ADVANCES IN MATHEMATICS 98, 27–64 (1993)

Determinants of Super-Schur Functions, Lattice Paths, and Dotted Plane Partitions

FRANCESCO BRENTI*

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109-1003

Contents.

1. INTRODUCTION

Let $\mathbf{x} = (x_1, x_2, ...)$ and $\mathbf{y} = (y_1, y_2, ...)$ be two sequences of independent variables and λ be a partition. We denote by

$$s_\lambda(x_1, x_2, .../y_1, y_2, ...)$$

the super-Schur function corresponding to λ in the variables $\mathbf{x}$ and $\mathbf{y}$. These functions arise naturally in the representation theory of Lie superalgebras [12, 13] and were also defined, independently, by Metropolis, Nicoletti, and Rota in [19], under the name of bisymmetric functions. Since then, they have been studied extensively and we refer the reader to [1, 2, 6] for their definition (they can be defined in several equivalent ways) and further information about them.

The purpose of the present work is to give combinatorial interpretations to the minors of the infinite matrix

$$\mathscr{S}(\mathbf{x}, \mathbf{y}) \overset{\text{def}}{=} (s_{(k)}(x_1, ..., x_n/y_1, ..., y_n))_{n, k \in \mathbf{N}}.$$

Our main results (Theorems 3.3 and 4.3) are proved combinatorially using lattice paths and are stated in terms of dotted and diagonal strict plane

* Partially supported by NSF Grant DMS 8903246.

0001-8708/93 $9.00

partitions, respectively. They also have many applications. As special cases we obtain combinatorial interpretations of determinants of homogeneous, elementary, and Hall–Littlewood symmetric functions, Schur's Q-functions, q-binomial coefficients, and q-Stirling numbers of both kinds. Other applications include the solution of a problem posed by Iwahori in [33] and the combinatorial interpretation of a class of symmetric functions first defined, algebraically, by Macdonald in [18]. Many of our results (including all those in Sections 7 and 8) are new even in the case $q=1$. Others are q-analogues of known results. Our main theorem also has several interesting applications to the theory of total positivity [14]. These are treated in [3].

We now collect some definitions, notation, and results that will be used in the rest of the paper. We let $\mathbf{P} =^{\text{def}} \{1, 2, 3, ...\}$ and $\mathbf{N} =^{\text{def}} \mathbf{P} \cup \{0\}$; for $a \in \mathbf{N}$ we let $[a] =^{\text{def}} \{1, 2, ..., a\}$ (where $[0] =^{\text{def}} \varnothing$). The cardinality of a set A will be denoted by $|A|$. For $i_1, ..., i_t \in \mathbf{P}$ we write $\{i_1, i_2, ..., i_t\}_<$ if $i_1 < i_2 < \cdots < i_t$. For $m, n \in \mathbf{P}$ with $m \leqslant n$ we let $[m, n] =^{\text{def}} [n] \setminus [m-1]$. Given a (finite) set S we denote by $\Pi(S)$ the set of all (set) partitions of S (see, e.g., [28, p. 33] for further information about partitions of a set). For $n \in \mathbf{P}$, we denote by S_n the symmetric group on n elements. Throughout this work, q will denote an indeterminate. For $n \in \mathbf{P}$ we let $[n]_q =^{\text{def}} 1 + q + q^2 + \cdots + q^{n-1}$, and $[0]_q =^{\text{def}} 0$. We follow [18, Chap. I] for notation and terminology related to partitions and symmetric functions. In particular, given a partition λ, we denote by λ' its conjugate, by $m_i(\lambda)$ the number of parts of λ that are equal to i, for $i \in \mathbf{P}$, and by s_λ (respectively e_λ, h_λ, p_λ) the *Schur* (respectively *elementary*, *complete homogeneous*, *power sum*) symmetric function associated to λ. We usually identify a partition $\lambda = (\lambda_1, ..., \lambda_r)$ with its diagram $\{(i, j) \in \mathbf{P} \times \mathbf{P} : 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant \lambda_i\}$.

We follow [26] for notation and terminology regarding plane partitions. However, we often need to work with more general objects than skew plane partitions, which we now define. Let $\lambda = (\lambda_1, ..., \lambda_r)$ and $\mu = (\mu_1, ..., \mu_r)$ be two partitions such that $\mu \subseteq \lambda$. A *shifted skew tabloid* of shape $\lambda \setminus \mu$ is an array $T = (T_{i,j})_{1 \leqslant i \leqslant r,\, \mu_i + i \leqslant j \leqslant \lambda_i + i - 1}$ of positive integers (we sometimes use the notation $sh(T) = \lambda \setminus \mu$). We call the sequence $(T_{1,1+\mu_1}, ..., T_{r,r+\mu_r})$ the *coshape* of T, and we denote by T_i the ith row of T, for $i = 1, ..., r$. We also let $|T| =^{\text{def}} \sum_{(i,j) \in sh(T)} T_{i,j}$. We call T a *shifted skew plane partition* if $T_{i,j} \geqslant T_{i+1,j}$ and $T_{i,j} \geqslant T_{i,j+1}$ whenever both sides of the inequality are defined. We say that T is *row strict* (respectively, *column strict*, *diagonal strict*) if $T_{i,j} > T_{i,j+1}$ (respectively, $T_{i,j} > T_{i+1,j}$, $T_{i,j} > T_{i+1,j+1}$) whenever both sides of the inequality are defined. Given a shifted skew plane partition T as above we let $\bar{T} =^{\text{def}} (T_{i,j})_{1 \leqslant i \leqslant r,\, \mu_i + i + 1 \leqslant j \leqslant \lambda_i + i - 1}$, $\bar{\bar{T}} =^{\text{def}} (T_{i,j})_{1 \leqslant i \leqslant r,\, \mu_i + i \leqslant j \leqslant \lambda_i + i - 2}$, and $\hat{T} =^{\text{def}} (T_{i,j} + 1)_{1 \leqslant i \leqslant r,\, \mu_i + i \leqslant j \leqslant \lambda_i + i}$ (where $T_{i,i+\lambda_i} =^{\text{def}} 0$, for $i = 1, ..., r$). If $\mu = \varnothing$ then we call T a *shifted plane partition* of shape λ. Note that we do *not* require the parts of λ to be distinct. Note that if both μ and λ have *distinct* parts then a shifted skew plane partition of shape $\lambda\backslash\mu$ is just a skew plane partition of shape $(\lambda_1, \lambda_2+1, ..., \lambda_r+r-1)\backslash(\mu_1, \mu_2+1, ..., \mu_r+r-1)$. Also note that a strict Gelfand pattern of length n (as defined, e.g., in [29], or [7]) with given first row, is equivalent to a diagonal strict shifted plane partition of shape $(n, n-1, ..., 3, 2, 1)$ and given coshape. Therefore diagonal strict shifted plane partitions are a natural generalization of strict Gelfand patterns.

Given an infinite matrix $M=(M_{n,k})_{n,k\in\mathbf{N}}$ (where $M_{n,k}$ is the entry in the nth *row* and kth *column* of M) and $\{n_1, ..., n_r\}_<$, $\{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ we let

$$M\begin{pmatrix} n_1, ..., n_r \\ k_1, ..., k_r \end{pmatrix} \stackrel{\text{def}}{=} \det[(M_{n_i,k_j})_{1\leqslant i,j\leqslant r}].$$

Given an infinite sequence $\{a_i\}_{i\in\mathbf{N}}$ we let

$$\{a_i\}_{i\in\mathbf{N}}\begin{pmatrix} n_1, ..., n_r \\ k_1, ..., k_r \end{pmatrix} \stackrel{\text{def}}{=} A\begin{pmatrix} n_1, ..., n_r \\ k_1, ..., k_r \end{pmatrix},$$

where $A =^{\text{def}} (a_{n-k})_{n,k\in\mathbf{N}}$ (where $a_i =^{\text{def}} 0$ if $i<0$).

Let $D=(V, A)$ be a *directed graph* (or, *digraph*, for short). We always assume that D has no loops or multiple edges, so that we can identify the elements of A with ordered pairs (u, v), with $u, v\in V$, $u\neq v$. A *path* in D is a sequence $\tau=u_1, u_2, ..., u_n$ of elements of V such that $(u_i, u_{i+1})\in A$ for $i=1, ..., n-1$, we then say that τ goes from u_1 to u_n. We say that D is *locally finite* if for every $u, v\in V$ there are only a finite number of paths from u to v. Note that this implies that D is acyclic. We say that D is *weighted* if there is a function $w\colon A\to R$, where R is some commutative $\mathbf{Q}$-algebra. Let $D=(V, A, w)$ be a locally finite, weighted, digraph. For a path $\pi=u_0u_1\cdots u_k$ in D we let

$$w(\pi) \stackrel{\text{def}}{=} \prod_{i=1}^{k} w(u_{i-1}, u_i),$$

and for $u, v\in V$ we let

$$P_D(u, v) \stackrel{\text{def}}{=} \sum_{\pi} w(\pi),$$

where the sum is over all paths π in D going from u to v. We adopt the convention that $P_D(u, u) =^{\text{def}} 1$ for all $u\in V$ (i.e., there is only one path, the empty path, from u to u and its weight is 1). We usually omit the subscript D when there is no danger of confusion. Given $\mathbf{u} =^{\text{def}} (u_1, ..., u_r)$, $\mathbf{v} =^{\text{def}} (v_1, ..., v_r)\in V^r$ we let

$$N(\mathbf{u}, \mathbf{v}) \stackrel{\text{def}}{=} \sum_{(\pi_1, ..., \pi_r)} w(\pi_1, ..., \pi_r),$$

where $w(\pi_1, ..., \pi_r) =^{\text{def}} \prod_{i=1}^{r} w(\pi_i)$ and where the sum is over all r-tuples of paths $(\pi_1, ..., \pi_r)$ from $\mathbf{u}$ to $\mathbf{v}$ (i.e., π_i is a path from u_i to v_i, for $i = 1, ..., r$) that are non-intersecting (i.e., π_i and π_j have no vertices in common if $i \neq j$). We say that $\mathbf{u}$ and $\mathbf{v}$ are *compatible* if, for every $\sigma \in S_r \setminus \{Id\}$, there are no r-tuples of paths from $(u_1, ..., u_r)$ to $(v_{\sigma(1)}, ..., v_{\sigma(r)})$ that are nonintersecting. The following fundamental result was first proved by Lindström in [17] and has later found numerous applications in enumerative combinatorics (see, e.g., [8, 9, 22, 31]). We refer the reader to [9, Corollary 2; 31, Theorem 1.2; 17, Lemma 1] for its proof.

