Journal of Sound and Vibration (1993) **162**(2), 311–322

DYNAMIC INSTABILITY OF LAYERED ANISOTROPIC CIRCULAR CYLINDRICAL SHELLS, PART I: THEORETICAL DEVELOPMENT

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(*Received* 12 *December* 1989, *and in final form* 25 *November* 1991)

A theoretical development is presented for the parametric resonance of layered anisotropic circular cylindrical shells. The shell's ends are clamped and subjected to axial loading consisting of a static part and a harmonic part. The shell is modelled by using linear shell theory; classical lamination theory is used to determine the stiffness of the overall composite shell structure. The shell's response is divided into a pre-instability (unperturbed) part and an incremental perturbation—which can be dynamically unstable. Rather than assuming the unperturbed state to be a static membrane state of stress, here unperturbed response inertia and spatial variations are retained. A successful solution strategy is developed by employing several Fourier expansions. By means of it, the equations of motion of the perturbed response are reduced to a system of Mathieu equations. The stability of such a system can be determined by known methods. Numerical results are presented in part II.

1. INTRODUCTION

A theoretical development is given for the parametric resonance response of a layered circular cylindrical shell with clamped supports. The layers are comprised of orthotropic material with arbitrary directions of orthotropy; the ordering of the layers is arbitrary. The governing equations and solution methodology are developed here; numerical results are given in part II [1].

Parametric resonance of a structural component refers to an unstable dynamic response brought about by periodic loading. This instability is of concern because it can occur at load magnitudes much less than the static buckling load, so a component designed to withstand static buckling may fail in a periodic loading environment. Also, the instability occurs over a range of forcing frequencies rather than at a single value.

Early work on dynamic stability of structural components can be found in the text by Bolotin [2], and the survey article by Evan-Iwanowski [3].

More recent work on composite plates has been reported in references [4–7]. Theoretical and experimental studies were recently made [8, 9] for non-linear isotropic plates.

The parametric resonance of isotropic circular cylindrical shells has been thoroughly studied. See, for example, references [10] and [11]. For anisotropic shells, the subject has received little treatment, but at the present time interest in the area is increasing and works are beginning to emerge. A fairly early study is described in reference [12]. There, the

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solution is obtained for symmetric and asymmetric cross-ply shells, neglecting axial and circumferential inertia. In reference [13], the parametric resonance of a composite shell due to harmonic pressure loading was studied. In it the shell was taken to be layered in such a way that the resulting structure is at most orthotropic. Recently, the problem (for harmonic axial loading) has been considered in reference [14] for a simply supported, shear deformable specially orthotropic thick shell.

In the present study, the shell is modelled by linear shell theory and classical lamination theory. Analogous to buckling theory, a shell element in the deformed configuration is used to derive the equations of motion. This approach yields a non-linear radial equation of motion. Linearization is achieved via a perturbation technique in which the response is separated into two parts. The basic assumption in this approach is that the shell's response is stable and axisymmetric up to the point of instability. At this point, the shell's response is assumed instantaneously to become unstable. The unstable response can be non-axisymmetric. The response prior to instability is termed the unperturbed response. The non-axisymmetric perturbation is termed the perturbed response. The linearization technique results in a radial equation of motion which has as coefficients of some of its terms the unperturbed response variables.

It has been shown in conjunction with static buckling of circular cylindrical shells (see, for example, reference [15]) that spatial variations of the known unperturbed response are frequently confined to regions near the shell ends. Thus, these deformations are often neglected; the unperturbed response variables are taken to be spatially constant. This was the approach taken in references [12–14] and [16].

The validity of such an assumption for a particular shell is highly dependent on its geometry and anisotropy. For isotropic shells, it is possible to relate the importance of the unperturbed response effects directly to a non-dimensional parameter (see reference [11]). For composites, because of the large number of laminate configurations possible, such precise trends are difficult to obtain. However, a study of the works of Jones and Hennemann [17] and Booton and Tennyson [18] reveals some slight tendencies as to the effects of unperturbed response spatial variations for the case of static buckling of composite shells. For example, the effect tends to be more important for clamped supports than for pinned supports. Also, Jones and Hennemann [17] have shown that for static buckling of pin-supported shells, the effect tends to be more important for isotropic shells than for composite shells. For clamped supported composite shells, the effect is significant, giving static buckling loads lower by as much as 15% than those obtained by neglecting the effect. To the authors' knowledge, the effect of such pre-instability spatial variation on parametric resonance of composite shell structures has not been studied before. Nor have the effects of unperturbed response inertia been studied for composite shells. In the present study, both the foregoing unperturbed response considerations have been included.

Since the shell's layering is assumed to be general, the equations of motion do not admit a solution in the form of single trigonometric functions (such as were used in references [13] and [14]); nor are some of the boundary conditions satisfied by such functions. Here a successful solution strategy is developed by means of Fourier series expansions with respect to the axial co-ordinate, and a complex periodic form with respect to the circumferential co-ordinate. This permits the governing equations to be reduced to a system of Mathieu equations. The dynamic instability regions may be determined from this system by any of a number of methods. Details and numerical results are given in part II.

