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The effective potential and the renormalisation group

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We discuss renormalisation-group improvement of the effective potential both in general and in the context of O(N) scalar ϕ^4 and the Standard Model. In the latter case we find that absolute stability of the electroweak vacuum implies that $m_{\mathrm{H}} \geqslant 1.95 m_{\mathrm{t}} - 189$ GeV, for $\alpha_3(M_{\mathrm{Z}}) = 0.11$. We point out that the lower bound on m_{H} *decreases* if $\alpha_3(M_{\mathrm{Z}})$ is increased.

1. Introduction

The effective potential $V(\phi)$ plays a crucial role in determining the nature of the vacuum in weakly coupled field theories, as was emphasised in the classic paper of Coleman and Weinberg (CW) [1]. The loopwise perturbation expansion of V is reliable only for a limited range of ϕ; but, as was recognised by CW, it is possible to extend the range of ϕ by exploiting the fact that V satisfies a renormalisation group (RG) equation. It is therefore possible to show that in massless $\lambda\phi^4$, $V(\phi)$ has a local minimum at $\phi = 0$, while massless scalar QED has a local minimum for $\phi \neq 0$.

Let us review briefly how V is calculated in perturbation theory, using the (functionally derived) elegant method of Jackiw [2]. In general one shifts scalar fields: $\phi(x) \to \phi + \phi_q(x)$, where ϕ is x-independent. Then $V(\phi)$ is given by the sum of vacuum graphs with ϕ-dependent propagators and vertices. It is not immediately obvious from this algorithm what the result for the one-loop calculation is; partly for this reason, some authors have preferred to consider graphs with one ϕ_q leg, which, it is easy to show, lead to a determination of $\partial V/\partial\phi$. All this is very familiar; not so well known, perhaps, is the following point. Jackiw's algorithm in conjunction with a specific subtraction scheme (such as MS or $\overline{\mathrm{MS}}$) leads to an expression for $V(\phi)$ such that $V(0)$ is well defined and calculable: and also, of course generally ignored. Our point is that unless $V(0)$ is specifically subtracted (or otherwise dealt with) then $V(\phi)$ fails to satisfy a RG equation of the usual form. This fact was noted, for example, in ref. [3], but has often been overlooked, leading to incorrect "solutions" to the RG equation. This happens because the form of the solution transmogrifies the apparently trivial $V(0)$ term into a ϕ-dependent quantity. We will see how this comes about in sects. 2 and 3 where we discuss various strategies for dealing with $V(0)$, and their consequences. We will also argue that it is in fact simpler to use the RG equation for $\partial V/\partial\phi$, since this leads to an "improved" form of V that removes the necessity of considering "improvement" of $V(0)$.

In subsequent sections we explore various forms for the RG equation for both V and $\partial V/\partial\phi$ for various field theories. We consider in detail scalar ϕ^4 theory, with particular emphasis on the impact of infrared divergences on the domain of validity of the solution. We also consider the standard model, where the behaviour of V at large ϕ is important since it can affect the stability of the electroweak vacuum. Here we improve (in principle) on previous treatments [4,5] by our use of a correct form of the RG solution, and also by use of a correct form of the 2-loop β-function for the Higgs self coupling [6]; but, as is easy to anticipate, the analysis of ref. [5] should not be materially affected. Interestingly, however, we find a dependence on $\alpha_3(M_Z)$ that differs significantly from that given in ref. [4].

2. The renormalisation group equation for V

In what follows we consider the RG equation in renormalisable field theories with a single renormalisation scale μ and couplings λ_i of dimension δ_i. Thus the set λ_i consists of all masses and coupling constants, both dimensionless and dimensionful. In general, V is a function $V(\mu, \lambda_i, \phi^a)$ where ϕ^a represents all the scalar fields. In many cases, however, symmetries may be exploited so that V may be calculated as a function of a single field ϕ. This is the case in the standard model, for example. In more complicated cases (involving supersymmetry, for instance) one frequently chooses to explore a specific direction in ϕ-space. Of course ultimately one must then be able to argue that the absolute minimum of V is indeed in the chosen direction. (This is not always a trivial matter [7].) In any event, we will assume for simplicity that it is sufficient to consider the case of a single ϕ-field only.

It is straightforward to derive the RG equation satisfied by V, but there is one subtlety. If we calculate V according to the procedure outlined in sect. 1, then the result $\hat{V}(\mu, \lambda_i, \phi)$ is such that $\hat{V}(\mu, \lambda_i, 0)$ is a non-trivial function that receives contributions from all orders in perturbation theory. (In fact $\hat{V}(\mu, \lambda_i, 0)$ may well have an imaginary part if $\phi = 0$ is not a local minimum of the tree potential, but let us imagine for the moment that this problem does not arise). Thus we may write

$$\hat{V}(\mu, \lambda_i, \phi) = \hat{V}(\mu, \lambda_i, 0) - \sum_{n=1}^{\infty} \frac{1}{n!} \phi^n \Gamma^{(n)}(p_i = 0), \tag{2.1}$$

where $\Gamma^{(n)}$ represents the 1PI Green function with n ϕ-legs and all external momenta set equal to zero. Then by virtue of the RG equation satisfied by $\Gamma^{(n)}$, we have

$$\mathscr{D}\hat{V} - \gamma\phi \frac{\partial \hat{V}}{\partial \phi} = \mathscr{D}\Omega. \tag{2.2}$$

where we have denoted $\hat{V}(\mu, \lambda_i, \phi)$ by $\hat{V}$ and $\hat{V}(\mu, \lambda_i, 0)$ by Ω. The operator $\mathscr{D}$ is

$$\mathscr{D} = \mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial \lambda_i}. \tag{2.3}$$

Ω is simply a contribution to the vacuum energy on which, outside of gravity, no observable can depend. Accordingly, we can make a ϕ-independent shift in V, i.e. $\hat{V} \to V = \hat{V} + \Omega'(\mu, \lambda_i)$ then by choosing Ω' so that

$$\mathscr{D}\Omega' + \mathscr{D}\Omega = 0, \tag{2.4}$$

we can arrange that

$$\mathscr{D}V - \gamma\phi \frac{\partial V}{\partial \phi} = 0, \tag{2.5}$$

which is the usual RG equation for the effective potential. Thus the RG equation restricts the form of the "cosmological constant" $\Omega + \Omega'$ and leads to observable consequences when we presently consider RG "improvement" of V.

