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Note on combinatorial optimization with max-linear objective functions

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Abstract

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We consider combinatorial optimization problems with a feasible solution set $S \subseteq \{0,1\}^n$ specified by a system of linear constraints in 0-1 variables. Additionally, several cost functions $c_1,\ldots,c_p$ are given. The *max-linear objective function* is defined by $f(x):=\max\{c^1x,\ldots,c^px\colon x\in S\}$ where $c^q:=(c_1^q,\ldots,c_n^q)$ is for $q=1,\ldots,p$ an integer row vector in $\mathbb{R}^n$.

The problem of minimizing $f(x)$ over S is called the *max-linear combinatorial optimization* (*MLCO*) *problem*.

We will show that MLCO is NP-hard even for the simplest case of $S=\{0,1\}^n$ and $p=2$, and strongly NP-hard for general p. We discuss the relation to multi-criteria optimization and develop some bounds for MLCO.

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1. Introduction

We consider combinatorial optimization problems with a feasible solution set $S \subseteq \{0,1\}^n$ specified by a system of linear constraints in 0-1 variables. Additionally, several cost functions $c_1, \dots, c_p$ are given. The max-linear objective function is defined by

$$f(\boldsymbol{x}) := \max\{\boldsymbol{c}^1\boldsymbol{x}, \dots, \boldsymbol{c}^p\boldsymbol{x} \colon \boldsymbol{x} \in S\}$$

$$\text{where } \boldsymbol{c}^q := (c_1^q, \dots, c_n^q) \text{ is an integer row vector in } \mathbb{R}^n,\ q = 1, \dots, p. \tag{1.1}$$

The problem of minimizing $f(\boldsymbol{x})$ over S is called the *max-linear combinatorial optimization (MLCO) problem*.

MLCO can always be modeled as an integer program by standard techniques (Nemhauser and Wolsey [15]). In the problem which we study in this paper the set S always has a special structure so that a single linear objective function can be optimized over it efficiently, i.e., in polynomial time. The focus of our investigation will be MLCO problems with $p \geq 2$ over such sets S.

MLCO plays a significant role in the assembly of printed circuit boards (see Drezner and Nof [6]). There, S is the set of all incidence vectors of maximum cardinality matchings in a bipartite graph. Other applications include partition problems (Garey and Johnson [8]), multi-processor scheduling problems and certain stochastic optimization problems (Granot and Zang [10]).

A special case of this problem is the ML matroid problem. The NP-completeness of this problem has been proved by Warburton [17] who also analyzes worst-case performances of some Greedy heuristics. Granot [9] introduces Lagrangean duals for the problem.

In Section 2 of this paper we show that even the case $S = \{0,1\}^n$ (the *unconstrained MLCO*) with $p = 2$ is NP-hard and strongly NP-hard for general p. Section 3 deals with the relation of MLCO and discrete multi-criteria optimization problems. Section 4 contains some remarks on branch and bound strategies for MLCO.

2. Complexity results

Elegant methods are available for minimizing convex functions over convex sets (see Fletcher [7], Luenberger [2]). However, this problem becomes hard even for simple discrete sets as the following example taken from Murty and Kabaldi [14] shows.

Let $d_0, d_1, \dots, d_n$ be given positive integers. Then the subset-sum problem is that of checking whether there exists a Boolean vector $\boldsymbol{x} \in \{0,1\}^n$ such that $d_1x_1 + \cdots + d_nx_n = d_0$. This problem is well known to be NP-complete (Garey and Johnson [8]). If we define the convex quadratic function $f(\boldsymbol{x}) := (d_1x_1 + \cdots + d_nx_n - d_0)^2$, then the subset-sum problem is equivalent to checking whether the optimal value in $\min\{f(x)\colon x \in \{0,1\}^n\}$ is 0 or strictly greater than 0.

In this section we show that even the unconstrained MLCO problem defined with the simplest convex functions—max-linear functions defined by two linear functions c^1 and c^2—leads to an NP-hard optimization problem.

Theorem 2.1. *The unconstrained MLCO with respect to two linear functions is* NP-*hard.*

Proof. Consider an instance of the interval subset-sum (ISS) problem: Let $a_1, \dots, a_m$ be positive integral weights, and let v and d be positive integers with $v \le d$. The ISS problem asks for a subset N of $\{1, \dots, m\}$ such that the sum of integral weights indexed by the elements of N is contained in the interval $[v, d]$. Since the subset-sum problem is a special case of the ISS problem (with $v = d$), ISS is NP-complete. We reduce ISS to the unconstrained MLCO, such that an instance of ISS has a feasible solution if and only if the optimum objective value in the corresponding instance of MLCO is strictly less than 0.