2. EQUATIONS OF MOTION

The co-ordinate system used is shown in Figure 1. The origin is at the shell's geometric middle surface, the radius of which is denoted by a. The displacement components in the

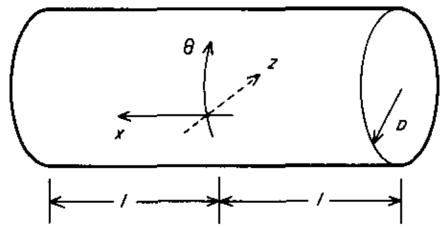

Figure 1. Shell geometry.

x, θ and z directions are denoted by u, v and w, respectively. Axial loading of the form $P^* = (1/2\pi a)(P_0 + P_d \cos \Omega t)$ is applied to the shell's ends.

The shell is comprised of arbitrarily ordered orthotropic layers. The Love–Kirchhoff hypothesis, linear strain displacement relations, and the shallow shell approximation are used to model the shell's kinematics. The stiffness properties of the overall composite shell structure are determined by classical lamination theory (see, for example, reference [19]).

Following standard buckling theory, an element of the shell in the deformed configuration is used to derive the equations of motion in terms of stress and moment resultants (see, for example, reference [15]). The resulting equation in the radial direction is nonlinear and is linearized by a perturbation procedure in which the response is split into two parts. One part, the unperturbed response (denoted by a subscript p), represents a stable axisymmetric motion of the shell. The assumption of axisymmetry in the unperturbed response is taken to be reasonable so long as large bending–stretching coupling does not exist in the composite (which is the case in the numerical examples treated in part II). The second part, the perturbed response (denoted by a subscript 1), represents a perturbation of the unperturbed state and can become unstable. By using this procedure, the equations of motion of the shell's perturbed response can be shown, after some manipulation, to be

$$D^{(1)}u_1 + D^{(2)}v_1 + D^{(3)}w_1 = \bar{m}\, \partial^2 u_1/\partial t^2, \qquad D^{(2)}u_1 + D^{(4)}v_1 + D^{(5)}w_1 = \bar{m}\, \partial^2 v_1/\partial t^2, \quad (1, 2)$$

$$\begin{aligned} D^{(6)}u_1 &+ D^{(7)}v_1 + D^{(8)}w_1 + (\partial N_{x\theta_p}/\partial x)\, \partial w_1/\partial \theta + 2N_{x\theta_p}\, \partial^2 w_1/\partial \theta\, \partial x \\ &+ (\partial w_p/\partial x)(D^{(1)}u_1 + D^{(2)}v_1 + D^{(3)}w_1) + a(\partial N_{xx_p}/\partial x)\, \partial w_1/\partial x + (1/a)N_{\theta\theta_p}\, \partial^2 w_1/\partial \theta^2 \\ &+ aN_{xx_p}\, \partial^2 w_1/\partial x^2 + a(\partial^2 w_p/\partial x^2)(D^{(9)}u_1 + D^{(10)}v_1 + D^{(11)}w_1) \\ &+ \bar{m}E_{43}\, \partial^3 u_1/\partial t^2\, \partial x + \bar{m}E_{44}\, \partial^3 v_1/\partial t^2\, \partial x = \bar{m}\, \partial^2 w_1/\partial t^2. \end{aligned} \quad (3)$$

Here the N's denote the usual stress resultants commonly employed in shell theory, and $\bar{m}$ is the shell's mass per unit length. The differential operators $D^{(1)}$–$D^{(11)}$ are given in Appendix 1, where A_{ij}, B_{ij}, and D_{ij} are the well known material stiffnesses as derived from lamination theory [19]; C_i and E_i represent material and geometric constants and are given in Appendix 2.

Note that equation (3) basically stems from applying Newton's law in the radial direction. However, in arriving at the final form, expressions for $\partial^3 u_1/\partial x^3$ and $\partial^3 v_1/\partial x^3$ were used which were obtained by taking the derivative with respect to x of equations (1) and (2). This leads to a form better suited to the Fourier series manipulations which follow.

The unperturbed response variables appear as coefficients of perturbed response variables in equation (3). Thus, the unperturbed response must be determined before the perturbed response equations of motion can be solved.

3. UNPERTURBED RESPONSE

The equations of motion governing the shell's unperturbed axisymmetric response can be shown to be:

$$aA_{11}\,\partial^2 u_p/\partial x^2 + C_1\,\partial^2 v_p/\partial x^2 - aB_{11}\,\partial^3 w_p/\partial x^3 - A_{12}\,\partial w_p/\partial x = 0, \tag{4}$$

$$C_2\,\partial^2 u_p/\partial x^2 + C_3\,\partial^2 v_p/\partial x^2 - C_4\,\partial^3 w_p/\partial x^3 - C_5\,\partial w_p/\partial x = 0, \tag{5}$$

$$E_{33}\,\partial^4 w_p/\partial x^4 + E_{34}\,\partial^2 w_p/\partial x^2 + A_{12}\,\partial u_p/\partial x + C_5\,\partial v_p/\partial x - C_8 w_p = \bar{m}\,\partial^2 w_p/\partial t^2. \tag{6}$$

For a clamped shell, one possible set of boundary conditions to be satisfied by the unperturbed response at $x = \pm l$ are

$$w_p = 0, \qquad \partial w_p/\partial x = 0, \qquad v_p = 0, \qquad N_{xx_p} = -P^*. \tag{7–10}$$

After some manipulation and application of boundary condition (10), equations (4), (5) and (6) can be shown to give

$$F_1\,\partial^4 w_p/\partial x^4 + F_2\,\partial^2 w_p/\partial x^2 + F_3 w_p + F_4 = \bar{m}\,\partial^2 w_p/\partial t^2, \tag{11}$$

where F_1, F_2 and F_3 are constants given in Appendix 2, and

$$F_4 = -F_{12}P^* + F_{13}K_2. \tag{12}$$

F_{12} and F_{13} are constants given in Appendix 2 and K_2 is an undetermined function of time which arises from integration of equation (5).