On the assumption that we want a potential that satisfies eq. (2.5), then what is the appropriate choice of Ω'? The obvious choice is of course

$$\text{(i)} \quad \Omega' = -\Omega. \tag{2.6}$$

This was advocated, for example, in ref. [3]. Its defect, however, is that as mentioned above V may have an imaginary part at the origin. A suitable generalisation to the case when the minimum of V lies at non-zero ϕ is given by [8]

$$\text{(ii)} \quad \Omega' = -\hat{V}(\phi)\,|_{\phi=v}, \tag{2.7}$$

where v is the value of ϕ at the minimum. (If V has more than one local minimum then any one will give a well defined V satisfying eq. (2.5)). It is a simple exercise to show that Ω' as given by eq. (2.7) satisfies the equation

$$\mathscr{D}\Omega' = \mathscr{D}\Omega - \left.\frac{\partial \hat{V}}{\partial \phi}\right|_{\phi=v} (\gamma v + \mathscr{D} v), \tag{2.8}$$

so that indeed Ω' satisfies eq. (2.4) since by definition

$$\left.\frac{\partial \hat{V}}{\partial \phi}\right|_{\phi=v} = \left.\frac{\partial V}{\partial \phi}\right|_{\phi=v} = 0. \tag{2.9}$$

Note that this choice of Ω' corresponds to setting the cosmological constant to zero order-by-order in perturbation theory.

A third possibility which is relevant to some recent work of Kastening [9,10] is to choose

$$\text{(iii)} \quad \Omega' = \Omega'(\lambda_i). \tag{2.10}$$

That is, to choose Ω' to be independent of μ. To leading order Ω' is therefore obtained by solving the equation

$$\beta_i \frac{\partial \Omega'}{\partial \lambda_i} = -\mu \frac{\partial \Omega}{\partial \mu} = \frac{1}{32\pi^2} \mathrm{STr}\, M^4 \Big|_{\phi=0}, \tag{2.11}$$

where STr is a spin-weighted trace and M^2 is the mass matrix for the quantum fields as a function of ϕ. In sect. 5 we will construct the solution to eq. (2.11) for the O(N) scalar case and compare the result with ref. [10].

3. Solutions to the renormalisation-group equation

In this section we consider the solution to various forms of the RG equation for V, and show how these solutions can be used to extend the domain of perturbative believability (in ϕ) of the result: or equivalently, sum the leading (and subleading...) logarithms. We suppose that V satisfies the equation

$$\mathscr{D}V - \gamma\phi \frac{\partial V}{\partial \phi} = 0. \tag{3.1}$$

Straightforward application of the method of characteristics leads to the solution

$$V(\mu, \lambda_i, \phi) = V(\mu(t), \lambda_i(t), \phi(t)), \tag{3.2}$$

where

$$\mu(t) = \mu\, e^{t}, \tag{3.3}$$

$$\phi(t) = \phi\xi(t), \tag{3.4}$$

and

$$\xi(t) = \exp\left(- \int_0^t \gamma(\lambda_i(t'))\, dt' \right). \tag{3.5}$$

$\lambda_i(t)$ are the usual running couplings and masses, determined by the equations

$$\frac{d\lambda_i(t)}{dt} = \beta_i(\lambda(t)) \tag{3.6}$$

subject to the boundary conditions $\lambda_i(0) = \lambda_i$. It is sometimes more convenient to use dimensional analysis to recast eq. (3.2) as follows:

$$\bar{\mathscr{D}}V + 4\bar{\gamma}V = 0, \tag{3.7}$$

where

$$\bar{\mathscr{D}} = \mu\frac{\partial}{\partial\mu} + \bar{\beta}_i\frac{\partial}{\partial\lambda_i}, \tag{3.8}$$

and

$$\begin{aligned} \bar{\beta}_i &= (\beta_i + \delta_i\lambda_i\gamma)/(1+\gamma), \\ \bar{\gamma} &= \gamma/(1+\gamma). \end{aligned} \tag{3.9}$$

Here δ_i is the dimension of the coupling λ_i. The solution of eq. (3.7) is

$$V(\mu, \lambda_i, \phi) = \bar{\xi}(t)^4 V\big(\mu(t), \bar{\lambda}_i(t), \phi\big) \tag{3.10}$$

where $\mu(t)$ is as in eq. (3.3). $\bar{\xi}(t)$ and $\bar{\lambda}_i(t)$ are defined as in eq. (3.5) and (3.6) but with $\gamma \to \bar{\gamma}$, $\beta \to \bar{\beta}$ and $\lambda(t) \to \bar{\lambda}(t)$. The absence of a $\partial/\partial\phi$ from eq. (3.7) accounts for the fact that ϕ rather than $\phi(t)$ appears on the right-hand side of eq. (3.10). Either form of the solution may be employed with equivalent results; let us focus for the moment on eq. (3.10). Let us denote $V(\mu(t), \bar{\lambda}_i(t), \Phi)$ as $V(t, \phi)$ for short. Now suppose we wish to calculate $V(\mu, \lambda_i, \phi)$ ($\equiv V(0, \phi)$) for some μ, say, 100 GeV. The key to the usefulness of the RG is that we can choose a value of t such that the perturbation series for $V(t, \phi)$ converges more rapidly (for certain ϕ) than the series for $V(0, \phi)$. Moreover, there is nothing to stop us choosing a different value of t for each value of ϕ. Now the perturbation series for V is characterised at large ϕ by powers of the parameter $\lambda\ \ln(\phi/\mu)$ where λ is some dimensionless coupling. Then clearly perturbation theory is improved if we choose t such that $\mu(t) \sim \phi$, as long as $\lambda(t)$ remains small. The precise domain of applicability of the solution for a given choice of t depends on the details of the theory: in sect. 5 we will consider in detail the case of $O(N)$ ϕ^4 theory.

Meanwhile, however, let us consider the relevance of the above discussion to the issue of the subtraction term $\Omega'(\mu, \lambda_i)$ introduced in sect. 2. The important point we wish to make here is that whichever procedure we use to define Ω', and whether we use the RG solution eq. (3.2) or eq. (3.10), a choice of t dependent on ϕ renders Ω' a function of ϕ and hence no longer a trivial subtraction. This point has been missed in some previous treatments of the RG solution and is implicit in the treatment of Kastening.