LEMMA 1.1. *Let $D = (V, A, w)$ be a locally finite, weighted, digraph and $\mathbf{u} =^{\text{def}} (u_1, ..., u_n)$, $\mathbf{v} =^{\text{def}} (v_1, ..., v_n) \in V^n$ be compatible. Then*

$$N(\mathbf{u}, \mathbf{v}) = \det[(P_D(u_i, v_j))_{1 \leqslant i, j \leqslant n}].$$

2. INVARIANT DIGRAPHS AND SUPER-SCHUR FUNCTIONS

Let $D = (\mathbf{N} \times \mathbf{N}, A, w)$ be a, locally finite, weighted digraph. We say that D is *weakly y-invariant* if

$$P_D((0, m), (n, k)) = P_D((0, 0), (n, k - m)) \tag{1}$$

for all $m, n, k \in \mathbf{N}$ with $k \geqslant m$. We say that D is *y-invariant* if the map $S_y \colon D \to D$ defined by

$$S_y((n, k)) \stackrel{\text{def}}{=} (n, k + 1)$$

is an isomorphism between D and $S_y(D)$ (as weighted digraphs). The following result follows easily from (1) and Lemma 1.1.

THEOREM 2.1. *Let D be a locally finite, weighted, digraph on the vertex set $\mathbf{N} \times \mathbf{N}$. Assume that D is planar and weakly y-invariant. Let $\{n_1, ..., n_r\}_<$, $\{k_1, ..., k_r, m\}_< \subseteq \mathbf{N}$, and $n \in \mathbf{N}$. Then*

$$\begin{aligned}(P_D((0, 0), (n, k)))_{n, k \in \mathbf{N}} &\begin{pmatrix} n_1, ..., n_r \\ k_1, ..., k_r \end{pmatrix} \\ &= \mathbf{N}(((0, m - k_1), ..., (0, m - k_r)), ((n_1, m), ..., (n_r, m)))\end{aligned} \tag{2}$$

and

$$\begin{aligned}\{P_D((0, 0), (n, k))\}_{k \in \mathbf{N}} &\begin{pmatrix} n_1, ..., n_r \\ k_1, ..., k_r \end{pmatrix} \\ &= N(((0, n_1), ..., (0, n_r)), ((n, k_1), ..., (n, k_r))).\end{aligned} \tag{3}$$

We now come to the connection between invariant digraphs and super-Schur functions.

PROPOSITION 2.2. *There exists a y-invariant, locally finite, weighted digraph D on* $\mathbf{N} \times \mathbf{N}$, *such that*

$$P_D((0, 0), (n, k)) = s_{(k)}(x_1, ..., x_n / y_1, ..., y_n)$$

for all $(n, k) \in \mathbf{N} \times \mathbf{N}$.

Proof. We construct D as follows. We put an edge from $(n-1, k)$ (respectively, $(n-1, k-1)$, $(n, k-1)$) to (n, k) with weight 1 (respectively, y_n, x_n) whenever $n \in \mathbf{P}$ and $(n-1, k)$ (respectively, $(n-1, k-1)$, $(n, k-1)$) is in $\mathbf{N} \times \mathbf{N}$. A portion of D is shown in Fig. 1. It is clear that D is y-invariant and locally finite.

Furthermore, it follows from our definition that

$$\begin{aligned} P_D((0, 0), (n, k)) = x_n P_D((0, 0), (n, k-1)) \\ + y_n P_D((0, 0), (n-1, k-1)) \\ + P_D((0, 0), (n-1, k)) \end{aligned} \tag{4}$$

for all $n \in \mathbf{P}$, $k \in \mathbf{N}$. Therefore

$$M_n(t) = t x_n M_n(t) + y_n t M_{n-1}(t) + M_{n-1}(t)$$

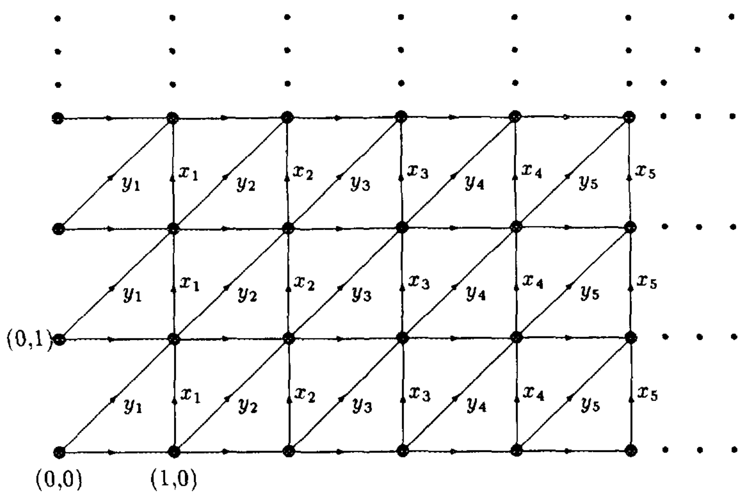

FIG. 1. The digraph D.

for $n \in \mathbf{P}$, where $M_n(t) =^{\text{def}} \sum_{k \geq 0} P_D((0,0),(n,k))t^k$, and hence

$$M_n(t) = \prod_{i=1}^{n} \frac{1+y_i t}{1-x_i t},$$

for $n \in \mathbf{N}$. On the other hand, it is well known (see, e.g., [27, p. 271, Eq. (9)]) that

$$s_{(k)}(x_1, \ldots, x_m/y_1, \ldots, y_n) = \sum_{j=0}^{k} h_j(x_1, \ldots, x_m) e_{k-j}(y_1, \ldots, y_n). \tag{5}$$

Hence, using Eqs. (2.2) and (2.5) of Chap. I of [18], we obtain that

$$\sum_{k \geq 0} s_{(k)}(x_1, \ldots, x_m/y_1, \ldots, y_n) t^k = \prod_{i=1}^{m} (1-x_i t)^{-1} \prod_{i=1}^{n} (1+y_i t), \tag{6}$$

and the result follows. ∎

From now on the letter D always denotes the weigted digraph defined in the proof of the preceding result. Proposition 2.2 and Theorem 2.1 combined already give a combinatorial interpretation for every minor of $\mathscr{S}(\mathbf{x}, \mathbf{y})$. However, we wish to find a more explicit combinatorial interpretation. This is done in the next section.

3. Determinants of Super-Schur Functions

A *dotted partition* is a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where, for each $i \in \mathbf{P}$ such that $m_i(\lambda) > 0$, the rightmost occurrence of i in λ may be dotted. So, for example, there are 8 dotted partitions of 3, namely

$$3 \quad \dot{3} \quad 21 \quad 2\dot{1} \quad \dot{2}\dot{1} \quad \dot{2}1 \quad 11\dot{1} \quad 111.$$

In general, there are $2^{d(\lambda)}$ ways to dot a partition λ and hence there are $\sum_{\lambda \vdash n} 2^{d(\lambda)}$ dotted partitions of n, where $d(\lambda)$ is the number of distinct parts of λ.

The importance of dotted partitions lies in the fact that they offer a very convenient way to encode paths in the digraph D. Given a path τ in D from (a, b) to (c, d) we let, for $i = a, \ldots, c$,

$$m_\tau(i) \stackrel{\text{def}}{=} \min\{j : (i, j) \in \tau\},$$

and

$$M_\tau(i) \stackrel{\text{def}}{=} \max\{j : (i, j) \in \tau\}.$$

Given two (possibly) dotted integers we write $a \doteq b$ to indicate that they are equal as *dotted* integers, and $a = b$ if they are only equal as integers (so that, for example, $2 = \dot{2}$, $2 \doteq 2$, $\dot{2} \doteq \dot{2}$). We also write $(a+b)^{\cdot}$ instead of the more cumbersome

$$\dot{\overbrace{a+b}}.$$

PROPOSITION 3.1. *Let $m, b, c, d \in \mathbf{N}$ with $m > b, d$. Then there is a bijection between directed paths in D from $(0, m-b)$ to $(c, m-d)$ and dotted partitions into $c+1$ parts with smallest part $\doteq (d+1)^{\cdot}$ and largest part $= b+1$.*

Proof. Let τ be a directed path in D from $(0, m-b)$ to $(c, m-d)$. For each $0 \leqslant i \leqslant c$ let

$$\lambda_i \stackrel{\text{def}}{=} m + 1 - M_\tau(i). \tag{7}$$

Clearly

$$b + 1 = \lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_{c-1} \geqslant \lambda_c = d + 1.$$

