\Omega)$, and such that

$$w_\varepsilon(t, x) - W(t, x, \varphi(t, x)/\varepsilon) = o(1) \qquad \text{in } L^\infty L^p \qquad (6.4.11)$$

with

$$W(t, x, \theta) = \sum_{i=1}^{N} W_i(t, x, \theta_i) \qquad (6.4.12)$$

$$W_i(t, x, \theta_i) = \frac{\partial \lambda_k}{\partial v_i}(t, x, 0)\, V_i(t, x, \theta_i). \qquad (6.4.13)$$

In these circumstances, Eq. (6.4.1) takes the simpler form

$$\partial_t u_\varepsilon + \varepsilon w_\varepsilon(t, x)\, \partial_x u_\varepsilon = f_\varepsilon. \qquad (6.4.14)$$

The corresponding equation for the profiles is

$$\partial_t U + E_k(W)\, D_k U = E_k F. \qquad (6.4.15)$$

We solve explicitly Eq. (6.4.14) by integration along the characteristics. The first step in the proof of Proposition 6.4.1, is to study the system defining the characteristic curves $s \to (s, x + \varepsilon y_\varepsilon(s; t, x))$ of $\partial_t + \varepsilon w_\varepsilon\, \partial_x$.

$$\begin{aligned} &\frac{d}{ds} y_\varepsilon(s; t, x) = w_\varepsilon(s, x + \varepsilon y_\varepsilon(s; t, x)) \\ &y_\varepsilon(t; t, x) = 0. \end{aligned} \qquad (6.4.16)$$

The characteristic curves of (6.4.15) $\partial_t + E_k(W)\, D_k$, are $s \to (s, x, \theta_k + Y(s; t, x, \theta_k)\, \partial_x \varphi_k)$, where the scalar function Y is solution to

$$\begin{aligned} &\frac{d}{ds} Y(s; t, x, \theta_k) = (E_k W)(s, x, \theta_k + Y(s; t, x, \theta_k)\, \partial_x \varphi_k(x)) \\ &Y(t; t, x, \theta_k) = 0. \end{aligned} \qquad (6.4.17)$$

Recall $\tilde{\Omega} = \{(s, t, x); 0 \leqslant s \leqslant t, (t, x) \in \Omega\}$.

PROPOSITION 6.4.2. *With notations as above, the family y_ε is bounded in $C^1_\varepsilon(\tilde{\Omega})$, $Y \in \mathscr{C}^1(\tilde{\Omega}; \Theta_k)$ and*

$$y_\varepsilon(s; t, x) - Y(s; t, x, \varphi_k(s, x)/\varepsilon) = o(1) \qquad \text{in } L^\infty(s, t; L^p(x)). \tag{6.4.18}$$

First, we need a lemma.

LEMMA 6.4.3. *Assume that y_ε is a bounded family in $C^1_\varepsilon(\tilde{\Omega})$, that $Y \in \mathscr{C}^1(\tilde{\Omega}; \Theta_k)$ and that (6.4.18) holds. Also assume that $1 + \varepsilon(\partial y_\varepsilon / \partial x)$ is bounded from below by a positive constant. Let v_ε be a bounded family in $L^\infty(\Omega)$, such that there is $V \in \mathscr{C}^0(\Omega; \Psi)$ satisfying (6.4.4). Let $w_\varepsilon(s; t, x) = v_\varepsilon(s, x + \varepsilon y_\varepsilon(s; t, x))$ and let $W(s; t, x, \theta) = V(s, x, \theta + Y(s; t, x, \theta_k) \cdot \partial_x \varphi(s, x))$. Then w_ε is bounded in $L^\infty(\tilde{\Omega})$, $W \in \mathscr{C}^0(\tilde{\Omega}; \Psi)$ and*

$$w_\varepsilon(s; t, x) - W(s; t, x, \varphi(s, x)/\varepsilon) = o(1) \qquad \text{in } L^\infty L^p. \tag{6.4.19}$$

Proof. Write $v_\varepsilon(s, x) = \bar{v}_\varepsilon(s, x) + a_\varepsilon(s, x)$, $y(s; t, x) = \bar{y}_\varepsilon(s; t, x) + z_\varepsilon(s; t, x)$, where $\bar{y}_\varepsilon(s; t, x) = Y(s; t, x, \varphi_k(x)/\varepsilon)$ and $\bar{v}_\varepsilon(t, x) = V(t, x, \varphi(t, x)/\varepsilon)$. We obtain

$$\varphi(s, x + \varepsilon y_\varepsilon(s; t, x)) = \varphi(s, x) + \varepsilon \bar{y}_\varepsilon(s; t, x)\, \partial_x \varphi(s, x) + \varepsilon e_\varepsilon(s; t, x) \tag{6.4.20}$$

with

$$|e_\varepsilon| \leqslant O(|\varepsilon y_\varepsilon|^2) + O(|z_\varepsilon|). \tag{6.4.21}$$

With these notations, we obtain

$$w_\varepsilon(s, t, x + \varepsilon y_\varepsilon) - W(s; t, x, \varphi(s, x)/\varepsilon) = a_\varepsilon(s, x + \varepsilon y_\varepsilon(s; t, x)) + B + C, \tag{6.4.22}$$

with

$$B := V(s, x + \varepsilon y_\varepsilon \varphi(s, x + \varepsilon y_\varepsilon(s; t, x))/\varepsilon) - V(s, x, \varphi(s, x + \varepsilon y_\varepsilon(s; t, x))/\varepsilon)$$

and

$$C := V(s, x, \varphi(s, x + \varepsilon y_\varepsilon(s; t, x))/\varepsilon) - V(s, x, \varphi(s, x + \varepsilon y_\varepsilon(s; t, x))/\varepsilon - e_\varepsilon(s; t, x)).$$

(1) Suppose $p = +\infty$.

Since $a_\varepsilon = o(1)$ in L^∞, $b_\varepsilon := a_\varepsilon(s; t, x + \varepsilon y_\varepsilon) = o(1)$ in L^∞. The term B is $O(\varepsilon)$ because $y_\varepsilon = O(1)$ in L^∞ and $V \in \mathscr{C}^0$. From (6.4.21), $e_\varepsilon = o(1)$ in L^∞. By uniform continuity of V in θ, $C = o(1)$ in L^∞. Lemma follows from (6.4.22).

(2) Suppose $p < +\infty$.

Since the Jacobian of $x \to x + \varepsilon y_\varepsilon$ is bounded from below, we have $\|b_\varepsilon(s;t)\|_{L^p} \leqslant c\, \|a_\varepsilon(s;t)\|_{L^p} = o(1)$. The term B is still $o(1)$ in L^∞ and the term C is bounded in L^∞. Furthermore, because of (6.4.21), for any $\delta > 0$, $\text{meas}(\{x; (s,x) \in \Omega, |e_\varepsilon(s,x)| > \delta\}$ tends to 0 as ε goes to 0, uniformly in $s \in [0, T]$. Because of uniform continuity of V in θ, there is a positive function $\delta \to w(\delta)$, converging to 0 when δ tends to 0, such that for any s, t we have

$$\{x; |C| > \omega(\delta)\} \subset \{x; |e_\varepsilon| > \delta\}. \tag{6.4.23}$$

From this follows that $C = o(1)$ in $L^\infty(s, t; L^p(x))$.

Proof of Proposition 6.4.2. By Picard's method, y_ε is the limit as $v \to +\infty$ of y_ε^v, where $y_\varepsilon^0 = 0$ and

$$y_\varepsilon^{v+1}(s;t,x) = \int_t^s w_\varepsilon(\tau, x + \varepsilon y_\varepsilon^v(\tau; t, x))\, d\tau. \tag{6.4.24}$$

The proof is in four steps.

(a) Since w_ε is bounded in $C^1(\Omega)$, (6.4.24) implies that the y_ε^v are uniformly bounded in $C_\varepsilon^1(\tilde{\Omega})$. Moreover $y_\varepsilon^v \to y_\varepsilon$ in $L^\infty(\tilde{\Omega})$ *uniformly in* ε.

Formula (6.4.24) also implies that if $|t-s|$ is small enough, say smaller than some $\delta > 0$, then $1 + \varepsilon(\partial y_\varepsilon/\partial x) \geqslant \frac{1}{2}$.

(b) On the other hand, an induction on v shows that if $|t-s| \leqslant \delta$, then

$$y_\varepsilon^v(s;t,x) - Y^v(s;t,x,\varphi_k(x)/\varepsilon) = o(1) \qquad \text{in } L^\infty L^p, \tag{6.4.25$_v$}$$

where the Y^v are defined by $Y^0 = 0$ and

$$Y^{v+1}(s;t,x,\theta_k) = \int_t^s (E_k W)(\tau; t, x, \theta_k + Y^v(\tau; t, x, \theta_k)\, \partial_x \varphi_k(x))\, d\tau. \tag{6.4.26}$$

Indeed, if (6.4.25)$_v$ holds, then by Lemma 6.4.3 we see that

$$w_\varepsilon(\tau, x + \varepsilon y_\varepsilon^v(\tau; t, x)) - G^v(\tau; t, x, \varphi(\tau, x)/\varepsilon) = o(1) \qquad \text{in } L^\infty L^p \tag{6.4.27}$$

with

$$G^v(\tau; t, x, \theta) = W(\tau, x, \theta + Y^v(\tau; t, x, \theta_k) \cdot \partial_x \varphi(\tau, x)). \tag{6.4.28}$$

Now, with (6.4.26), (6.4.13), we can apply Theorem 4.4.2 to the integral (6.4.24) to find that (6.4.25)$_{v+1}$ holds with

$$Y^{v+1}(s;t,x,\theta_k) = \int_t^s (E_k G^v)(\tau; t, x, \theta_k)\, d\tau. \tag{6.4.29}$$

Because Y^ν depends only on θ_k, Lemma 4.1.4 implies that

$$(E_k G^\nu)(\tau; t, x, \theta_k) = (E_k W)(\tau; t, x, \theta_k + Y^\nu(\tau; t, x, \theta_k)\,\partial_x \varphi_k(x)) \tag{6.4.30}$$

and formula (6.4.29) is (6.4.26), as claimed.