It is evident that, with regard to extending the domain of perturbative calculability, one must take into account the behaviour of $\Omega'(\mu(t), \lambda_i(t))$ although, since it depends on ϕ only through t, this is unlikely to pose a problem at large ϕ, for example. But we can, in fact, finesse this issue altogether by beginning with the RG equation for $V' \equiv \partial V/\partial\phi$ instead of the one for $V(\phi)$, the point being that

$$\frac{\partial V}{\partial \phi}(\mu, \lambda_i, \phi) = \frac{\partial \hat{V}}{\partial \phi}(\mu, \lambda_i, \phi), \tag{3.11}$$

so that the Ω' term simply does not arise. The analog to eq. (3.1) is

$$\mathscr{D}V' - \gamma\phi\frac{\partial V'}{\partial \phi} = \gamma V', \tag{3.12}$$

with solution

$$V'(\mu, \lambda_i, \phi) = \xi(t) V'(\mu(t), \lambda_i(t), \phi(t)), \tag{3.13}$$

while the analog to eq. (3.10) is simply

$$V'(\mu, \lambda_i, \phi) = \bar{\xi}(t)^4 V'(\mu(t), \bar{\lambda}_i(t), \phi), \tag{3.14}$$

since V' evidently obeys an RG equation of the same form as eq. (3.7).

4. ϕ^4 theory: the $N = 1$ case

In this section we apply the formalism developed in sects. 2 and 3 to the case of massive $\lambda\phi^4$ theory, defined by the lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\lambda\phi^4. \tag{4.1}$$

$\hat{V}(\phi)$ is given by the loopwise expansion

$$\hat{V}(\phi) = \hat{V}_0 + \hat{V}_1 + \hat{V}_2 + \dots, \tag{4.2}$$

where

$$\hat{V}_0 = \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{24}\lambda\phi^4, \tag{4.3}$$

and

$$\hat{V}_1 = \kappa\tfrac{1}{4}H^2\left(\ln\frac{H}{\mu^2} - \frac{3}{2}\right). \tag{4.4}$$

In eq. (4.4), $H = m^2 + \frac{1}{2}\lambda\phi^2$, $\kappa \equiv (16\pi^2)^{-1}$, and we are using $\overline{\mathrm{MS}}$ as we do throughout. (The result for $\hat{V}_2$ may be found in ref. [11].)

At the one-loop level the relevant RG functions are given by

$$\beta_\lambda^{(1)} = 3\lambda^2\kappa, \qquad \beta_{m^2}^{(1)} = m^2\lambda\kappa, \qquad \gamma^{(1)} = 0. \tag{4.5a,b,c}$$

By virtue of eq. (4.5c) the two forms of the RG solution are identical, and we have

$$V(\mu, \lambda, m^2, \phi) = \Omega'(\mu(t), \lambda(t), m^2(t)) + \tfrac{1}{2}m^2(t)\phi^2 + \tfrac{1}{24}\lambda(t)\phi^4$$
$$+ \frac{\kappa}{4}H^2(t)\left(\ln\frac{H(t)}{\mu^2(t)} - \frac{3}{2}\right) + \dots, \tag{4.6}$$

where $H(t) = m^2(t) + \frac{1}{2}\lambda(t)\phi^2$,

$$\lambda(t) = \lambda(1 - 3\lambda t\kappa)^{-1} \tag{4.7}$$

and

$$m^2(t) = m^2(1 - 3\lambda t\kappa)^{-1/3}. \tag{4.8}$$

The function Ω' depends on the choice made to achieve a V satisfying the RG equation as explained in sect. 2. With choice (iii), i.e. $\Omega'(\mu, \lambda, m^2) = \Omega'(\lambda, m^2)$ it is easy to show using eq. (2.11) that

$$\Omega'(\lambda, m^2) = -\frac{m^4}{2\lambda} + cm^4\lambda^{-2/3}, \tag{4.9}$$

where c is an arbitrary constant. Notice, that when m^2 and λ become t-dependent in accordance with eq. (4.6)–(4.8) the c term above remains t-independent and therefore harmless; so we may set $c = 0$. Note that this choice of Ω' has the curious feature that in the free-field limit ($\lambda \to 0$) it corresponds to an *infinite* vacuum subtraction. We will return later to the consequences of choice (ii) for Ω'; for the time being let us persist with eq. (4.9). With this Ω', in fact, eq. (4.6) essentially reproduces the leading logarithms sum of Kastening (eq. (25) of ref. [10]). The natural choice of t from the point of view of eq. (4.6) is given by the equation

$$\mu^2(t) = \mu^2 e^{2t} = m^2(t) + \tfrac{1}{2}\lambda(t)\phi^2, \tag{4.10}$$

since this evidently removes the $\ln(H/\mu^2)$ terms to all orders. An alternative choice which enables us to make contact with Kastening's work is to choose *

$$\mu^2(t) = \mu^2 e^{2t/\hbar} = m^2 + \tfrac{1}{2}\lambda\phi^2, \tag{4.11}$$

which is a less implicit definition of t inasmuch as now

$$t = \frac{\hbar}{2}\ln\frac{m^2 + \frac{1}{2}\lambda\phi^2}{\mu^2}. \tag{4.12}$$

Now we show how the various leading logarithm (subleading logarithm ...) sums collected in Kastening's functions f_1, f_2 etc. are in fact subsumed in our solution. (We choose now to work with eq. (3.2) rather than eq. (3.10).) We need to expand the solution $V(\mu(t), \lambda(t), m^2(t), \phi(t))$ in powers of $\hbar$ but retaining all orders in t. Thus, from the expression for β_λ incorporating two-loop corrections

$$\frac{d\lambda(t)}{dt} = 3\lambda^2(t)\kappa - \tfrac{17}{3}\hbar\lambda^3(t)\kappa^2 + \dots, \tag{4.13}$$

it is easy to show that

$$\lambda(t) = \lambda(1 - 3\lambda t\kappa)^{-1} + \tfrac{17}{9}\hbar\lambda^2\kappa(1 - 3\lambda t\kappa)^{-2}\ln(1 - 3\lambda t\kappa) + \mathcal{O}(\hbar^2). \tag{4.14}$$