Whitney and Sun [20] in a work on static buckling of composite shells used a Fourier series method to satisfy spatial dependence. This is also the approach used in a later section to reduce the perturbed response equations of motion to a system of Mathieu equations. This is done by expanding the perturbed response variables in terms of Fourier series in the axial co-ordinate, leading to an infinite system of equations in terms of trigonometric functions and the associated Fourier coefficients. For the equations to be satisfied, coefficients of like trigonometric functions are separately equated.

The terms in equation (3) which involve products of unperturbed response quantities with perturbed response quantities present a problem when the unperturbed response is a function of space—as it is here. Products of the unperturbed response solution functions with the trigonometric functions of the perturbed response would prevent the grouping of coefficients of like terms as described above.

This problem can be circumvented if the *products* can be expressed in terms of Fourier series. This is accomplished by first expressing the unperturbed response solution in terms of a Fourier series, as opposed to using the exact analytic solution (which exists in this case). The products in equation (3) are then each a product of Fourier series. These can each be expressed in terms of one Fourier series by means of a theorem for the product of two infinite series.

The details of determining the product of Fourier series will be described later. Here the solution of the unperturbed response in terms of Fourier series is given.

The solution of (11) is taken in the form

$$w_p = \frac{f_0}{2} + \sum_{n=1}^{\infty}\left(f_n \cos\frac{n\pi x}{l} + g_n \sin\frac{n\pi x}{l}\right), \tag{13}$$

where f_0, f_n and g_n are time dependent.

As discussed by Green [21] (see also reference [20]), the derivative of the Fourier series of a function defined on a given interval can be determined term by term only if the periodic extension of the function is continuous. If this is not the case, then the derivative of the Fourier series depends on the values of the function at the endpoints of the interval (see reference [22]). In the present case, the derivatives of equation (13) are given by

$$\frac{\partial w_p}{\partial x}=\sum_{n=1}^{\infty}\left(\frac{n\pi}{l}g_n\cos\frac{n\pi x}{l}-\frac{n\pi}{l}f_n\sin\frac{n\pi x}{l}\right), \tag{14}$$

$$\frac{\partial^2 w_p}{\partial x^2}=\sum_{n=1}^{\infty}\left[-\left(\frac{n\pi}{l}\right)^2 f_n\cos\frac{n\pi x}{l}-\left(\frac{n\pi}{l}\right)^2 g_n\sin\frac{n\pi x}{l}\right], \tag{15}$$

$$\frac{\partial^3 w_p}{\partial x^3}=\tfrac{1}{2}c_w^{(3)}+\sum_{n=1}^{\infty}\left\{\left[-\left(\frac{n\pi}{l}\right)^3 g_n+(-1)^n c_w^{(3)}\right]\cos\frac{n\pi x}{l}+\left(\frac{n\pi}{l}\right)^3 f_n\sin\frac{n\pi x}{l}\right\}, \tag{16}$$

$$\frac{\partial^4 w_p}{\partial x^4}=\tfrac{1}{2}c_w^{(4)}+\sum_{n=1}^{\infty}\left\{\left[\left(\frac{n\pi}{l}\right)^4 f_n+(-1)^n c_w^{(4)}\right]\cos\frac{n\pi x}{l}\right.$$
$$\left.+\left[\left(\frac{n\pi}{l}\right)^4 g_n-\frac{n\pi}{l}(-1)^n c_w^{(3)}\right]\sin\frac{n\pi x}{l}\right\}, \tag{17}$$

where

$$c_w^{(3)}=(1/l)(w_1''(l)-w_1''(-l)),\qquad c_w^{(4)}=(1/l)(w_1'''(l)-w_1'''(-l)), \tag{18, 19}$$

where the prime symbol (′) denotes partial differentiation with respect to x.

Note that since equation (11) involves only even derivatives, g_n and $c_w^{(3)}$ are zero in subsequent steps. Substitution into equation (11) gives

$$\tfrac{1}{2}F_1c_w^{(4)}+\tfrac{1}{2}F_3f_0+F_4=\tfrac{1}{2}\bar{m}\ddot{f}_0 \tag{20}$$

$$F_1(-1)^n c_w^{(4)}+[F_1(n\pi/l)^4-F_2(n\pi/l)^2+F_3]f_n=\bar{m}\ddot{f}_n,\qquad n=1,2,3,\ldots, \tag{21}$$

where an overdot denotes time differentiation.