Now let $j \in \mathbf{P}$ be a part of $\lambda_0, ..., \lambda_c$ and let $i =^{\text{def}} \max\{i : \lambda_i = j\}$ (so that λ_i is the rightmost occurrence of j in λ) then we dot λ_i if and only if either $i = c$ or the step of τ leaving $(i, M_\tau(i))$ is a diagonal one. This defines a dotted partition satisfying the given conditions. Conversely, let $\lambda = (b + 1 = \lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_c \doteq (d+1)^{\cdot})$ be a dotted partition. For $i = 1, ..., c$ we let

$$m_i \stackrel{\text{def}}{=} \begin{cases} m + 1 - \lambda_{i-1} + 1, & \text{if } \lambda_{i-1} \text{ is dotted in } \lambda, \\ m + 1 - \lambda_{i-1}, & \text{otherwise,} \end{cases} \tag{8}$$

and $m_o =^{\text{def}} m - b$. Note that this implies that

$$m_i \leqslant m + 1 - \lambda_i$$

for $i = 0, ..., c$ (since if λ_{i-1} is dotted in λ then $\lambda_i < \lambda_{i-1}$). Hence we may define a path τ by

$$\tau \stackrel{\text{def}}{=} \bigcup_{i=0}^{c} \{(i, j) : m_i \leqslant j \leqslant m + 1 - \lambda_i\}.$$

It is then straightforward to verify that τ is indeed a path in D from $(0, m-b)$ to $(c, m-d)$ and that the two maps constructed are inverses of each other and this completes the proof. ∎

We illustrate the above construction with an example. Suppose $d=0$, $b=c=5$, $m=7$, and let τ be the path from $(0, 2)$ to $(5, 7)$ depicted in Fig. 2. Then the corresponding dotted partition is $\dot{6}554\dot{3}\dot{1}$.

Given a partition $\lambda=(\lambda_1, ..., \lambda_r)$ a *shifted dotted plane partition* of shape λ is an array of (possibly dotted) positive integers $\pi=(\pi_{i,j})_{1 \leqslant i \leqslant r, i \leqslant j \leqslant i+\lambda_i-1}$ where each row is a dotted partition and $\pi_{i,j} > \pi_{i+1,j}$ whenever $\pi_{i,j}$ and $\pi_{i+1,j}$ are both defined and $\pi_{i+1,j}$ is *not* dotted. Note that we *do not* require the parts of λ to be distinct. For example,

$$\begin{array}{cccccc} 5 & 5 & 3 & 3 & \dot{3} & \\ & \dot{5} & \dot{3} & 2 & 2 & \dot{2} \\ & & \dot{3} & \dot{2} & 1 & \\ & & & 1 & & \end{array}$$

is a shifted dotted plane partition of shape $(5, 5, 3, 1)$. Let π be a shifted dotted plane partition as above. For $k=1, ..., \lambda_1$ we let

$$t_k(\pi) \stackrel{\text{def}}{=} \sum_{i=1}^{d_k(\pi)} \pi_{i,i+k-1}, \tag{9}$$

where $d_k(\pi) =^{\text{def}} |\{i \in \mathbf{P} : \pi_{i,i+k-1} > 0\}|$, and

$$\dot{d}_k(\pi) \stackrel{\text{def}}{=} |\{i \in \mathbf{P} : \pi_{i,i+k-1} \text{ is dotted}\}|.$$

So $d_k(\pi)$ is the size of the kth diagonal of π, $\dot{d}_k(\pi)$ is the number of dots on the kth diagonal, and $t_k(\pi)$ is the sum of the parts on the kth diagonal. Note that $d_1(\pi) \geqslant d_2(\pi) \geqslant \cdots \geqslant d_{\lambda_1}(\pi) > 0$ is the conjugate

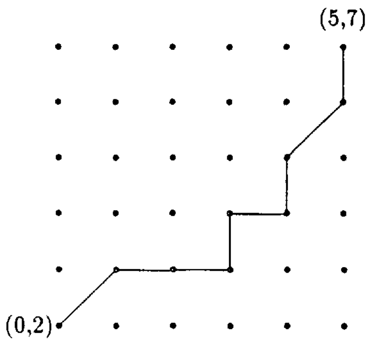

FIGURE 2

partition to $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$. Also, given π as above we let $\tilde{\pi} =^{\text{def}} (\pi_{ij})_{1 \leqslant i \leqslant r;\, i \leqslant j \leqslant i+\lambda_i-2}$. Note that this implies that

$$d_i(\tilde{\pi}) = d_{i+1}(\pi) \tag{10}$$

for all $i \in \mathbf{N}$. We also let

$$t(\pi) \stackrel{\text{def}}{=} (t_1(\pi), ..., t_{\lambda_1}(\pi), 0, 0, ...),$$
$$d(\pi) \stackrel{\text{def}}{=} (d_1(\pi), ..., d_{\lambda_1}(\pi), 0, 0, ...),$$

and

$$\dot{d}(\pi) \stackrel{\text{def}}{=} (\dot{d}_1(\pi), ..., \dot{d}_{\lambda_1}(\pi), 0, 0, ...).$$

Given a set of variables $\mathbf{x} = (x_1, x_2, x_3, ...)$ and a vector $d = (d_1, d_2, d_3, ...)$ of integers we let

$$\mathbf{x}^d \stackrel{\text{def}}{=} \prod_{i \geqslant 1} x_i^{d_i},$$

and $S(d) =^{\text{def}} (d_2, d_3, ...)$. Finally, we define the *weight* of π to be

$$w(\pi) \stackrel{\text{def}}{=} \mathbf{y}^{\dot{d}(\tilde{\pi})} \mathbf{x}^{t(\tilde{\pi}) - \dot{d}(\tilde{\pi}) - S(t(\pi))}.$$

For example, if π is

$$\begin{array}{ccccccc} 8 & \dot{8} & 7 & 6 & \dot{5} & \dot{4} & \dot{1} \\ & \dot{6} & 5 & 5 & 4 & \dot{2} & \dot{1} \\ & & \dot{5} & 2 & \dot{1} & & \end{array} \tag{11}$$

then $t(\pi) = (19, 15, 13, 10, 7, 5, 1, 0, 0, ...)$,

$$\tilde{\pi} = \begin{array}{cccccc} 8 & \dot{8} & 7 & 6 & \dot{5} & \dot{4} \\ & \dot{6} & 5 & 5 & 4 & \dot{2}, \\ & & \dot{5} & 2 & & \end{array}$$

$$d(\tilde{\pi}) = (3, 3, 2, 2, 2, 1, 0, 0, ...),$$
$$\dot{d}(\tilde{\pi}) = (2, 1, 0, 0, 2, 1, 0, 0, ...),$$

and

$$t(\tilde{\pi}) = (19, 15, 12, 10, 7, 4, 0, 0, ...).$$

Hence $S(t(\pi)) = (15, 13, 10, 7, 5, 1, 0, 0, ...)$,

$$t(\tilde{\pi}) - \dot{d}(\tilde{\pi}) - S(t(\pi)) = (2, 1, 2, 3, 0, 2, 0, 0, 0, ...),$$

and

$$w(\pi) = x_1^2 x_2 x_3^2 x_4^3 x_6^2 y_1^2 y_2 y_5^2 y_6.$$

Shifted dotted reverse plane partitions (which are defined by reversing the inequalities in every row and column) have already been considered in the literature under the name of shifted $\mathbf{P}'$-tableaux (or shifted circled tableaux) in connection with the projective representations of the symmetric group (see, e.g., [30, 21, 2]) a fact which is not surprising since the super-Schur functions are connected to the characters of these representations (see (29) and the comments following Theorem 5.8). It is interesting to note, however, that they arise here from purely combinatorial considerations.

THEOREM 3.2. *Let* $\{b_1, ..., b_r\}_> \subseteq \mathbf{N}$ *and* $c_1 \geqslant \cdots \geqslant c_r$, m, $d_1 \geqslant \cdots \geqslant d_r \in \mathbf{N}$ *with* $m > \max\{b_1, d_1\}$ $(r \in \mathbf{P})$. *Then there is a weight preserving bijection between non-intersecting paths from* $((0, m-b_1), ..., (0, m-b_r))$ *to* $((c_1, m-d_1), ..., (c_r, m-d_r))$ *in* D *and shifted dotted plane partitions of shape* $(c_1+1, ..., c_r+1)$ *in which the* i*th row has smallest part* $\doteq (d_i+1)^{\cdot}$ *and largest part* $= b_i + 1$, *for* $i = 1, ..., r$.