(c) As shown in the proof of Lemma 6.3.2, the sequence Y^ν is bounded in $\mathscr{C}^1(\tilde{\Omega}; \Theta_k)$, and converges in $L^\infty(\tilde{\Omega}; \Theta_k)$ to the solution $Y \in \mathscr{C}^1(\tilde{\Omega}; \Theta_k)$ of (6.4.17).

(d) Because $y_\varepsilon^\nu \to y_\varepsilon$ uniformly in ε, and because $Y^\nu(s; t, x, \varphi_k(x)/\varepsilon) \to Y(s; t, x, \varphi_k(x)/\varepsilon)$ in L^∞, estimate (6.4.18) for $|t-s| \leqslant \delta$ follows from (6.4.25). Taking into account the semigroup properties of the solutions of the ODE's (6.4.16) and (6.4.17), estimate (6.4.18) for general t and s in $[0, T]$ follows.

Proof of Proposition 6.4.1. The solution of (6.4.14) with the initial condition (6.4.2) is given by the formula

$$u_\varepsilon(t, x) = h_\varepsilon(x + \varepsilon y_\varepsilon(0; t, x)) + \int_0^t f_\varepsilon(\tau, x + \varepsilon y_\varepsilon(\tau; t, x))\, d\tau. \tag{6.4.31}$$

Making use of Proposition 6.4.2 and Lemma 6.4.3 (we know that the mapping $x \to x + \varepsilon y_\varepsilon(s; t, x)$ is invertible with inverse $x \to x + \varepsilon y_\varepsilon(t; s, x)$) we obtain the asymptotics for $f_\varepsilon(\tau, x + \varepsilon y_\varepsilon(\tau; t, x))$. Next, applying Theorem 4.4.2, we obtain the approximation (6.4.9) with

$$\begin{aligned} U(t, x, \theta_k) &= H(x, \theta_k + Y(0, t, x, \theta_k)\,\partial_x \varphi_k(x)) \\ &\quad + \int_0^t (E_k F)(\tau; t, x, \theta_k + Y(\tau; t, x, \theta_k)\,\partial_x \varphi_k(x))\, d\tau. \end{aligned} \tag{6.4.32}$$

Now, $s \to (s, x, \theta_k + Y(s; t, x, \theta_k)\,\partial_x \varphi_k(x))$ are precisely the integral curves of $\partial_t + (E_k W)(t, x, \theta_k)\, D_k$ (see (6.4.17)) and thus (6.4.32) is the explicit form of the solution U of

$$X_k U + E_k(W)\, D_k U = E_k(F_k) \tag{6.4.33}$$

$$U|_{t=0}(x, \theta_k) = H_k(x, \theta_k). \tag{6.4.34}$$

It remains to check that $E_k(W) = \gamma_k(V)$ and Proposition 6.4.1 will be proved. Taking into account formulas (6.4.12), (6.4.13), and Definition (6.3.3), this is a consequence of the following lemma, whose second part will be needed later on.

LEMMA 6.4.4. *Recall the Definition* (2.9.10) *of the coefficients* $\Gamma^k_{i,j}$. *Then*

(i) *One has* $\Gamma^k_{i,k} = (\partial\lambda_k/\partial v_i)(t, x, 0)$.

(ii) *If* $P(t, x, v)$ *denote a matrix which diagonalizes* A *as in* (6.2.23), (6.2.24), *then for* $k \neq j$, *one has* $\Gamma^k_{i,j} = (\lambda_{j,0} - \lambda_{k,0})(\partial P_{k,j}/\partial v_i)(t, x, 0)$.

Proof. Differentiate relation (6.2.23) and evaluate at $v = 0$, taking (6.2.24) into account.

6.5. $\mathscr{C}^0$ *Asymptotics: Linear System with Diagonal Principal Part*

In this section, we study the semilinear system

$$\partial_t u_\varepsilon + A(t, x, \varepsilon v_\varepsilon)\,\partial_x u_\varepsilon + m(t, x, \varepsilon v_\varepsilon, w_\varepsilon)\,u_\varepsilon + Q(t, x, \varepsilon v_\varepsilon, u_\varepsilon) = f_\varepsilon \tag{6.5.1}$$

$$u_\varepsilon|_{t=0} = h_\varepsilon, \tag{6.5.2}$$

where we assume that m is a $N \times N$ matrix which depends smoothly on the variables (t, x, v) and linearly on w

$$m(t, x, v, w) = \sum m^i(t, x, v)\,w_i \tag{6.5.3}$$

and $Q(t, x, v, u)$ is a quadratic form in the variables $u \in \mathbb{R}^N$, valued in $\mathbb{R}^N$, with coefficients depending smoothly on $(t, x, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$. We denote by Q_k the kth component of Q

$$Q_k(t, x, v, u) = \sum_{j,i} Q_k^{i,j}(t, x, v)\,u_i u_j. \tag{6.5.4}$$

First we assume that

$$v_\varepsilon,\ w_\varepsilon,\ f_\varepsilon \text{ and } h_\varepsilon \text{ are bounded in } C^1_\varepsilon(\Omega),\ C^0(\Omega),\ C^0(\Omega), \text{ and } C^0_\varepsilon([y_-, y_+]), \text{ respectively.} \tag{6.5.5}$$

We also assume that there are $\mathbb{R}^N$-valued $V \in \mathscr{C}^1(\Omega; \Psi)$, and $H \in \mathscr{C}^0([y_-, y_+]; \Psi)$, such that V_k, H_k depend only on θ_k, F is of the form $\sum F_{i,j}(t, x, \theta_i, \theta_j)$ and

$$v_\varepsilon(t, x) - V(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty L^p \tag{6.5.6}$$

$$f_\varepsilon(t, x) - F(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty L^p \tag{6.5.7}$$

$$h_\varepsilon(x) - H(x, \varphi(0, x)/\varepsilon) = o(1) \quad \text{in } L^p, \tag{6.5.8}$$

where $p = +\infty$ [resp. $p < +\infty$] if condition $(\mathscr{T}q)$ [resp. $(w - \mathscr{T}q)$] holds.

Finally, we suppose that there is a function $W \in \mathscr{C}^0(\Omega; \Psi)$ of the form $\sum W_k(t, x, \theta_k)$ such that

$$w_\varepsilon(t, x) - W(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty L^p. \tag{6.5.9}$$

Corresponding to (6.5.1), (6.5.2), there is the following system for the profiles

$$X_k U_k + \gamma_k(V) D_k U_k + E_k \left(\sum_{i,j} \bar{m}^i_{k,j} W_i U_j \right) + E_k \left(\sum_{i,j} \bar{Q}^{i,j}_k U_i U_j \right) = E_k(F_k) \qquad (6.5.10)_k$$

$$U_k |_{t=0} = H_k(x, \rho_k \theta_k) \qquad (6.5.11)_k$$

with $\bar{m}^i_{k,j}(t, x) = m^i_{k,j}(t, x, 0)$ and $\bar{Q}^{i,j}_k(t, x) = Q^{i,j}_k(t, x, 0)$.

In the linear case ($Q \equiv 0$), we know from Lemma 6.2.4 that there exists a unique solution $u_\varepsilon \in C^0(\Omega)$ of (6.5.1), (6.5.2) which is the limit in $L^\infty(\Omega)$ of the solutions u^ν_ε of (6.2.14); moreover the limit is uniform in ε. On the other hand, there exists a unique solution $U \in \mathscr{C}^0(\Omega; \Psi)$ of (6.5.10), (6.5.11) which is the limit in $L^\infty(\Omega \times \Psi)$ of the solutions U^ν of (6.3.21).

In the semilinear case ($Q \neq 0$), Lemmas 6.2.5 and 6.3.7 imply that there exists $T' > 0$, independent of ε, such that the same conclusions hold on $\Omega' = \Omega \cap \{t \leqslant T'\}$.

PROPOSITION 6.5.1. *With notations as above, one has on Ω in the linear case, or on Ω' in the semilinear case*

$$u_\varepsilon(t, x) - U(t, x, \varphi(t, x)/\varepsilon) = o(1) \qquad \textit{in } L^\infty L^p. \qquad (6.5.12)$$

Proof. Since u_ε is the uniform limit in $L^\infty(\Omega)$ of the u^ν_ε and because U is the limit in $L^\infty(\Omega \times \Psi)$ of the U^ν it suffices to show that for each ν

$$u^\nu_{\varepsilon,k}(t, x) = U^\nu_k(t, x, \varphi_k(t, x)/\varepsilon) + o(1) \qquad \text{in } L^\infty L^p. \qquad (6.5.13)_\nu$$

For $\nu = 0$ the estimate is trivial. If we assume $(6.5.13)_\nu$, then with (6.5.6)–(6.5.9) we see that the right hand side f^ν_ε of (6.2.20) satisfies (6.4.5) with the profile F^ν

$$F^\nu_k = F_k - \sum_{i,j} \bar{m}^i_{k,j} W_i U^\nu_j - \sum_{i,j} \bar{Q}^{i,j}_k U^\nu_i U^\nu_j. \qquad (6.5.14)$$

Equation (6.2.20) [or (6.2.14) when $Q = 0$] is a decoupled system of N scalar equations and Proposition 6.4.1 implies that $(6.5.13)_{\nu+1}$ holds with the profile $U^{\nu+1}_\#$ which is the solution of

$$X_k U^{\nu+1}_{\#,k} + \gamma_k(V) D_k U^{\nu+1}_{\#,k} = E_k(F_k) - E_k \left(\sum_{i,j} \bar{m}^i_{k,j} W_i U^\nu_j - \sum_{i,j} \bar{Q}^{i,j}_k U^\nu_i U^\nu_j \right) \qquad (6.5.15)$$

$$U^{\nu+1}_{\#,k} |_{t=0}(x, \theta_k) = H_k(x, \theta_k). \qquad (6.5.16)$$

Here we recognize Eq. (6.3.28) [or (6.3.21)] with initial condition (6.3.22). Thus $U_{\#}^{\nu+1} = U^{\nu+1}$ and $(6.5.13)_{\nu+1}$ holds, and Proposition 6.5.1 is proved.