Similarly we can evaluate $m^2(t)$, $\phi(t)$ and $\xi(t)$ through two loops. The relevant two-loop contributions to the RG functions are

$$\beta_\lambda^{(2)} = -\tfrac{17}{3}\hbar^2\lambda^3\kappa^2, \qquad \beta_{m^2}^{(2)} = -\tfrac{5}{6}m^2\hbar^2\lambda^2\kappa^2, \qquad \gamma^{(2)} = \tfrac{1}{12}\hbar^2\lambda^2\kappa^2. \tag{4.15a,b,c}$$

Using these results we get

$$\begin{aligned} m^2(t) &= m^2(1 - 3\lambda t\kappa)^{-1/3} \\ &\quad + \hbar m^2(1 - 3\lambda t\kappa)^{-4/3}\left[\tfrac{17}{27}\kappa\lambda\ln(1 - 3\lambda t\kappa) + \tfrac{19}{17}\lambda^2 t\kappa^2\right] + \mathcal{O}(\hbar^2), \\ \phi(t) &= \phi - \tfrac{1}{12}\hbar\lambda^2 t\kappa\phi(1 - 3\lambda t\kappa)^{-2} + \mathcal{O}(\hbar^2), \\ \xi(t) &= 1 - \tfrac{1}{12}\hbar\lambda^2 t\kappa(1 - 3\lambda t\kappa)^{-2} + \mathcal{O}(\hbar^2). \end{aligned} \tag{4.16}$$

* For the purposes of this discussion we found it convenient to write in the factors of $\hbar$ explicitly.

Using the formulae for $\lambda(t)$, $m^2(t)$, $\phi(t)$ and $\xi(t)$ together with eq. (3.2) or (3.13) one can sum the leading (subleading...) logarithms in $V(\phi)$ or $V'(\phi)$ respectively. The sum of the leading logarithms is given by the $\hbar^0$ term in (3.2)

$$L_1 = \tfrac{1}{2}m^2\phi^2(1-3\lambda t\kappa)^{-1/3} + \tfrac{1}{24}\lambda\phi^4(1-3\lambda t\kappa)^{-1} - \frac{m^4}{2\lambda}(1-3\lambda t\kappa)^{1/3}. \quad (4.17)$$

With t defined as in eq. (4.12) this is identical to the result of ref. [9]. To sum the subleading logarithms one simply takes the $\mathcal{O}(\hbar)$ contribution to (3.2). (Note that we would need to calculate the one-loop contribution to Ω').

We have gone through this exercise to demonstrate how the results of refs. [9,10] may be recovered directly from the solution of the RG equation. The analysis is founded on choice (iii) for Ω', which, as we have already indicated, we find somewhat artificial, particularly with regard to the free-field limit. In addition, in more complicated theories with many couplings the determination of the $\Omega'(\lambda_i)$ satisfying eq. (2.11) becomes onerous. We could choose to adopt choice (ii); it is easy to see, however, that the result will then include terms of the form $H'^2 \ln H'/\mu^2(t)$ where $H' = m^2(t) + \frac{1}{2}\lambda(t)\langle\phi\rangle^2$. Although such terms are not dangerous at large ϕ since they do not grow as ϕ^4, they do lead to an unwieldy form of the solution. With a view to more complicated theories, it appears to us simpler, as we indicated already, to work with $V' = \partial V/\partial\phi$. Then through one loop we have (from either eq. (3.13) or (3.14)) simply

$$V' = m^2(t)\phi + \tfrac{1}{6}\lambda(t)\phi^3 + \frac{\kappa}{2}\lambda(t)\phi H(t)\left(\ln\frac{H(t)}{\mu^2(t)} - 1\right) + \dots \quad (4.18)$$

We now evaluate V' and hence (numerically) V with t defined as in eq. (4.10). (Note that since t depends nontrivially on ϕ, the result for V differs from that obtained from the equivalent RG equation for V itself). For $m^2 > 0$ and sufficiently small λ, the result differs insignificantly from the tree result for $\phi < \mathcal{O}(\mu e^{1/\lambda\kappa})$, which corresponds to the approach of $\lambda(t)$ to the Landau pole. For $m^2 < 0$ there is the fact that for $H = m^2 + \frac{1}{2}\lambda\phi^2 < 0$ the "unimproved" potential develops an imaginary part, and there is no solution for t to eq. (4.10). Discussion of the imaginary part notwithstanding, it is clear that perturbation theory is not to be trusted for $H \to 0$, as follows. If we consider the higher-order graphs constructed from the cubic interaction only, then using dimensional analysis these contribute to $V(\phi)$ terms of the general form $(\lambda\phi)^4\eta^{L-3}$ where

$$\eta = \frac{\kappa\lambda^2\phi^2}{m^2 + \frac{1}{2}\lambda\phi^2} \quad (4.19)$$

and L is the number of loops. Since $\eta \to \infty$ as $H \to 0$ we clearly have perturbative break-down in this region. This sort of infrared problem is characteristic of super-renormalisable interactions and is important, of course, in calculations of V at finite temperature. Note that in the neighbourhood of the tree minimum, $m^2 + \frac{1}{6}\lambda\phi^2 \approx 0$, we have $\eta \sim \lambda$ so perturbative calculability requires merely $\kappa\lambda(t) \ll 1$ as we have already assumed.

Finally let us consider briefly the massless case, $m^2 = 0$. As originally indicated by CW, V then remains well defined and perturbatively calculable for $\phi \to 0$, so that $\phi = 0$ remains a local minimum (and the global one, modulo the fact that as before V can not be calculated in the neighbourhood of the Landau pole).

5. O(N) ϕ^4 theory

Here we generalise sect. 4 to the case of massive O(N) symmetric ϕ^4 theory, defined by the lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\lambda(\phi^2)^2, \tag{5.1}$$

where $\phi^2 = \Sigma_{i=1}^N \phi^i\phi^i$. Including one-loop corrections the effective potential is given by

$$\begin{aligned} V(\phi) &= \Omega' + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{24}\lambda\phi^4 \\ &\quad + \frac{\kappa}{4}H^2\left(\ln\frac{H}{\mu^2} - \frac{3}{2}\right) + \frac{\kappa}{4}(N-1)G^2\left(\ln\frac{G}{\mu^2} - \frac{3}{2}\right), \end{aligned} \tag{5.2}$$

where $G = m^2 + \lambda\phi^2/6$, and we have exploited the O(N) invariance to write V as a function of a single field ϕ. Once again the two-loop corrections may be found in ref. [11].