Proof. Let $(\tau_1, ..., \tau_r)$ be a set of non-intersecting paths in D from $((0, m-b_1), ..., (0, m-b_r))$ to $((c_1, m-d_1), ..., (c_r, m-d_r))$ (so τ_i goes from $(0, m-b_i)$ to $(c_i, m-d_i)$, for $i = 1, ..., r$). By Proposition 3.1 we can associate to τ_i a dotted partition $\lambda(\tau_i)$ into c_i+1 parts with smallest part $\doteq (d_i+1)^{\cdot}$ and largest part $= b_i+1$. We form a shifted dotted plane partition π by letting $\lambda(\tau_i)$ be the ith row of it and shifting the resulting array. We only have to check that the conditions on the columns of π are satisfied. For this note that the paths τ_i and τ_{i+1} are non-intersecting if and only if

$$m_{\tau_{i+1}}(j) > M_{\tau_i}(j) \tag{12}$$

for each $1 \leqslant j \leqslant c_i$. But, by our definitions,

$$m_{\tau_{i+1}}(j) = \begin{cases} M_{\tau_{i+1}}(j-1), & \text{if } \lambda_{j-1}(\tau_{i+1}) \text{ is not dotted,} \\ M_{\tau_{i+1}}(j-1)+1, & \text{otherwise.} \end{cases}$$

Hence we conclude from (12) that

$$M_{\tau_i}(j) < \begin{cases} M_{\tau_{i+1}}(j-1), & \text{if } \lambda_{j-1}(\tau_{i+1}) \text{ is not dotted,} \\ M_{\tau_{i+1}}(j-1)+1, & \text{otherwise.} \end{cases}$$

But, by (7), this means that

$$\lambda_j(\tau_i) > \begin{cases} \lambda_{j-1}(\tau_{i+1}), & \text{if } \lambda_{j-1}(\tau_{i+1}) \text{ is not dotted,} \\ \lambda_{j-1}(\tau_{i+1})-1, & \text{otherwise,} \end{cases} \tag{13}$$

for $1 \leqslant j \leqslant c_i$. But, by the definition of π, $\pi_{ij} \doteq \lambda(\tau_i)_{j-i}$ for $i \leqslant j \leqslant i + c_i$, hence (13) is exactly the condition for π to be a dotted plane partition. Hence $(\tau_1, ..., \tau_r)$ are non-interesting if and only if π is a dotted plane partition. To prove that this bijection is weight preserving fix $1 \leqslant j \leqslant r$ and observe that

$$w(\tau_j) = \left(\prod_{i \in S_j} y_i\right) \prod_{i=1}^{c_j} x_i^{M_{\tau_j}(i) - m_{\tau_j}(i)}, \tag{14}$$

where S_j is the set of those $i \in [c_j]$ such that the step of τ_j leaving $(i-1, M_{\tau_j}(i-1))$ is a diagonal one. Then, by our definitions we have that $S_j = \{i \in [c_j] : \pi_{j, j+i-1} \text{ is dotted}\}$ and, by (7) and (8), there follows that

$$M_{\tau_j}(i) - m_{\tau_j}(i) = \begin{cases} \pi_{j, j+i-1} - \pi_{j, j+i} - 1, & \text{if } \ i \in S_j, \\ \pi_{j, j+i-1} - \pi_{j, j+i}, & \text{otherwise,} \end{cases}$$

for $i = 1, ..., c_j$. Hence, by (14),

$$w(\tau_j) = \prod_{i \in S_j} \frac{y_i}{x_i} \prod_{i=1}^{c_j} x_i^{\pi_{j, j+i-1} - \pi_{j, j+i}}.$$

Therefore

$$\begin{aligned} w(\tau_1, ..., \tau_r) &= \prod_{j=1}^{r} w(\tau_j) \\ &= \prod_{j=1}^{r} \left(\prod_{i=1}^{c_j} x_i^{\pi_{j, j+i-1} - \pi_{j, j+i}} \prod_{i \in S_j} \frac{y_i}{x_i} \right) \\ &= \prod_{i=1}^{c_1} \left(x_i^{t_i(\tilde{\pi}) - t_{i+1}(\pi)} \left(\frac{y_i}{x_i}\right)^{d_i(\tilde{\pi})} \right), \end{aligned}$$

since

$$\sum_{j=1}^{d_i(\tilde{\pi})} (\pi_{j, j+i-1} - \pi_{j, j+i}) = \sum_{j=1}^{d_i(\tilde{\pi})} \pi_{j, j+i-1} - \sum_{j=1}^{d_{i+1}(\pi)} \pi_{j, j+i} = t_i(\tilde{\pi}) - t_{i+1}(\pi),$$

by (9) and (10). Therefore

$$\begin{aligned} w(\tau_1, ..., \tau_r) &= \prod_{i \geqslant 1} x_i^{t_i(\tilde{\pi}) - t_{i+1}(\pi) - \dot{d}_i(\tilde{\pi})} y_i^{\dot{d}_i(\tilde{\pi})} \\ &= \mathbf{x}^{t(\tilde{\pi}) - \dot{d}(\tilde{\pi}) - S(t(\pi))} \mathbf{y}^{\dot{d}(\tilde{\pi})} \\ &= w(\pi), \end{aligned}$$

as desired. ∎

We illustrate the preceding theorem with an example. Let $m = 8$ and π be the shifted dotted plane partition given in (11). Then the three non-intersecting paths corresponding to π are those depicted in Fig. 3.

Combining Theorems 3.2 and 2.1 and Proposition 2.2 we obtain the main results of this section.

THEOREM 3.3. *Let* $\{n_1, ..., n_r\}_>$, $\{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$\mathscr{S}(\mathbf{x}, \mathbf{y}) \begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} = \sum_{\pi} \mathbf{x}^{t(\bar{\pi}) - \dot{d}(\bar{\pi}) - S(t(\pi))} \mathbf{y}^{\dot{d}(\bar{\pi})}, \tag{15}$$

where the sum is over all shifted dotted plane partitions π *of shape* $(n_1 + 1, ..., n_r + 1)$ *in which the ith row has smallest part* $\doteq \dot{1}$ *and largest part* $= k_i + 1$ *for* $i = 1, ..., r$.

THEOREM 3.4. *Let* $\{m_1, ..., m_r\}_<$, $\{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ *and* $m, n \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$\{s_{(k)}(x_1, ..., x_n / y_1, ..., y_n)\}_{k \in \mathbf{N}} \begin{pmatrix} m_1, ..., m_r \\ k_1, ..., k_r \end{pmatrix}$$

$$= \sum_{\pi} \mathbf{x}^{t(\bar{\pi}) - \dot{d}(\bar{\pi}) - S(t(\pi))} \mathbf{y}^{\dot{d}(\bar{\pi})}, \tag{16}$$

where the sum is over all shifted dotted plane partitions π *of shape* $((n+1)^r)$ *in which the ith row has smallest part* $\doteq (m - k_i + 1)^{\cdot}$ *and largest part* $=$ $m - m_i + 1$, *for* $i = 1, ..., r$.

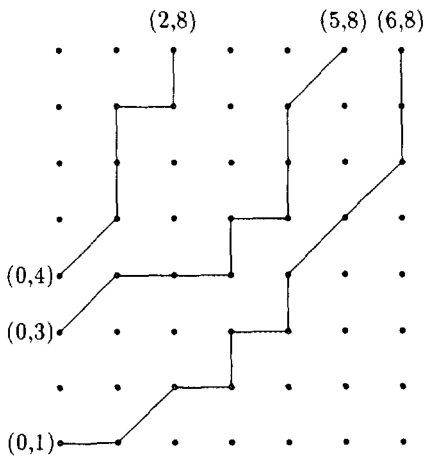

FIGURE 3

Since the minor on the LHS of (16) is just the skew super-Schur function corresponding to the skew shape $(m-m_1+1, ..., m-m_r+r)/(m-k_1+1, ..., m-k_r+r)$, Theorem 3.4 gives a combinatorial interpretation for these skew super-Schur functions. Other combinatorial interpretations have been obtained by Berele and Regev [2], Balantekin and Bars [1], Dondi and Jarvis [6], and Stanley [27] (regarding Stanley's combinatorial interpretation see also Theorem 4.4 of the present work).

4. The Combinatorics of Dotted Plane Partitions

In this section we derive some equivalent formulations of Theorems 3.3 and 3.4 which are sometimes more convenient to use. These will be obtained by a combinatorial analysis of shifted dotted plane partitions. First, it will be convenient to introduce some notation.

Let $T=(T_{i,j})_{1\leqslant i\leqslant r,\, i\leqslant j\leqslant i+n_i}$ be a shifted plane partition of shape $(n_1+1, ..., n_r+1)$. We call a part $T_{i,j}$, of T, *free* if $T_{i-1,j}>T_{i,j}>T_{i,j+1}$ (the inequalities being vacuously satisfied if either one of $T_{i-1,j}$ and $T_{i,j+1}$ are undefined). We let

$$\mathscr{F}(T) \stackrel{\text{def}}{=} \{(i,j)\in sh(T) : T_{i,j} \text{ is free}\},$$

and call $\mathscr{F}(T)$ the *free set* of T. Given T as above we define

$$\mathscr{U}(T) \stackrel{\text{def}}{=} \{(i,j)\in sh(T) : (i-1,j)\in sh(T),\ T_{i-1,j}=T_{i,j}\},$$

and call $\mathscr{U}(T)$ the *upper set* of T. We also let

$$\tilde{\mathscr{F}}(T) \stackrel{\text{def}}{=} \{(i,j)\in \mathscr{F}(T) : j<n_i+i\},$$

and

$$\tilde{\mathscr{U}}(T) \stackrel{\text{def}}{=} \{(i,j)\in \mathscr{U}(T) : j<n_i+i\}.$$

Note that, in general, $\tilde{\mathscr{F}}(T)\neq \mathscr{F}(\tilde{T})$ and $\tilde{\mathscr{U}}(T)\neq \mathscr{U}(\tilde{T})$. Given any subset $S\subseteq sh(T)$ and $k\in \mathbf{P}$ we let

$$S_k \stackrel{\text{def}}{=} |\{(i,j)\in S : j-i+1=k\}|.$$

Finally, we let

$$F(T) \stackrel{\text{def}}{=} (\mathscr{F}(T)_1, \mathscr{F}(T)_2, ...),$$
$$U(T) \stackrel{\text{def}}{=} (\mathscr{U}(T)_1, \mathscr{U}(T)_2, ...),$$

and define $\tilde{F}(T)$, $\tilde{U}(T)$ analogously. For example, if

$$T = \begin{array}{ccccc} 5 & 4 & 4 & 2 & \\ & 4 & 3 & 2 & 1 \\ & & 2 & 2 & \end{array}$$

then $F(T) = (1, 1, 1, 2, 0, \ldots)$, $U(T) = (1, 1, 1, 0, \ldots)$, $\tilde{F}(T) = (1, 1, 1, 0, \ldots)$, and $\tilde{U}(T) = (1, 0, 1, 0, \ldots)$.

