Note on combinatorial optimization with max-linear objective functions

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Abstract

We consider combinatorial optimization problems with a feasible solution set $S \subseteq \{0,1\}^n$ specified by a system of linear constraints in 0-1 variables. Additionally, several cost functions $c_1, \ldots, c_p$ are given. The max-linear objective function is defined by $f(x) = \max\{c_1^T x, \ldots, c_p^T x : x \in S\}$ where $c_q = (c_{q1}, \ldots, c_{qn})$ is for $q = 1, \ldots, p$ an integer row vector in $\mathbb{R}^n$.

The problem of minimizing $f(x)$ over $S$ is called the max-linear combinatorial optimization (MLCO) problem.

We will show that MLCO is NP-hard even for the simplest case of $S = \{0,1\}^n$ and $p = 2$, and strongly NP-hard for general $p$. We discuss the relation to multi-criteria optimization and develop some bounds for MLCO.

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1. Introduction

We consider combinatorial optimization problems with a feasible solution set $S \subseteq \{0,1\}^n$ specified by a system of linear constraints in 0-1 variables. Additionally, several cost functions $c_1, \ldots, c_p$ are given. The max-linear objective function is defined by

$$f(x) := \max \{c_1^T x, \ldots, c_p^T x : x \in S\}$$

where $c^q := (c_1^q, \ldots, c_p^q)$ is an integer row vector in $\mathbb{R}^n$, $q = 1, \ldots, p$. \hfill (1.1)

The problem of minimizing $f(x)$ over $S$ is called the max-linear combinatorial optimization (MLCO) problem.

MLCO can always be modeled as an integer program by standard techniques (Nemhauser and Wolsey [15]). In the problem which we study in this paper the set $S$ always has a special structure so that a single linear objective function can be optimized over it efficiently, i.e., in polynomial time. The focus of our investigation will be MLCO problems with $p \geq 2$ over such sets $S$.

MLCO plays a significant role in the assembly of printed circuit boards (see Drezner and Nof [6]). There, $S$ is the set of all incidence vectors of maximum cardinality matchings in a bipartite graph. Other applications include partition problems (Garey and Johnson [8]), multi-processor scheduling problems and certain stochastic optimization problems (Granot and Zang [10]).

A special case of this problem is the ML matroid problem. The NP-completeness of this problem has been proved by Warburton [17] who also analyzes worst-case performances of some Greedy heuristics. Granot [9] introduces Lagrangean duals for the problem.

In Section 2 of this paper we show that even the case $S = \{0,1\}^n$ (the unconstrained MLCO) with $p = 2$ is NP-hard and strongly NP-hard for general $p$. Section 3 deals with the relation of MLCO and discrete multi-criteria optimization problems. Section 4 contains some remarks on branch and bound strategies for MLCO.

2. Complexity results

Elegant methods are available for minimizing convex functions over convex sets (see Fletcher [7], Luenberger [2]). However, this problem becomes hard even for simple discrete sets as the following example taken from Murty and Kabaldi [14] shows.

Let $d_0, d_1, \ldots, d_n$ be given positive integers. Then the subset-sum problem is that of checking whether there exists a Boolean vector $x \in \{0,1\}^n$ such that $d_1 x_1 + \cdots + d_n x_n = d_0$. This problem is well known to be NP-complete (Garey and Johnson [8]). If we define the convex quadratic function $f(x) := (d_1 x_1 + \cdots + d_n x_n - d_0)^2$,
then the subset-sum problem is equivalent to checking whether the optimal value in
\( \min \{ f(x) : x \in \{0, 1\}^n \} \) is 0 or strictly greater than 0.

In this section we show that even the unconstrained MLCO problem defined with
the simplest convex functions—max-linear functions defined by two linear functions
\( c^1 \) and \( c^2 \)—leads to an NP-hard optimization problem.

**Theorem 2.1.** The unconstrained MLCO with respect to two linear functions is
NP-hard.

**Proof.** Consider an instance of the interval subset-sum (ISS) problem: Let \( a_1, \ldots, a_m \)
be positive integral weights, and let \( u \) and \( d \) be positive integers with \( u \leq d \). The ISS
problem asks for a subset \( N \) of \( \{1, \ldots, m\} \) such that the sum of integral weights
indexed by the elements of \( N \) is contained in the interval \([u, d]\). Since the subset-sum
problem is a special case of the ISS problem (with \( u = d \)), ISS is NP-complete. We
reduce ISS to the unconstrained MLCO, such that an instance of ISS has a feasible
solution if and only if the optimum objective value in the corresponding instance
of MLCO is strictly less than 0.