At the one-loop level the relevant RG functions are

$$\beta_\lambda^{(1)} = \frac{N+8}{3}\lambda^2\kappa, \qquad \beta_{m^2}^{(1)} = \frac{N+2}{3}m^2\lambda\kappa, \qquad \gamma^{(1)} = 0. \tag{5.3a,b,c}$$

As explained in previous sections, we prefer to deal with the RG equation for V' but we note for completeness that if we choose to define V à la Kastening then writing $\Omega' = m^4 f(\lambda)$ we have from eq. (2.11) that

$$\lambda\frac{\mathrm{d}f}{\mathrm{d}\lambda} + 2\frac{N+2}{N+8}f = \frac{3N}{2(N+8)\lambda}, \tag{5.4}$$

with a solution

$$f = \begin{cases} 3N[2(N-4)\lambda]^{-1} + c\lambda^{-2(N+2)/(N+8)}, & \text{if } N \neq 4 \\ (\ln\lambda)/(2\lambda) + c/\lambda, & \text{if } N = 4. \end{cases} \tag{5.5}$$

As in sect. 4 the c-terms in Ω' are in fact t-independent in the RG solution so we may set $c = 0$. It is easy to see that for $N \neq 4$ eq. (5.5) corresponds to eq. (15) of ref. [10] (with $t = 0$).

Returning now to V', we have from eq. (3.13) that

$$V'(\mu, m^2, \lambda, \phi) = m^2(t)\phi + \tfrac{1}{6}\lambda(t)\phi^3 + \frac{\kappa}{2}\lambda\phi H(t)\left(\ln\frac{H(t)}{\mu^2(t)} - 1\right)$$
$$+ \frac{\kappa}{6}(N-1)\lambda\phi G(t)\left(\ln\frac{G(t)}{\mu^2(t)} - 1\right) + \dots \tag{5.6}$$

Evidently there is no choice of t which eliminates the logarithms to all orders: but if our concern is to control the behaviour of V at large ϕ then any choice such that $\mu^2(t) \sim \phi^2$ will do. With (say) $t = \ln(\phi/\mu)$, it is a simple matter to compute V' as defined by eq. (5.6) and hence (numerically) $V(\mu, m^2, \lambda, \phi)$. For $\kappa\lambda \ll 1$, the result differs little from the tree approximation out to $\phi \sim \mu e^{1/(\kappa\lambda)}$ just as in the $N = 1$ case.

As in the $N = 1$ case perturbation theory will break down (for $m^2 < 0$) in the region $H \approx 0$. We now, however, have also to consider whether there are also IR problems at $G \approx 0$: i.e. at the tree minimum. Evidently for $G < 0$, V becomes complex: but how closely can we approach $G = 0$ from above and retain perturbative calculability? In fact there is no problem as $G \to 0$; this is evident explicitly at one and two [11] loops. To extend this result to higher loops, note that we have in general cubic vertices of the type H^3 and HGG but not G^3. Consider some graph consisting of HGG vertices only: if it is singular as $G \to 0$, then it will still be so if we "shrink" every H-propagator by the substitution $1/(k^2 + H) \to 1/H$. But the diagram will then consist of G^4 vertices only, with the effective coupling $\lambda^2\phi^2/H$. Then by dimensional analysis, or simply by noting that G^4 is a renormalisable (not a super-renormalisable) vertex, it is clear that the graph will not be singular as $G \to 0$. The significance of the fact that $\partial^2 V/\partial\phi^2$ is singular at $G = 0$ is not precisely clear to us; at the true minimum, of course (calculated consistently to any order in $\hbar$), the matrix $\partial^2 V/\partial\phi^i\partial\phi^j$ has no singularities and $N - 1$ zeroes corresponding to the would-be Goldstones.

6. The standard model

In this section we consider $V(\phi)$ in the standard model (SM) from the RG point of view, with emphasis on the question of vacuum stability. As in the O(N) scalar case we can exploit gauge invariance to write V as a function of a single field ϕ. We must also choose a gauge; the 't Hooft–Landau gauge is the most convenient. In this gauge the W, Z and γ are transverse, and the associate ghosts are massless and couple only to the gauge fields; the would-be Goldstone bosons $G^{\pm}$, G have a common mass deriving from the scalar potential only. Moreover, the gauge parameter is not renormalised in this gauge so it does not enter the RG equation.

Calculating V through one loop yields

$$V(\phi) = \Omega'(\mu, m^2, h, \lambda, g, g') + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{24}\lambda\phi^4$$
$$+ \kappa\left[\tfrac{1}{4}H^2\left(\ln\frac{H}{\mu^2} - \frac{3}{2}\right) + \tfrac{3}{4}G^2\left(\ln\frac{G}{\mu^2} - \frac{3}{2}\right) + \tfrac{3}{2}W^2\left(\ln\frac{W}{\mu^2} - \frac{5}{6}\right)\right.$$
$$\left. + \tfrac{3}{4}Z^2\left(\ln\frac{Z}{\mu^2} - \frac{5}{6}\right) - 3T^2\left(\ln\frac{T}{\mu^2} - \frac{3}{2}\right)\right] + \dots, \tag{6.1}$$

where

$$H = m^2 + \tfrac{1}{2}\lambda\phi^2, \qquad T = \tfrac{1}{2}h^2\phi^2, \qquad G = m^2 + \tfrac{1}{6}\lambda\phi^2,$$
$$W = \tfrac{1}{4}g^2\phi^2, \qquad Z = \tfrac{1}{4}(g^2 + g'^2)\phi^2.$$

Here h is the top quark Yukawa coupling (we neglect other Yukawa couplings throughout).

The occurrence of the logarithms of H, G, T, W and Z in the perturbation expansion means of course that no choice of t will eliminate the logarithms altogether. As indicated in the O(N) scalar case, however, it is clear that as long as the initial values of the dimensionless couplings are small and they remain small on evolution then as long as we choose $\mu(t) \sim \phi$, our RG solution eq. (3.14), say, will be perturbatively believable for all ϕ.

The essential feature that distinguishes gauge theories in general from the pure scalar cases discussed in the previous two sections is the fact that $\lambda = 0$ is no longer a fixed point in the evolution of the quartic scalar coupling $\lambda(t)$. Evolution of λ with ϕ may therefore drive λ negative and hence cause V to develop a second local minimum * at large ϕ; if this minimum is deeper than the (radiatively corrected) tree minimum then it will result in the destabilisation of the electroweak vacuum. Requiring stability (or at least longevity) of the electroweak vacuum results in an upper limit on m_t (for a given m_H). The existence of this limit and related issues has been explored in a series of papers by Sher et al. [4] (for a clear and comprehensive review see ref. [5]).