Proposition 4.1. *Let $k_1 > \cdots > k_r$, $b_1 \geqslant \cdots \geqslant b_r$, $n_1 \geqslant \cdots \geqslant n_r$ be positive integers. Then there is a bijection between shifted dotted plane partitions π of shape $(n_1 + 1, \ldots, n_r + 1)$ in which the ith row has largest part $= k_i$ and smallest part $\doteq b_i$, for $i = 1, \ldots, r$; and pairs (T, S) where T is a diagonal strict shifted plane partition of shape $(n_1 + 1, \ldots, n_r + 1)$ in which the ith row has largest part $= k_i$ and smallest part $= b_i$ for $i = 1, \ldots, r$, and $S \subseteq \tilde{\mathscr{F}}_{\neq}(T)$.*

Proof. Let $\pi = (\pi_{i,j})_{1 \leqslant i \leqslant r,\, i \leqslant j \leqslant i + n_i}$ be a shifted dotted plane partition as in the statement of the proposition. We let the corresponding T be the shifted plane partition obtained by undotting π and

$$S(\pi) \stackrel{\text{def}}{=} \{(i, j) \in \tilde{\mathscr{F}}(T) : \pi_{i,j} \text{ is dotted}\}.$$

It is not hard to see that this map $\pi \mapsto (T, S(\pi))$ is a bijection between the desired sets. ▮

Let $T = (T_{i,j})_{1 \leqslant i \leqslant r,\, b_i + i \leqslant j \leqslant k_i + i - 1}$ be a shifted skew plane partition of shape $(k_1, \ldots, k_r) \backslash (b_1, \ldots, b_r)$. For $i \in \mathbf{P}$ we let $c_i(T)$ (respectively $r_i(T)$) be the number of columns (respectively rows) of T that contain at least one part equal to i, and $m_i(T)$ be the number of parts of T that are equal to i. We then let

$$c(T) \stackrel{\text{def}}{=} (c_1(T), c_2(T), \ldots),$$
$$r(T) \stackrel{\text{def}}{=} (r_1(T), r_2(T), \ldots),$$

and

$$m(T) \stackrel{\text{def}}{=} (m_1(T), m_2(T), \ldots).$$

For example, if T is

$$\begin{array}{ccccccc} & 4 & 4 & 3 & 3 & 2 & 1 \\ 5 & 4 & 4 & 2 & 2 & & \\ & 4 & 2 & 1 & & & \\ & & 2 & 1 & 1 & & \end{array}$$

then $c(T) = (3,4,2,2,1,0,\ldots)$, $r(T) = (3,4,1,3,1,0,\ldots)$, and $m(T) = (4,5,2,5,1,0,\ldots)$.

THEOREM 4.2. *Let $k_1 > \cdots > k_r$, $n_1 \geqslant \cdots \geqslant n_r$, $b_1 \geqslant \cdots \geqslant b_r$ be positive integers. Then there is a bijection between diagonal strict shifted plane partitions π of shape $(n_1+1, \ldots, n_r+1)$ in which the ith row has largest part $= k_i$ and smallest part $= b_i$, for $i = 1, \ldots, r$, and diagonal strict shifted skew plane partitions T of shape $(k_1, \ldots, k_r)\backslash(b_1, \ldots, b_r)$ in which the ith row has largest part $\leqslant n_i$ for $i = 1, \ldots, r$, and $\geqslant n_{i+1}+1$ if $b_i = b_{i+1}$, for $i = 1, \ldots, r-1$. Furthermore, if π and T correspond under the above bijection then*

$$t(\tilde{\pi}) - S(t(\pi)) = m(T), \tag{17}$$

$$\tilde{U}(\pi) + \tilde{F}(\pi) = r(T), \tag{18}$$

$$\tilde{U}(\pi) = m(T) - c(T). \tag{19}$$

Proof. Let π be a diagonal strict shifted plane partition satisfying the conditions in the statement of the theorem. We construct the corresponding T by first conjugating each row of π (so as to keep the resulting array, call it π', shifted) and then deleting the first b_i parts of the ith row of π', for $i = 1, \ldots, r$. It is then not hard to see that the resulting T has all the required properties.

Conversely, given a diagonal strict shifted skew plane partition $T = (T_{i,j})_{1 \leqslant i \leqslant r,\, b_i+i \leqslant j \leqslant k_i+i-1}$ such that $T_{i,i+b_i} \leqslant n_i$ for $i = 1, \ldots, r$, and $T_{i,i+b_i} \geqslant n_{i+1}+1$ if $b_i = b_{i+1}$, for $i = 1, \ldots, r-1$, we construct the corresponding $\pi = (\pi_{i,j})_{1 \leqslant i \leqslant r,\, i \leqslant j \leqslant i+n_i}$ by first adding b_i parts $= n_i+1$ at the beginning of the ith row of T, for $i = 1, \ldots, r$, and then conjugating each row (so as to keep the resulting array shifted). The only property of π that needs some care in order to be verified is diagonal strictness. For this note that it follows from our construction that, for $1 \leqslant i \leqslant r$, $i \leqslant j \leqslant i+n_i$,

$$\pi_{i,j} = |\{k : T_{i,k} \geqslant j-i+1\}| + b_i. \tag{20}$$

So fix $1 \leqslant i \leqslant r-1$ and $i \leqslant j \leqslant n_{i+1}+i$. We show that

$$\pi_{i,j} > \pi_{i+1,j+1}. \tag{21}$$

Assume first that $\{k : T_{i+1,k} \geqslant j-i+1\} \neq \varnothing$ and let $s =^{\text{def}} \max\{k : T_{i+1,k} \geqslant j-i+1\}$ (note that this implies that $s \leqslant k_{i+1}+i \leqslant k_i+i-1$), then by (20), $\pi_{i+1,j+1} = s-(b_{i+1}+i)+b_{i+1} = s-i$. If $s-i < b_i$ then (21) follows; so assume that $s-i \geqslant b_i$, then $T_{i,s} \geqslant T_{i+1,s} \geqslant j-i+1$, hence $|\{k : T_{i,k} \geqslant j-i+1\}| \geqslant s-(b_i+i-1)$, therefore $\pi_{i,j} \geqslant s-i+1$, and (21) follows. Assume now that $\{k : T_{i+1,k} \geqslant j-i+1\} = \varnothing$. If $b_i > b_{i+1}$ then (21) follows; so assume that $b_i = b_{i+1}$, then $T_{i,b_i+i} \geqslant n_{i+1}+1$, hence $T_{i,b_i+i} \geqslant j-i+1$, therefore $|\{k : T_{i,k} \geqslant j-i+1\}| \geqslant 1$ and (21) follows.

Now let $\pi=(\pi_{i,j})_{1\leqslant i\leqslant r,\, i\leqslant j\leqslant i+n_i}$ and $T=(T_{i,j})_{1\leqslant i\leqslant r,\, b_i+i\leqslant j\leqslant k_i+i-1}$ correspond under the above bijection. It follows from (20) that, for $1\leqslant i\leqslant r$ and $1\leqslant j\leqslant n_i$,

$$\pi_{i,j+i-1}-\pi_{i,j+i}=|\{k:T_{i,k}=j\}|, \tag{22}$$

and (17) follows. Also, it follows from (22) that $\pi_{i,j+i-1}>\pi_{i,j+i}$ if and only if there is at least a part equal to j in the ith row of T. Hence $\tilde{\mathscr{U}}(\pi)_j+\tilde{\mathscr{F}}(\pi)_j=r_j(T)$ and (18) also follows. Finally, let $2\leqslant i\leqslant r$, $i\leqslant j\leqslant i+n_i-1$, and $\max\{b_{i-1}, b_i+1\}\leqslant t\leqslant k_i$. We claim that

$$\pi_{i,j}=\pi_{i-1,j}=t, \tag{23}$$

if and only if

$$T_{i,t+i-1}=T_{i-1,t+i-1}=j-i+1. \tag{24}$$

In fact, if (23) holds, then from (20) we conclude that $T_{i,i+t-1}\geqslant j-i+1$, and $T_{i-1,t+i-1}\leqslant j-i+1$, and (24) follows. Conversely, if (24) holds, then this implies that $|\{k:\pi_{i,k}\geqslant t\}|=|\{k:\pi_{i-1,k}\geqslant t+1\}|=j-i+1$, hence $\pi_{i,j}\geqslant t$, and $\pi_{i-1,j}\leqslant t$, and (23) follows. This implies that

$$\tilde{\mathscr{U}}(\pi)_k=|\{(i,j)\in\mathscr{U}(T):T_{i,j}=k\}|,$$

and (19) follows. ∎

We illustrate the above bijection with an example. Suppose that

$$\pi=\begin{array}{cccccc} 9 & 8 & 6 & 5 & 4 & \\ & 8 & 6 & 5 & 3 & 2 \\ & & 6 & 4 & 3 & 2 \\ & & & 3 & 3 & 1 \end{array}$$

then

$$T=\begin{array}{cccccc} & 4 & 3 & 2 & 2 & 1 \\ 4 & 3 & 3 & 2 & 1 & 1 \\ & 3 & 2 & 1 & 1 & \\ & 2 & 2 & & & \end{array}$$

$$\begin{aligned} t(\tilde{\pi})-S(t(\pi)) &= (26, 21, 14, 8, 0, ...)-(21, 15, 10, 6, 0, ...) \\ &= (5, 6, 4, 2, 0, ...)=m(T), \\ \tilde{U}(\pi)+\tilde{F}(\pi) &= (2, 2, 2, 0, ...)+(1, 2, 1, 2, 0, ...) \\ &= (3, 4, 3, 2, 0, ...)=r(T), \end{aligned}$$

and

$$\begin{aligned} m(T) - c(T) &= (5, 6, 4, 2, 0, \ldots) - (3, 4, 2, 2, 0, \ldots) \\ &= (2, 2, 2, 0, \ldots) = \tilde{U}(\pi). \end{aligned}$$

We can now prove the main results of this section.