Set $n := m+1$, $c_i^1 := a_i$, $c_i^2 := a_i$, $i = 1, \dots, m$, $c_{m+1}^1 := -d-1$ and $c_{m+1}^2 := v-1$. Given any $x \in \{0,1\}^{(m+1)}$ the following two cases can occur.

Case 1: $x_{m+1} = 0$. Then the objective value of x in the unconstrained MLCO is greater than or equal to 0, since $a_i > 0$, $i = 1, \dots, m$.

Case 2: $x_{m+1} = 1$. The objective value of x in the unconstrained MLCO is less than 0 if and only if

$$\sum_{i \in N} a_i < d+1 \quad \text{and} \quad \sum_{i \in N} (-a_i) < -(v-1)$$

where $N = \{i\colon 1 \le i \le m \text{ and } x_i = 1\}$.

In this case, N is a feasible solution to the given instance of the interval subset-sum problem. □

Theorem 2.2. *The unconstrained MLCO problem with p cost functions $c_1, \dots, c_p$ is strongly* NP-*hard.*

Proof. Let A be a (0,1) matrix with m rows and n columns and let $\boldsymbol{e}$ be the vector with each of its m components being equal to 1. The *set-partitioning problem* (*SPP*) is the problem to find some $x \in \{0,1\}^n$ such that $Ax = \boldsymbol{e}$. This problem is strongly NP-hard (see Garey and Johnson [8]). □

In order to reduce SPP to MLCO we denote with A_i the ith row of matrix A, define $p = 2m$, and introduce $n+1$ variables $x_1, \dots, x_n, x_{n+1}$, and p cost vectors $\boldsymbol{c}_q$, each with $n+1$ components, defined by

$$\boldsymbol{c}_q = \begin{cases} (-A_p, 0) & \text{for } q = 1, \dots, m, \\ (A_{q-m}, -2) & \text{for } q = m+1, \dots, 2m. \end{cases}$$

Then it can easily be verified that SPP has a feasible solution if and only if the optimum objective value in this MLCO is strictly less than 0.

3. Relation to multi-criteria problems

In multi-criteria optimization (MCO) we also consider several cost functions $c^1, \ldots, c^p$. The goal in MCO is to find *efficient solutions*, i.e., solutions $x \in S$ with the following property.

> If $y \in S$ and $c^q y < c^q x$ for all $q = 1, \ldots, p$, then none of these inequalities is strict.

If $x, y \in S$, $c^q y \leq c^q x$ for all $q = 1, \ldots, p$, and at least one of these inequalities is strict, then we call x *dominated* by y.

Theorem 3.1. *For any instance of MLCO there is an optimum solution which is efficient with respect to the cost functions $c^1, \ldots, c^p$.*

Proof. Suppose x is an optimum solution to a given MLCO, and let x be dominated by $y \in S$. Then $c^1 y \leq c^q x$ for all $q = 1, \ldots, p$ implies

$$\max\{c^1 y, \ldots, c^p y\} < \max\{c^1 x, \ldots, c^1 x\}.$$

Hence y is also optimal for MLCO. □

As a consequence of Theorem 3.1 we can solve MLCO by only considering the efficient solutions of the corresponding multi-criteria combinatorial optimization problem. Therefore for $p = 2$ a solution of the following with problem parameters $\sigma \in \{\min_{x \in S} c^2 x, \max_{x \in S} c^2 x\}$ will solve MLCO.

$$\begin{aligned} &\text{minimize} \quad c^1 x, \\ &\text{subject to} \quad x \in S \quad \text{and} \quad c^2 x \leq \sigma. \end{aligned}$$

If S is the set of bases of a matroid, then the latter problem is a matroidal knapsack problem discussed in Camerini and Vercellis [3] and Camerini et al. [2]. These papers applied to this particular MLCO problem give an alternative approach to the ones taken by Granot [9] or Warburton [17].

4. Branch and bound approach

We first discuss some general bounding strategies.