Now (as in fact essentially recognised by Sher in ref. [5]) the form of the RG "improved" V used in ref. [4] is not completely satisfactory, inasmuch as it is not in

* If one chooses to identify this "new" minimum with the true electroweak vacuum then it is easy to see that this results in the "Coleman–Weinberg" vacuum with a concomitant experimentally disfavoured prediction for the Higgs mass [5].

general a solution of the RG equation for all values of the Higgs (mass)^2 parameter m^2. In fact, however, because the false minimum, if present, occurs at large t (and hence $\phi \gg M_Z$) this should make little difference. Provided a choice of t is made such that $\mu(t) \sim \phi$ (at large ϕ), contributions to V from the Ω' term and subleading logarithms neglected in ref. [5] are very small for values of m_t and m_H in the range of interest. In fact it is easy to convince oneself that in terms of the solution eq. (3.10), for example, the question of the existence of a false (deep) minimum at some scale is simply the question of whether $\lambda(t)$ goes negative as t increases. Even for very small negative λ, the fact that this happens at $\phi/M_Z \gg 1$ means that the tree term $\lambda\phi^4/24$ drives V well below the electroweak minimum. Thus although we now have available the two-loop corrections to V [6] for the SM, they will have a negligible effect on the outcome. The importance of the evolution of λ to the stability of the vacuum was in fact recognised in ref. [12] and the calculation performed using the one-loop SM beta functions. The main question we resolve in this section is the effect of 2-loop corrections on this calculation. (Previous calculations of this correction are unreliable due to typographical error in the expression for $\beta_\lambda^{(2)}$ given in ref. [13].) In fact we have also calculated the evolution of m^2 through two loops and hence the improved V as a function of ϕ but, as anticipated above, the requirement that the electroweak vacuum remains stable turns out to essentially identical to the requirement that λ remains positive.

We give the SM β-functions through two loops in appendix A. It only remains to discuss boundary conditions. At $\mu = M_Z$ we use input values for g, g', α_3, λ, h, m^2, as follows:

$$\begin{aligned} g &= 0.650, \\ g' &= 0.358 \\ \alpha_3 &= 0.1,\ 0.11,\ 0.12,\ 0.13, \\ \lambda &= \lambda_0, \\ h &= h_0, \\ m^2 &= m_0^2. \end{aligned} \tag{6.2}$$

In order to translate the results into a limit on m_t, m_H we use the tree results

$$\begin{aligned} m_t &= \frac{1}{\sqrt{2}} h_0 v, \\ m_H^2 &= -2m_0^2 = \tfrac{1}{3}\lambda_0 v^2, \end{aligned} \tag{6.3}$$

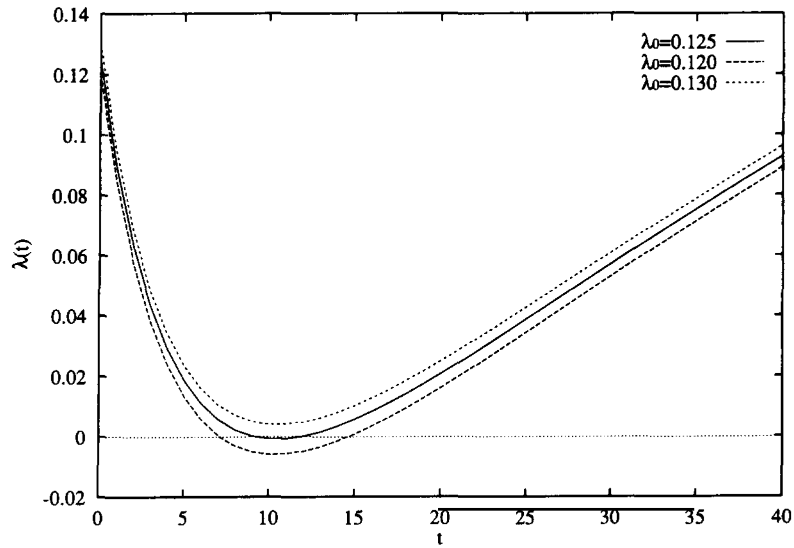

Fig. 1. Plot of the running coupling $\lambda(t)$ for $m_t = 120$ GeV and λ_0 just above, at and just below its critical value (0.125).

where $v^2 = -6m_0^2/\lambda_0$ $(=(246\ \text{GeV})^2)$. (Of course these relationships are themselves subject to radiative corrections which we could include in principle).

Because the $-36h^4$ term in $\beta_\lambda^{(1)}$ tends to drive λ negative, the result of the evolution is a lower limit on λ_0 (and hence m_H) for a given h_0 (and hence m_t). Now the evolution equation for h (see q. (A.1)) includes a contribution from α_3; increasing the input value of α_3 causes h to decrease faster as t increases, and so we would expect the lower bound on m_H to *decrease* with increasing α_3.

In fig. 1 we display the evolution of λ against t for $m_t = 120$ GeV and three values of λ_0. For $\lambda_0 \approx 0.120$, $\lambda(t)$ goes negative but remains small and becomes positive again for $t \sim 15$; but nevertheless because it is negative (albeit small) for $t \sim 10$ this results in a very deep minimum at large ϕ. The value $\lambda_0 = 0.125$ is the critical value, corresponding to $m_H = 50.3$ GeV.

In fig. 2 we display the critical m_H as a function of m_t for $\alpha_3 = 0.11$, as obtained in the one- and two-loop approximations, respectively. We see that the two-loop corrections are not very large; typically they decrease the lower bound on m_H by 2–4 GeV or so.

In fig. 3 we present the critical curve for four input values of α_3. The dependence on α_3 is quite marked, and as anticipated above, the lower bound on m_H decreases as α_3 increased. This conclusion is at variance to that of ref. [4], where the sensitivity to α_3 was indeed noted, but the bound on M_H was found to *increase* as α_3 increases *. We find, for example that for $m_t = 130$ GeV, the bound on m_H is given by 70.1 GeV if $\alpha_3 = 0.1$, but 59.6 GeV if $\alpha_3 = 0.13$.