THEOREM 4.3. *Let* $\{n_1, \ldots, n_r\}_>$, $\{k_1, \ldots, k_r\}_> \subseteq \mathbf{N}$. *Then*

$$\mathscr{S}(\mathbf{x}, \mathbf{y}) \binom{n_r, \ldots, n_1}{k_r, \ldots, k_1} = \sum_T \mathbf{y}^{m(T) - c(T)} \mathbf{x}^{m(T) - r(T)} (\mathbf{y} + \mathbf{x})^{r(T) + c(T) - m(T)},$$

where the sum is over all diagonal strict shifted plane partitions T of shape $(k_1, \ldots, k_r)$ in which the ith row has largest part $\leqslant n_i$ and $\geqslant n_{i+1} + 1$, for $i = 1, \ldots, r$ (where $n_{r+1} \stackrel{\text{def}}{=} -1$).

Proof. By Theorem 3.3 and Proposition 4.1 we have that

$$\begin{aligned} \mathscr{S}(\mathbf{x}, \mathbf{y}) \binom{n_r, \ldots, n_1}{k_r, \ldots, k_1} &= \sum_T \mathbf{x}^{\iota(\tilde{T}) - S(\iota(T))} \sum_{S \subseteq \mathscr{F}(T)} \prod_{i \geqslant 1} \left(\frac{y_i}{x_i}\right)^{\tilde{\mathscr{U}}(T)_i + S_i} \\ &= \sum_T \mathbf{y}^{\tilde{\mathscr{U}}(T)} \mathbf{x}^{\iota(\tilde{T}) - S(\iota(T)) - \tilde{\mathscr{U}}(T)} \sum_{S \subseteq \mathscr{F}(T)} \prod_{i \geqslant 1} \left(\frac{y_i}{x_i}\right)^{S_i} \\ &= \sum_T \mathbf{y}^{\tilde{\mathscr{U}}(T)} \mathbf{x}^{\iota(\tilde{T}) - S(\iota(T)) - \tilde{\mathscr{U}}(T)} \prod_{i \geqslant 1} \left(1 + \frac{y_i}{x_i}\right)^{\mathscr{F}(T)_i}, \end{aligned}$$

where the sum is over all diagonal strict shifted plane partitions T of shape $(n_1 + 1, \ldots, n_r + 1)$ in which the ith row has smallest part $= 1$ and largest part $= k_i + 1$, for $i = 1, \ldots, r$. The thesis now follows from Theorem 4.2. ∎

Reasoning as in the proof of the preceding theorem but using Theorem 3.4 we obtain the following result.

THEOREM 4.4. *Let* $\{m_1, \ldots, m_r\}_<$, $\{k_1, \ldots, k_r\}_< \subseteq \mathbf{N}$ *and* $m, n \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$\begin{aligned} &\{s_{(k)}(x_1, \ldots, x_n / y_1, \ldots, y_n)\}_{k \in \mathbf{N}} \binom{m_1, \ldots, m_r}{k_1, \ldots, k_r} \\ &\quad = \sum_T \mathbf{y}^{m(T) - c(T)} \mathbf{x}^{m(T) - r(T)} (\mathbf{y} + \mathbf{x})^{r(T) + c(T) - m(T)}, \end{aligned}$$

where the sum is over all diagonal strict plane partitions T of shape $(m - m_1 + 1, \ldots, m - m_r + r) \backslash (m - k_1 + 1, \ldots, m - k_r + r)$ with largest part $\leqslant n$.

In the case that $k_i = i$, for $i = 1, \ldots, r$, the preceding theorem first appeared, though without proof, in [27, Theorem 5.2].

5. Determinants of Symmetric Functions

In this section we consider various determinants of symmetric functions. We begin by considering determinants of elementary symmetric functions. The next two results follow immediately from Theorems 4.3 and 4.4, and the fact that, by (5),

$$e_k(y_1, ..., y_n) = s_{(k)}(\mathbf{0}/y_1, ..., y_n).$$

Corollary 5.1. *Let* $\{n_1, ..., n_r\}_>$, $\{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$(e_k(y_1, ..., y_n))_{n,\, k \in \mathbf{N}} \begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} = \sum_T \mathbf{y}^{m(T)},$$

where the sum is over all row strict shifted plane partitions T of shape $(k_1+1, ..., k_r+1)$ and coshape $(n_1+1, ..., n_r+1)$.

Corollary 5.2. *Let* $\{m_1, ..., m_r\}_<$, $\{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ *and* $m, n \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$\{e_k(y_1, ..., y_n)\}_{k \in \mathbf{N}} \begin{pmatrix} m_1, ..., m_r \\ k_1, ..., k_r \end{pmatrix} = \sum_T \mathbf{y}^{m(T)}, \tag{25}$$

where the sum is over all row strict skew plane partitions T of shape $(m-m_1+1, ..., m-m_r+r)\backslash(m-k_1+1, ..., m-k_r+r)$ in which every part is $\leqslant n$.

The reader will recognize that the above corollary is the "conjugate" Jacobi–Trudi identity (see, e.g., [18, p. 40, Eq. (5.5)]) since the RHS of (25) is just the Schur function corresponding to the skew shape that is the conjugate of $(m-m_1+1, ..., m-m_r+r)\backslash(m-k_1+1, ..., m-k_r+r)$.

We now consider determinants of complete homogeneous symmetric functions. By (5),

$$h_k(x_1, ..., x_n) = s_{(k)}(x_1, ..., x_n/\mathbf{0}),$$

hence the following two results follow immediately from Theorems 4.3 and 4.4, respectively.

Corollary 5.3. *Let* $\{n_1, ..., n_r\}_>$, $\{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$(h_k(x_1, ..., x_n))_{n,\, k \in \mathbf{N}} \begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} = \sum_T \mathbf{x}^{m(T)},$$

where the sum is over all column strict shifted plane partitions T of shape $(k_1+1, ..., k_r+1)$ and coshape $(n_1, ..., n_r)$.

COROLLARY 5.4. *Let* $\{m_1, ..., m_r\}_<$, $\{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ *and* $m, n \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$\{h_k(x_1, ..., x_n)\}_{k \in \mathbf{N}} \binom{m_1, ..., m_r}{k_1, ..., k_r} = \sum_T \mathbf{x}^{m(T)},$$

where the sum is over all column strict skew plane partitions T *of shape* $(m-m_1+1, ..., m-m_r+r)\backslash(m-k_1+1, ..., m-k_r+r)$ *with largest part* $\leqslant n$.

The preceding result is the well known Jacobi–Trudi identity (see, e.g., [18, p. 40, Eq. (5.4)]).

We now want to interpret combinatorially the minors of the matrix $(h_{n-k}(x_1, ..., x_k))_{n, k \in \mathbf{N}}$. For this we find it convenient to use Theorem 3.3 instead of the more elegant Theorem 4.3.

THEOREM 5.5. *Let* $\{n_1, ..., n_r\}_>$, $\{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$(h_{n-k}(x_1, ..., x_k))_{n, k \in \mathbf{N}} \binom{n_r, ..., n_1}{k_r, ..., k_1} = \frac{1}{\mathbf{x}^{(k_1, ..., k_r)'}} \sum_T \mathbf{x}^{\Delta(t(T))}$$

where the sum is over all row strict shifted plane partitions T *of shape* $(k_1, .., k_r)$ *and coshape* $(n_1, ..., n_r)$.

Proof. It follows from (5) that

$$h_{n-k}(x_1, ..., x_k) = \left[y_1 \cdots y_k s_{(n)}\left(x_1, ..., x_k \Big/ \frac{1}{y_1}, ..., \frac{1}{y_k}\right)\right]_{|\mathbf{y}=\mathbf{0}}, \tag{26}$$

for all $n, k \in \mathbf{N}$. But, by Theorem 3.3 we have that

$$\begin{aligned}\prod_{i=1}^{r} & (y_1 \cdots y_{k_i})\, \mathscr{S}(\mathbf{x}, \mathbf{y}^{-1})^t \binom{n_r, ..., n_1}{k_r, ..., k_1} \\ &= \prod_{i=1}^{r} (y_1 \cdots y_{k_i}) \sum_\pi \mathbf{x}^{t(\tilde{\pi}) - \dot{d}(\tilde{\pi}) - S(t(\pi))} \mathbf{y}^{-\dot{d}(\tilde{\pi})} \\ &= \sum_\pi \mathbf{y}^{d(\tilde{\pi}) - \dot{d}(\tilde{\pi})} \mathbf{x}^{t(\tilde{\pi}) - \dot{d}(\tilde{\pi}) - S(t(\pi))},\end{aligned} \tag{27}$$

where the sum is over all shifted dotted plane partitions π of shape $(k_1+1, ..., k_r+1)$ in which the ith row has smallest part $\doteq \dot{1}$ and largest part $= n_i + 1$, for $i = 1, ..., r$. Letting now $\mathbf{y} = \mathbf{0}$ in (27) we see that the only π which give a non-zero contribution are those for which $d_i(\tilde{\pi}) = \dot{d}_i(\tilde{\pi})$, for $i \geqslant 1$. These are exactly those π for which every part of $\tilde{\pi}$ (and hence of π) is dotted, and hence those π which are row strict. The result now follows from (26). ∎

Reasoning as in the proof of the preceding theorem, but using Theorem 3.4, we obtain the following result which, though less elegant than the equivalent Corollary 5.4, will be needed in Section 8 (see Theorem 8.2).