Since $g_n=0$, the boundary conditions (8) are identically satisfied (see equation (14)); substitution into the boundary conditions (7) gives

$$\tfrac{1}{2}f_0+\sum_{n=1}^{\infty}f_n(-1)^n=0. \tag{22}$$

To satisfy boundary conditions (9) v_p is first expressed in terms of w_p by using equations (4) and (5). This gives

$$v_p=\frac{\alpha_7}{\alpha_5}w_p'+\frac{\alpha_6}{\alpha_5}\int w_p\,\mathrm{d}x+\frac{C_1}{A_{11}\alpha_5}P^*x+\frac{K_2}{\alpha_5}x+K_3, \tag{23}$$

where K_3 is an unknown function of time. Substitution of equations (13) and (14) into equation (23), followed by application of the two boundary conditions (9), can be shown, after some manipulation, to give

$$K_3=0,\qquad K_2=-(C_1/A_{11})P^*-(\alpha_6/2)f_0.$$

Then equation (20) can be expressed in the form

$$\tfrac{1}{2}F_1c_w^{(4)}+\tfrac{1}{2}(F_3-F_{13}\alpha_6)f_0-(F_{12}+F_{13}C_1/A_{11})P^*=\tfrac{1}{2}\bar{m}\ddot{f}_0. \tag{24}$$

Equations (21), (22) and (24) form a system of $2n+1$ ordinary differential equations for the unperturbed response quantities f_0, $c_w^{(4)}$ and f_n. Upon elimination of f_0 and $c_w^{(4)}$ from the system, the steady state solution is readily determined to have the form

$$f_i = A_i P_0 + B_i P_d \cos \Omega t, \tag{25}$$

where A_i and B_i are constants. Substitution into the n governing equations leads to the following two systems of algebraic equations for A_i and B_i:

$$\alpha(n)A_n - 4F_1^I(-1)^n \sum_{m=1}^{\infty} A_m(-1)^m = 2F_2^I P_0(-1)^n, \qquad n = 1, 2, 3, \ldots, \tag{26}$$

$$B_n(\alpha(n) - \bar{m}\Omega^2) - (2\bar{m}\Omega^2 + 4F_1^I)(-1)^n \sum_{m=1}^{\infty} B_m(-1)^m = 2F_2^I P_d(-1)^n, \qquad n = 1, 2, 3, \ldots. \tag{27}$$

Here F_1^I and F_2^I are constants given in Appendix 2, and

$$\alpha(n) = F_2(n\pi/l)^2 - F_3 - F_1(n\pi/l)^4.$$

Equations (26) and (27) can be solved numerically for A_n and B_n. It should be noted that equations (26) and (27) depend on the forcing frequency, Ω. Upon solution, for a given Ω, the unperturbed response is described by equation (13) with $g_n = 0$ and f_n given by equation (25), and then f_0 is given by equation (22). In addition, the unperturbed response stress resultants can be shown to be

$$N_{xx_p} = -P^*, \qquad N_{x\theta_p} = F_{19}\,\partial^2 w_p/\partial x^2 + F_{20}w_p - F_3^I P^* + F_4^I K_2, \tag{28, 29}$$

$$N_{\theta\theta_p} = F_{22}\,\partial^2 w_p/\partial x^2 + F_{23}w_p - F_5^I P^* + F_6^I K_2, \tag{30}$$

where F_3^I, F_4^I, F_5^I, F_6^I, F_{19}, F_{20}, F_{22} and F_{23} are constants given in Appendix 2.

4. PERTURBED RESPONSE

The equations of motion describing the perturbed response of the shell will now be reduced to a system of Mathieu equations by means of Fourier series expansions of the perturbed response variables. A technique similar to the one used here has been used to solve static stability problems by Green [21] for isotropic plates and, as mentioned before, by Whitney and Sun [20] for composite shells. Very recently, the method has been described in reference [23] in the context of static response of anisotropic doubly curved panels.

The solution of equations (1), (2), and (3) is assumed in the form:

$$\begin{pmatrix} u_1(x,\theta,t) \\ v_1(x,\theta,t) \\ w_1(x,\theta,t) \end{pmatrix} = \left\{ \frac{1}{2}\begin{pmatrix} A_0(t) \\ C_0(t) \\ f_0(t) \end{pmatrix} + \sum_{n=1}^{\infty} \left\{ \begin{pmatrix} A_n(t) \\ C_n(t) \\ f_n(t) \end{pmatrix} \cos\frac{n\pi x}{l} + \begin{pmatrix} B_n(t) \\ D_n(t) \\ g_n(t) \end{pmatrix} \sin\frac{n\pi x}{l} \right\} \right\} e^{ik\theta}, \tag{31}$$

where k is the circumferential wavenumber and i is the imaginary unit. The time parameters A_0, A_n, B_n, C_0, etc., are, in general, complex. This choice of solution form is motivated by the following factors. Since the shell is circumferentially closed, the displacement functions must be periodic with respect to θ. Also, the generality of the shell's anisotropy does not allow the equations of motion to be satisfied by any solution involving only single trigonometric functions in each displacement component.