* See note added in proof.

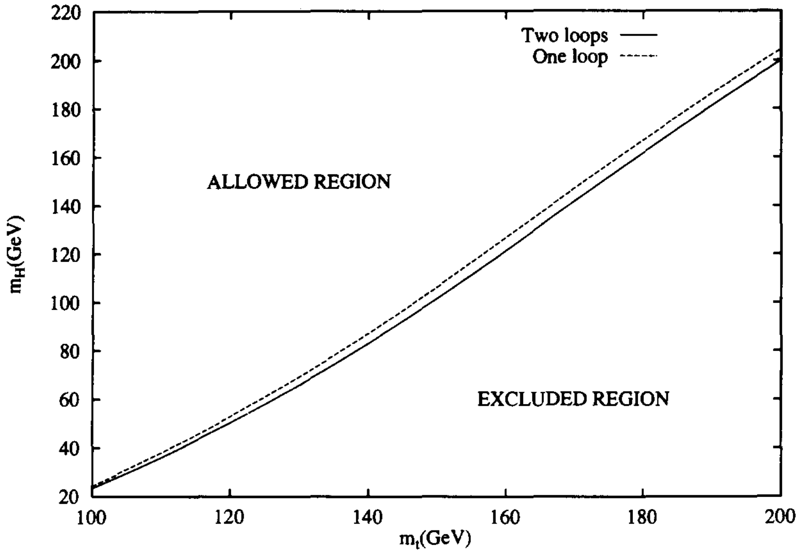

Fig. 2. Plot of the critical value of m_H for vacuum stability against m_t, for $\alpha_3(M_Z) = 0.11$, showing one- and two-loop approximations.

For $m_t \geqslant 140$ GeV the curves are to a very good approximation linear, and stability of the electroweak vacuum corresponds in this region to the relationship (for $\alpha_3 = 0.11$, for example)

$$m_H \geqslant 1.95 m_t - 189 \text{ GeV} \tag{6.4}$$

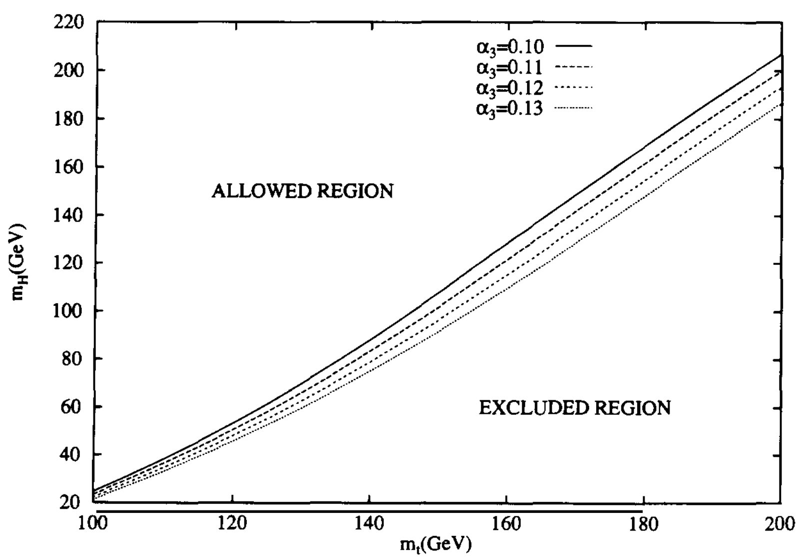

Fig. 3. Plot of the critical value of m_H for vacuum stability against m_t, for $\alpha_3(M_Z) = 0.1, 0.11, 0.12$ and 0.13.

This differs somewhat from the linear approximation given by Sher (ref. [5] p. 331), which corresponds to $m_H \geqslant 1.7 m_t - 160$ GeV. The reason for this discrepancy is that the latter result is based on an extrapolation of the results for low Higgs masses.

Let us consider briefly our results in the light of recent predictions [14] for m_t and m_H based on analysis of LEP data including radiative corrections:

$$m_t = 124^{+26}_{-28} \text{ GeV}, \tag{6.5}$$

and

$$m_H = 25^{+275}_{-19} \text{ GeV}. \tag{6.6}$$

With $m_t = 120$ GeV, for instance, we have from fig. 3 that (again with $\alpha_3 = 0.11$) $m_H \geqslant 50.3$ GeV (52.6 GeV from a one-loop analysis). So with this value of m_t we are already assured of vacuum stability by the direct search limit on m_H, $m_H \geqslant 59$ GeV. For $m_t = 140$ GeV, we have from fig. 3 that $m_H \geqslant 83.2$ GeV. Discovery of the Higgs (with this value of m_t) in the interval 59 GeV $\leqslant m_H \leqslant$ 83 GeV would strongly suggest the existence of physics beyond the standard model, since the obvious means to rescue electroweak stability would be by new physics at a scale heavy enough to have negligible impact on the radiative corrections responsible for the results eqs. (6.5) and (6.6). It is also clear that refinement of the value of $\alpha_3(M_Z)$ would be helpful in reducing the uncertainty in the critical curve.

7. Conclusions

The renormalisation group expresses the simple fact that observables are independent of the renormalisation scale μ. Consequently, an adroit choice of μ leads to improved perturbation theory by removing large logarithms in processes characterised by a single momentum scale *. Application of the RG to the effective potential is quite analogous, except now it is the region of large (or small) ϕ that becomes accessible. In this paper we hope we have elucidated the issues that arise; in particular the relationship between the usual RG approach and the analysis of refs. [9,10]. We have also reconsidered the RG improvement of the SM potential, with a result for the electroweak stability bound on m_H that differs somewhat from the original analysis of Sher et al. [4,5], particularly with regard to the dependence on $\alpha_3(M_Z)$. With the discovery of the top quark generally expected to be imminent, it will be interesting to see whether the direct search limit on m_H leaves a "window of instability", as discussed in sect. 6. Among further applications of the RG to the effective potential, we might consider extension to the supersymmetric

* Processes with several scales may benefit from a multiscale RG approach: see ref. [15].

SM, and also whether the RG improved potential has any bearing on the issue of triviality of non-asymptotically free theories.

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Note added in proof

We thank Marc Sher for confirming that the lower bound on m_{H} indeed decreases as $\alpha_3(M_Z)$ increases. The contrary result of M. Lindner et al. (ref. [4]) was due to a printing error.