THEOREM 5.6. *Let* $\{m_1, ..., m_r\}_<$, $\{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ *and* $m, k \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$(h_{n-k}(x_1, ..., x_k))_{n \in \mathbf{N}} \binom{m_1, ..., m_r}{k_1, ..., k_r} = \frac{1}{(x_1 \cdots x_k)^r} \sum_T \mathbf{x}^{t(\bar{T}) - S(t(T))},$$

where the sum is over all row strict shifted plane partitions T of shape $((k+1)^r)$ *in which the ith row has smallest part* $= m - k_i + 1$ *and largest part* $= m - m_i + 1$.

We now consider determinants of certain symmetric functions that are essentially, the Hall–Littlewood symmetric functions. Let $\alpha \in \mathbf{R}$ and $P_\lambda(x_1, ..., x_n; \alpha)$ denote the Hall–Littlewood symmetric function corresponding to the partition λ (we refer the reader to [18, Chap III] for the definition and further information about Hall–Littlewood symmetric functions). Following [18, p. 106] we let, for $n \in \mathbf{P}$,

$$q_0(x_1, ..., x_n; \alpha) \stackrel{\text{def}}{=} 1$$

and

$$q_k(x_1, ..., x_n; \alpha) \stackrel{\text{def}}{=} (1 - \alpha) P_{(k)}(x_1, ..., x_n; \alpha)$$

for $k \in \mathbf{P}$.

THEOREM 5.7. *Let* $\{n_1, ..., n_r\}_>$, $\{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$(q_k(x_1, ..., x_n; \alpha))_{n, k \in \mathbf{N}} \binom{n_r, ..., n_1}{k_r, ..., k_1} = \sum_T \mathbf{x}^{m(T)} (-\alpha)^{|\mathscr{U}(T)|} (1 - \alpha)^{|\mathscr{F}(T)|}, \quad (28)$$

where the sum is over all diagonal strict shifted plane partitions T of shape $(k_1, ..., k_r)$ *in which the ith row has largest part* $\leqslant n_i$ *and* $\geqslant n_{i+1} + 1$, *for* $i = 1, ..., r$ (*where* $n_{r+1} \stackrel{\text{def}}{=} -1$).

Proof. It is known (see [18, p. 106, Eq.(2.10)]) that

$$\sum_{k \geqslant 0} q_k(x_1, ..., x_n; \alpha) t^k = \prod_{i=1}^{n} \frac{1 - \alpha x_i t}{1 - x_i t}.$$

Hence, we conclude from (6) that

$$q_k(x_1, ..., x_n; \alpha) = s_{(k)}(x_1, ..., x_n / -\alpha x_1, ..., -\alpha x_n). \quad (29)$$

Therefore

$$(q_k(x_1, ..., x_n; \alpha))_{n, k \in \mathbf{N}} \begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} = \mathscr{S}(\mathbf{x}, -\alpha\mathbf{x}) \begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix}$$

and the thesis follows from Theorem 4.3. ∎

Reasoning as in the proof of the preceding theorem but using Theorem 4.4, we obtain the following result.

THEOREM 5.8. *Let* $\{m_1, ..., m_r\}_<$, $\{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ *and* $m, n \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$\{q_k(x_1, ..., x_n; \alpha)\}_{k \in \mathbf{N}} \begin{pmatrix} m_1, ..., m_r \\ k_1, ..., k_r \end{pmatrix} = \sum_T \mathbf{x}^{m(T)}(-\alpha)^{|\mathscr{U}(T)|} (1-\alpha)^{|\mathscr{F}(T)|}, \tag{30}$$

where the sum is over all diagonal strict skew plane partitions T *of shape* $(m - m_1 + 1, ..., m - m_r + r) \backslash (m - k_1 + 1, ..., m - k_r + r)$ *with largest part* $\leqslant n$.

Note that the symmetric function on the LHS of (30) is just the symmetric function $S_{\lambda/\mu}(\mathbf{x}; \alpha)$ defined (in the case that $\mu = \varnothing$) by Macdonald in [18, p. 116, Eq. (4.5)], where $\lambda =^{\text{def}} (m - m_1 + 1, ..., m - m_r + r)$ and $\mu =^{\text{def}} (m - k_1 + 1, ..., m - k_r + r)$. Therefore Theorem 5.8 gives a combinatorial interpretation of these symmetric functions. The symmetric functions $S_\lambda(\mathbf{x}; \alpha)$ possess many interesting properties including that of being a basis of $\Lambda[\alpha]$ orthogonal to the basis of the Schur functions, with respect to the inner product defined by

$$\langle p_\lambda, p_\mu \rangle \stackrel{\text{def}}{=} \delta_{\lambda,\mu} z_\lambda \prod_{i \geqslant 1} (1 - \alpha^{\lambda_i})^{-1}, \tag{31}$$

where $z_\lambda = \prod_{i \geqslant 1} i^{m_i(\lambda)} m_i(\lambda)!$. We refer the reader to [18, Chap. III, Sect. 4] for details.

A particularly interesting case of the preceding two theorems occurs when $\alpha = -1$. In fact, it follows from [18, p. 134, Ex. 8] (see also [31, Sect. 6]) that

$$q_k(x_1, ..., x_n; -1) = Q_{(k)}(x_1, ..., x_n),$$

where $Q_\lambda(x_1, ..., x_n)$ denotes the Schur's Q-function associated to the partition λ. These functions were originally defined by Schur [25] in connection with the study of the projective representations of the symmetric group and have since been widely studied (see, e.g., [11, 21, 31]).

6. q-Binomial Determinants

In this section we consider various determinants of q-binomial coefficients. These are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \stackrel{\text{def}}{=} \prod_{i=1}^{k} \frac{(1-q^{n-i+1})}{(1-q^i)}$$

(with the convention that $[{n \atop 0}]_q =^{\text{def}} 1$ for all $n \in \mathbf{N}$). We let

$$B(q) \stackrel{\text{def}}{=} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q \right)_{n,k \in \mathbf{N}}.$$

THEOREM 6.1. *Let* $\{n_1, ..., n_r\}_>$, $\{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$B(q) \begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} = q^{n((k_1+1, ..., k_r+1)')} \sum_T q^{|T|},$$

where the sum is over all row strict shifted plane partitions T *of shape* $(k_1+1, ..., k_r+1)$ *and coshape* $(n_1+1, ..., n_r+1)$ (*where for a partition* $\lambda = (\lambda_1, \lambda_2, ...)$, $n(\lambda) =^{\text{def}} \sum_{i \geq 1} (i-1)\lambda_i$).

Proof. It is well known (see, e.g., [18, p. 18, Ex. 3]) that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = h_{n-k}(1, q, q^2, ..., q^k). \tag{32}$$

Hence from Theorem 5.5 we obtain that

$$\begin{aligned} B(q) \begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} &= \sum_T \left(\prod_{i \geq 1} q^{-(i-1)d_i(T)} \right) \prod_{i \geq 1} q^{(i-1)(t_i(T) - t_{i+1}(T))} \\ &= \sum_T q^{\sum_{i \geq 1} (i-1)d_i(T)} q^{\sum_{i \geq 1} (i-1)(t_i(T) - t_{i+1}(T))}, \end{aligned}$$

where the sum is over all row strict shifted plane partitions T of shape $(k_1+1, ..., k_r+1)$ and coshape $(n_1+1, ..., n_r+1)$, and the thesis follows. ∎

In the case $q=1$ the preceding theorem was first proved (though stated in a slightly different way) by Gessel and Viennot [8, Corollary 11].

THEOREM 6.2. *Let* $\{m_1, ..., m_r\}_<$, $\{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ *and* $m, k \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$\left\{ \begin{bmatrix} n \\ k \end{bmatrix}_q \right\}_{n \in \mathbf{N}} \begin{pmatrix} m_1, ..., m_r \\ k_1, ..., k_r \end{pmatrix} = q^{-\sum_{i=1}^{r} (m_i + k - k_i)} \sum_T q^{|T|},$$

where the sum is over all column strict plane partitions T of shape $(m-m_1+1, \ldots, m-m_r+r)\setminus(m+k-k_1+1, \ldots, m+k-k_r+r)$ and largest part $\leqslant k+1$.

