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THE RESTRICTED $P + 2$ BODY PROBLEM†

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Abstract—In this paper we study a special case of the restricted n-body problem, called by us the restricted $P + 2$ body problem. The equilibrium configuration which the $P + 1$ bodies with mass form consists of one central mass encircled by a ring of P equally spaced particles of equal mass, the ring rotating at a specific angular velocity. We briefly discuss the stability of this configuration. We consider the dynamics of an infinitesimal mass under the influence of such a configuration. First the equilibrium points will be discussed, then the zero-velocity curves. We show that there are $3P$, $4P$ or $5P$ equilibrium points, depending on the ratio of the ring particle mass to the central body mass. Next motion about the equilibrium points is considered. We show that if the ring particle mass is small enough there will be P stable equilibrium points. Also if the number of particles, P, is large enough and the ratio of the ring particle mass to the central body mass is large enough there will be P different stable equilibrium points. Finally an analysis of the dynamics of the infinitesimal mass will be performed under the restriction that the particle does not cross or come close to the ring and lies in the plane of the ring. Under this restriction an approximate potential can be found which can be made arbitrarily close to the real potential under some circumstances. The dynamics of the particle under the approximate potential are integrable. We find a periodic orbit in this case with the Poincaré–Lindstedt method using the mass of the ring as a small parameter. The predictions from this approximate solution of the problem compare well with numerical integrations of the actual system.

1. EQUILIBRIUM CONFIGURATION

In normalized units the equilibrium configuration is described as follows. Consider a central body of mass 1. A ring consisting of P bodies, each with the same mass μ, are equally spaced around the central mass at a radius of 1. The angle between two consecutive bodies is 2θ where $\theta = \pi/P$ and is the vertex half-angle of the polygon formed by the P bodies. This ring has a specified angular velocity ω defined as a function of P and μ:

$$\omega^2 = 1 + \frac{\mu}{4} S \tag{1}$$

$$S = \sum_{j=1}^{P-1} \csc(j\theta). \tag{2}$$

The configuration described is an equilibrium solution to the n-body problem where $n = P + 1$ and $P \geqslant 2$.

The linear stability of this configuration under small perturbations was first considered by Maxwell in connection to his theory on the rings of Saturn [2, pp. 310–319]. Others have returned to this problem using improved formulations [3–5]. A sufficient condition for the above configuration to be linearly stable is [4]:

$$\mu \leqslant \mu^s_{\text{ring}} \qquad P \geqslant 7 \tag{3}$$

where:

$$\mu^s_{\text{ring}} = \frac{16\pi^3}{7(13 + 4\sqrt{10})I(3)P^3} = \frac{2.29\ldots}{P^3} \tag{4}$$

$$I(3) = \sum_{k=1}^{+\infty} \frac{1}{k^3} = 1.20205\ldots \tag{5}$$

If P is less than 7 the equilibrium configuration will be unstable independent of the mass of the ring particles. The inequality can be improved upon, especially for P relatively small [4].

2. THE RESTRICTED PROBLEM

In inertial cylindrical coordinates the equations of motion of a particle with infinitesimal mass under the influence of a potential force field are stated as:

$$\begin{aligned} \rho'' - \rho\eta'^2 &= U_\rho \\ \rho^2\eta'' + 2\rho\rho'\eta' &= U_\eta \\ \zeta'' &= U_\zeta \end{aligned} \tag{6}$$

where:

- ρ is the radial distance from the central body
- η is the angular distance from a fixed line
- ζ is the normal distance from the ring/central body plane
- $'$ denotes differentiation with respect to time
- $U(\rho, \eta, \zeta)$ is the force potential.

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If the particle is under the influence of the equilibrium configuration described above, the force potential is given as:

$$U(\rho, \eta, \zeta) = \frac{1}{\sqrt{\rho^2 + \zeta^2}} + \mu \sum_{k=1}^{P} \frac{1}{\sqrt{1 + \rho^2 + \zeta^2 - 2\rho \cos(2\theta k + \omega t - \eta)}} \tag{7}$$

where:

$\theta = \pi/P$ as before
ω is the angular velocity of the ring as before
t is the time.

This particular form of the equations of motion will be referenced in the final section of the paper.

Now transform to a coordinate system rotating at the angular rate ω via the transformation:

$$\eta = \omega t + \phi. \tag{8}$$

Then the equations of motion become:

$$\begin{aligned} \rho'' - \rho\phi'(2\omega + \phi') &= V_\rho \\ \rho^2\phi'' + 2\rho\rho'(\omega + \phi') &= V_\phi \\ \zeta'' &= V_\zeta. \end{aligned} \tag{9}$$

The force potential $V(\rho, \phi, \zeta)$ is now:

$$V(\rho, \phi, \zeta) = \tfrac{1}{2}\omega^2\rho^2 + U(\rho, \phi, \zeta) \tag{10}$$

$$U(\rho, \phi, \zeta) = \frac{1}{\sqrt{\rho^2 + \zeta^2}} + \mu \sum_{k=1}^{P} \frac{1}{\sqrt{1 + \rho^2 + \zeta^2 - 2\rho \cos(2\theta k - \phi)}} \tag{11}$$

where ϕ is the angle measured from a line rotating with angular rate ω in the plane.

In the rotating coordinates the equations of motion are time invariant and hence the Jacobi Integral exists and is:

$$\tfrac{1}{2}(\rho'^2 + \rho^2\phi'^2 + \zeta'^2) = V(\rho, \phi, \zeta) - C \tag{12}$$

where C is the Jacobi constant.

A simplification of the equations of motion (9) can be made using a certain symmetry property of the potential in eqn (10). We first note that the left-hand side of eqn (9) is invariant under the transformation $\phi \rightarrow \phi + a$ where a is a constant. Now observe that the potential $V(\rho, \phi, \zeta)$ is invariant under the transformation:

$$\begin{gathered} \phi \rightarrow \phi + 2k\theta \\ k = 0, \pm 1, \pm 2, \ldots. \end{gathered} \tag{13}$$

Then the partial derivatives of $V(\rho, \phi, \zeta)$ are also invariant under the transformation (13) and thus the equations of motion (9) are invariant under transformation (13). We use this result to identify the two points $\phi = \theta$ and $\phi = -\theta$. This is consistent as we see that:

$$V(\rho, \theta + \phi, \zeta) = V(\rho, -\theta + \phi, \zeta). \tag{14}$$

Thus we need only consider the equation of motion (9) on the reduced configuration space:

$$\begin{gathered} 0 < \rho < \infty \\ -\theta < \phi \leqslant \theta \\ -\infty < \zeta < \infty. \end{gathered} \tag{15}$$

This simplification is especially useful in the evaluation of the equilibrium points and zero-velocity curves.

Now a particular form of the potential is discussed. In either the inertial or rotating coordinate frames the contribution of the ring to the potential may be written in the form:

$$F = \mu \sum_{k=1}^{P} \frac{1}{\sqrt{1 + \rho^2 + \zeta^2 - 2\rho \cos(2\theta k - \chi)}} \tag{16}$$

where χ is some quantity independent of the index k. If we assume that the particle orbit lies in the plane of the ring and central body ($\zeta \equiv 0$), then the potential F may be expanded into Laplace coefficients so long as $\rho \neq 1$. In the following statements we have made use of the fact that:

$$\sum_{k=1}^{P} \cos(2m\theta k - m\chi) = \begin{cases} 0 & m \neq nP \\ P\cos(m\chi) & m = nP. \end{cases} \tag{17}$$

Then, following well known procedures [1, p. 495], we may state:

$$F = \mu P \begin{cases} \tfrac{1}{2} b_{1/2}^{(0)}(\rho) + \sum_{n=1}^{\infty} b_{1/2}^{(nP)}(\rho)\cos(nP\chi) & \rho < 1 \\ \dfrac{1}{\rho}\left[\tfrac{1}{2} b_{1/2}^{(0)}(1/\rho) + \sum_{n=1}^{\infty} b_{1/2}^{(nP)}(1/\rho)\cos(nP\chi)\right] & \rho > 1 \end{cases} \tag{18}$$

where

$$b_{1/2}^{(j)}(\alpha) = \frac{4}{\pi}\alpha^j \int_0^{\pi/2} \frac{\sin^{2j}(\sigma)\, d\sigma}{\sqrt{1 - \alpha^2 \sin^2(\sigma)}} \tag{19}$$

and

$$b_{1/2}^{(0)}(\alpha) = \frac{4}{\pi} K(\alpha)$$

$K(-)$ is the complete elliptic integral of the first kind.