As described in the previous section, the required derivatives with respect to x are determined according to the theorem given in reference [22] and are given below:

$$\begin{pmatrix} u_1'(x,\theta,t) \\ v_1'(x,\theta,t) \\ w_1'(x,\theta,t) \end{pmatrix} = e^{ik\theta}\left\{\frac{1}{2}\begin{pmatrix} c_u^{(1)} \\ c_v^{(1)} \\ c_w^{(1)} \end{pmatrix} + \sum_{n=1}^{\infty}\left\{\left[\frac{n\pi}{l}\begin{pmatrix} B_n(t) \\ D_n(t) \\ g_n(t) \end{pmatrix} + (-1)^n \begin{pmatrix} c_u^{(1)} \\ c_v^{(1)} \\ c_w^{(1)} \end{pmatrix}\right]\cos\frac{n\pi x}{l} - \frac{n\pi}{l}\begin{pmatrix} A_n(t) \\ C_n(t) \\ f_n(t) \end{pmatrix}\sin\frac{n\pi x}{l}\right\}\right\}, \quad (32)$$

$$\begin{pmatrix} u_1''(x,\theta,t) \\ v_1''(x,\theta,t) \\ w_1''(x,\theta,t) \end{pmatrix} = e^{ik\theta}\left\{\frac{1}{2}\begin{pmatrix} c_u^{(2)} \\ c_v^{(2)} \\ c_w^{(2)} \end{pmatrix} + \sum_{n=1}^{\infty}\left\{\left[-\left(\frac{n\pi}{l}\right)^2\begin{pmatrix} A_n(t) \\ C_n(t) \\ f_n(t) \end{pmatrix} + (-1)^n \begin{pmatrix} c_u^{(2)} \\ c_v^{(2)} \\ c_w^{(2)} \end{pmatrix}\right]\cos\frac{n\pi x}{l} - \frac{n\pi}{l}\left[\frac{n\pi}{l}\begin{pmatrix} B_n(t) \\ D_n(t) \\ g_n(t) \end{pmatrix} + (-1)^n\begin{pmatrix} c_u^{(1)} \\ c_v^{(1)} \\ c_w^{(1)} \end{pmatrix}\right]\sin\frac{n\pi x}{l}\right\}\right\}, \quad (33)$$

$$w_1'''(x,\theta,t) = e^{ik\theta}\left\{\tfrac{1}{2}c_w^{(3)} + \sum_{n=1}^{\infty}\left\{\left[-\left(\frac{n\pi}{l}\right)^2\left(\frac{n\pi}{l}g_n + (-1)^n c_w^{(1)}\right) + (-1)^n c_w^{(3)}\right]\cos\frac{n\pi x}{l} - \frac{n\pi}{l}\left[-\left(\frac{n\pi}{l}\right)^2 f_n + (-1)^n c_w^{(2)}\right]\sin\frac{n\pi x}{l}\right\}\right\}, \quad (34)$$

$$w_1''''(x,\theta,t) = e^{ik\theta}\left\{\tfrac{1}{2}c_w^{(4)} + \sum_{n=1}^{\infty}\left\{\left[-\left(\frac{n\pi}{l}\right)^2\left(-\left(\frac{n\pi}{l}\right)^2 f_n + (-1)^n c_w^{(2)}\right) + (-1)^n c_w^{(4)}\right]\cos\frac{n\pi x}{l} - \frac{n\pi}{l}\left[-\left(\frac{n\pi}{l}\right)^2\left(\frac{n\pi}{l}g_n + (-1)^n c_w^{(1)}\right) + (-1)^n c_w^{(3)}\right]\sin\frac{n\pi x}{l}\right\}\right\}. \quad (35)$$

Here

$$\begin{pmatrix} c_u^{(1)} \\ c_v^{(1)} \\ c_w^{(1)} \end{pmatrix} = \frac{1}{l}\begin{pmatrix} u_1(l) - u_1(-l) \\ v_1(l) - v_1(-l) \\ w_1(l) - w_1(-l) \end{pmatrix}, \qquad \begin{pmatrix} c_u^{(2)} \\ c_v^{(2)} \\ c_w^{(2)} \end{pmatrix} = \frac{1}{l}\begin{pmatrix} u_1'(l) - u_1'(-l) \\ v_1'(l) - v_1'(-l) \\ w_1'(l) - w_1'(-l) \end{pmatrix}, \quad (36, 37)$$

$$c_w^{(3)} = (1/l)(w_1''(l) - w_1''(-l)), \qquad c_w^{(4)} = (1/l)(w_1'''(l) - w_1'''(-l)). \quad (38, 39)$$

Note that the derivatives of the Fourier series of the functions require the endpoint values of the functions. These values are either related to the boundary conditions in a specified way or are treated as unknowns. Since the solution is assumed in complex form, each of the boundary constants, $c_u^{(1)}$, $c_v^{(1)}$, etc., are, in general, complex.

