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The quantum coordinate ring of the special linear group

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Abstract

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We prove that, even under the multiparameter definition of Artin, Schelter and Tate, the quantum coordinate ring $\mathcal{O}_{\mathrm{q}}(\mathrm{SL}_n(k))$ of the special linear group $\mathrm{SL}_n(k)$ satisfies most of the standard ring-theoretic properties of the classical coordinate ring $\mathcal{O}(\mathrm{SL}_n(k))$.

The results

Fix a field k. Let $\mathcal{O}_{\mathrm{q}} = \mathcal{O}_{\mathrm{q}}(\mathrm{SL}_n(k))$ be the (multiparameter) quantum coordinate ring of the special linear group $\mathrm{SL}_n(k)$ and let $\mathcal{M}_{\mathrm{q}} = \mathcal{O}_{\mathrm{q}}(M_n(k))$ be the corresponding quantum coordinate ring of all $n \times n$ matrices, as defined in [2]. (The definition of these and other concepts used in this introduction are given in the next section.) By definition, $\mathcal{O}_{\mathrm{q}} = \mathcal{M}_{\mathrm{q}}/(\Delta_{\mathrm{q}} - 1)$, where Δ_{q} is a central element in $\mathcal{M}_{\mathrm{q}}$ called the 'quantum determinant'. One would like to assert that the standard properties of the classical coordinate ring $\mathcal{O}(\mathrm{SL}_n(k))$, for example integrality and finite global homological dimension, also hold for $\mathcal{O}_{\mathrm{q}}$. This is particularly true since it is easy to show that these properties do hold for $\mathcal{M}_{\mathrm{q}}$ (this follows from the fact that, as is proved in [2, pp. 890–891], $\mathcal{M}_{\mathrm{q}}$ is an iterated Ore extension of k in the sense of [4, Section 12.2]). However, it is typically hard (and in the abstract

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impossible) to show that such properties pass to factor rings. The main aim of this note is to make the following observation, giving a different method for obtaining $\mathcal{O}_{\mathbf{q}}$ from $\mathcal{M}_{\mathbf{q}}$:

Proposition. *Set* $\mathcal{O}_{\mathbf{q}}(\mathrm{GL}_n(k)) = \mathcal{O}_{\mathbf{q}}(M_n(k))[\Delta_{\mathbf{q}}^{-1}]$ *and let* z *be a central indeterminate. Then*

$$\mathcal{O}_{\mathbf{q}}(\mathrm{SL}_n(k)) \otimes_k k[z, z^{-1}] \cong \mathcal{O}_{\mathbf{q}}(\mathrm{GL}_n(k)) .$$

One should interpret this result as a 'quantum' analogue of the well-known fact that $\mathrm{SL}_n(k) \times k^* \cong \mathrm{GL}_n(k)$. Once stated, this proposition is almost trivial to prove. Its significance, however, is that many desirable properties pass from a ring to a central localization. Thus, for example, the Proposition allows one to prove the following:

Corollary. (i) $\mathcal{O}_{\mathbf{q}}$ *is a domain and a maximal order in its division ring of fractions.*
(ii) $\mathrm{GKdim}(\mathcal{O}_{\mathbf{q}}) = \mathrm{gldim}(\mathcal{O}_{\mathbf{q}}) = n^2 - 1$.
(iii) $\mathcal{O}_{\mathbf{q}}$ *is Auslander regular and CM.*
(iv) $K_0(\mathcal{O}_{\mathbf{q}}) = \mathbb{Z}$.

At least for the standard one-parameter version of $\mathcal{O}_{\mathbf{q}}$, the fact that $\mathcal{O}_{\mathbf{q}}$ is a domain can be proved in several other ways, but all of them seem to require considerably more knowledge about the structure of $\mathcal{O}_{\mathbf{q}}$. For example, for most values of the quantum parameter and for any classical group G, a proof that $\mathcal{O}_q(G)$ is a domain can be obtained by combining [6] and [8, Lemme 9.11], while for $G = \mathrm{SL}_n$ it follows from the appendix to [1]. Also, D. Jordan and the authors have shown independently that $\mathcal{M}_{\mathbf{q}}/(\Delta_{\mathbf{q}})$ is a domain, from which it follows that $\mathcal{O}_{\mathbf{q}}$ is a domain.