The Laplace coefficient form of the potential holds so long as $\rho \neq 1$. Should $\rho = 1$ at some point, a new formulation of the potential must be made in the neighborhood of this point.

The Laplace coefficient form of the potential simplifies several computations needed later in the paper. Note that the force potential F is now multiplied by μP, which is the total mass of the ring, instead of just being multiplied by μ, the mass of one ring particle.

3. EQUILIBRIUM POINTS

In the equations of motion (9) for the rotating system, the condition for equilibrium points to exist are simply stated as:

$$V_\rho(\rho,\phi,\zeta)=0$$
$$V_\phi(\rho,\phi,\zeta)=0$$
$$V_\zeta(\rho,\phi,\zeta)=0. \quad (20)$$

We can use some symmetry properties of the potential $V(\rho,\phi,\zeta)$ to simplify the discussion.

First note that if some function $g(x)$ is even about some point x_0, then the first derivative of the function $g(x)$ at the point x_0 will be zero. It can be easily shown that the potential $V(\rho,\phi,\zeta)$ is even in ζ about 0 and is even in ϕ about 0. Coupled with the symmetry mentioned previously we see that the potential function is also even in ϕ about θ. Thus we may make the immediate observations:

$$V_\zeta(\rho,\phi,0)=0 \quad (21)$$
$$V_\phi(\rho,0,\zeta)=0 \quad (22)$$
$$V_\phi(\rho,\theta,\zeta)=0. \quad (23)$$

A simple computation will show that $V_\zeta(\rho,\phi,\zeta)$ is zero only at $\zeta=0$, thus $\zeta=0$ is the only equilibrium point in ζ. Numerical computations of V_ϕ show that in general $V_\phi\neq 0$ for $\phi\neq 0,\ \theta$. Thus we confine our search for equilibrium points to the problem of finding the roots of the equations:

$$V_\rho(\rho,0,0)=0 \quad (24)$$
$$V_\rho(\rho,\theta,0)=0. \quad (25)$$

We will consider each equation separately. In the following discussions we assume that $\zeta=0$ unless otherwise indicated.

For $\phi=0$ eqn (24) is:

$$V_\rho(\rho,0)=\omega^2\rho-\frac{1}{\rho^2}-\mu\sum_{k=1}^{P}\frac{\rho-\cos(2\theta k)}{(1+\rho^2-2\rho\cos(2\theta k))^{3/2}}. \quad (26)$$

We immediately discount the point $\rho=1$ due to the singularity from the ring particle. See Table 1 for a list of the function $V_\rho(\rho,0)$ evaluated at the ends of the intervals $(0,1)$ and $(1,\infty)$.

If we state $V(\rho,0,0)$ in the Laplace coefficient form we have:

$$V=\tfrac{1}{2}\omega^2\rho^2+\frac{1}{\rho}+\mu P\begin{cases}\tfrac{1}{2}b_{1/2}^{(0)}(\rho)+\sum_{n=1}^{\infty}b_{1/2}^{(nP)}(\rho) & \rho<1\\[4pt] \frac{1}{\rho}\left[\tfrac{1}{2}b_{1/2}^{(0)}(1/\rho)+\sum_{n=1}^{\infty}b_{1/2}^{(nP)}(1/\rho)\right] & \rho>1.\end{cases} \quad (27)$$

Simple computations will show that $\tfrac{1}{2}\omega^2\rho^2$, $1/\rho$, $b_{1/2}^{(nP)}(\rho)$ and $1/\rho b_{1/2}^{(nP)}(1/\rho)$ are all convex functions of ρ in the appropriate intervals $(0,1)$ and $(1,\infty)$. Thus $V(\rho,0,0)$ is a convex function in these intervals. Combining this with the values given in Table 1 we see that there is one and only one equilibrium point in each interval. We call the equilibrium points $E1$ and $E2$.

Table 1. Limiting points of $V_\rho(\rho,0)$

ρ	V_ρ
0^+	$-\infty$
1^-	$+\infty$
1^+	$-\infty$
$+\infty$	$+\infty$

The equilibrium point $E1$ lies in the interval $(0,1]$ and, depending on the mass μ, may take on any position in this interval. Note that if $\mu\to 0$ then $\rho_1\to 1$ and if $\mu\to\infty$ then $\rho_1\to 0$. For small μ we can solve for the position of $E1$:

$$\rho_1=1-(\mu/3)^{1/3}+\tfrac{1}{3}(\mu/3)^{2/3}+\tfrac{1}{9}(\mu/3)^{3/3}+\cdots. \quad (28)$$

The equilibrium point $E2$ lies in the interval $[1,+\infty)$. In this case an upper bound on ρ_2 can be found, independent of the mass ratio μ. Recalling that $\rho>1$ and that $\omega^2=1+\mu S/4$ we see from eqn (26) that a sufficient condition for $V_\rho>0$ is:

$$\rho S\geqslant 4\sum_{k=1}^{P}\frac{\rho-\cos(2\theta k)}{(1+\rho^2-2\rho\cos(2\theta k))^{3/2}}. \quad (29)$$

Given that:

$$\frac{1}{(\rho-1)^3}\geqslant\frac{1}{(1+\rho^2-2\rho\cos(2\theta k))^{3/2}} \quad (30)$$

it is easy to verify that the position of $E2$, ρ_2, is less than the quantity:

$$\rho_M=1+\left(\frac{4P}{S}\right)^{1/3} \quad (31)$$

and thus lies in the interval $[1,\rho_M)$. See Table 2 for some values of ρ_M as a function of P.

For $P\gg 1$ we have the following asymptotic result [4, Appendix]:

$$S=\frac{2P}{\pi}\log\left(\frac{2P}{\pi}\exp\gamma\right)-\frac{\pi}{36P}+O(1/P^2) \quad (32)$$

where $\gamma=0.57721\ldots$ and is Euler's Constant. Thus for large P we have the asymptotic result:

$$\rho_M=1+\left[\frac{2\pi}{\log\left(\frac{2P}{\pi}\exp\gamma\right)}\right]^{1/3}. \quad (33)$$

Table 2. Upper bound on radius of equilibrium points $E2$ and $E3$ vs P

P	ρ_M
2	3.00
3	2.73
5	2.54
50	2.16
500	2.00
5000	1.90

Note that ρ_M is only an upper bound on ρ_2. Also note that if $P \to \infty$ then $\rho_M \to 1^+$ independent of the mass μ. As before we also have that if $\mu \to 0$ then $\rho_2 \to 1$. For small μ we can again solve for the position of $E2$:

$$\rho_2 = 1 + (\mu/3)^{1/3} + \tfrac{1}{3}(\mu/3)^{2/3} - \tfrac{1}{9}(\mu/3)^{3/3} + \cdots . \quad (34)$$

For $\phi = \theta$ eqn (25) is:

$$V_\rho(\rho, \theta) = \omega^2 \rho - \frac{1}{\rho^2} - \mu \sum_{k=1}^{P} \frac{\rho - \cos\theta(2k-1)}{(1 + \rho^2 - 2\rho\cos\theta(2k-1))^{3/2}} . \quad (35)$$

The roots of this equation prove to be much more complex than the previous case. See Table 3 for $V_\rho(\rho, \theta)$ evaluated at $\rho = 1$ and at its limiting points.