Since the combinatorial optimization problem under consideration can be solved in polynomial time for a single objective we can efficiently compute

$$\delta^q := \min\{c^q x\colon x \in S\}, \quad q=1,\dots,p. \tag{4.1}$$

Let x^q be the solution in which δ^q is attained. Then

$$L(S) = \max\{\delta^q\colon q=1,\dots,m\} \tag{4.2}$$

is a lower bound for the MLCO problem. An upper bound is obtained by setting

$$U(S) := \min\{f(x^q): q=1,\dots,p\}. \tag{4.3}$$

Let y be one of the solutions x^q such that δ^q is equal to $L(S)$ and let z be one of the solutions x^r such that $f(x^r) = U(S)$. Let T be the union of all variables which are equal to 1 either in y or z or both. One of the variables in T will be selected as branching variable: For each $t \in T$ let $S(t) := \{x \in S\colon x_t = 0\}$. Compute $L(S(t))$ and $U(S(t))$. Then take the t with the smallest $U(S(t)) - L(S(t))$ and x_t as the branching variable.

The lower bound (4.2) can be improved by using Lagrangean relaxation: An LP-formulation of MLCO is

$$\begin{array}{ll} \text{minimize} & z \\ \text{subject to} & z - c^1 x \ge 0, \\ & z - c^2 x \ge 0, \\ & \dots \\ & z - c^p x \ge 0, \\ & x \in S, \\ & z \text{ unrestricted.} \end{array} \tag{4.4}$$

Let $\pi_1, \dots, \pi_p$ be nonnegative Lagrange multipliers associated with the constraints $z - c^q x \ge 0$, $q = 1,\dots,p$ in (4.4). Then a lower bound of MLCO is obtained by maximizing over all $\pi_1,\dots,\pi_p \ge 0$ the function

$$\min\{z - \pi_1(z - c^1 x) - \cdots - \pi_p(z - c^p x)\colon z \text{ unrestricted, } x \in S\}. \tag{4.5}$$

Since (4.5) can be written as

$$\min\{(1-\pi_1-\cdots-\pi_p)z + (\pi_1 c^1 + \cdots + \pi_p c^p)x\colon z \text{ unrestricted, } x \in S\}, \tag{4.6}$$

we can restrict ourselves to $\pi_1,\dots,\pi_p$ satisfying $\pi_1 + \cdots + \pi_p = 1$. Thus we get the following result.

Theorem 4.1.

$$L_1(S) := \max_{\pi_1+\cdots+\pi_p=1} \min_{x \in S} (\pi_1 c^1 + \cdots + \pi_p c^p) x$$

is a lower bound for the MLCO. However L_1 improves the bound of (4.2), *i.e.,* $L(S) \le L_1(S)$.

Proof. $L_1(S)$ is a lower bound since it is the optimal objective value of the

Lagrangean dual of LP (4.4). Since π with $\pi_q=1$ for exactly one $q\in\{1,\dots,p\}$ is feasible the result follows. □

If the set S is specified by a unimodular system of linear constraints in (0,1)-variables it can be solved through LP techniques. In this case (4.6) is a piecewise linear concave function over the set of all nonnegative π_q, $q=1,\dots,p$, and can be computed efficiently by using techniques of nondifferentiable concave programming (see, for instance, Shapiro [16]). For the case $p=2$ one can use algorithms for solving parametric combinatorial optimization problems with respect to a single parameter (see Carstensen [4,5], Hamacher and Foulds [11], etc.) or use efficient approximation techniques for its solution (see Burkard et al. [1]).

If a linear description $S=\{x\colon Ax=b,\ x_j=0 \text{ or } x_j=1,\ j=1,\dots,n\}$ of S is given, it is well known (see, for instance, Murty [13]) that the bound $L_1(S)$ can be further improved by replacing in Theorem 4.1 the set S by $S_{\text{lin}}:=\{x\colon Ax=b,\ 0\le x_j\le 0,\ j=1,\dots,n\}$. Hence

$$L_2(S):=L_1(S):=\max_{\pi_1+\cdots+\pi_p=1}\ \min_{x\in S_{\text{lin}}}\ (\pi_1 c^1+\cdots+\pi_p c^p)x$$

is a lower bound such that the optimal solution x^* of MLCO satisfies

$$L(S)\le L_1(S)\le L_2(S)\le f(x^*)\le U(S). \tag{4.7}$$

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