The proofs

Given a ring C, write C^* for the set of units of C. Let $\mathbf{q} = \{\lambda, q_{ij}: 1 \le i, j \le n\} \subset k^*$ be fixed, non-zero scalars that satisfy $\lambda \ne -1$ and $q_{ij} = q_{ji}^{-1}$ and $q_{ii} = 1$, for all $1 \le i, j \le n$. Define $\mathcal{O}_{\mathbf{q}}(k^{(n)})$ to be the k-algebra with generators $\{x_i: 1 \le i \le n\}$ and relations $x_j x_i = q_{ji} x_i x_j$, for all $1 \le i, j \le n$. Define $\mathbf{p} = \{\lambda, p_{ij}\}$ by $p_{ij} = \lambda^{-1} q_{ij}$, for all $i > j$, and, as before, $p_{ji} = p_{ij}^{-1}$ and $p_{ii} = 1$. Then $\mathcal{M}_{\mathbf{q}} = \mathcal{O}_{\mathbf{q}}(M_n(k))$ is defined to be the universal bi-algebra having $\mathcal{O}_{\mathbf{q}}(k^{(n)})$ as a left comodule algebra and $\mathcal{O}_{\mathbf{p}}(k^{(n)})$ as a right comodule algebra, in the sense of [10, Section 5]. Thus, $\mathcal{M}_{\mathbf{q}}$ is the k-algebra with generators $\{x_{ij}: 1 \le i, j \le n\}$ and relations defined by [2, equation (8)]. The precise definition of these relations is not important, here, except they are of the following form:

$$x_{ij}x_{lm} = \begin{cases} \alpha_{ijlm}x_{lm}x_{ij} + (\lambda - 1)\lambda^{-1}q_{im}x_{lj}x_{im} \\ \quad \text{if } i > l \text{ and } j > m , \\ \alpha_{ijlm}x_{lm}x_{ij} \\ \quad \text{otherwise} , \end{cases} \tag{1}$$

for some $\alpha_{ijlm} \in k^*$. We remark that the restrictions on the scalars $\mathbf{q}$ and $\mathbf{p}$ given above are precisely what is required for $\mathcal{M}_{\mathbf{q}}$ to have the same Hilbert series as a polynomial ring in n^2 variables (see [2, Theorem 1]). The universal argument of [10, Section 8] shows that there exists a *quantum determinant*

$$\Delta_{\mathbf{q}} = \sum_{\pi \in S_n} \alpha_\pi x_{1,\pi(1)}x_{2,\pi(2)} \cdots x_{n,\pi(n)} , \tag{2}$$

where the $\alpha_\pi \in k^*$ are certain scalars and are defined in [2, equation (15)]. By [2, Theorem 3],

$$\Delta_{\mathbf{q}} \text{ is central} \quad \Leftrightarrow \quad \lambda^j \prod_{m=1}^{n} q_{jm} = \lambda^k \prod_{m=1}^{n} q_{km} \text{ for all } j,k . \tag{3}$$

Thus, $\mathcal{O}_{\mathbf{q}}(\mathrm{SL}_n(k)) = \mathcal{M}_{\mathbf{q}}/(\Delta_{\mathbf{q}} - 1)$ is defined precisely when (3) holds. For k sufficiently large, this gives a $\binom{n-1}{2} + 1$ parameter family of deformations of $\mathcal{O}(\mathrm{SL}_n(k))$ (see [7, Section 14]). If $\mathcal{S} = \{\Delta_{\mathbf{q}}^r \colon r \geq 1\}$, we define $\mathcal{O}_{\mathbf{q}}(\mathrm{GL}_n(k)) = (\mathcal{M}_{\mathbf{q}})_{\mathcal{S}} = \mathcal{M}_{\mathbf{q}}[\Delta_{\mathbf{q}}^{-1}]$.

Finally, if C is a commutative k-algebra, we define $\mathcal{O}_{\mathbf{q}}(C) = \mathcal{O}_{\mathbf{q}}(\mathrm{SL}_n(k)) \otimes_k C$ and $\mathcal{M}_{\mathbf{q}}(C) = \mathcal{O}_{\mathbf{q}}(M_n(k)) \otimes_k C$. The results of this note actually hold if k is taken to be any Noetherian, commutative domain (in which case it is unnecessary to define $\mathcal{O}_{\mathbf{q}}(C)$) but, in order to prove this, one first needs to prove the corresponding generalization of [2].

The standard, one-parameter quantum coordinate ring $\mathcal{O}_q(\mathrm{SL}_n(k))$ of $\mathrm{SL}_n(k)$, as for example defined in [10] or [14], is obtained by taking $\lambda = q^2$ and $q_{ij} = q$ for all $i > j$.