6.6. $\mathscr{C}^0$ *Asymptotics: Case of a General Linear System*

In this section, we study the following linear system, which is of the form (6.2.21), (6.2.22) and which corresponds to taking $Q = 0$ in (6.5.1)

$$\partial_t u_\varepsilon + A(t, x, \varepsilon v_\varepsilon)\, \partial_x u_\varepsilon + m(t, x, \varepsilon v_\varepsilon, w_\varepsilon)\, u_\varepsilon = f_\varepsilon \tag{6.6.1}$$

$$u_\varepsilon|_{t=0} = h_\varepsilon. \tag{6.6.2}$$

We still assume that m is a $N \times N$ matrix of the form (6.5.3), which depends smoothly on the variables (t, x, v) and linearly in w.

We assume (6.5.5) and that there are $\mathbb{R}^N$-valued $V \in \mathscr{C}^1(\Omega; \Psi)$, $F \in \mathscr{C}^0(\Omega; \Psi)$, $W \in \mathscr{C}^0(\Omega; \Psi)$, and $H \in \mathscr{C}^0([y_-, y_+]; \Psi)$, such that $V_k H_k$ depend only on θ_k, F is a sum of terms $F^{i,j}(t, x, \theta_i, \theta_j)$ and W is a sum of terms $W_k(t, x, \theta_k)$. We still assume (6.5.7), (6.5.8), and (6.5.9) but we now strengthen (6.5.6), assuming that

$$v_\varepsilon(t, x) - V(t, x, \varphi(t, x)/\varepsilon) = o(1) \qquad \text{in } L^\infty W_\varepsilon^{1,p}, \tag{6.6.3}$$

where $p = +\infty$ [resp. $p < +\infty$] if condition $(\mathscr{T}q)$ [resp. $(w - \mathscr{T}q)$] holds.

From Lemma 6.2.6, we know that there is a unique solution $u_\varepsilon \in C^0(\Omega)$ of (6.6.1), (6.6.2) which is bounded in $L^\infty(\Omega)$.

On the other hand, we know from Lemma 6.3.5 that there is a unique $\mathbb{R}^N$-valued solution $U \in \mathscr{C}^0(\Omega; \Psi)$, with U_k depending only on θ_k, of the system

$$X_k U_k + \gamma_k(V)\, D_k U_k + \mathscr{E}'_k(V, U) + E_k\left(\sum_{i,j} \bar{m}^i_{k,j} W_i U_j\right) = E_k(F_k) \tag{$(6.6.4)_k$}$$

$$U_k|_{t=0} = H_k(x, \theta_k) \tag{6.6.5}$$

with $\bar{m}^i_{k,j}(t, x) = m^i_{k,j}(t, x, 0)$.

Proposition 6.6.1. *With notations as above, one has*

$$u_\varepsilon(t, x) - U(t, x, \varphi(t, x)/\varepsilon) = o(1) \qquad \textit{in } L^\infty L^p_{\mathrm{loc}}. \tag{6.6.6}$$

Proof. As in (6.2.25), we introduce $\hat{u}_\varepsilon(t, x) \in C^0(\Omega)$ the solution to (6.2.26), (6.2.27). Because of (6.6.3), we have estimates of the form

$$\varepsilon\, \partial_x v_{i,\varepsilon}(t, x) = D_i V_i(t, x, \varphi_i(t, x)/\varepsilon) + o(1) \qquad \text{in } L^\infty L^p. \tag{6.6.7}$$

Because $X_i \varphi_i = 0$, we also have

$$\varepsilon\, \partial_t v_{i,\varepsilon}(t, x) = -\lambda_{i,0}(t, x)\, D_i V_i(t, x, \varphi_i(t, x)/\varepsilon) + o(1) \qquad \text{in } L^\infty L^p. \tag{6.6.8}$$

With notations as in (6.2.28), one has for $\partial = \partial_t$ or ∂_x

$$\partial P_\varepsilon = \sum_{j=1}^{N} \frac{\partial P}{\partial v_j}(t, x, 0) \cdot \varepsilon\, \partial v_{\varepsilon, j} + O(\varepsilon).$$

Thus we deduce from (6.6.7) and (6.6.8), that $\hat{m}_\varepsilon$ as defined in (6.2.28) satisfies

$$\hat{m}_\varepsilon(t, x) = M(t, x, \varphi(t, x)/\varepsilon) + o(1) \qquad \text{in } L^\infty L^p \tag{6.6.9}$$

with a matrix M whose entries are

$$M_{k,\ell}(t, x, \theta) = \sum_{i=1}^{N} (\lambda_{k,0} - \lambda_{i,0}) \frac{\partial P_{k,\ell}}{\partial v_i} D_i V_i + \bar{m}^i_{k,\ell} W_i. \tag{6.6.10}$$

In this formula, $\partial P_{k,\ell}/\partial v_i$ is evaluated at $(t, x, 0)$.

Apply Proposition 6.5.1 to the system (6.2.26) to find that

$$\hat{u}_{\varepsilon,k}(t, x) = U_k(t, x, \varphi_k(t, x)/\varepsilon) + o(1) \qquad \text{in } L^\infty L^p, \tag{6.6.11}$$

where U is solution to

$$X_k U_k + \gamma_k(V)\, D_k U_k + E_k\left(\sum_\ell M_{k,\ell} U_\ell\right) = E_k(F_k) \tag{6.6.12}$$

$$U_k|_{t=0}(x, \theta_k) = H_k(x, \theta_k). \tag{6.6.13}$$

Using formulas (6.6.10), (6.3.5), (6.3.6), and Lemma 6.4.4, a straightforward computation shows that

$$E_k\left(\sum_\ell M_{k,\ell} U_\ell\right) = \mathscr{E}'_k(V, U) + E_k\left(\sum_{i,j} \bar{m}^i_{k,j} W_i U_j\right)$$
$$+ E_k\left(\sum_i (\lambda_{k,0} - \lambda_{i,0}) \frac{\partial P_{k,k}}{\partial v_i} (D_i V_i)\right) U_k. \tag{6.6.14}$$

Because $E_k\{(D_i V_i)\} = 0$ when $i \neq k$, the third term of the sum vanishes. Therefore, U is indeed the solution of (6.6.4), (6.6.5) and, because $P(t, x, 0) =$ Id, formula (6.6.11) implies (6.6.6) and Proposition 6.6.1 is proved.

6.7. *Asymptotics of Derivatives for a Linear System. Proof of Proposition 6.1.3*

Consider the system

$$\partial_t u_\varepsilon + A(t, x, \varepsilon v_\varepsilon)\, \partial_x u_\varepsilon = f_\varepsilon \tag{6.7.1}$$

$$u_\varepsilon|_{t=0} = h_\varepsilon, \tag{6.7.2}$$

where

$$v_\varepsilon \text{ and } f_\varepsilon \text{ are bounded in } C^1_\varepsilon(\Omega) \text{ and } h_\varepsilon \text{ is bounded in } C^1_\varepsilon([y_-, y_+]). \tag{6.7.3}$$

Assume that there are $\mathbb{R}^N$-valued $V \in \mathscr{C}^1(\Omega; \Psi)$, $F \in \mathscr{C}^1(\Omega; \Psi)$, and $H \in \mathscr{C}^1([y_-, y_+]; \Theta^0)$, such that $V_k H_k$ depends only on θ_k, $F = \sum F_{i,j}(t, x, \theta_i, \theta_j)$ and

$$v_\varepsilon(t, x) - V(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty W^{1,p}_\varepsilon \tag{6.7.4}$$

$$f_\varepsilon(t, x) - F(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty W^{1,p}_\varepsilon \tag{6.7.5}$$

$$h_\varepsilon(x) - H(x, \varphi^0(x)/\varepsilon) = o(1) \quad \text{in } W^{1,p}_\varepsilon, \tag{6.7.6}$$

where $p = +\infty$ [resp. $p < +\infty$] if condition $(\mathscr{T}q)$ [resp. $(w - \mathscr{T}q)$] holds.

From Lemma 6.2.7 we know that under these assumptions, the solutions u_ε are bounded in $C^1_\varepsilon(\Omega)$. On the other hand, we know from Lemma 6.3.8 that the solution U of (6.3.1), (6.3.2) belongs to $\mathscr{C}^1(\Omega; \Psi)$.

PROPOSITION 6.7.1. *With notations as above, one has*

$$\varepsilon\, \partial_x u_\varepsilon(t, x) - DU(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \textit{in } L^\infty L^p. \tag{6.7.7}$$

Proof. The proof is in two steps.