Set \( n := m + 1 \), \( c^1_i := a_i \), \( c^2_i := a_i \), \( i = 1, \ldots, m \), \( c^1_{m+1} := -d - 1 \) and \( c^2_{m+1} := v - 1 \).

Given any \( x \in \{0, 1\}^{m+1} \) the following two cases can occur.

Case 1: \( x_{m+1} = 0 \). Then the objective value of \( x \) in the unconstrained MLCO is
greater than or equal to 0, since \( a_i > 0 \), \( i = 1, \ldots, m \).

Case 2: \( x_{m+1} = 1 \). The objective value of \( x \) in the unconstrained MLCO is less
than 0 if and only if

\[
\sum_{i \in N} a_i < d + 1 \quad \text{and} \quad \sum_{i \in N} (-a_i) < -(v - 1)
\]

where \( N = \{i : 1 \leq i \leq m \text{ and } x_i = 1\} \).

In this case, \( N \) is a feasible solution to the given instance of the interval subset-
sum problem. \( \square \)

**Theorem 2.2.** The unconstrained MLCO problem with \( p \) cost functions \( c_1, \ldots, c_p \) is
strongly NP-hard.

**Proof.** Let \( A \) be a \((0,1)\) matrix with \( m \) rows and \( n \) columns and let \( e \) be the vector
with each of its \( m \) components being equal to 1. The set-partitioning problem (SPP)
is the problem to find some \( x \in \{0, 1\}^n \) such that \( Ax = e \). This problem is strongly
NP-hard (see Garey and Johnson [8]). \( \square \)

In order to reduce SPP to MLCO we denote with \( A_i \) the \( i \)th row of matrix \( A \),
define \( p = 2m \), and introduce \( n + 1 \) variables \( x_1, \ldots, x_n, x_{n+1} \), and \( p \) cost vectors \( c_q \),
each with \( n + 1 \) components, defined by

\[
c_q = \begin{cases} (-A_p, 0) & \text{for } q = 1, \ldots, m, \\ (A_{q-m}, -2) & \text{for } q = m+1, \ldots, 2m. \end{cases}
\]
Then it can easily be verified that SPP has a feasible solution if and only if the optimum objective value in this MLCO is strictly less than 0.

3. Relation to multi-criteria problems

In multi-criteria optimization (MCO) we also consider several cost functions $c^1, \ldots, c^p$. The goal in MCO is to find efficient solutions, i.e., solutions $x \in S$ with the following property.

If $y \in S$ and $c^q y < c^q x$ for all $q = 1, \ldots, p$, then none of these inequalities is strict.

If $x, y \in S$, $c^q y \leq c^q x$ for all $q = 1, \ldots, p$, and at least one of these inequalities is strict, then we call $x$ dominated by $y$.

**Theorem 3.1.** For any instance of MLCO there is an optimum solution which is efficient with respect to the cost functions $c^1, \ldots, c^p$.

**Proof.** Suppose $x$ is an optimum solution to a given MLCO, and let $x$ be dominated by $y \in S$. Then $c^q y \leq c^q x$ for all $q = 1, \ldots, p$ implies

$$\max\{c^1 y, \ldots, c^p y\} < \max\{c^1 x, \ldots, c^p x\}.$$ 

Hence $y$ is also optimal for MLCO. $\square$

As a consequence of Theorem 3.1 we can solve MLCO by only considering the efficient solutions of the corresponding multi-criteria combinatorial optimization problem. Therefore for $p = 2$ a solution of the following with problem parameters $\sigma \in \{\min_{x \in S} c^2 x, \max_{x \in S} c^2 x\}$ will solve MLCO.

minimize $c^1 x$,
subject to $x \in S$ and $c^2 x \leq \sigma$.

If $S$ is the set of bases of a matroid, then the latter problem is a matroidal knapsack problem discussed in Camerini and Vercellis [3] and Camerini et al. [2]. These papers applied to this particular MLCO problem give an alternative approach to the ones taken by Granot [9] or Warburton [17].

4. Branch and bound approach

We first discuss some general bounding strategies.