In Table 3 the quantity S_0 is defined as:

$$S_0 = \sum_{k=1}^{P} \csc \tfrac{1}{2}\theta(2k-1) \quad (36)$$

Now $S_0 > S$ for all P so $V_\rho(1, \theta) < 0$. For $P \gg 1$ we find asymptotically [4, Appendix]:

$$S_0 = \frac{2P}{\pi} \log\left(\frac{8P}{\pi} \exp\gamma\right) + \frac{\pi}{36P} + O(1/P^2) \quad (37)$$

$$V_\rho(1, \theta) = -\frac{\mu P}{\pi} \log 2 + \cdots . \quad (38)$$

In this case the nice convex properties of V disappear for $\rho \leqslant 1$ and remain for $\rho > 1$ only outside of a neighborhood of $\rho = 1$, the neighborhood being a function of P. Nonetheless, one can construct an argument that for $\rho \geqslant 1$ there is one and only one equilibrium point, called $E3$. Using the analysis from equilibrium point $E2$ we see that $E3$ lies in the interval $[1, \rho_M)$, with ρ_M the same as in eqn (31). Again if $\mu \to 0$ then $\rho_3 \to 1$. For small μ we can solve for the position of $E3$:

$$\rho_3 = 1 + \frac{\mu}{12}(S_0 - S) + \cdots . \quad (39)$$

The equilibrium points $E1$, $E2$ and $E3$ exist for all $P \geqslant 2$.

From Table 3 we see that in the interval (0, 1) there are either no equilibrium points or an even number of equilibrium points. Inspection of eqn (35) along with the fact that $V_\rho(1, \theta) < 0$ makes it obvious that we may always choose μ small enough so that there are no equilibrium points in this interval. Conversely one might suspect that if μ is chosen large enough that at least two equilibrium points may appear, this is indeed the case. These two new equilibrium points will be referred to as $E4$ and $E5$.

Table 3. Limiting points of $V_\rho(\rho, \theta)$

ρ	V_ρ
0^+	$-\infty$
1	$-\mu(S_0 - S)/4$
$+\infty$	$+\infty$

A necessary and sufficient condition for any equilibrium point to appear in the interval (0, 1) is that at some ρ^* and μ^* we have:

$$V_\rho(\rho^*, \theta; \mu^*) = 0$$
$$V_{\rho\rho}(\rho^*, \theta; \mu^*) = 0$$
$$0 < \rho^* < 1$$
$$\mu^* > 0. \quad (40)$$

This condition will occur at some point ρ^* as μ is increased from zero. For $P = 2$ the condition never occurs and hence $E4$ and $E5$ do not exist for $P = 2$. The condition will always occur for $P \geqslant 3$ for large enough μ. For $P \geqslant 4$ it can be shown that a necessary condition for eqn (40) to occur is that first the condition $V_{\rho\rho}(1, \theta) = 0$ must occur at some $\mu < \mu^*$. This condition will never occur for $P = 3$ but may always occur for $P \geqslant 4$. For small μ we can show that $V_{\rho\rho}(1, \theta) > 0$. Thus a necessary condition for the new equilibrium points to appear when $P \geqslant 4$ is that:

$$V_{\rho\rho}(1, \theta) \leqslant 0. \quad (41)$$

Evaluating the above condition yields the inequality:

$$48 \leqslant \mu[N_0 - 5S_0 - 4S] \quad (42)$$

where:

S_0 and S are defined as before

$$N_0 = \sum_{k=1}^{P} \frac{1 + \cos^2 \frac{1}{2}\theta(2k-1)}{\sin^3 \frac{1}{2}\theta(2k-1)}$$

Again for large P we have [4, Appendix]:

$$N_0 = \frac{28 I(3)}{\pi^3} P^3 + \cdots . \quad (43)$$

Where $I(3)$ has been defined previously.

Using this expansion we state that a necessary condition for the equilibrium points $E4$ and $E5$ to appear for $P \geqslant 4$ is that:

$$\mu \geqslant \bar{\mu} = \frac{12\pi^3}{7 I(3) P^3} = \frac{44.2188\ldots}{P^3} . \quad (44)$$

Comparing this to inequality (3) for the stability of the ring we see that $E4$ and $E5$ appear only for unstable rings.

Given P it is possible to solve eqns (40) numerically to find ρ^* and μ^*. We have performed the analysis and present the results in Fig. 1 for $P \geqslant 4$. Note that the plotted value of μ^* has been normalized by $\bar{\mu}$ [eqn (44)]. For $P = 3$ the bifurcation point is:

$$\mu^*/\bar{\mu} = 44.411\ldots$$
$$\rho^* = 0.32437\ldots .$$

Note from Fig. 1 that even though these equilibrium points appear only for unstable rings, the necessary mass of the ring for these points to appear may still be vanishingly small for large enough P.

We label the two new equilibrium points as follows. $E4$ is the equilibrium point closest to $\rho = 1$. For large P if $\mu \to +\infty$ then $\rho_4 \to 1^-$, but ρ_4 never arrives at 1.

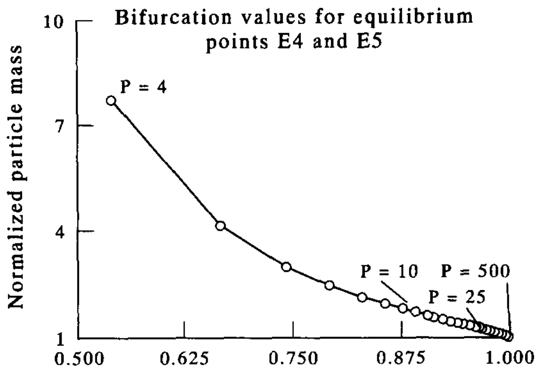

Fig. 1. Normalized mass ($\mu^*/\bar{\mu}$) and radius (ρ^*) for bifurcation of $E4$ and $E5$ as a function of P.

This is clear as $V_\rho(1, \theta) < 0$. There is no convenient solution for ρ_4 in terms of the mass μ.

$E5$ is the equilibrium point closest to $\rho = 0$. If $\mu \to +\infty$ then $\rho_5 \to 0^+$. If μ is large enough and $P \geqslant 3$ we can solve for the position of $E5$:

$$\rho_5 = \left[\frac{4}{\mu(S+2P)}\right]^{1/3} + \cdots . \tag{45}$$

Figure 2 depicts the intervals where the equilibrium points may be found. A more geometrical interpretation of the equilibrium points appears in the zero-velocity curve discussion.

4. THE ZERO-VELOCITY CURVES

The zero-velocity curves are defined using the energy integral [eqn (12)] evaluated such that the total velocity in the rotating coordinate frame is zero:

$$V(\rho, \phi) = C \tag{46}$$

where C is a constant. If we imagine the energy C to vary, then eqn (46) defines a surface in the 3-space (ρ, ϕ, C). The zero-velocity curves are just horizontal sections of this surface for a given value of C. Motion is possible if the particle is on or below this surface, impossible if above this surface. Due to the symmetry properties of $V(\rho, \phi)$ we need only consider the curves defined on: $0 < \rho < \infty$ and $0 \leqslant \phi \leqslant \theta$.

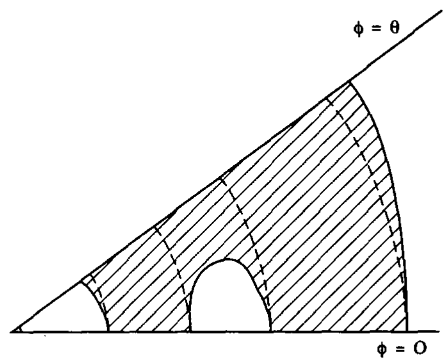

Fig. 3. Zero-velocity curves for large energy ($C \gg 1$). Note that the curve shapes and sizes are exaggerated for clarity.

For $C \gg 1$ the asymptotic shape of the curves are known and are shown in general in Fig. 3. Note that the dimensions and variations of the curve are exaggerated for clarity. As C is decreased the curves will keep the same approximate shape, the $\phi = 0$ point of the curves will "lead" the $\phi \neq 0$ points of the curves until the curves intersect. Every intersection of the curves is an equilibrium point. Note that in all zero-velocity curve figures the angle θ has been normalized to a convenient value for plotting. In all the figures motion is not possible in the hatched areas.

Figures 4–6 depict the zero-velocity curves for $\mu < \mu^*$ as the energy C decreases, thus the equilibrium points $E4$ and $E5$ are not present. Note that in this case $E3$ is the global minimum of the zero-velocity surface. If $C < V_{E3}$ then motion is possible throughout the plane.