Step 1. Approximation of $\varepsilon\, \partial_x u_\varepsilon$. Let

$$z_\varepsilon = \varepsilon\, \partial_x u_\varepsilon \in C^0(\Omega). \tag{6.7.8}$$

As already noted in Section 6.2, z_ε is the unique $L^\infty(\Omega)$ (weak) solution of (6.2.33), (6.2.34) which is of the form (6.6.1), with $\varepsilon\, \partial_x f_\varepsilon$ on the right hand side, $m(t, x, v, w) = (\partial A/\partial x)(t, x, v)\, w_0 + \sum_{i=1}^N (\partial A/\partial v_i)(t, x, v)\, w_i$ and $w_0 = 1$, $w_i = \varepsilon\, \partial_x v_i$ for $i \geqslant 1$. We deduce from (6.7.5) that

$$\varepsilon\, \partial_x f_\varepsilon - F'(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty L^p \tag{6.7.9}$$

with

$$F'(t, x, \theta) = DF(t, x, \theta) \tag{6.7.10}$$

which is still of the form $\sum \{D_i + D_j\}\, F^{i,j}(t, x, \theta_i, \theta_j)$. Similarly, (6.7.6) implies that

$$\varepsilon\, \partial_x h_{\varepsilon,k} - H'_k(x, \varphi_k(0, x)/\varepsilon) = o(1) \quad \text{in } L^p \tag{6.7.11}$$

with $H'_k(t, \theta_k) = D_k\{H_k(x, \theta_k)\}$.

Using (6.6.7), we see that the assumptions (6.5.7), (6.5.8), (6.5.9) are fullfilled for the equation satisfied by z_ε. Thus Proposition 6.6.1 yields

$$z_\varepsilon(t, x) - Z(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty L^p \tag{6.7.12}$$

with Z_k depending only on θ_k satisfying

$$X_k Z_k + \gamma_k(V)\, D_k Z_k + \mathscr{E}'_k(V, Z) + E_k\left(\sum_{i,j} \bar{m}^i_{k,j}(D_i V_i)\, Z_j\right)$$
$$+ E_k\left(\sum_j \bar{m}^0_{k,j} Z_j\right) = E_k(DF_k) \tag{6.7.13)$_k$ ($$

$$Z_k|_{t=0} = D_k H_k(x, \theta_k), \tag{6.7.14)$_k$ ($$

where

$$\bar{m}^i_{k,j}(t, x) = \frac{\partial A_{k,j}}{\partial v_i}(t, x, 0) = \Gamma^k_{i,j} \quad \text{for} \quad 1 \leqslant i \leqslant N \tag{6.7.15}$$

and

$$\bar{m}^0_{k,j} = \frac{\partial A_{k,j}}{\partial x}(t, x, 0) = \delta_{k,j}\, \partial_x \lambda_{k,0}(t, x) \tag{6.7.16}$$

the last equality being a consequence of the fact that $A(t, x, 0)$ is assumed to be diagonal.

We next perform several simplifications in (6.7.13). First we can discard the terms $\bar{m}^i_{k,j} E_k\{(D_i V_i)\, Z_j)$, where $j = k$, $i \neq k$, since $E_k(D_i V_i) = 0$. Only the term $\Gamma^k_{k,k}(D_k V_k)\, Z_k$ remains. Next, we consider the terms where $j \neq k$ and $i = k$. They yield a term we denote by $\mathscr{H}_k(DV, Z)$

$$\mathscr{H}_k(W, Z) = \sum_{j \neq k} \Gamma^k_{k,j} W_k E_k(Z_j). \tag{6.7.17}$$

Finally, we group the remaining terms $i \neq k$, $j \neq k$ with the corresponding ones from $\mathscr{E}'_k(V, Z)$ and we obtain a sum denoted $\mathscr{F}_k(DV, Z)$, where

$$\mathscr{F}_k(W, Z) = E_k\left(\sum_{i \neq k, j \neq k} \{\Pi^k_{i,j} + \Gamma^k_{i,j}\}\, W_i Z_j\right)$$
$$= E_k\left(\sum_{i \neq k, j \neq k} \frac{\lambda_{j,0} - \lambda_{i,0}}{\lambda_{j,0} - \lambda_{k,0}}\, \Gamma^k_{i,j} W_i Z_j\right). \tag{6.7.18}$$

With these notations, we can rewrite Eq. (6.7.13) as

$$X_k Z_k + \gamma_k(V)\, D_k Z_k + \mathscr{F}_k(DV, Z) + \mathscr{H}_k(DV, Z)$$
$$+ \{\Gamma^k_{k,k}(D_k V_k) + \partial_x \lambda_{k,0}\}\, Z_k = E_k(DF_k). \tag{6.7.19}$$

Step 2. We show that $Z_k = D_k U_k$.

By uniqueness, if suffices to show that $Z' = D_k U_k$ satisfies the system (6.7.19), (6.7.14). The comparison of the initial conditions is trivial. On the other hand, because we know that $U \in \mathscr{C}^1$ one can differentiate (6.3.7) with respect to D_k.

(a) The commutator $[D_k, X_k + \gamma_k(V) D_k]$ is

$$[D_k, X_k + \gamma_k D_k] = \{(D_k \gamma_k) + \partial_x \lambda_{k,0}\} D_k, \tag{6.7.20}$$

where we have used that $X_k \varphi_k = 0$ so that $X_k(\partial_x \varphi_k) = -\partial_x \lambda_{k,0} \cdot \partial_x \varphi_k$.

Moreover, from the Definition (6.3.3) of $\gamma_k(V)$,

$$\gamma_k(V) = E_k\left(\sum_i V_i \Gamma^k_{i,k}\right).$$

We claim that

$$D_k \gamma_k = E_k\left(\sum_i (D_i V_i) \Gamma^k_{i,k}\right) = \Gamma^k_{k,k}(D_k V_k). \tag{6.7.21}$$

Indeed, if $i = k$, then $E_k(V_k) = V_k$ and the commutator of E_k and D_k is zero. If $i \neq k$, then by Example 2.6.4, because V_i depends only on θ_i, $E_k(V_i)$ does not depend on θ_k and $D_k\{E_k(V_i)\} = 0$. On the other hand, it also follows from (2.6.9) that $E_k(D_i V_i) = 0$.

(b) By Corollary 4.3.4, we obtain that

$$\begin{aligned} D_k \mathscr{E}'_k(V, U) &= \sum_{i \neq k, j \neq k} \frac{\lambda_{j,0} - \lambda_{i,0}}{\lambda_{k,0} - \lambda_{i,0}} \Pi^k_{i,j} E_k\{(D_i V_i)(D_j U_j)\} \\ &= \sum_{i \neq k, j \neq k} \frac{\lambda_{j,0} - \lambda_{i,0}}{\lambda_{j,0} - \lambda_{k,0}} \Gamma^i_{k,j} E_k\{(D_i V_i) Z'_j\} \\ &= \mathscr{F}_k(DV, Z'), \end{aligned} \tag{6.7.22}$$

where $\mathscr{F}$ is defined in (6.7.18).

(c) Taking (6.7.20) and (6.7.22) into account, we deduce from (6.3.1) that

$$X_k Z'_k + \gamma_k D_k Z'_k + \mathscr{F}_k(DV, Z') + (D_k \gamma_k) Z'_k + (\partial_x \lambda_{k,0}) Z'_k = D_k\{E_k(F_k)\}. \tag{6.7.23}$$

By Lemma 4.3.1, we know that $D_k\{E_k(F_k)\} = E_k(DF_k)$. Furthermore, because $Z'_j = D_j U_j$, then $E_k(Z'_j) = 0$ when $j \neq k$, and from the Definition (6.7.17) we see that $\mathscr{H}_k(V, Z') = 0$, so we can freely add this term to the left hand side of (6.7.23) and it is now proved that Z' satisfies (6.7.19), (6.7.14). By uniqueness, $Z' = Z$ and the proposition follows.

PROPOSITION 6.7.2. *With assumptions as in Proposition* 6.7.1, *one has*

$$u_\varepsilon(t, x) - U(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty W^{1,p}_\varepsilon. \tag{6.7.24}$$

Proof. Thanks to Proposition 6.7.1, it remains to show that

$$\varepsilon\, \partial_t u_\varepsilon(t, x) - D'U(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty L^p, \tag{6.7.25}$$

where $D' := \partial_t \varphi \cdot \partial_\theta$. Consider the kth equation in system (6.7.1). Because $A(t, x, 0)$ is diagonal and u_ε, v_ε are bounded in C^1_ε, one has

$$\varepsilon\, \partial_t u_{\varepsilon,k} + \lambda_{k,0} \varepsilon\, \partial_x u_{\varepsilon,k} = O(\varepsilon) \quad \text{in } L^\infty(\Omega).$$

Therefore (6.7.25) is a consequence of (6.7.7) and of the eikonal equations (1.3.2).

Proof of Proposition 6.1.3. Under the assumptions of Proposition 6.1.3, we have shown that there exists a $T > 0$, such that the sequence u^ν_ε is well defined and bounded in $C^1_\varepsilon(\Omega)$ (Section 6.2) and that the sequence U^ν is defined and bounded in $\mathscr{C}^1(\Omega; \Psi)$ (Section 6.3). We prove by induction on ν that

$$u^\nu_\varepsilon(t, x) - U^\nu(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty W^{1,p}_\varepsilon. \tag{6.7.26$_\nu$}$$

Assume that $(6.7.26)_\nu$ holds. We check that $f^\nu_\varepsilon(t, x) = b(t, x, \varepsilon u^\nu_\varepsilon)\, u^\nu_\varepsilon$ is bounded in $C^1_\varepsilon(\Omega)$, that (6.7.5) holds for f^ν_ε and that

$$F^\nu(t, x, \theta) = b(t, x, 0)\, U^\nu(t, x, \theta). \tag{6.7.27}$$

Therefore one can use Proposition 6.7.2, and comparing the corresponding Eq. (6.3.1) with (6.1.3), we obtain that $(6.7.26)_{\nu+1}$ is also satisfied. The proof of Proposition 6.1.3 is now complete.