Proof of the Proposition. Let C be a commutative k-algebra and $\mu \in C^*$. Then, as a C-algebra, $\mathcal{M}_{\mathbf{q}}(\mathscr{C}) = \mathcal{O}_{\mathbf{q}}(M_n(k)) \otimes_k C$ is still defined by the relations given in (1). The important point to note about these relations is that they are homogeneous in the set of variables $\{x_{1j} \colon 1 \leq j \leq n\}$. In other words, if a given relation from (1) has r occurrences of elements from the set $\{x_{1j}\}$ occurring in one monomial, then every monomial in that relation has r occurrences of elements from $\{x_{1j}\}$. Thus, there is an C-algebra automorphism σ_μ of $\mathcal{M}_{\mathbf{q}}(C)$ defined by

$$\sigma_\mu(x_{1j}) = \mu^{-1}x_{1j} , \quad \sigma_\mu(x_{ij}) = x_{ij} , \qquad \text{for all } 1 \leq j \leq n,\ 2 \leq i \leq n .$$

By the description of $\Delta_{\mathbf{q}}$ in (2), one sees that $\sigma_\mu(\Delta_{\mathbf{q}}) = \mu^{-1}\Delta_{\mathbf{q}}$. Now assume that $C = k[z, z^{-1}]$, for an indeterminate z. Then:

$$\begin{aligned}&\mathcal{O}_{\mathbf{q}}(\mathrm{SL}_n(k))[z, z^{-1}]\\ &\quad\cong \mathcal{O}_{\mathbf{q}}(\mathrm{SL}_n(k)) \otimes_k k[z, z^{-1}] \cong \mathcal{M}_{\mathbf{q}} \otimes_k k[z, z^{-1}]/(\Delta_{\mathbf{q}} - 1)\\ &\quad\overset{\sigma_z}{\cong} \mathcal{M}_{\mathbf{q}} \otimes_k k[z, z^{-1}]/(\Delta_{\mathbf{q}} - z) \cong \mathcal{M}_{\mathbf{q}}[\Delta_{\mathbf{q}}^{-1}]\,.\end{aligned}$$

Thus, $\mathcal{O}_{\mathbf{q}}(\mathrm{SL}_n(k))[z, z^{-1}] \cong \mathcal{O}_{\mathbf{q}}(\mathrm{GL}_n(k))$. $\square$

Another way of viewing this result is as follows: Under the isomorphism $\mathcal{O}_{\mathbf{q}}[z, z^{-1}] \cong \mathcal{O}_{\mathbf{q}}(\mathrm{GL}_n(k))$, the element z maps to $\Delta_{\mathbf{q}}$, and so, by inverting $\mathscr{C} = k[z]^*$, respectively $\mathscr{D} = k[\Delta_{\mathbf{q}}]^*$, we obtain $\mathcal{O}_{\mathbf{q}}(\mathrm{SL}_n(k(z))) \cong (\mathcal{M}_{\mathbf{q}})_{\mathscr{D}}$.

Let M be a finitely generated module over a Noetherian k-algebra A. Then the Gelfand–Kirillov and homological dimensions of M will be denoted by $\mathrm{GKdim}(M)$, respectively $\mathrm{hd}(M)$. The global homological dimension of A will be written $\mathrm{gldim}(A)$. If the injective dimensions of ${}_AA$ and A_A are finite, then they are equal, by [16, Lemma A], and this integer will be denoted by $\mathrm{injdim}(A)$. If $\mathrm{injdim}(A) < \infty$, then A is called *Auslander–Gorenstein* if A satisfies the following condition: For any integers $0 \le i < j$ and finitely generated (right) A-module M, one has $\mathrm{Ext}^i_A(N, A) = 0$ for all (left) A-submodules N of $\mathrm{Ext}^j_A(M, A)$. If A is an Auslander–Gorenstein ring of finite global dimension, then A is called *Auslander-regular*. Set $j(M) = \min\{j\colon\ \mathrm{Ext}^j(M, A) \neq 0\}$. The ring A is *CM* if $j(M) + \mathrm{GKdim}(M) = \mathrm{Gkdim}(A)$ holds for all finitely generated A-modules M.

Before proving the Corollary, we need the following result that provides some more-or-less well-known facts about these conditions.