Figures 7–9 depict the zero-velocity curves for $\mu \gg \mu^*$ as the energy C decreases. Thus in this case $E4$ and $E5$ exist, additionally they are the global finite maximum and global minimum respectively of the zero-velocity surface.

Between the cases $\mu < \mu^*$ and $\mu \gg \mu^*$ the order in which the equilibrium points appear as the energy is decreased is shown in Table 4 for $P = 25$ in particular. Note that for large P Table 4 presents the usual case.

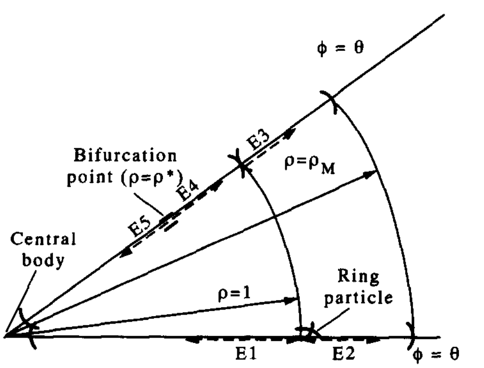

Fig. 2. Intervals on the plane where the equilibrium points may be found.

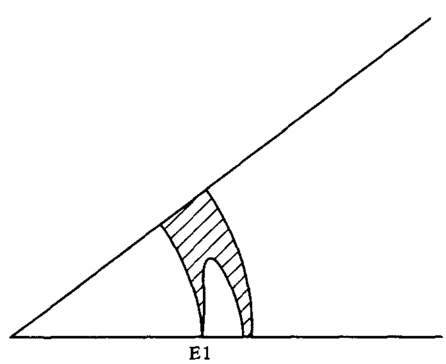

Fig. 4. Zero-velocity curves for $\mu < \mu^*$; $P = 25$, $\mu = 0.003$, $C = 1.6440$.

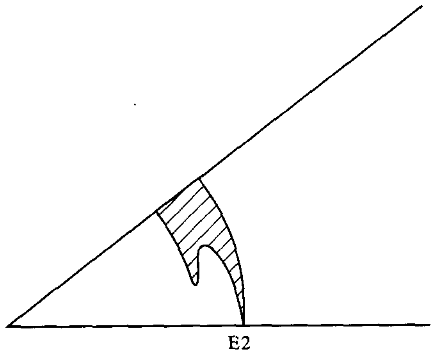

Fig. 5. Zero-velocity curves for $\mu < \mu^*$; $P = 25$, $\mu = 0.003$, $C = 1.6424$.

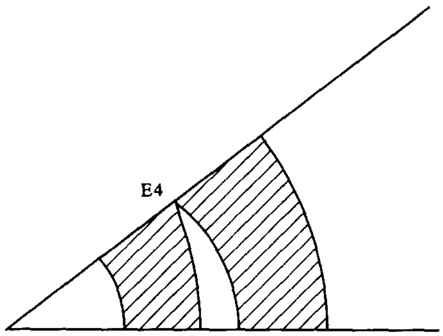

Fig. 7. Zero-velocity curves for $\mu \gg \mu^*$; $P = 25$, $\mu = 0.03$, $C = 2.8295$.

Thus we see that the zero-velocity surface is a complicated function of μ. In Figs 4–9 we have used $P = 25$, which is a good representation of the cases which occur for small P. It should be noted that for P relatively small the zero-velocity surface will not vary as in Table 4. Instead the surface will in general have different variations in the order of appearance of the equilibrium points and may not have some of the situations listed in Table 4. As an example, for $P = 10$ there are a total of nine variations in the order of appearance of the equilibrium points, as compared with seven variations in Table 4. For $P = 4$ there are only five variations.

Extending the analysis to the full problem we have the results given in Table 5 for $P \geqslant 3$. For $P = 2$ there are always only 6 equilibrium points.

Note that μ may increase in two ways. The individual ring particles may increase their mass relative to the central body mass, thus the ring grows heavier. Or the central body mass may decrease relative to the ring mass, thus the central body has less effect on the ring.

5. SMALL MOTIONS

Small motions close to the equilibrium points are studied via the equations of variations formed about the equilibrium points. The characteristic equation of the variation equations provides information as to the stability of motion close to the equilibrium points (stability meaning harmonic motion). At a given equilibrium point the equations of variation may be stated as:

$$(\Delta\rho/\rho_0)'' - 2\omega\,\Delta\phi' = V_{\rho\rho|0}(\Delta\rho/\rho_0) + \frac{1}{\rho_0} V_{\rho\phi|0}\,\Delta\phi + \frac{1}{\rho_0} V_{\rho\zeta|0}\,\Delta\zeta \quad (47)$$

$$\Delta\phi'' + 2\omega(\Delta\rho/\rho_0)' = \frac{1}{\rho_0} V_{\phi\rho|0}(\Delta\rho/\rho_0) + \frac{1}{\rho_0^2} V_{\phi\phi|0}\,\Delta\phi + \frac{1}{\rho_0^2} V_{\phi\zeta|0}\,\Delta\zeta \quad (48)$$

$$\Delta\zeta'' = V_{\zeta\rho|0}\,\Delta\rho + V_{\zeta\phi|0}\,\Delta\phi + V_{\zeta\zeta|0}\,\Delta\zeta \quad (49)$$

where:

$|\Delta\rho|, |\Delta\phi|, |\Delta\zeta| \ll 1$ and are perturbations from the equilibrium points
ρ_0 is the equilibrium point radial coordinate
ϕ_0 is the equilibrium point angular coordinate
ζ_0 is the equilibrium point out-of-plane coordinate

As can be easily verified, $V_{\rho\phi|0} = V_{\rho\zeta|0} = V_{\rho\zeta|0} = 0$ at all the equilibrium points $E1$–$E5$.

It is easy to verify that $V_{\zeta\zeta}(\rho, \phi, 0) < 0$ for all ρ and ϕ. Thus we see immediately that the decoupled motion in ζ is always harmonic and thus considered stable. We now consider the planar case only.

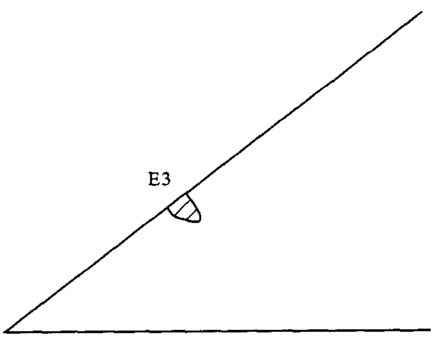

Fig. 6. Zero-velocity curves for $\mu < \mu^*$; $P = 25$, $\mu = 0.003$, $C = 1.6400$.

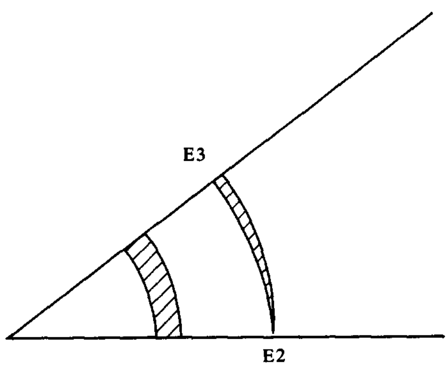

Fig. 8. Zero-velocity curves for $\mu \gg \mu^*$; $P = 25$, $\mu = 0.03$, $C = 2.6561$.

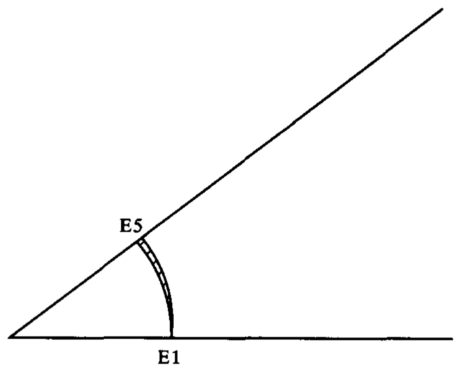

Fig. 9. Zero-velocity curves for $\mu \gg \mu^*$; $P = 25$, $\mu = 0.03$, $C = 2.6395$.