6.8. *Asymptotics of Derivatives for a Quasilinear System*: *Proof of Proposition* 6.1.4

Let u_ε be solution to (6.0.1), (6.0.2). We have just checked that $u_\varepsilon \in C^1(\Omega)$ and that the u_ε form a bounded family in $C^1_\varepsilon(\Omega)$. Propositions 6.1.1–6.1.3 imply that there is $U \in \mathscr{C}^1(\Omega; \Psi)$ with U_k depending only on θ_k, such that

$$u_\varepsilon(t, x) - U(t, x, \varphi(t, x)/\varepsilon) = o(1) \quad \text{in } L^\infty L^p. \tag{6.8.1}$$

PROPOSITION 6.8.1. *There exists $T' > 0$ such that the following estimate is valid on $\Omega' = \Omega \cap \{t \leq T'\}$.*

$$\varepsilon\, \partial_x u_\varepsilon(t, x) - \partial_x \varphi \cdot \partial_\theta U(t, x, \varphi(t, x)/\varepsilon) = o(1) \qquad \textit{in } L^\infty L^p_{\text{loc}}. \quad (6.8.2)$$

Before starting the proof, we remark that this result implies Proposition 6.4.1. Indeed, using Eq. (6.0.1) one can estimate $\varepsilon\, \partial_t u_\varepsilon$ and deduce from (6.8.2) that

$$\varepsilon\, \partial_t u_\varepsilon(t, x) - D'U(t, x, \varphi(t, x)/\varepsilon) = o(1) \qquad \text{in } L^\infty L^p \qquad (6.8.3)$$

with $D' := \partial_t \varphi \cdot \partial_\theta$. Then (6.8.1)–(6.8.3) imply that

$$u_\varepsilon(t, x) - U(t, x, \varphi(t, x)/\varepsilon) = o(1) \qquad \text{in } L^\infty W^{1,p}_\varepsilon \qquad (6.8.4)$$

which is the conclusion of Proposition 6.1.4.

Proof of Proposition 6.8.1. (a) Because we know that u_ε is C^1, one can differentiate (6.0.1), and more precisely, if we introduce $z_\varepsilon(t, x) = \varepsilon P^{-1}(t, x, \varepsilon u_\varepsilon(t, x))\, \partial_x u_\varepsilon(t, x) \in C^0(\Omega)$, then, as in Lemma 6.2.6, z_ε is the L^∞ solution of the semilinear equation (6.2.48), (6.2.49).

This system is of the form (6.5.1), with Q and $Q_k^{i,j}$ given by (6.2.52) while $m(t, x, \varepsilon u_\varepsilon)\, z$ is defined by formula (6.2.51). Moreover

$$f_\varepsilon = f_\varepsilon(t, x, \varepsilon u_\varepsilon) = P^{-1}(\partial_x b) \cdot \varepsilon u_\varepsilon. \qquad (6.8.5)$$

We note that $f_\varepsilon(t, x, 0) = 0$ and because $\partial_t P(t, x, 0) = \partial_x P(t, x, 0) = 0$, we also have

$$\bar{m} = m(t, x, 0) = -b(t, x, 0) + \partial_x A(t, x, 0). \qquad (6.8.6)$$

Similarly, we see that

$$\bar{Q}_k^{i,j} = \delta_{k,j} \frac{\partial \lambda_j}{\partial v_i} + (\lambda_{j,0} - \lambda_{i,0}) \frac{\partial P_{k,j}}{\partial v_i}. \qquad (6.8.7)$$

Moreover, the initial condition (6.2.49) clearly satisfies

$$z_\varepsilon(0, x) - H'(x, \varphi(0, x)/\varepsilon) = o(1) \qquad \text{in } L^p \qquad (6.8.8)$$

with

$$H'_k(x, \theta_k) = D_k H(x, \theta_k). \qquad (6.8.9)$$

(b) Because we know that u_ε is bounded in $C^1_\varepsilon(\Omega)$ and that (6.8.1) holds, we can apply Proposition 6.5.1 to show that

$$z_\varepsilon(t, x) - Z(t, x, \varphi(t, x)/\varepsilon) = o(1) \qquad \text{in } L^\infty L^p \qquad (6.8.10)$$

with $Z \in \mathscr{C}^0(\Omega; \Psi)$, Z_k depending only on θ_k, satisfying

$$X_k Z_k + \gamma_k(U)\, D_k Z_k + E_k\left(\sum_j \bar{m}_{k,j} Z_j\right) + E_k\left(\sum_{i,j} \bar{Q}_k^{i,j} Z_i Z_j\right) = 0 \tag{6.8.11}$$

$$Z_k|_{t=0} = H'_k. \tag{6.8.12}$$

Because of (6.8.6), the third term on the left hand side is $\mathscr{B}_k(Z) + (\partial_x \lambda_{k,0})\, Z_k$, where

$$\mathscr{B}_k(Z) = -E_k\left(\sum_j \bar{b}_{k,j} Z_j\right). \tag{6.8.13}$$

Next, we split the last term of (6.8.11) into three parts. The first one corresponds to the indices $i = j = k$

$$\frac{\partial \lambda_k}{\partial v_k}(t, x, 0)(Z_k)^2 = \Gamma^k_{k,k}(Z_k)^2 \tag{6.8.14}$$

the last equality being a consequence of Lemma 6.4.4(i).

The second part corresponds to the indices $j \neq k$ and $i \neq k$. With Lemma 6.4.4(ii), it can be written

$$\sum_{i \neq k,\, j \neq k} \frac{\lambda_{j,0} - \lambda_{i,0}}{\lambda_{j,0} - \lambda_{k,0}} \Gamma^k_{i,j} E_k(Z_i Z_j) = \mathscr{F}_k(Z, Z), \tag{6.8.15}$$

where we recognize the functional $\mathscr{F}$ that was introduced in (6.7.18).

The third part corresponds to the indices $j = k$ and $i \neq k$ or $j \neq k$ and $i = k$. We denote it by $\mathscr{K}_k(Z, Z)$.

$$\mathscr{K}_k(Z, Z) = \sum_{i \neq k} \bar{Q}_k^{i,k} Z_k E_k(Z_i) + \sum_{j \neq k} \bar{Q}_k^{k,j} Z_k E_k(Z_j). \tag{6.8.16}$$

With these notations, Z is solution to

$$\begin{aligned} X_k Z_k + \gamma_k(U)\, D_k Z_k + (\partial_x \lambda_{k,0})\, Z_k + \Gamma^k_{k,k}(Z_k)^2 \\ + \mathscr{F}_k(Z, Z) + \mathscr{K}_k(Z, Z) + \mathscr{B}_k(Z) = 0. \end{aligned} \tag{6.8.17}$$

(c) On the other hand, we know that $U \in \mathscr{C}^1(\Omega; \Psi)$ satisfies (6.1.5), (6.1.6), and we can differentiate these equations. Introduce $Z' = DU = (Z'_k)_{1 \leqslant k \leqslant N}$ with $Z'_k = D_k U_k$.

Equation (6.1.5) is of the form (6.3.1) or preferably of the form (6.3.7), with $V = U$ and $F = \bar{b}U$. We make use of the computations made in Section 6.7 (second step of the proof of Proposition 6.7.1). Because

$DV = DU = Z'$, from (6.7.21) we see that $D_k \gamma_k(V) = \Gamma^k_{k,k} Z'_k$ and formula (6.7.23) shows that Z' satisfies

$$X_k Z'_k + \gamma_k(U)\, D_k Z'_k + (\partial_x \lambda_{k,0})\, Z'_k + \Gamma^k_{k,k}(Z'_k)^2 + \mathscr{F}_k(Z', Z') = D_k\{E_k(F_k)\}. \tag{6.8.18}$$

Since $F = \bar{b}U$ it is easily checked (see Lemma 4.3.1) that $D_k\{E_k(F_k)\} = E_k(D(\bar{b}U)) = -\mathscr{B}_k(Z')$. On the other hand, because $i \neq k$ and $j \neq k$ in the sum of (6.8.16), $\mathscr{K}_k(Z', Z') = 0$ and (6.8.18) shows that Z' is solution of the same equation (6.8.17) as Z. Furthermore, (6.1.6) shows that the two solutions are equal to H' defined in (6.8.9) at $t = 0$. By uniqueness (see Lemma 6.3.7) we can conclude that $Z = Z' = DU$.

(d) Since $P(t, x, 0) = \mathrm{Id}$ and $z_\varepsilon = O(1)$ in $L^\infty(\Omega)$ then $\varepsilon\, \partial_x u_\varepsilon - z_\varepsilon = O(\varepsilon)$ in $L^\infty(\Omega)$. Estimate (6.8.10) together with the fact that $Z = DU$, implies (6.8.2), and Proposition 6.8.1 follows.

7. Life Span of Solutions

This section is devoted to the proof of Theorems 2.10.2 and 2.10.4.

7.1. *The Semilinear Case. Proof of Theorem* 2.10.2

Let Ω_0 be a domain of the form (2.1.3) and consider the Cauchy problem (2.8.1) with Cauchy data h_ε which satisfy (2.8.2) and (2.8.3) with $p = +\infty$. We assume that conditions ($\mathscr{C}$) and ($\mathscr{T}$) hold. Moreover, as in Section 5, we assume that L is diagonal. Recall the definition (2.10.6) of $T_*(\delta)$ as well as the definition of T_*. The proof is in two steps.

Step 1. We use

LEMMA 7.1.1. $T_* \leqslant \lim_{\delta \to 0} T_*(\delta)$ *and the approximation* (2.8.4) *is valid in* $\Omega(T) := \Omega_0 \cap \{t \leqslant T\}$ *for all* $T < T_*$.