Lemma. *Suppose that R is a Noetherian ring that is Auslander-regular and CM. Let $S = R[x; \sigma, \delta]$ be an Ore extension, in the sense of* [4, *Section* 12.2]. *Then*:

(i) *S is Auslander-regular.*

(ii) *Assume that $R = \bigoplus_{i\ge 0} R_i$ is a connected graded k-algebra (thus $R_0 = k$) such that $\sigma(R_i) \subseteq R_i$ for each $i \ge 0$. Then S is CM.*

(iii) *Let f be a central, regular element of R. Then R/fR is Auslander–Gorenstein and CM.*

Proof. (i) This follows from [5, Theorem 4.2].

(ii) Filter S by degree in x and note that the corresponding graded ring $\mathrm{gr}(S)$ is isomorphic to $R[y; \sigma]$. The hypotheses on R ensure that $\mathrm{gr}(S)$ has the structure of a connected graded ring, defined by $\mathrm{gr}(S)_n = \bigoplus_{i+j=n} R_i y^j$. Moreover, y is a normal, homogeneous element in $\mathrm{gr}(S)$ and $\mathrm{gr}(S)/y\mathrm{gr}(S) \cong R$. Hence, by [9, Theorem 3.6], $\mathrm{gr}(S)$ is (graded) CM and, by part (i) $\mathrm{gr}(S)$ is Auslander-regular. If M is a finitely generated S-module, give M a good filtration and consider the associated graded $\mathrm{gr}(S)$-module $\mathrm{gr}(M)$. Then, by [3, Theorem 4.3], $j_S(M) = j_{\mathrm{gr}(S)}(\mathrm{gr}(M))$ while, by [13, Theorem 1.3], $\mathrm{GKdim}_S(M) = \mathrm{GKdim}_{\mathrm{gr}(S)}(\mathrm{gr}(M))$. Thus,

$$j_S(M) = \mathrm{GKdim}(\mathrm{gr}(S)) - \mathrm{GKdim}(\mathrm{gr}(M)) = \mathrm{GKdim}(S) - \mathrm{GKdim}(M)$$

and S is CM.

(iii) To avoid triviality, assume that f is not a unit. Set $\bar{R} = R/fR$. Since $j_R(\bar{R}) = 1$, the CM condition implies that $\mathrm{GKdim}(\bar{R}) = \mathrm{GKdim}(R) - 1$. Let M be a finitely generated $\bar{R}$-module. By the Rees Lemma [15, Theorem 9.37], $\mathrm{Ext}^j_R(M, R) = \mathrm{Ext}^{j-1}_{\bar{R}}(M, \bar{R})$, for each $j \geq 1$. It follows that $\bar{R}$ is Auslander–Gorenstein and CM. □

The extra conditions in part (ii) of the Lemma are necessary since, in general, $\mathrm{GKdim}(M) \neq \mathrm{GKdim}(\mathrm{gr}(M))$. For example, suppose that $R = \mathbb{C}[z, z^{-1}, y]$, where z and y are central indeterminates, and σ is the $\mathbb{C}$-automorphism of R defined by $\sigma(z) = z$ but $\sigma(y) = zy$. Then, let $S = R[x; \sigma, 0]$ and set $M = S/(x-1)S$ and $N = S/xS$ (thus, $N = \mathrm{gr}(M)$ in the notation of the proof of part (ii) of the Lemma). It follows from [13, Proposition 3.4] that $j(M) = j(N) = 1$ but $\mathrm{GKdim}(M) = 3 > 2 = \mathrm{GKdim}(N)$.

Proof of the Corollary. Order the generators x_{ij} of $\mathcal{M}_q$ lexicographically and consider the corresponding chain of rings

$$(*) \qquad k\langle x_{11}\rangle \subset k\langle x_{11}\rangle\langle x_{12}\rangle \subset \cdots \subset \mathcal{M}_q .$$

If $R \subset S = R\langle x\rangle$ is a successive pair of rings from this chain, then [2, pp. 890–891] shows that there exists a k-algebra automorphism τ and a τ-derivation δ of R such that S is isomorphic to the Ore extension $R[x; \tau, \delta]$. Thus $\mathcal{M}_q$ is an iterated Ore extension.