In the planar case we assume harmonic motion with a frequency λ and form the characteristic equation:

$$\sigma^2 - 2(2 - \mathcal{M})\sigma + \mathcal{K} = 0 \qquad (50)$$

where:

$$\sigma = (\lambda/\omega)^2 \qquad (51)$$

$$\mathcal{M} = \frac{1}{2\omega^2}\left(V_{\rho\rho|0} + \frac{1}{\rho_0^2}V_{\phi\phi|0}\right) \qquad (52)$$

$$\mathcal{K} = \frac{1}{\omega^4\rho_0^2}V_{\phi\phi|0}V_{\rho\rho|0}. \qquad (53)$$

For stable motion the following conditions must be true:

$$\mathcal{K} > 0 \qquad (54)$$

$$2 > \mathcal{M} \qquad (55)$$

$$(2 - \mathcal{M})^2 > \mathcal{K}. \qquad (56)$$

From a general consideration of the zero-velocity surface we can construct Table 6, which gives the signs of the surface curvature $V_{\rho\rho}$ and $V_{\phi\phi}$ and of $\mathcal{K}$ at the equilibrium points.

From Table 6 we see immediately that $E1$, $E2$ and $E4$ violate condition (54) as they are saddle points of the potential surface. For these points condition (56) is always satisfied and motion about these points decouples into a pure harmonic motion and a pure hyperbolic motion.

As $E3$ and $E5$ are local minima of the potential surface, condition (54) is always true and conditions (55) and (56) must be checked. We first consider $E3$.

For $E3$ condition (55) translates into that μ cannot be too large. Then, assuming that μ is small and P large we substitute eqn (39) for ρ_3 into condition (56), simplify and ignore higher orders of μ and $1/P$ to find that a sufficient condition for stability of $E3$ is that:

$$\mu < \mu^s_{E_3} \qquad (57)$$

$$\mu^s_{E3} = \frac{4\pi^3}{7(13 + 4\sqrt{10})I(3)P^3} = \frac{0.572\ldots}{P^3} \leqslant \frac{1}{4}\mu^s_{\text{ring}}$$

where μ^s_{ring} is the stability bound for the ring [eqn (3)]. Thus we see that it is possible for $E3$ to be stable if the mass μ is small enough. Note that it is possible for $E3$ to be unstable while the ring configuration is stable. Figure 10 shows limiting mass for stability of $E3$ vs P. The mass is normalized by $\mu^s_{E_3}$.

Now we consider $E5$. Remarkably, it is possible under some circumstances for $E5$ to be stable, although the ring configuration itself will be unstable. Under analysis it turns out that condition (55) is the controlling condition for stability in virtually all cases for $E5$. This is primarily due to the fact that $\mathcal{K}$ becomes vanishingly small (although positive) at $E5$. This result is easily verified using the Laplace form of the ring potential.

Assuming that μ is large, we substitute eqn (45) for ρ_5 into condition (55), disregard higher orders and find the condition:

$$(\mu P)^{2/3}(S/P - 6) > \frac{27}{4}\left[\frac{4}{2 + S/P}\right]^{2/3}. \qquad (58)$$

Table 4. Relative energy values of equilibrium points as μ varies

	Order of appearance as C decreases
Increasing	$E1$, $E2$, $E3$ ($E4$ and $E5$ not present)
μ	$E1$, $E2$, $E4$, $E5$, $E3$
↓	$E1$, $E4$, $E2$, $E5$, $E3$
	$E4$, $E1$, $E2$, $E5$, $E3$
	$E4$, $E2$, $E1$, $E5$, $E3$
	$E4$, $E2$, $E1$, $E3$, $E5$
	$E4$, $E2$, $E3$, $E1$, $E5$

Table 5. Total numbers of equilibrium points as a function of μ ($P \geqslant 3$)

Relative size of mass	Number of equilibrium points
$\mu < \mu^*$	$3P$
$\mu = \mu^*$	$4P$
$\mu > \mu^*$	$5P$

Table 6. Sign of surface curvature at the equilibrium points

E point	$V_{\rho\rho}$	$V_{\phi\phi}$	$\mathcal{K}$
1	+	−	−
2	+	−	−
3	+	+	+
4	−	+	−
5	+	+	+

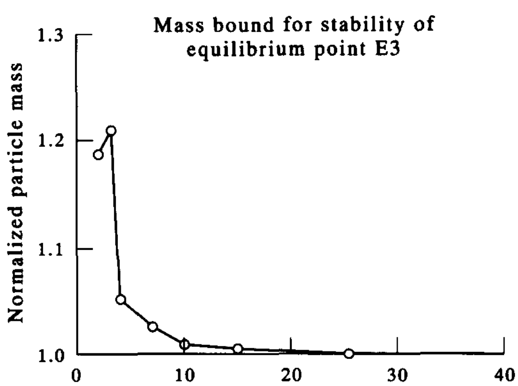

Fig. 10. Normalized maximum mass for $E3$ stability vs P.

We can derive two necessary conditions from this inequality. First we need:

$$S - 6P > 0. \tag{59}$$

If we assume that P is large and use the asymptotic expansion for S we find that:

$$P > \frac{\pi}{2}\exp(3\pi - \gamma) = 10{,}928.67\ldots. \tag{60}$$

This has been verified numerically where we find that condition (55) is satisfied only when $P \geqslant 10{,}930$. Given that this condition is satisfied, we see that the mass μ must satisfy the condition:

$$\mu > \frac{4}{2P + S}\left[\frac{27}{4(S/P - 6)}\right]^{3/2}. \tag{61}$$

From numerical computations we see that this is only a necessary condition and that the actual minimum value of μ for stability is greater than this bound. In the conditions checked the actual bound and the above bound have been of the same order of magnitude.

6. A SPECIAL CASE

Now we discuss a particular class of motions of the particle. We restrict the particle orbit to lie in the plane of the ring and we assume that the particle radius stays outside a neighborhood of the value $\rho = 1$. In other words we assume that the particle orbit does not intersect the ring. Under these restrictions we may use the Laplace coefficient form of the force potential [eqn (18)] for the inertial equations of motion (6). Assuming $\zeta = 0$ we have:

$$\rho'' - \rho\eta'^2 = U_\rho \tag{62}$$

$$\rho^2\eta'' + 2\rho\rho'\eta' = U_\eta \tag{63}$$

$$U = \frac{1}{\rho} + F \tag{64}$$

$$F = \delta\begin{cases} \frac{1}{2}b_{1/2}^{(0)}(\rho) + \sum_{n=1}^{\infty} b_{1/2}^{(nP)}(\rho)\cos(nP(\eta - \omega t)) & \rho < 1 \\ \frac{1}{\rho}\left[\frac{1}{2}b_{1/2}^{(0)}(1/\rho) + \sum_{n=1}^{\infty} b_{1/2}^{(nP)}(1/\rho)\cos(nP(\eta - \omega t))\right] & \rho > 1 \end{cases} \tag{65}$$

where we replace μP by δ, which is the total mass of the ring.

A simplifying assumption we will introduce is to retain only the zeroth order Laplace coefficient and disregard all higher order Laplace coefficients. This will lead to an approximate potential:

$$\bar{U}(\rho) = \frac{1}{\rho} + \frac{2\delta}{\pi}\begin{cases} K(\rho) & \rho < 1 \\ \frac{1}{\rho}K(1/\rho) & \rho > 1 \end{cases} \tag{66}$$

where $K(\text{—})$ is the complete elliptic integral of the first kind.

We note that $\bar{U}(\rho)$ is independent of the time t and of the angle η. Thus the system with the approximate potential will have its energy and angular momentum conserved. The presence of two integrals indicates that the system is in fact an integrable system. We will investigate the motion of the system under this approximate potential.