Proof. To begin with we make a remark. Theorem 2.8.1 provides us with a time T such that (2.8.4) holds on Ω_T. Moreover the proofs of Proposition 5.1.2 and 5.1.4 show that this time T can be estimated from below by $\inf\{T_0, \mathrm{t}(m, \mu(\varepsilon_0))\}$, where t is a positive function of

$$m = \|H\|_{L^\infty([y_-, y_+] \times \Psi)} \qquad \text{and} \qquad \mu(\varepsilon_0) = \sup_{\varepsilon \leqslant \varepsilon_0} \|h_\varepsilon\|_{L^\infty([y_-, y_+])} \tag{7.1.1}$$

Fix $T < T_* \leqslant T_0$ and let $M := \|U\|_{L^\infty(\Omega(T) \times \Psi)}$ and $\mathrm{t} := \mathrm{t}(M, M+1)$. Because of (2.8.3), there is $\varepsilon_1 > 0$ such that $\mu(\varepsilon_1) \leqslant M + 1$ and therefore we have $T_*(\varepsilon_1) \geqslant \mathrm{t}_1 = \inf\{T_0, \mathrm{t}\}$. Suppose now that $\mathrm{t} \leqslant T$. At time t we have $\|U(\mathrm{t}, \cdot, \cdot)\|_{L^\infty} \leqslant M$, by definition of M. Furthermore, because estimate

(2.8.4) is valid up to time t, there is $\varepsilon_2 > 0$ such that $\|u_\varepsilon(\mathrm{t}, \cdot)\|_{L^\infty} \leqslant M+1$ if $\varepsilon \leqslant \varepsilon_2$. So we can now apply Theorem 2.8.1 with Cauchy data on the line $t = \mathrm{t}$ to conclude first that $T_*(\varepsilon_2) \geqslant \mathrm{t}_2 = \mathrm{Inf}\{T_0, 2\mathrm{t}\}$ and second that the approximation (2.8.4) is valid on $\Omega(\mathrm{t}_2)$. An easy induction shows that, after a finite number of steps, we obtain an $\varepsilon_p > 0$ such that $T_*(\varepsilon_p) \geqslant \mathrm{t}_p = \mathrm{Inf}\{T_0, p\mathrm{t}\} \geqslant T$ and the approximation (2.8.4) holds on $\Omega(\mathrm{t}_p)$. The lemma follows.

Step 2. It remains to prove

LEMMA 7.1.2. *For all* $\delta > 0$, $T_* \geqslant T_*(\delta)$.

Proof. Assume that for some $\delta > 0$, $T_* < T_*(\delta) \leqslant T_0$. Then by definition, the u_ε are defined and uniformly bounded, say by M, on $\Omega(T_*)$. For any $t < T_*$, we know from Lemma 7.1.1 that the approximation (2.8.4) holds on $\Omega(t)$. Making use of Theorem 2.10.5, we obtain that

$$\forall t < T_*, \qquad \|U(t, \cdot, \cdot)\|_{L^\infty} \leqslant M. \tag{7.1.2}$$

Consider now, for $T < T_*$ the Cauchy problem (2.8.5) for the profiles, with Cauchy data $U(T, \cdot, \cdot)$ on the line $t = T$. This problem has a solution as stated in Proposition 5.1.4 on an interval $[T, T+\tau]$ with $\tau > 0$, and τ can be bounded from below by some positive function t of the L^∞-norm of the Cauchy data. Because of (7.1.2) we get that τ is larger than $\mathrm{t}(M)$ which is independent of $T < T_*$. Therefore, choosing T close enough to T_* this shows that the solution U of (2.8.5) can be continued after T_*, which contradicts the definition of T_*. Hence $T_* \geqslant T_*(\delta)$ and the lemma is proved.

7.2. *The Quasilinear Case. Proof of Theorem* 10.2.4

Fix $\rho > 0$ and a domain Ω^ρ as in (2.9.2). Consider the Cauchy problem (2.9.1) with Cauchy data h_ε which satisfy (2.9.3) and (2.9.4) with $p = +\infty$. We assume that the strong conditions $(\mathscr{C}q)$ and $(\mathscr{T}q)$ hold. Moreover, as in Section 6 we assume that L_0 is diagonal. Recall the Definition (2.10.10) of $T_*(\delta)$ as well as the definition of T_*.

Theorem 2.9.1 provides us with an ε_1, and a time T such (2.9.5) holds on $\Omega(T) := \Omega^\rho \cap \{t \leqslant T\}$; but the proofs of Propositions 6.1.1, 6.1.2, and 6.1.4 show that this time T can be estimated from below by $\inf\{T_0, \mathrm{t}(m, \mu(\varepsilon_0))\}$, where t is a function of

$$m := \|H\|_{L^\infty([y_-, y_+] \times \Psi)} + \|\nabla H\|_{L^\infty([y_-, y_+] \times \Psi)} \tag{7.2.1}$$

and

$$\mu(\varepsilon_0) := \sup_{\varepsilon \leqslant \varepsilon_0} \{\|h_\varepsilon\|_{L^\infty([y_-, y_+])} + \varepsilon \|\nabla h_\varepsilon\|_{L^\infty([y_-, y_+])}\}. \tag{7.2.2}$$

LEMMA 7.2.1. *$T_* \leqslant \lim_{\delta \to 0} T_*(\delta)$ and the approximation* (2.9.5) *is valid in $\Omega(T)$ for all $T < T_*$.*

Proof. It is quite similar to Step 1 in Section 7.1. Fix $T < T_* \leqslant T_0$ and let $M := \|U\|_{L^\infty(\Omega(T) \times \Psi)} + \|\nabla U\|_{L^\infty(\Omega(T) \times \Psi)}$. Let $\mathrm{t} := \mathrm{t}(M, M+1)$.

Because of (2.9.4), there is $\delta_1 > 0$ such that $m(\varepsilon_1) \leqslant M+1$ and therefore Theorem 2.9.1 implies that there is $\varepsilon_1 \in]0, \delta_1]$ such that $T_*(\varepsilon_1) \geqslant \mathrm{t}_1 = \inf\{T_0, \mathrm{t}\}$ and (2.9.5) is valid on $\Omega(\mathrm{t}_1)$. Suppose that $\mathrm{t} \leqslant T$. At time t we have $\|U(\mathrm{t}, \cdot, \cdot)\|_{L^\infty} + \|\nabla U(\mathrm{t}, \cdot, \cdot)\|_{L^\infty} \leqslant M$, by definition of M. Furthermore, because estimate (2.9.5) is valid up to time t, there is $\delta_2 > 0$ such that $\|u_\varepsilon(\mathrm{t}, \cdot)\|_{L^\infty} + \varepsilon \|\nabla u_\varepsilon(\mathrm{t}, \cdot)\|_{L^\infty} \leqslant M+1$ if $\varepsilon \leqslant \delta_2$. So we can now apply Theorem 2.9.1 with Cauchy data on the line $t = \mathrm{t}$ and we can conclude that there is $\varepsilon_2 \in]0, \delta_2]$ such that $T_*(\varepsilon_2) \geqslant \mathrm{t}_2 = \mathrm{Inf}\{T_0, 2\mathrm{t}\}$ and the approximation (2.9.5) is valid on $\Omega_{\rho, \mathrm{t}_2}$. The lemma follows by the same induction as in Section 7.1.

Before proving the analogue of Step 2 in the last section, we need a refined estimate of the life span of solutions of the equations of profiles. It is based on the following remarks. First, the only nonlinear terms involved in (6.1.5) are of the form $U_j(D_i U_l)$. Second, the vector fields D_i nicely commute with the equations as was shown in (6.7.20). Therefore the life span of the solutions to (6.1.5) is governed by the blow-up of DU and not by the blow-up of the full gradient ∇U.

LEMMA 7.2.2. *Consider the Cauchy problem* (6.1.5) *for the profiles U_k. There exists a positive function $\mathrm{t}(\cdot)$ such that the solution U exists on $\Omega(T) \times \Psi$ and belongs to $\mathscr{C}^1(\Omega(T); \Psi)$ for $T = \mathrm{Inf}\{T_0; \mathrm{t}(m)\}$ with*

$$m := \|H\|_{L^\infty([y_-, y_+] \times \Psi)} + \|DH\|_{L^\infty([y_-, y_+] \times \Psi)}. \tag{7.2.3}$$

Proof. Step 1. Consider the iteration scheme (6.1.3), that we write in the form (6.3.37). Let $m_\nu(t) := \|U^\nu\|_{L^\infty(\Omega(t) \times \Psi)} + \|DU^\nu\|_{L^\infty(\Omega(t) \times \Psi)}$. From Lemma 6.3.2, we deduce that

$$\|U^{\nu+1}(t)\|_{L^\infty} \leqslant \|H\|_{L^\infty} + C \int_0^t m_\nu(s)\{1 + \|U^{\nu+1}(s)\|_{L^\infty}\}\, ds, \tag{7.2.4}$$

with a constant C which is an upper bound of the various coefficients which only depend on the given unperturbed solution u_0.