(i) By [12, Theorem 1.2.9], $\mathcal{M}_q$ is a Noetherian domain. Since $\mathcal{O}_q = \mathcal{M}_q/(\Delta_q - 1)$, certainly $\mathcal{O}_q$ is Noetherian. By the Proposition, $\mathcal{O}_q[z, z^{-1}] \cong (\mathcal{M}_q)_{\mathcal{S}}$ is a domain, and hence so is its subring $\mathcal{O}_q$. By [11, Proposition V.2.5], $\mathcal{M}_q$ is a maximal order in its division ring of fractions D; that is, if $\mathcal{M}_q \subseteq T \subseteq D$, for some ring T such that $aTb \subseteq \mathcal{M}_q$ for some non-zero elements $a,b \in \mathcal{M}_q$, then $T = \mathcal{M}_q$. By [11, Proposition IV.2.1], $(\mathcal{M}_q)_{\mathcal{S}}$ is also a maximal order in D. It follows easily from the Proposition that $\mathcal{O}_q$ is a maximal order. This proves part (i) of the corollary.

(ii) and (iii) By [2, Proposition 2 and its proof], $\mathrm{GKdim}(\mathcal{M}_q) = \mathrm{gldim}(\mathcal{M}_q) = n^2$. Thus, by the Proposition and [12, Theorem 7.5.3(iv)],

$$\begin{aligned}\mathrm{gldim}(\mathcal{O}_q) &= \mathrm{gldim}(\mathcal{O}_q[z, z^{-1}]) - 1 = \mathrm{gldim}((\mathcal{M}_q)_{\mathcal{S}}) - 1 \\ &\leq \mathrm{gldim}(\mathcal{M}_q) - 1 = n^2 - 1 .\end{aligned}$$

By (1), $\mathcal{M}_q$ has the structure of a connected graded ring, by giving each x_{ij} degree one. If $R \subset S = R[x; \sigma, \delta]$ are a pair of successive rings in the chain $(*)$, then this induces, on R, the structure of connected graded ring $R = \bigoplus_{i\geq 0} R_i$ and implies that $\sigma(R_j) \subseteq R_j$, for each j. Thus, by part (ii) of the Lemma and induction, $\mathcal{M}_{\mathbf{q}}$ is Auslander-regular and CM. Now regard $\mathcal{O}_{\mathbf{q}}$ as $\mathcal{M}_{\mathbf{q}}/(\Delta_{\mathbf{q}} - 1)$. Then, part (iii) of the Lemma implies that $\mathcal{O}_{\mathbf{q}}$ is Auslander–Gorenstein and CM, with $\mathrm{GKdim}(\mathcal{O}_{\mathbf{q}}) = \mathrm{GKdim}(\mathcal{M}_{\mathbf{q}}) - 1 = n^2 - 1$. Since $\mathrm{gldim}(\mathcal{O}_{\mathbf{q}}) < \infty$, this implies that $\mathcal{O}_{\mathbf{q}}$ is Auslander-regular.

It remains to show that $\mathrm{gldim}(\mathcal{O}_{\mathbf{q}}) \geq n^2 - 1$. Consider the factor ring

$$A = \mathcal{O}_{\mathbf{q}}/(x_{ij}\colon i \neq j,\ x_{ll} - 1\colon l \neq 1)\,.$$

The description of the relations of $\mathcal{O}_{\mathbf{q}}$ in (1) and (2) imply that $A \cong k[x_{11}]/(x_{11} - \gamma)$, for some $\gamma \in k^*$. Therefore, both A and $\mathcal{O}_{\mathbf{q}}$ have a 1-dimensional module S. (This also follows from [2, Theorem 3].) But, by the CM condition, $j(S) = \mathrm{GKdim}(\mathcal{O}_{\mathbf{q}}) - \mathrm{GKdim}(S) = n^2 - 1$. Thus, $\mathrm{gldim}(\mathcal{O}_{\mathbf{q}}) \geq n^2 - 1$. This completes the proof of parts (ii) and (iii) of the Corollary.

(iv) By the proof of (ii), $\mathrm{gldim}(\mathcal{M}_{\mathbf{q}}) < \infty$. Thus, by [12, Corollary 12.3.6 and Theorem 12.6.13], $K_0(\mathcal{M}_{\mathbf{q}}) = \mathbb{Z}$. Therefore, by the Proposition and [12, Proposition 12.1.12], $K_0(\mathcal{O}_{\mathbf{q}}[z, z^{-1}]) = K_0((\mathcal{M}_{\mathbf{q}})_{\mathcal{S}}) = \mathbb{Z}$. By [12, Corollary 12.3.6], this implies that $K_0(\mathcal{O}_{\mathbf{q}}) = \mathbb{Z}$. $\square$

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