A measure of the error between the approximate and actual potentials and forces may be found using the definition of the Laplace coefficients. The errors are found to be bounded by:

$$|U - \bar{U}| \leqslant \frac{4}{\pi}\delta\begin{cases} \frac{\rho^P}{1 - \rho^P}K(\rho) & \rho < 1 \\ \frac{1}{\rho^P - 1}\frac{1}{\rho}K(1/\rho) & \rho > 1 \end{cases} \tag{67}$$

$$|U_\rho - \bar{U}_\rho| \leqslant \frac{4}{\pi}\delta\begin{cases} \frac{\rho^{P-1}}{(1 - \rho^P)^2}\left[(P - 1 + \rho^P)K(\rho) + \frac{1 - \rho^P}{1 - \rho^2}E(\rho)\right] & \rho < 1 \\ \frac{1}{(\rho^P - 1)^2}\left[P\rho^{P-2}K(1/\rho) + \frac{\rho^P - 1}{\rho^2 - 1}E(1/\rho)\right] & \rho > 1 \end{cases} \tag{68}$$

$$|U_\eta - \bar{U}_\eta| \leqslant \frac{4}{\pi}\delta\begin{cases} \frac{P\rho^P}{(1 - \rho^P)^2}K(\rho) & \rho < 1 \\ \frac{P\rho^P}{(\rho^P - 1)^2}\frac{1}{\rho}K(1/\rho) & \rho > 1 \end{cases} \tag{69}$$

where $E(\text{—})$ is the complete elliptic integral of the second kind.

It is clear in all cases that if $P \to \infty$ all the errors go to zero. We also see that the further ρ is from 1 the smaller the errors are. Finally, if $\rho \to \pm 1$ the errors will become arbitrarily large. We note that if we restrict to a stable ring, the total mass δ must satisfy $\delta < 2.29/P^2$ [eqn (3)], and thus goes to zero as $P \to \infty$.

The approximate potential may also be derived in a non-rigorous fashion. Assume that a continuous, uniform, circular, one-dimensional ring exists around a central body. The ring has total mass δ and is situated at a radius of 1 from the central body. Then the potential energy of the ring on a test particle in the plane of the ring is expressed as the integral:

$$\bar{F}(\rho) = \frac{\delta}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{1 - 2\rho\cos\theta + \rho^2}}. \quad (70)$$

Introducing the Laplace coefficients and considering the cases $\rho > 1$ and $\rho < 1$ separately we find:

$$\bar{F}(\rho) = \frac{2\delta}{\pi} \begin{cases} K(\rho) & \rho < 1 \\ \frac{1}{\rho} K(1/\rho) & \rho > 1. \end{cases} \quad (71)$$

This corresponds exactly with our approximate potential due to the ring.

With the approximate potential the equations of motion become:

$$\rho'' - \rho\eta'^2 = \bar{U}_\rho \quad (72)$$

$$\frac{d}{dt}(\rho^2\eta') = 0. \quad (73)$$

Thus the quantity $\rho^2\eta' = h$, the angular momentum, is conserved. Substituting for η' in the above equation yields the one degree of freedom system:

$$\rho'' = \frac{h^2}{\rho^3} + \bar{U}_\rho. \quad (74)$$

This system has an energy integral:

$$\tfrac{1}{2}\rho'^2 = \bar{U}(\rho) - \frac{h^2}{2\rho^2} - C \quad (75)$$

where the constant C corresponds to the previously found constant C in eqn (12) when the same approximating assumptions are introduced into that system.

Eliminating the time in favor of the inertial angle η, we can reduce the solution of the system to the quadrature:

$$\eta - \eta_0 = \int_{\rho_0}^{\rho} \frac{\pm h/\rho^2 \, d\rho}{\sqrt{2(\bar{U}(\rho) - h^2/2\rho^2 - C)}}. \quad (76)$$

We will, instead, use a perturbation method to find the periodic orbit of this system to the first order in δ using the Poincaré–Lindstedt method.

We are interested only in those orbits of the particle which do not intersect the ring. By consideration of the energy integral (75) we may narrow our focus of the types of motion considered and develop a necessary and sufficient condition for when non-intersecting orbits exist.

Considering the energy integral, we see that given h^2 and C the position ρ of the particle must be such that:

$$G(\rho) = \bar{U}(\rho) - \frac{h^2}{2\rho^2} \geqslant C \quad (77)$$

where $G(\rho)$ is the zero-velocity energy function. If, for a given h^2, we plot $G(\rho)$, then for any given C motion will be possible on all parts of the graph of $G(\rho)$ which lie above the line C. Furthermore, where the line C intersects the graph of $G(\rho)$ the velocity ρ' is zero, if C is tangent to the graph of $G(\rho)$ then there is an equilibrium point present. If we let $\delta = 0$ the graph assumes the general shape as shown in Fig. 11.

Considering Fig. 11 we see that if $C \leqslant 0$ then $\rho \to \infty$, and if $0 < C \leqslant 1/(2h^2)$ then the radius ρ will vary between two bounds, we note especially in this case that ρ will always pass through the point $\rho = h^2$.

To consider the effect of the ring, let δ be very small. Then the main effect is a pole at $\rho = 1$ extending to $+\infty$, the rest of the curve will not be appreciably affected except in this neighborhood. Then some general types of orbits may be shown to intersect with the ring. If $h^2 = 1$ all bounded orbits will intersect the ring. If $h^2 > 1$ then all orbits with $\rho < 1$ initially will intersect the ring. If $h^2 < 1$ then all bounded orbits with $\rho > 1$ initially will intersect the ring. Thus for non-intersecting orbits to exist we must enforce the following restrictions: $h^2 \neq 1$, if $h^2 > 1$ then $\rho > 1$, if $h^2 < 1$ then $\rho < 1$. These restrictions are only necessary conditions for a bounded orbit to not intersect the ring.

For δ not necessarily small, under the above necessary conditions we can develop necessary and sufficient conditions for the existence of orbits which do not intersect the ring. Given $h^2 > 1$ ($h^2 < 1$) and $\rho > 1$ ($\rho < 1$) a necessary and sufficient condition for bounded non-intersecting orbits to exist is that there is either a local maximum and local minimum or a saddle point on the $\rho > 1$ ($\rho < 1$) portion of the graph of $G(\rho)$.

It is easy to see that given δ small enough the above necessary and sufficient conditions will be met and there will be a local minimum and a local maximum on the graph of $G(\rho)$. As δ is increased there is a point (δ^*) at which the local maximum and minimum will merge and become a saddle point on the curve. For $\delta > \delta^*$ all bounded orbits will intersect the ring. Associated with δ^* is the position of the saddle point ρ^*. Both of these are a function of h^2. We are interested in δ^*, as $\delta \leqslant \delta^*$ is a necessary and sufficient condition for non-intersecting bounded orbits to exist. To compute δ^* we must solve the equations:

$$G_\rho(\rho^*; \delta^*, h^2) = 0 \quad (78)$$

$$G_{\rho\rho}(\rho^*; \delta^*, h^2) = 0. \quad (79)$$

In these equations h^2 is a parameter. We are interested in expressing the relation δ^* vs h^2. Thus we

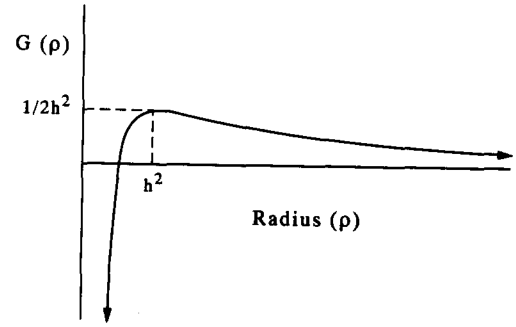

Fig. 11. Zero-velocity curve ($G(\rho)$ vs ρ) for $\delta = 0$.

instead view ρ as a parameter in the equations and find the parametric curve for h^{*2} and δ^*:

$$\delta^* = \frac{\pi}{2}\begin{cases} \dfrac{(1-\rho^2)^2}{\rho[2E(\rho)-(2-\rho^2)(1-\rho^2)K(\rho)]} & \rho < 1 \\[2ex] \dfrac{(\rho^2-1)^2}{\rho^2[2E(1/\rho)-(\rho^2-1)K(1/\rho)]} & \rho > 1 \end{cases} \tag{80}$$

$$h^{*2} = \rho\begin{cases} \dfrac{(1+\rho^2)E(\rho)-(1-\rho^2)K(\rho)}{2E(\rho)-(2-\rho^2)(1-\rho^2)K(\rho)} & \rho < 1 \\[2ex] \dfrac{(\rho^2+1)E(1/\rho)-(\rho^2-1)K(1/\rho)}{2E(1/\rho)-(\rho^2-1)K(1/\rho)} & \rho > 1. \end{cases} \tag{81}$$

See Figs 12 and 13 for the plot of the above curve. Asymptotically we have the following relations. If $h^{*2}\to 0$ then $\delta^*\to\infty$, if $h^{*2}\to\infty$ then $\delta^*\to\infty$ and if $h^{*2}\to 1^{\pm}$ then $\delta^*\to 0$.