Next, we can differentiate $(6.3.37)_k$ with respect to D_k as we did in Section 6.7, Step 2 of the proof of Proposition 6.7.1. We obtain that $Z = DU^{\nu+1}$ is solution to (6.7.19) with $V = U^\nu$ and $F = \bar{b}U^\nu$ and satisfies the initial conditions $Z_k|_{t=0}=D_k H_k$. Therefore, Lemma 6.3.2 also implies that

$$\|DU^{\nu+1}(t)\|_{L^\infty}\leqslant\|DH\|_{L^\infty}+C\int_0^t m_\nu(s)\{1+\|DU^{\nu+1}(s)\|_{L^\infty}\}\,ds. \tag{7.2.5}$$

Adding (7.2.4) and (7.2.5), we obtain

$$m_{\nu+1}(t)\leqslant m+C\int_0^t m_\nu(s)\{1+m_{\nu+1}(s)\}\,ds \tag{7.2.6}$$

and therefore

$$m_{\nu+1}(t)\leqslant me^{C\tilde{\mu}(t)}+e^{C\tilde{\mu}(t)}-1 \qquad \text{with} \quad \tilde{\mu}(t)=\int_0^t m_\nu(s)\,ds. \tag{7.2.7}$$

By induction on ν, we conclude that for

$$t\leqslant \mathrm{t}(m)=\frac{1}{2C}\,\mathrm{Ln}\left(1+\frac{1}{2m+1}\right)<\mathrm{Ln}\left(1+\frac{1}{m}\right) \tag{7.2.8}$$

one has

$$m_\nu(t)\leqslant\frac{me^{Ct}}{m+1-me^{Ct}}\leqslant 2m. \tag{7.2.9}$$

Step 2. We show that for $\mathrm{t}=\mathrm{t}(m)$, the sequence U^ν is uniformly bounded in $\mathscr{C}^1(\Omega(\mathrm{t})\times\Psi)$. In fact, we already know that $U^\nu\in\mathscr{C}^1$ and if we differentiate (6.3.37) with respect to (x,θ,τ) we see that $Z^{\nu+1}=\partial_{(x,\theta,\tau)}U^{\nu+1}$ satisfies a system of the form

$$X_k Z_k^{\nu+1}+\gamma_k(U^\nu)\,D_k Z_k^{\nu+1}=G_k^{\nu+1}$$

where $G^{\nu+1}$ is the sum of a linear term in Z^ν and of a bilinear term in U^ν and $U^{\nu+1}$ and their derivatives. Now, because of the form (6.3.3) and (6.3.5) of γ_k and $\mathscr{E}'_k$, it turns out as in the proof of Corollary 4.3.4 that this bilinear term only involves expressions of the form

$$Z^\nu\cdot DU^{\nu+1}, \qquad DU^\nu\cdot Z^{\nu+1} \qquad \text{or} \quad Z^\nu\cdot U^{\nu+1}$$

Taking (7.2.9) into account, Lemma 6.3.2 yields the estimate

$$\|Z^{\nu+1}(t)\|_{L^\infty}\leqslant\|\nabla H\|_{L^\infty}+C(1+m)\int_0^t \{\|Z^{\nu+1}(s)\|_{L^\infty}+\|Z^\nu(s)\|_{L^\infty}\}\,ds \tag{7.2.10}$$

which implies that $\|Z^{\nu+1}(t)\|_{L^\infty}$ is uniformly bounded for $t \leqslant \mathrm{t}$. The ∂_t derivatives are estimated from the equation.

Step 3. The moduli of continuity of $\nabla U^{\nu+1}$ are uniformly estimated for $t \leqslant \mathrm{t}$. Indeed this is a consequence of Lemma 6.3.9.

Step 4. We now repeat the end of the proof of Proposition 6.1.2. Step 1 suffices to ensure that $U^\nu \to U$ in $L^\infty(\Omega(\mathrm{t}) \times \Psi)$; Steps 2 and 3 imply that the convergence also holds in C^1 on any compact subset of $\Omega(\mathrm{t}) \times \Psi$ and therefore, using Step 3, that ∇U is uniformly continuous on $\Omega(\mathrm{t}) \times \Psi$. Then by Lemma 4.1.2, $U \in \mathscr{C}^1(\Omega(\mathrm{t}); \Psi)$, U is solution to (6.1.5) and the lemma is proved.

LEMMA 7.2.3. *For all* $\delta > 0$, $T_* \geqslant T_*(\delta)$.

Proof. Assume that for some $\delta > 0$, $T_* < T_*(\delta) \leqslant T_0$. Then by definition, the u_ε are defined and uniformly bounded in C^1_ε, say by M, on $\Omega(T_*)$. For any $t < T_*$, we know from Lemma 7.2.1 that the approximation (2.9.5) holds on $\Omega(t)$. More precisely, we know that $u_\varepsilon(t, x) = U(t, x, \varphi(t, x)/\varepsilon) + o(1)$ and, as stated in Proposition 6.8.1, that $\varepsilon \partial_x u_\varepsilon(t, x) = DU(t, x, \varphi(t, x)/\varepsilon) + o(1)$. Hence Theorem 2.10.5 implies that

$$\forall t < T_*, \qquad \|U(t, \cdot, \cdot)\|_{L^\infty} + \|DU(t, \cdot, \cdot)\|_{L^\infty} \leqslant M. \qquad (7.1.11)$$

Lemma 7.2.2 shows that the Cauchy problem (2.9.6) with Cauchy data $U(T, \cdot, \cdot)$ on the line $t = T < T_*$ has a solution on an interval $[T, T + \mathrm{t}]$ with $\mathrm{t} = \mathrm{t}(M)$ independent of $T < T_*$. Therefore, choosing T close enough to T_* we obtain the solution U of (2.9.6) can be continued after T_*, which contradicts the definition of T_*. Hence $T_* \geqslant T_*(\delta)$ and the lemma is proved. This finishes the proof of Theorem 2.10.4.

8. THE SUM LAW

This section is devoted to the proof of Theorem 2.11.1. The extension to quasilinear systems is discussed in section 8.2.

8.1. *The Semilinear Case*

Let Ω_0 be a domain of the form (2.1.3) and $\Omega = \Omega_0 \cap \{t \leqslant T\}$ a fixed domain on which the solution U to (2.11.1) is defined. For simplicity we assume that the restricted closure property $(r - \mathscr{C})$ holds and that the profiles do not depend on τ. The operator L is supposed diagonal with entries X_k. Recall the definitions of Section 2.11.

The first step in the proof of Theorem 2.11.1 is a modification of Proposition 5.1.4 about the profile equation when the data are assumed C^1_{pp} in θ.

PROPOSITION 8.1.1. *Assume the Cauchy data H_k are of class $\sigma_k^0 \geqslant 1$ in θ_k on $[y_-, y_+]$. Then the solution U to* (2.11.1) *is of class* 1 *in θ at every point of Ω.*

Proof. Start with the Picard iterates (5.1.4), which, as shown in Proposition 5.1.4, converge in $\mathscr{C}^0(\Omega; \Psi)$ to U. Assuming U^ν of class 1 in Ω, we have to show that $U^{\nu+1}$ is of class 1 and that U^ν converges to U in $C^0(\Omega; C^1_{pp}(\Psi))$.

Fix k and differentiate $(5.1.4)_k$ with respect to any constant coefficient vector fields ∂_θ, tangent to Ψ. Such operators obviously commute with X_k and also with E_k as noted in Proposition 4.1.3(iv). Note also that these operators involve differentiations with respect to θ_n variables with $n \neq k$, reflecting the abuse of notation (2.6.4), that is the fact that the isomorphism between Θ_k and Ψ/Ψ_k is not the identity.

We are led to equations of the form

$$\begin{aligned} X_k\, \partial_\theta U_k^{\nu+1} &= E_k b_k'(t, x, U^\nu)\, \partial_\theta U^\nu \\ \partial_\theta U_k^{\nu+1}|_{t=0} &= \partial_\theta H_k. \end{aligned} \qquad (8.1.1)_k$$

It follows that the $\partial_\theta U_k^\nu$ converge to $\partial_\theta U_k$ in $\mathscr{C}^0(\Omega'; \Psi)$, Ω' corresponding to a $T' \leqslant T$. Second, the right hand side in (8.1.1) is linear in $\partial_\theta U$, thus the life span of $\partial_\theta U$ is, by the classical argument of continuation, the same as the life span of U, thus proving Proposition 8.1.1.

The second step in the proof of Theorem 2.11.1 involves an induction on the values of the indices of regularity of U. It relies on the following Proposition 8.1.2 which is more than an improvement of Proposition 8.1.1. The key argument is the sum law lemma of Section 4.2 about averaging operators, which allows us, following [RR] to gain step by step the regularity up to the index σ. Proof of Theorem 2.11.1 is an immediate consequence of the two Propositions 8.1.1, 8.1.2.

For $\sigma = (\sigma_1, ..., \sigma_N)$ and $l \in \mathbb{N}$, the notation $\sigma \wedge l$ is used to denote the N-tuple $(\sigma_1 \wedge l, ..., \sigma_N \wedge l)$. We say that $U(t, x, \theta) = (U_1(t, x, \theta_1), ..., U_N(t, x, \theta_N))$ is of class $\sigma = (\sigma_1, ..., \sigma_N)$ in θ at some point (t, x) if for every $k = 1, ..., N$, U_k is of class σ_k in θ_k at the same point (t, x).

PROPOSITION 8.1.2. *Assume the Cauchy data H_k are of class $\sigma_k^0 \geqslant 1$ in θ_k on $[y_-, y_+]$. Assume also that the solution U is of class $\sigma \wedge \ell$ at every point of Ω, where σ is the N-tuple of indices σ_k defined by* (2.11.3, 4, 4) *or* (2.11.6) *and ℓ is an integer greater than or equal to* 1. *Then the solution U to* (2.11.1) *is of class $\sigma \wedge (\ell + 1)$ in θ at every point of Ω.*

Proof. We have to show that for any k, U_k is of class $\sigma_k \wedge (\ell + 1)$, assuming U to be of class $\sigma \wedge \ell$. Fix $k = 1$ and $\bar{p}$ a point in Ω.