Appendix A

We list the RG functions for the SM (see sect. 6 for notation and conventions) through two loops.

The one-loop RG functions are

$$\begin{aligned}
\kappa^{-1}\gamma^{(1)} &= 3h^2 - \tfrac{9}{4}g^2 - \tfrac{3}{4}g'^2, \\
\kappa^{-1}\beta_\lambda^{(1)} &= 4\lambda^2 + 12\lambda h^2 - 36h^4 - 9\lambda g^2 - 3\lambda g'^2 \\
&\quad + \tfrac{9}{4}g'^4 + \tfrac{9}{2}g^2 g'^2 + \tfrac{27}{4}g^4, \\
\kappa^{-1}\beta_h^{(1)} &= \tfrac{9}{2}h^3 - 8g_3^2 h - \tfrac{9}{4}g^2 h - \tfrac{17}{12}g'^2 h, \\
\kappa^{-1}\beta_g^{(1)} &= -\tfrac{19}{6}g^3, \\
\kappa^{-1}\beta_{g'}^{(1)} &= \tfrac{41}{6}g'^3, \\
\kappa^{-1}\beta_{g_3}^{(1)} &= -7g_3^3, \\
\kappa^{-1}\beta_{m^2}^{(1)} &= m^2\left(2\lambda + 6h^2 - \tfrac{9}{2}g^2 - \tfrac{3}{2}g'^2\right).
\end{aligned} \tag{A.1}$$

The two-loop contributions to the RG functions are given by

$$\kappa^{-2}\gamma^{(2)} = \tfrac{1}{6}\lambda^2 - \tfrac{27}{4}h^4 + 20g_3^2h^2 + \tfrac{45}{8}g^2h^2 + \tfrac{85}{24}g'^2h^2 - \tfrac{271}{32}g^4 + \tfrac{9}{16}g^2g'^2 + \tfrac{431}{96}g'^4,$$

$$\kappa^{-2}\beta_\lambda^{(2)} = -\tfrac{26}{3}\lambda^3 - 24\lambda^2h^2 + 6\lambda^2(3g^2 + g'^2) - 3\lambda h^4 + 80\lambda g_3^2h^2 + \tfrac{45}{2}\lambda g^2h^2 + \tfrac{85}{6}\lambda g'^2h^2 - \tfrac{73}{8}\lambda g^4 + \tfrac{39}{4}\lambda g^2g'^2 + \tfrac{629}{24}\lambda g'^4 + 180h^6 - 192h^4g_3^2 - 16h^4g'^2 - \tfrac{27}{2}h^2g^4 + 63h^2g^2g'^2 - \tfrac{57}{2}h^2g'^4 + \tfrac{915}{8}g^6 - \tfrac{289}{8}g^4g'^2 - \tfrac{559}{8}g^2g'^4 - \tfrac{379}{8}g'^6,$$

$$\kappa^{-2}\beta_h^{(2)} = h\Big(-12h^4 + h^2\big(\tfrac{131}{16}g'^2 + \tfrac{225}{16}g^2 + 36g_3^2 - 2\lambda\big) + \tfrac{1187}{216}g'^4 - \tfrac{3}{4}g^2g'^2 + \tfrac{19}{9}g'^2g_3^2 - \tfrac{23}{4}g^4 + 9g^2g_3^2 - 108g_3^4 + \tfrac{1}{6}\lambda^2\Big),$$

$$\kappa^{-2}\beta_g^{(2)} = g^3\big(\tfrac{3}{2}g'^2 + \tfrac{35}{6}g^2 + 12g_3^2 - \tfrac{3}{2}h^2\big),$$

$$\kappa^{-2}\beta_{g'}^{(2)} = g'^3\big(\tfrac{199}{18}g'^2 + \tfrac{9}{2}g^2 + \tfrac{44}{3}g_3^2 - \tfrac{17}{6}h^2\big),$$

$$\kappa^{-2}\beta_{g_3}^{(2)} = g_3^3\big(\tfrac{11}{6}g'^2 + \tfrac{9}{2}g^2 - 26g_3^2 - 2h^2\big),$$

$$\kappa^{-2}\beta_{m^2}^{(2)} = 2m^2\Big(-\tfrac{5}{6}\lambda^2 - 6\lambda h^2 + 2\lambda(3g^2 + g'^2) - \tfrac{27}{4}h^4 + 20g_3^2h^2 + \tfrac{45}{8}g^2h^2 + \tfrac{85}{24}g'^2h^2 - \tfrac{145}{32}g^4 + \tfrac{15}{16}g^2g'^2 + \tfrac{157}{96}g'^4\Big). \qquad (A.2)$$

References

[1] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888
[2] R. Jackiw, Phys. Rev. D9 (1974) 1686
[3] M.B. Einhorn and D.R.T. Jones, Nucl. Phys. B211 (1983) 29
[4] M.J. Duncan, R. Phillipe and M. Sher, Phys. Lett. B153 (1985) 165;
M. Sher and H.W. Zaglauer, Phys. Lett. B206 (1988) 527;
M. Lindner, M. Sher and H.W. Zaglauer, Phys. Lett. B288 (1989) 139;
J. Ellis, A. Linde and M. Sher, Phys. Lett. B252 (1990) 203
[5] M. Sher, Phys. Rep. 179 (1989) 274
[6] C. Ford, I. Jack and D.R.T. Jones, Nucl. Phys. B387 (1992) 373
[7] J.M. Frère, D.R.T. Jones and S. Raby, Nucl. Phys. B222 (1983) 11
[8] M.B. Einhorn and D.R.T. Jones, preprint NSF-ITP-92-105, Nucl. Phys. B. (in press)
[9] B. Kastening, Phys. Lett. B283 (1992) 287
[10] B. Kastening, preprint UCLA/92/TEP/26
[11] C. Ford and D.R.T. Jones, Phys. Lett. B274 (1992) 409 [Erratum: B285 (1992) 399]
[12] N. Cabibbo, L. Maiani, G. Parisi and R. Petronzio, Nucl. Phys. B158 (1979) 295
[13] M.E. Machacek and M.T. Vaughn, Nucl. Phys. B222 (1983) 83
[14] J. Ellis, G.L. Fogli and E. Lisi, Phys. Lett. B292 (1992) 427
[15] M.B. Einhorn and D.R.T. Jones, Nucl. Phys. B230 (1984) 261