The boundary conditions to be satisfied by the perturbed response at $x = \pm l$ are

$$w_1 = 0, \qquad \partial w_1/\partial x = 0, \qquad u_1 = 0, \qquad v_1 = 0. \quad (40\text{–}43)$$

Applying condition (40) gives $c_w^{(1)} = 0$ and

$$\tfrac{1}{2}f_0 + \sum_{n=1}^{\infty} f_n(-1)^n = 0. \quad (44)$$

Likewise, applying condition (41) gives $c_w^{(2)}=0$ and

$$\sum_{n=1}^{\infty} \frac{n\pi}{l} g_n(-1)^n=0. \tag{45}$$

Applying conditions (42) and (43) gives $c_u^{(1)}=0$ and $c_v^{(1)}=0$,

$$\tfrac{1}{2}A_0+\sum_{n=1}^{\infty} A_n(-1)^n=0 \qquad \text{and} \qquad \tfrac{1}{2}C_0+\sum_{n=1}^{\infty} C_n(-1)^n=0. \tag{46, 47}$$

Since the time parameters A_0, A_n, etc., are in general complex, each of the conditions (44)–(47) has a real and an imaginary part. Thus, these conditions represent eight equations.

The coefficients in equation (3) are functions of x in the form of Fourier series, and t. Thus, when one writes w_p in the form of equation (13) products of Fourier series arise. Each product can be re-expressed as follows.

In general, the product of two Fourier series,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

and

$$F(x)=\frac{A_0}{2}+\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l}+B_n\sin\frac{n\pi x}{l}\right)$$

can be expressed as a single Fourier series

$$f(x)F(x)=\frac{\alpha_0'}{2}+\sum_{n=1}^{\infty}\left(\alpha_n'\cos\frac{n\pi x}{l}+\beta_n'\sin\frac{n\pi x}{l}\right), \tag{48}$$

where (see reference [22])

$$\alpha_0'=\frac{a_0A_0}{2}+\sum_{n=1}^{\infty}(a_nA_n+b_nB_n), \tag{49}$$

$$\alpha_n'=\frac{a_0A_n}{2}+\tfrac{1}{2}\sum_{m=1}^{\infty}[a_m(A_{m+n}+A_{m-n})+b_m(B_{m+n}+B_{m-n})], \tag{50}$$

$$\beta_n'=\frac{a_0B_n}{2}+\tfrac{1}{2}\sum_{m=1}^{\infty}[a_m(B_{m+n}-B_{m-n})-b_m(A_{m+n}-A_{m-n})]. \tag{51}$$

In equations (50) and (51) the following two relations apply:

$$A_{-k}=A_k, \qquad B_{-k}=-B_k. \tag{52, 53}$$

The assumed solution form (31), and appropriate derivatives are substituted into equations (1), (2) and (3). Upon analytical re-expression of the products as described, this results in three equations in terms of the Fourier time parameters, each having a real and an imaginary part. Separating out the real and imaginary parts of each equation results in six equations. In each of these six equations, coefficients of sines, coefficients of cosines, and constant (in x) terms are equated separately, giving a total of $12n+6$ equations. These manipulations, which involve many terms and so can become lengthy, are performed by using a REDUCE symbolic manipulator program.

The resulting infinite sequence of equations is of the form

$$\underline{m}\, \mathrm{d}^2\bar{f}/\mathrm{d}t^2 + (\underline{R} - P_0\underline{S}_1 - P_d\underline{S}_2 \cos \Omega t)\bar{f} = 0, \tag{54}$$

where $\underline{m}$ is a mass matrix, $\underline{R}$, $\underline{S}_1$ and $\underline{S}_2$ are matrices containing material and geometric constants, and $\bar{f}$ is the vector of unknowns. Included in equations (54) are the eight equations (44)–(47) (real and imaginary parts) reflecting the boundary conditions. Thus there are $12n + 14$ equations for the $12n + 14$ unknowns.

By using this system, the static buckling loads and free vibration natural frequencies of the statically loaded shell can be calculated through the solution of eigenvalue problems. In addition, the instability regions of the shell can be determined by any of the methods previously described. Specific details of the numerical procedures, results describing the shell's stability, and a study of the various unperturbed response effects are given in part II [1].

5. CONCLUSIONS

A theoretical development has been given for the parametric resonance response of a layered anisotropic circular cylindrical shell having clamped supports. Pre-instability inertia and spatial variations have been retained in the formulation leading to a system of equations of motion having coefficients dependent on space and time. Solution methodology involving Fourier series expansions in the axial co-ordinate and a complex periodic form in the circumferential co-ordinate has been used to reduce these equations to a system of Mathieu equations.

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APPENDIX 1

Differential operators used in the text are as follows:

$$D^{(1)}=aA_{11}\frac{\partial^2}{\partial x^2}+\frac{A_{44}}{a}\frac{\partial^2}{\partial\theta^2}+2A_{14}\frac{\partial^2}{\partial\theta\,\partial x},\qquad D^{(2)}=C_1\frac{\partial^2}{\partial x^2}+\frac{C_5}{a}\frac{\partial^2}{\partial\theta^2}+E_1\frac{\partial^2}{\partial\theta\,\partial x},$$

$$D^{(3)}=-aB_{11}\frac{\partial^3}{\partial x^3}-A_{12}\frac{\partial}{\partial x}-\frac{B_{24}}{a^2}\frac{\partial^3}{\partial\theta^3}-\frac{A_{24}}{a}\frac{\partial}{\partial\theta}+E_2\frac{\partial^3}{\partial\theta^2\,\partial x}-3B_{14}\frac{\partial^3}{\partial\theta\,\partial x^2},$$