We have to find a neighborhood ω of $\bar{p}$ such that U_1 is, on this neighborhood, a continuous function valued in $C_{pp}^{\sigma_1 \wedge (\ell+1)}(\Theta_1)$. Let $\bar{p}_0 \in [y_-, y_+]$ denote the intersection of the backward 1-characteristic starting from $\bar{p}$ with $\{t=0\}$. Let (ω_n) denote a finite covering of the characteristic $[\bar{p}_0, \bar{p}]$ by open sets in Ω such that for some $q_n \in \omega_n \cap [\bar{p}_0, \bar{p}]$ the assumed regularity of U leads to

$$U_j \in C^0(\omega_n; C_{pp}^{\sigma_j(q_n) \wedge \ell}(\Psi)), \qquad j \neq k. \tag{8.1.2}$$

We suppose $\bar{p}$ is one of the q_n. Next we choose ω to be any open 1-characteristic tube containing $[\bar{p}_0, \bar{p}]$ and contained in $\bigcup \omega_n$. If p is in ω the whole 1-characteristic segment $[p_0, p]$ lies in ω.

We solve the equations for U_1 by writing the iterative scheme which starts from 0 and reads

$$U_1^{\nu+1}(p) = H_1(p_0) + \int_{p_0}^{p} \sum_n \chi_n(q)[E_1 b(U_1^\nu(q), U'(q))]\, dq. \tag{8.1.3}$$

In the above formula, p is a point in ω, χ_n a partition of unity subordinated to ω_n and U' denote the collection of U_j for $j \neq 1$. The key idea is to apply Theorem 4.2.1 to the integrand in (8.1.3) to achieve a gain of one derivative.

LEMMA 8.1.3. *Assume $\sigma_j \geq 1$, $j = 1, \ldots, N$. For every n, we have the estimates*

$$\|\chi_n[E_1 b(U_1^\nu, U')]\|_{\sigma_1(\bar{p}) \wedge (\ell+1)} \leq C(\|U'\|_{\sigma'(q_n) \wedge \ell}) \cdot \|U_1^\nu\|_{\sigma_1(\bar{p}) \wedge (\ell+1)}, \tag{8.1.4}$$

where $\|\cdot\|_s$ denotes the norm in $C^0(\omega; C_{pp}^s(\Psi))$ and σ' the regularity index of the U' variables.

Proof. For every n, we have by Theorem 4.2.1

$$\|\chi_n[E_1 b(U_1^\nu, U')]\|_\mu \leq C(\|U'\|_{\sigma'(q_n) \wedge \ell}) \cdot \|U_1^\nu\|_{\sigma_1(\bar{p}) \wedge (\ell+1)} \tag{8.1.5}$$

with μ defined by

$$\mu = \mathrm{Min}(\sigma_1(\bar{p}) \wedge (\ell+1); \mathrm{Inf}\Big(\sum_{j \in J \setminus \{1\}} \sigma_j(q_n) \wedge \ell; J \in \mathfrak{J}(1)\Big). \tag{8.1.6}$$

The lemma follows from the observation that for any $J \in \mathfrak{J}(1)$,

$$\sum_{j \in J \setminus \{1\}} \sigma_j(q_n) \wedge \ell \geq \sigma_1(q_n) \wedge (\ell+1) \geq \sigma_1(\bar{p}) \wedge (\ell+1). \tag{8.1.7}$$

The second inequality is a direct consequence of (2.11.4): σ_1 does not increase along the 1-characteristics. Let us prove the first one, distinguishing three possibilities.

Case 1. For all $j \in J$, $\sigma_j \leqslant \ell$. Result follows from (2.11.5).

Case 2. For all $j \in J$, $\sigma_j > \ell$. We obtain $\sum_{j \in J \setminus \{1\}} \sigma_j(q_n) \wedge \ell = (|J| - 1)\ell \geqslant 2\ell \geqslant \ell + 1$, since, first, the order of a resonance is at least 3, second, $\ell \geqslant 1$.

Case 3. There exist j_0, j_1 in J such that $1 \leqslant \sigma_{j_0} \leqslant \ell$, $\ell < \sigma_{j_1}$. The sum is still greater than or equal to $\ell + 1$ since $\sigma_j \geqslant 1$ for every j.

Lemma 8.1.3 is applied to the source terms in (8.1.3). Application of the lemma is possible since the assumption $\sigma^0 \geqslant 1$ implies that $\sigma \geqslant 1$. As the constants C which appear in the lemma do not depend on v, standard arguments show the iterates converge in $C^0(\omega; C_{pp}^{\sigma_1(p) \wedge (\ell + 1)}(\Psi))$. Proposition 8.1.2 follows.

8.2. *The Quasilinear Case*

Write (6.1.5) in the form

$$X_k U_k + \gamma_k(U) D_k U_k = E_k \left(\bar{b} \cdot U + \sum_{i, j \neq k} \Gamma_{i,j}^k U_i D_j U_j \right), \tag{8.2.1}$$

where

$$\gamma_k(U) = E_k \left(\sum_i \Gamma_{i,k}^k U_i \right). \tag{8.2.2}$$

Denote by $\mathfrak{C}_k : s \to (s, \xi_k(s; t, x), \theta_k + \mu_k(s; t, x, \theta_k))$ the characteristic curve of $X_k + \gamma_k(U) D_k$ through the point (t, x, θ_k). It was proved in Lemma 6.3.2 that μ_k belongs to $\mathscr{C}^1(\Omega; \Theta_k)$ provided that $U \in \mathscr{C}^1(\Omega; \Theta)$. It is not hard to see that further regularity of U_k in θ_k leads to corresponding regularity of μ_k in θ_k. This follows from differentiable properties of ordinary differential equations, noting that ξ_k does not depend on θ_k, and Proposition 4.1.2. Using this we obtain

PROPOSITION 8.2.1. *Let $F_k \in \mathscr{C}^1(\Omega; \Theta_k)$, $U_j \in \mathscr{C}^1(\Omega; \Theta_k)$, $j = 1, ..., N$. If F_k and U_l are of class $\sigma \geqslant 1$ in θ in Ω, the solution $V \in \mathscr{C}^1(\Omega; \Theta_k)$ to*

$$\begin{gathered} X_k V + \gamma_k(U) D_k V = F_k \\ V|_{t=0} = 0 \end{gathered} \tag{8.2.3}$$

is of class σ in θ_k in Ω.

Proof. From (8.2.2) we deduce that $\gamma_k(U)\in\mathscr{C}^1(\Omega;\Theta_k)$ is of class σ in Ω. The solution V to (8.2.3) reads

$$V(t,x,\theta_k)=\int_0^t F_k(\mathfrak{C}_k(s))\,ds. \tag{8.2.4}$$

Proposition 8.2.1 follows from the chain rule, differentiation under the integral and Proposition 4.1.2.

As in the preceeding section, we first establish regularity of the solution to (8.2.1) from which the induction will proceed.

PROPOSITION 8.2.2. *Assume $H\in\mathscr{C}^1([y_-,y_+]\times\Theta)$ to be of class 2 in θ on $[y_-,y_+]$. Then U, the solution to* (8.2.1) *with H as Cauchy data, is of class 2 in θ in Ω.*

Proof. Consider the Picard iterates defined by

$$X_k U_k^{\nu+1}+\gamma_k(U^\nu)\,D_k U_k^{\nu+1}=E_k\left(\bar{b}\cdot U^\nu+\sum_{i,j\neq k}\Gamma^k_{i,j}U_i^\nu D_j U_j^\nu\right) \tag{8.2.5}$$

with H_k as initial condition. Proposition 8.2.1 shows that U^ν of class 2 implies that $U^{\nu+1}$ is of class 2 if one shows that the right hand side of (8.2.5) is of class 2. But this follows from (4.3.8) as in the proof of Lemma 4.3.5. With arguments as in Section 6.3, one can then show, that the moduli of continuity of the second derivatives in θ of the iterates are bounded, thus proving that their limit U is of class 2.

The last step in the proof of Theorem 2.11.1 for quasilinear systems is analogous to Proposition 8.1.2.

PROPOSITION 8.2.3. *Assume the Cauchy data H_k of class $1+\sigma_k^0\geq 2$ in θ_k on $[y_-,y_+]$. Assume also that the solution U to* (8.2.1) *is of class $1+\sigma\wedge\ell$ in θ at every point of Ω, where σ is the N-tuple of indices σ_k defined by* (2.11.3)–(2.11.5) *or* (2.11.6) *with only third order resonances and ℓ is an integer greater than or equal to* 1. *Then U is of class $1+\sigma\wedge(\ell+1)$ in θ at every point of Ω.*

Proof. We write the solution as the limit of the iterates defined by

$$U_k^{\nu+1}(t,x;\theta_k)=H_k(\mathfrak{C}^\nu(0))+\int_0^t E_k\left(\bar{b}\cdot U^\nu+\sum_{i,j\neq k}\Gamma^k_{i,j}U_i D_j U_j\right)(\mathfrak{C}^\nu(s))\,ds, \tag{8.2.6}$$

where $\mathfrak{C}^\nu$ is the characteristic curve through (t,x,θ_k) of $X_k+\gamma_k(U^\nu)\,D_k$. We then use arguments analogous to those in the proof of Proposition 8.1.2 except that one replaces σ by $1+\sigma$ and just has to add 1 to the inequality (8.1.3) to take into account the fact that the bilinear terms are products $U_i D_j U_j$.

An induction in ℓ yields quasilinear analogue of Theorem 2.11.1 stated at the end of Section 2.11.

ACKNOWLEDGMENTS

We thank Jean Martinet and J.-P. Bourguignon for their bibliographical help with the theory of webs, D. Cerveau who cheerfully contributed to Lemma 4.5.2, and R. Gay, J. Hunter, A. Majda, and R. Rosales for numerous fruitful discussions.

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