Since the combinatorial optimization problem under consideration can be solved in polynomial time for a single objective we can efficiently compute
\[ \delta^q := \min \{ c^q x : x \in S \}, \quad q = 1, \ldots, p. \] (4.1)

Let \( x^q \) be the solution in which \( \delta^q \) is attained. Then

\[ L(S) = \max \{ \delta^q : q = 1, \ldots, m \} \] (4.2)

is a lower bound for the MLCO problem. An upper bound is obtained by setting

\[ U(S) := \min \{ f(x^q) : q = 1, \ldots, p \}. \] (4.3)

Let \( y \) be one of the solutions \( x^q \) such that \( \delta^q \) is equal to \( L(S) \) and let \( z \) be one of the solutions \( x^r \) such that \( f(x^r) = U(S) \). Let \( T \) be the union of all variables which are equal to 1 either in \( y \) or \( z \) or both. One of the variables in \( T \) will be selected as branching variable: For each \( t \in T \) let \( S(t) := \{ x \in S : x_t = 0 \} \). Compute \( L(S(t)) \) and \( U(S(t)) \). Then take the \( t \) with the smallest \( U(S(t)) - L(S(t)) \) and \( x_t \) as the branching variable.

The lower bound (4.2) can be improved by using Lagrangean relaxation: An LP-formulation of MLCO is

\[
\begin{align*}
\text{minimize} & \quad z \\
\text{subject to} & \quad z - c^1 x \geq 0, \\
& \quad z - c^2 x \geq 0, \\
& \quad \ldots \\
& \quad z - c^p x \geq 0, \\
& \quad x \in S, \\
& \quad z \text{ unrestricted.}
\end{align*}
\] (4.4)

Let \( \pi_1, \ldots, \pi_p \) be nonnegative Lagrange multipliers associated with the constraints \( z - c^q x \geq 0, \quad q = 1, \ldots, p \) in (4.4). Then a lower bound of MLCO is obtained by maximizing over all \( \pi_1, \ldots, \pi_p \geq 0 \) the function

\[ \min \{ z - \pi_1 (z - c^1 x) - \cdots - \pi_p (z - c^p x) : z \text{ unrestricted, } x \in S \}. \] (4.5)

Since (4.5) can be written as

\[ \min \{(1 - \pi_1 - \cdots - \pi_p) z + (\pi_1 c^1 + \cdots + \pi_p c^p) x : z \text{ unrestricted, } x \in S \}, \] (4.6)

we can restrict ourselves to \( \pi_1, \ldots, \pi_p \) satisfying \( \pi_1 + \cdots + \pi_p = 1 \). Thus we get the following result.

**Theorem 4.1.**

\[ L_1(S) := \max_{\pi_1 + \cdots + \pi_p = 1} \min_{x \in S} (\pi_1 c^1 + \cdots + \pi_p c^p) x \]

is a lower bound for the MLCO. However \( L_1 \) improves the bound of (4.2), i.e., \( L(S) \leq L_1(S) \).

**Proof.** \( L_1(S) \) is a lower bound since it is the optimal objective value of the
Lagrangean dual of LP (4.4). Since $\pi$ with $\pi_q = 1$ for exactly one $q \in \{1, \ldots, p\}$ is feasible the result follows.

If the set $S$ is specified by a unimodular system of linear constraints in $(0,1)$-variables it can be solved through LP techniques. In this case (4.6) is a piecewise linear concave function over the set of all nonnegative $\pi_q$, $q = 1, \ldots, p$, and can be computed efficiently by using techniques of nondifferentiable concave programming (see, for instance, Shapiro [16]). For the case $p = 2$ one can use algorithms for solving parametric combinatorial optimization problems with respect to a single parameter (see Carstensen [4,5], Hamacher and Foulds [11], etc.) or use efficient approximation techniques for its solution (see Burkard et al. [1]).

If a linear description $S = \{x: Ax = b, x_j = 0 \text{ or } x_j = 1, j = 1, \ldots, n\}$ of $S$ is given, it is well known (see, for instance, Murty [13]) that the bound $L_1(S)$ can be further improved by replacing in Theorem 4.1 the set $S$ by $S_{\text{lin}} := \{x: Ax = b, 0 \leq x_j \leq 0, j = 1, \ldots, n\}$. Hence

$$L_2(S) := L_1(S) := \max_{\pi_1 + \cdots + \pi_p = 1} \min_{x \in S_{\text{lin}}} (\pi_1 c^1 + \cdots + \pi_p c^p) x$$

is a lower bound such that the optimal solution $x^*$ of MLCO satisfies

$$L(S) \leq L_1(S) \leq L_2(S) \leq f(x^*) \leq U(S). \quad (4.7)$$

References