To actually find the non-intersecting orbits we follow the following procedure. Given h^2 compute δ^*. Then, if $\delta < \delta^*$ find the local maximizer ρ_{max} and the local minimizer ρ_{min} of the zero-velocity energy function and the corresponding zero-velocity energy values C_{max} and C_{min}. Then the non-intersecting bounded orbits will be those whose energy C is such that $C_{min} \leqslant C \leqslant C_{max}$ and whose initial position ρ_0 is such that $\rho_0 \geqslant \rho_{min}$ if $h^2 > 1$ or $\rho_0 \leqslant \rho_{max}$ if $h^2 < 1$. If $\delta = \delta^*$ then the only non-intersecting bounded orbit is the circular orbit at ρ^*.

Guaranteed of the existence of non-intersecting orbits under the above conditions, we now compute the periodic orbit to the first power of δ using the Poincaré–Lindstedt method.

Introduce the following well known transformations to the system:

$$dt = \frac{\rho^2}{h}\,d\eta \tag{82}$$

$$\rho = 1/u. \tag{83}$$

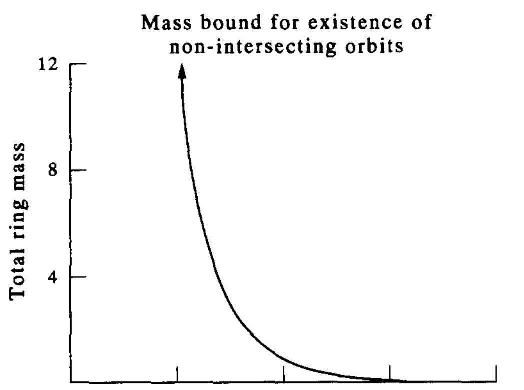

Fig. 12. Curve defining where non-intersecting orbits may exist: δ^* vs h^{*2} ($\rho < 1$).

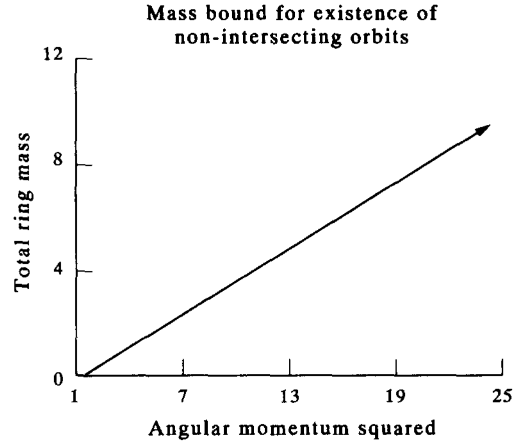

Fig. 13. Curve defining where non-intersecting orbits may exist: δ^* vs h^{*2} ($\rho > 1$).

Then the system assumes the form:

$$\frac{d^2u}{d\eta^2} + u = \frac{1}{h^2}\bar{U}_u \tag{84}$$

where $\bar{U}$ is now viewed as a function of u:

$$\bar{U}(u) = u + \delta\bar{F}(u) \tag{85}$$

$$\bar{F}(u) = \frac{2}{\pi}\begin{cases} K(1/u) & u > 1 \\ uK(u) & u < 1. \end{cases} \tag{86}$$

Note that we have redefined the ring potential $\bar{F}$ so that it is now factored by δ.

Now introduce the new angle variable τ via:

$$\frac{d\eta}{d\tau} = k = 1 + \delta p_1 + \cdots. \tag{87}$$

So that the equation of motion becomes:

$$\ddot{u} + k^2 u = \frac{k^2}{h^2}[1 + \delta\bar{F}_u] \tag{88}$$

where $(\dot{-})$ denotes differentiation with respect to τ and k is the modified frequency of the system, and is defined by the unspecified constants $p_1, p_2, \ldots$.

Now we assume a solution of the form:

$$u(\tau) = u_0(\tau) + \delta u_1(\tau) + \cdots \tag{89}$$

where the $u_i(\tau)$ are to be found as periodic functions of τ with period 2π.

Substitute this solution into eqn (88) and equate powers of δ to find the sequence of equations:

$$\ddot{u}_0 + u_0 = \frac{1}{h^2} \tag{90}$$

$$\ddot{u}_1 + u_1 = 2p_1\left(\frac{1}{h^2} - u_0\right) + \frac{1}{h^2}\bar{F}_u(u_0) \tag{91}$$

$$\vdots$$

with the imposed conditions at an apsis:

$$u_0(0) = u_0 \quad \dot{u}_0(0) = 0 \tag{92}$$

$$u_i(0) = 0 \quad \dot{u}_i(0) = 0 \quad i = 1, 2, \ldots. \tag{93}$$

The zeroth order solution is:

$$u_0(\tau) = \frac{1}{h^2}[1 + (h^2u_0 - 1)\cos(\tau)]. \quad (94)$$

We introduce two new constants a and e via:

$$h^2 = a(1 - e^2) \quad (95)$$

$$u_0 = \frac{1}{a(1-e)}. \quad (96)$$

These are normally defined as the semi-major axis and the eccentricity in the two body problem. The initial condition u_0 is then the inverse of the periapsis of the initial orbit. We will use h^2 and $a(1-e^2)$ interchangeably in the following discussions and equations, depending on the context. Thus we have the familiar form for the zeroth order solution:

$$u_0(\tau) = \frac{1}{h^2}[1 + e\cos(\tau)]. \quad (97)$$

The first order equation now becomes:

$$\ddot{u}_1 + u_1 = \frac{-2p_1 e}{h^2}\cos(\tau) + \frac{1}{h^2}\bar{F}_u(u_0). \quad (98)$$

The forcing term $\bar{F}_u(u_0)$ is now periodic in τ and can be expanded into a cosine series:

$$\bar{F}_u(u_0) = \begin{cases} \sum_{k=0}^{\infty} A_k \cos(k\tau) & u < 1 \\ \sum_{k=0}^{\infty} B_k \cos(k\tau) & u > 1. \end{cases} \quad (99)$$

See the Appendix for the derivations and forms of these coefficients. In the following results we use the notation:

$$C_k = \begin{cases} A_k & u < 1 \\ B_k & u > 1. \end{cases} \quad (100)$$

Then the first order periodic solution becomes:

$$u_1(\tau) = \frac{1}{h^2}C_0[1 - \cos(\tau)] - \frac{1}{h^2}\sum_{k=2}^{\infty}\frac{C_k}{k^2-1}[\cos(k\tau) - \cos(\tau)] \quad (101)$$

The first order term in the frequency is obtained by casting out the $\cos(\tau)$ term in the forcing function in eqn (98):

$$p_1 = \frac{C_1}{2e}. \quad (102)$$

We have performed numerical simulations of orbits for the full equations of motion (6). Integrations start at periapsis and proceed to the next periapsis. This corresponds to a one period motion in the Poincaré–Lindstedt form of the approximate equations with the above initial conditions. The following quantities are easily extracted from the simulation: advance of the argument of the periapsis, period of motion, change in periapsis radius and change in the instantaneous angular momentum h at periapsis.

In the approximate equations of motion the periapsis radius and the angular momentum remain constant. In the numerical integrations we see that both the periapsis radius and the angular momentum at periapsis remain constant to a high degree of accuracy. These results become less true if P is small or if ρ is close to 1 at some point in the orbit. In general the angular momentum does vary in time, although for larger P the variations may become vanishingly small.