$$D^{(4)}=C_3\,\partial^2/\partial x^2+E_6\,\partial^2/\partial\theta^2+E_3\,\partial^2/\partial\theta\,\partial x,$$

$$D^{(5)}=-C_4\,\partial^3/\partial x^3-C_5\,\partial/\partial x+E_8\,\partial^3/\partial\theta^3+E_7\,\partial/\partial\theta+E_5\,\partial^3/\partial\theta\,\partial x^2+E_4\,\partial^3/\partial\theta^2\,\partial x,$$

$$D^{(6)}=E_{39}\,\partial^3/\partial\theta\,\partial x^2+E_{35}\,\partial^3/\partial\theta^2\,\partial x+A_{12}\,\partial/\partial x+(B_{24}/a^2)\,\partial^3/\partial\theta^3+(A_{24}/a)\,\partial/\partial\theta,$$

$$D^{(7)}=E_{40}\,\partial^3/\partial\theta\,\partial x^2+E_{36}\,\partial^3/\partial\theta^2\,\partial x+C_5\,\partial/\partial x+E_8\,\partial^3/\partial\theta^3-E_7\,\partial/\partial\theta,$$

$$D^{(8)}=E_{33}\,\partial^4/\partial x^4+E_{34}\,\partial^2/\partial x^2+E_{37}\,\partial^4/\partial\theta^3\,\partial x+E_{38}\,\partial^2/\partial\theta\,\partial x+E_{41}\,\partial^4/\partial\theta^2\,\partial x^2$$
$$+E_{42}\,\partial^4/\partial\theta\,\partial x^3-C_8-(D_{22}/a^3)\,\partial^4/\partial\theta^4-2(B_{22}/a^2)\,\partial^2/\partial\theta^2,$$

$$D^{(9)}=A_{11}\frac{\partial}{\partial x}+\frac{A_{14}}{a}\frac{\partial}{\partial\theta},\qquad D^{(10)}=\left(\frac{A_{12}}{a}-\frac{B_{12}}{a^2}\right)\frac{\partial}{\partial\theta}+\frac{C_1}{a}\frac{\partial}{\partial x},$$

$$D^{(11)}=-(A_{12}/a)-B_{11}\,\partial^2/\partial x^2-(B_{12}/a^2)\,\partial^2/\partial\theta^2-2(B_{14}/a)\,\partial^2/\partial\theta\,\partial x.$$

APPENDIX 2

Constants used in the text are as follows:

$$C_1=(aA_{14}-B_{14}),\qquad C_3=(aA_{44}-2B_{44}+D_{44}/a),\qquad C_4=(aB_{14}-D_{14}),$$

$$C_5=(A_{24}-B_{24}/a),\qquad C_8=A_{22}/a,\qquad E_1=A_{12}-(B_{12}/a)+A_{44}-(B_{44}/a),$$

$$E_2=-(B_{12}/a)-(2B_{44}/a),\qquad E_3=2A_{24}+(2D_{24}/a^2)-(4B_{24}/a),$$

$$E_4=\frac{-3B_{24}}{a}+\frac{3D_{24}}{a^2},\qquad E_5=-2B_{44}-B_{12}+\frac{2D_{44}}{a}+\frac{D_{12}}{a},\qquad E_6=\frac{A_{22}}{a}-\frac{2B_{22}}{a^2}+\frac{D_{22}}{a^3},$$

$$E_7=\frac{-A_{22}}{a}+\frac{B_{22}}{a^2},\qquad E_8=\frac{-B_{22}}{a^2}+\frac{D_{22}}{a^3},\qquad E_9=\frac{-2D_{12}}{a}-\frac{4D_{44}}{a},\qquad E_{10}=\frac{3B_{24}}{a}-\frac{3D_{24}}{a^2},$$

$$E_{11}=C_3-\frac{C_1^2}{aA_{11}},\qquad E_{12}=\frac{C_1B_{11}}{A_{11}}-C_4,\qquad E_{13}=\frac{C_1A_{12}}{aA_{11}}-C_5,\qquad E_{14}=\frac{C_5}{a}-\frac{C_1A_{44}}{a^2A_{11}},$$

$$E_{15}=E_6-\frac{C_1C_5}{a^2A_{11}},\qquad E_{16}=E_8+\frac{C_1B_{24}}{a^3A_{11}},\qquad E_{17}=E_7+\frac{C_1A_{24}}{a^2A_{11}},$$

$$E_{18}=E_1-\frac{2C_1A_{14}}{aA_{11}},\qquad E_{19}=E_3-\frac{C_1E_1}{aA_{11}},\qquad E_{20}=E_4-\frac{C_1E_2}{aA_{11}},$$

$$E_{21}=E_5+\frac{3C_1B_{14}}{aA_{11}},\qquad E_{22}=aB_{11}+C_1\frac{E_{12}}{E_{11}},\qquad E_{23}=A_{12}+C_1\frac{E_{13}}{E_{11}},$$

$$E_{24}=-\frac{A_{44}}{a}+C_1\frac{E_{14}}{E_{11}},\qquad E_{25}=-\frac{C_5}{a}+C_1\frac{E_{15}}{E_{11}},\qquad E_{26}=\frac{B_{24}}{a^2}+C_1\frac{E_{16}}{E_{11}},$$

$$E_{27}=\frac{A_{24}}{a}+C_1\frac{E_{17}}{E_{11}},\qquad E_{28}=-2A_{14}+C_1\frac{E_{18}}{E_{11}},\qquad E_{29}=-E_1+C_1\frac{E_{19}}{E_{11}},$$

$$E_{30}=-E_2+C_1\frac{E_{20}}{E_{11}},\qquad E_{31}=3B_{14}+C_1\frac{E_{21}}{E_{11}},\qquad E_{32}=\frac{C_1^2}{aA_{11}E_{11}}+1,$$

$$E_{33}=-aD_{11}+\frac{B_{11}E_{22}}{A_{11}}-\frac{C_4E_{12}}{E_{11}},\qquad E_{34}=\frac{B_{11}E_{23}}{A_{11}}-\frac{C_4E_{13}}{E_{11}}-2B_{12},$$

$$E_{35}=\frac{B_{11}E_{24}}{A_{11}}-\frac{C_4E_{14}}{E_{11}}-E_2,\qquad E_{36}=\frac{B_{11}E_{25}}{A_{11}}-\frac{C_4E_{15}}{E_{11}}+E_{10},$$

$$E_{37}=\frac{B_{11}E_{26}}{A_{11}}-\frac{C_4E_{16}}{E_{11}}-\frac{4D_{24}}{a^2},\qquad E_{38}=\frac{B_{11}E_{27}}{A_{11}}-\frac{C_4E_{17}}{E_{11}}-\frac{4B_{24}}{a},$$

$$E_{39}=\frac{B_{11}E_{28}}{A_{11}}-\frac{C_4E_{18}}{E_{11}}+3B_{14},\qquad E_{40}=\frac{B_{11}E_{29}}{A_{11}}-\frac{C_4E_{19}}{E_{11}}-E_5,\qquad E_{41}=\frac{B_{11}E_{30}}{A_{11}}-\frac{C_4E_{20}}{E_{11}}+E_9,$$

$$E_{42}=\frac{B_{11}E_{31}}{A_{11}}-\frac{C_4E_{21}}{E_{11}}-4D_{14},\qquad E_{43}=\frac{B_{11}E_{32}}{A_{11}}-\frac{C_4C_1}{aE_{11}A_{11}},\qquad E_{44}=\frac{C_4}{E_{11}}-\frac{C_1B_{11}}{E_{11}A_{11}},$$

$$\alpha_5=C_3-\frac{C_1^2}{aA_{11}},\qquad \alpha_6=C_5-\frac{C_1A_{12}}{aA_{11}},\qquad \alpha_7=C_4-\frac{C_1B_{11}}{A_{11}},\qquad \alpha_1=\frac{B_{11}}{A_{11}}-\frac{C_1\alpha_7}{aA_{11}\alpha_5},$$

$$\alpha_2=\frac{A_{12}}{aA_{11}}-\frac{C_1\alpha_6}{aA_{11}\alpha_5},\qquad \alpha_3=\frac{1}{A_{11}}+\frac{C_1^2}{aA_{11}^2\alpha_5},\qquad \alpha_4=\frac{C_1}{aA_{11}\alpha_5},\qquad F_1=E_{33},$$

$$F_2=E_{34}+A_{12}\alpha_1+C_5\frac{\alpha_7}{\alpha_5},\qquad F_3=A_{12}\alpha_2+\frac{C_5\alpha_6}{\alpha_5}-C_8,\qquad F_{12}=A_{12}\alpha_3-\frac{C_5C_1}{\alpha_5A_{11}},$$

$$F_{13}=\frac{C_5}{\alpha_5}-\alpha_4A_{12},\qquad F_1^I=\tfrac{1}{2}(F_3-F_{13}\alpha_6),\qquad F_2^I=F_{12}+\frac{F_{13}C_1}{A_{11}},$$

$$F_3^I = A_{14}\alpha_3 - \left(A_{44} - \frac{B_{44}}{a}\right)\frac{C_1}{A_{11}\alpha_5}, \qquad F_4^I = -A_{14}\alpha_4 + \left(A_{44} - \frac{B_{44}}{a}\right)\frac{1}{\alpha_5}, \qquad F_5^I = A_{12}\alpha_3 - \frac{C_7 C_1}{A_{11}\alpha_5},$$

$$F_6^I = \frac{C_7}{\alpha_5} - A_{12}\alpha_4, \qquad F_{19} = A_{14}\alpha_1 + \left(A_{44} - \frac{B_{44}}{a}\right)\frac{\alpha_7}{\alpha_5} - B_{14},$$

$$F_{20} = A_{14}\alpha_2 + \left(A_{44} - \frac{B_{44}}{a}\right)\frac{\alpha_6}{\alpha_5} - \frac{A_{24}}{a}, \qquad F_{22} = A_{12}\alpha_1 + C_5\frac{\alpha_7}{\alpha_5} - B_{12},$$

$$F_{23} = A_{12}\alpha_2 + C_5\frac{\alpha_6}{\alpha_5} - \frac{A_{22}}{a}.$$