The advance in the argument of the periapsis, $\Delta\omega_o$ (not to be confused with the angular velocity of the ring ω), and the period of motion, T, can be extracted from the approximate solution just found to the first order in δ.

The advance in the argument of the periapsis is easily computed by comparing the total inertial angle η the orbit travels through in one period of motion with 2π, the total inertial angle the orbit would travel through if $\delta = 0$. From eqns (87) and (102) we see that the advance in the argument of the periapsis to the first order in δ is:

$$\Delta\omega_o = \frac{\pi C_1}{e}\delta. \quad (103)$$

If we expand the above to the order of $1/h^8$ or a^4 we have:

$\rho > 1 \quad (h^2 > 1)$

$$\Delta\omega_o = \tfrac{3}{2}\pi\delta\frac{1}{h^4}\left[1 + \frac{3}{8}\frac{1}{h^4}(1 + \tfrac{3}{4}e^2) + \frac{31}{64}\frac{1}{h^8}(1 + \tfrac{5}{2}e^2 + \tfrac{5}{8}e^4) + \cdots\right] \quad (104)$$

$\rho < 1 \quad (a < 1)$

$$\Delta\omega_o = \tfrac{3}{2}\pi\delta\sqrt{1-e^2}a^3[1 + \tfrac{15}{8}a^2(1 + \tfrac{3}{4}e^2) + \tfrac{175}{64}a^4(1 + \tfrac{5}{2}e^2 + \tfrac{5}{8}e^4) + \cdots]. \quad (105)$$

The period of motion must be computed by combining eqns (82) and (87) and then integrating over τ from 0 to 2π using eqns (89), (97) and (101) to express $\rho = 1/u$ as a function of τ. The resulting expression is a complicated formula. Keeping terms to the order of $1/h^4$, a^3 and e^4 we have:

$\rho > 1 \quad (h^2 > 1)$

$$T = 2\pi a^{3/2}\left[1 - \frac{\delta}{1-e}\left(2 + e + \frac{3}{4h^4}(3 + 3e + 2e^2 + e^3/2)\right) + \cdots\right] \quad (106)$$

$\rho < 1 \quad (a < 1)$

$$T = 2\pi a^{3/2}\left[1 + \frac{\delta a^3}{\sqrt{1-e^2}} \times (\tfrac{11}{4} + 3e + \tfrac{5}{4}e^2 - \tfrac{17}{32}e^4 + \cdots)\right]. \quad (107)$$

Table 7. Comparisons between numerical and analytical results

δ	h^2	e	$\Delta\omega_0^c$	$\Delta\omega_0^a$	% Error	ΔT^c	ΔT^a	% Error
0.01	10	0.02	4.832 E-4	4.730 E-4	2.11	−4.021	−4.144	−3.06
0.01	25	0.5	7.639 E-5	7.545 E-5	1.23	−57.745	−60.607	−4.96
0.01	0.5	0.1	1.070 E-2	1.055 E-2	1.36	8.639 E-3	8.942 E-3	−3.51
0.0001	10	0.1	4.918 E-6	4.730 E-6	3.81	−4.725 E-2	−4.762 E-2	−0.78
0.0001	0.5	0.1	1.061 E-4	1.055 E-4	0.54	8.568 E-5	8.942 E-5	−4.37

Table 7 gives some comparison values between the approximate solution predictions and the numerical solution. The superscript c denotes the numerically computed values and the superscript a denotes the analytically computed values. The notation ΔT is defined as $\Delta T = T - 2\pi a^{3/2}$. The percentage error is computed by $100[(*)^c - (*)^a]/(*)^c$.

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APPENDIX

The force term for eqn (98) is expressed as:

$$F_u(u) = \frac{2}{\pi}\begin{cases} \frac{1}{u}K(1/u) - \frac{u}{u^2-1}E(1/u) & u > 1 \\ \frac{1}{1-u^2}E(u) & u < 1. \end{cases} \tag{108}$$

Using power series representations we have:

$$K(\alpha) = \frac{\pi}{2}\sum_{n=0}^{\infty} d_n^2 \alpha^{2n} \tag{109}$$

$$E(\alpha) = \frac{\pi}{2}\sum_{n=0}^{\infty} d_n^2 \frac{\alpha^{2n}}{2n-1} \tag{110}$$

$$d_n = \frac{(2n)!}{2^{2n}(n!)^2}. \tag{111}$$

Using the above we find that:

$$F_u(u) = \frac{2}{\pi}\begin{cases} -\sum_{n=1}^{\infty}(2n)d_n^2\frac{1}{u^{2n+1}} & u > 1 \\ \sum_{n=0}^{\infty}(2n+1)d_n^2 u^{2n} & u < 1. \end{cases} \tag{112}$$

Substituting the zeroth order solution $u_0(\tau)$ into the above series makes F_u an even periodic function in τ. Thus, it may be expressed in terms of a cosine series. To do that it is necessary to develop the following cosine series:

$$(1 + e\cos\tau)^{2n} = \sum_{m=0}^{\infty} a_m^{2n}\cos(m\tau) \tag{113}$$

$$\frac{1}{(1+e\cos\tau)^{2n+1}} = \sum_{m=0}^{\infty} b_m^{2n+1}\cos(m\tau) \tag{114}$$

where the coefficients are defined as follows:

$$a_0^n = c_0^n \tag{115}$$

$$a_k^n = 2\left(\frac{e}{2}\right)^k c_k^n \tag{116}$$

$$b_0^n = \frac{\sqrt{1-e^2}}{(1-e^2)^n} f_0^n \tag{117}$$

$$b_k^n = (-1)^k 2\left(\frac{e}{2}\right)^k \frac{\sqrt{1-e^2}}{(1-e^2)^n} f_k^n. \tag{118}$$

The coefficients c_k^n and f_k^n have the general definitions:

$$c_k^n = \begin{cases} \sum_{l=0}^{[(n-k)/2]} \frac{n!}{l!(l+k)!(n-k-2l)!}\left(\frac{e}{2}\right)^{2l} & n \geq k \\ 0 & n < k \end{cases} \tag{119}$$

$$f_k^{n+1} = \begin{cases} \sum_{l=0}^{[(n-k)/2]} \frac{(n-k)!(n+k)!}{n!l!(l+k)!(n-k-2l)!}\left(\frac{e}{2}\right)^{2l} & n+1 > k \\ \frac{n-k}{n}(1-e^2)f_k^n + 2f_{k-1}^{n+1} & n+1 \leq k \end{cases} \tag{120}$$

$$f_k^1 = \left(\frac{2}{1+\sqrt{1-e^2}}\right)^k \tag{121}$$

where $[a]$ denotes the integer part of a. Note that to compute f_k^{n+1} for $n+1 \leq k$ we must solve the recursion relation stated above.

Inserting these coefficients into eqn (112) and rearranging, we find the form:

$$F_u(u_0) = \begin{cases} \sum_{n=0}^{\infty} B_k\cos(k\tau) & u > 1 \\ \sum_{n=0}^{\infty} A_k\cos(k\tau) & u < 1. \end{cases} \tag{122}$$

Where the coefficients are defined as:

$$A_0 = \sum_{n=0}^{\infty}\frac{(2n+1)d_n^2}{h^{4n}}c_0^{2n} \tag{123}$$

$$B_0 = \sqrt{1-e^2}\sum_{n=1}^{\infty}(2n)d_n^2 a^{2n+1} f_0^{2n+1} \tag{124}$$

$$A_k = 2\left(\frac{e}{2}\right)^k \sum_{[k/2]}^{\infty}\frac{(2n+1)d_n^2}{h^{4n}}c_k^{2n} \tag{125}$$

$$B_k = (-1)^k 2\left(\frac{e}{2}\right)^k\sqrt{1-e^2}\sum_{n=1}^{\infty}(2n)d_n^2 a^{2n+1} f_k^{2n+1}. \tag{126}$$

Or in terms of order of magnitudes:

$$A_k = O((e/h^2)^k) \tag{127}$$

$$B_k = O(e^k). \tag{128}$$