Proof. The result clearly holds if $k_1<k$, so assume that $k_1 \geqslant k$. From (32) we conclude that

$$\left\{\begin{bmatrix} n \\ k \end{bmatrix}_q\right\}_{n\in\mathbf{N}} \begin{pmatrix} m_1, \ldots, m_r \\ k_1, \ldots, k_r \end{pmatrix} = \{h_n(1, q, \ldots, q^k)\}_{n\in\mathbf{N}} \begin{pmatrix} m_1, \ldots, m_r \\ k_1-k, \ldots, k_r-k \end{pmatrix}$$

and the thesis follows from Corollary 5.4. ∎

We illustrate the above theorem with an example. Suppose $k=3$, $\{m_1, m_2, m_3\}_< \stackrel{\text{def}}{=} \{1, 2, 3\}$, $\{k_1, k_2, k_3\}_< \stackrel{\text{def}}{=} \{5, 6, 7\}$, and $m=8$. Then we are looking for column strict skew plane partitions of shape $(1, 1, 1)$ and largest part $\leqslant 4$. Clearly, there are 4 such plane partitions, namely

$$\begin{matrix} 4 \\ 3, \\ 2 \end{matrix} \quad \begin{matrix} 4 \\ 3, \\ 1 \end{matrix} \quad \begin{matrix} 4 \\ 2, \\ 1 \end{matrix} \quad \begin{matrix} 3 \\ 2. \\ 1 \end{matrix}$$

Therefore

$$\frac{\sum_T q^{|T|}}{q^{\sum_{i=1}^{r}(m_i+k-k_i)}} = \frac{q^6+q^7+q^8+q^9}{q^3} = q^3+q^4+q^5+q^6,$$

and hence,

$$\left\{\begin{bmatrix} n \\ 3 \end{bmatrix}_q\right\}_{n\in\mathbf{N}} \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} = \det \begin{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q & \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q & \begin{bmatrix} 6 \\ 3 \end{bmatrix}_q \\ \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q & \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q & \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q \\ 0 & \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q & \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q \end{bmatrix}$$

$$= q^3+q^4+q^5+q^6.$$

In [33] Iwahori asked for a combinatorial interpretation of the minors of the matrix

$$F(q) \stackrel{\text{def}}{=} \left(\begin{bmatrix} n+k \\ k \end{bmatrix}_q\right)_{n,\, k\in\mathbf{N}}$$

(we use the letter F since the matrix $(\binom{n+k}{k})_{n,\, k\in\mathbf{N}}$ is sometimes called the *Fermat matrix*, see, e.g., [5, p. 171]), and whether its principal minors form a periodic sequence. We can easily answer these questions using (32) and Corollary 5.3.

THEOREM 6.3. *Let* $\{n_1, ..., n_r\}_>, \{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$F(q)\begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} = q^{-\sum_{i=1}^{r} n_i} \sum_{T} q^{|\bar{T}|},$$

where the sum is over all column strict shifted plane partitions T *of shape* $(n_1+1, ..., n_r+1)$ *and coshape* $(k_1+1, ..., k_r+1)$.

In particular, we obtain the following result.

COROLLARY 6.4. *Let* $n \in \mathbf{N}$. *Then*

$$F(q)\begin{pmatrix} 0, 1, ..., n \\ 0, 1, ..., n \end{pmatrix} = q^{n(n+1)(2n+1)/6}.$$

Other determinants of q-binomial coefficients have been considered in [4, 15].

7. q-STIRLING DETERMINANTS OF THE FIRST KIND

In this section we consider determinants of q-Stirling numbers of the first kind. These were introduced by Gould in [10] and have been the subject of considerable research in recent years (see, e.g., [16, 20, 22, 24]). They are defined inductively by letting

$$c[n, k]_q \stackrel{\text{def}}{=} c[n-1, k-1]_q + [n-1]_q\, c[n-1, k]_q$$

for $n \in \mathbf{P}$, and $c[0, k]_q =^{\text{def}} \delta_{0,k}$ for $k \in \mathbf{N}$ (with the convention that $c[n, k]_q =^{\text{def}} 0$ if either $n < 0$ or $k < 0$). We let

$$C(q) \stackrel{\text{def}}{=} (c[n+1, k+1]_q)_{n,k \in \mathbf{N}}.$$

All the results in this section are new even in the case $q = 1$.

Let $T = (T_{i,j})_{1 \leqslant i \leqslant r,\, i+\mu_i \leqslant j \leqslant \lambda_i + i - 1}$ be a row strict shifted skew plane partition. We define the *row complement* of T, denoted $RC(T)$, to be the array having as ith row the elements of $[T_{i,i+\mu_i}] \setminus \{T_{i,i+\mu_i}, T_{i,i+\mu_i+1}, ..., T_{i,i+\lambda_i-1}\}$, in decreasing order, for $i = 1, ..., r$. For example, if

$$T = \begin{array}{ccccccc} 10 & 9 & 7 & 5 & 4 & 3 & 1 \\ & 9 & 7 & 5 & 4 & 1 & \\ & & 5 & 2 & & & \end{array}, \tag{33}$$

then

$$RC(T) = \begin{array}{cccc} 8 & 6 & 2 & \\ 8 & 6 & 3 & 2. \\ 4 & 3 & 1 & \end{array}$$

THEOREM 7.1. *Let* $\{n_1, ..., n_r\}_>, \{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$C(q)\binom{n_r, ..., n_1}{k_r, ..., k_1} = \sum_T \prod_{i \in RC(T)} [i]_q,$$

where the sum is over all row strict shifted plane partitions T *of shape* $(k_1+1, ..., k_r+1)$ *and coshape* $(n_1+1, ..., n_r+1)$.

Proof. It is well known (see, e.g., [22]) that

$$C[n+1, k+1]_q = e_{n-k}([1]_q, [2]_q, ..., [n]_q).$$

Hence, by Corollary 5.1,

$$\begin{aligned} &C(q)\binom{n_r, ..., n_1}{k_r, ..., k_1} \\ &\quad = \prod_{i=1}^{r}\left(\prod_{j=1}^{n_i}[j]_q\right)(e_k([1]_q^{-1}, ..., [n]_q^{-1}))_{n,k \in \mathbf{N}}\binom{n_r, ..., n_1}{k_r, ..., k_1} \\ &\quad = \prod_{i=1}^{r}\prod_{j=1}^{n_i}[j]_q \sum_T \prod_{i \geqslant 1}[i]_q^{-m_i(T)}, \end{aligned}$$

where the sum is over all row strict shifted plane partitions T of shape $(k_1+1, ..., k_r+1)$ and coshape $(n_1+1, ..., n_r+1)$, and the thesis follows. ∎

We illustrate the above theorem with an example. Suppose that $\{k_1, k_2, k_3\}_> =^{\text{def}} \{2, 1, 0\}$, and $\{n_1, n_2, n_3\}_> =^{\text{def}} \{5, 4, 3\}$. Then there is only one row strict shifted plane partition T of shape $(3, 2, 1)$ and coshape $(6, 5, 4)$, namely

$$\begin{array}{ccc} 6 & 5 & 4 \\ & 5 & 4 \\ & & 4 \end{array}$$

Its row complement is

$$\begin{array}{ccc} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{array}$$

and therefore

$$\prod_{i \in RC(T)} [i]_q = ([3]_q [2]_q [1]_q)^3$$
$$= q^9 + 6q^8 + 18q^7 + 35q^6 + 48q^5 + 48q^4$$
$$+ 35q^3 + 18q^2 + 6q + 1. \tag{34}$$

Hence

$$C(q)\begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \end{pmatrix} = \det \begin{bmatrix} c[4, 1]_q & c[4, 2]_q & c[4, 3]_q \\ c[5, 1]_q & c[5, 2]_q & c[5, 3]_q \\ c[6, 1]_q & c[6, 2]_q & c[6, 3]_q \end{bmatrix}$$
$$= q^9 + 6q^8 + 18q^7 + 35q^6 + 48q^5$$
$$+ 48q^4 + 35q^3 + 18q^2 + 6q + 1. \tag{35}$$

Reasoning as in the proof of the preceding theorem, but using Corollary 5.2, yields the following result.

THEOREM 7.2. *Let* $\{m_1, ..., m_r\}_<$, $\{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ *and* $m, n \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$\{c[n+1, k+1]_q\}_{k \in \mathbf{N}} \begin{pmatrix} m_1, ..., m_r \\ k_1, ..., k_r \end{pmatrix} = \sum_T \prod_{i \in RC(T)} [i]_q,$$

where the sum is over all row strict skew plane partitions T *of shape* $(m - m_1 + 2, ..., m - m_r + r + 1) \setminus (m - k_1 + 1, ..., m - k_r + r)$ *and coshape* $((n+1)^r)$.

Even though the preceding theorems are very elegant, they are not "natural" in the sense that they do not involve permutations, as the usual combinatorial interpretation of the q-Stirling numbers of the first kind does (see, e.g., [16]). To obtain such a "natural" combinatorial interpretation a little more work is required.

Given a permutation $\sigma \in S_n$ having k cycles $C_1, ..., C_k$ we let $S(\sigma) =^{\text{def}} \{\min(C_1), ..., \min(C_k), n+1\}$, and $\{\sigma^{(1)}, ..., \sigma^{(k+1)}\}_> =^{\text{def}} S(\sigma)$. We say that σ is written in *normal form* if:

(i) each cycle of σ is written with its smallest element first;

(ii) the cycles are written in increasing order of their first elements.

The *normal representation* of σ is the word obtained from the normal form of σ by erasing all the parentheses. The number of *inversions* of σ, denoted by $\operatorname{inv}(\sigma)$, is the number of inversions in the normal representation of σ. More precisely,

$$\operatorname{inv}(\sigma) \stackrel{\text{def}}{=} |\{(i, j) \in [n] \times [n] : i > j, a_i < a_j\}|,$$

where $a_1 \cdots a_n$ is the normal representation of σ. For example, if $\sigma = (23)(1)(87)(9465)$ then $S(\sigma) = \{2, 1, 7, 4, 10\}$, $\sigma^{(5)} = 1$, $\sigma^{(4)} = 2$, $\sigma^{(3)} = 4$, $\sigma^{(2)} = 7$, $\sigma^{(1)} = 10$, the normal form of σ is $(1)(23)(4659)(78)$, its normal representation is 123465978, and $\operatorname{inv}(\sigma) = 3$.

LEMMA 7.3. *Let $n \in \mathbf{P}$ and $S \subseteq [n+1]$, $1, n+1 \in S$. Then*

$$\sum_{\{\sigma \in S_n : S(\sigma) = S\}} q^{\operatorname{inv}(\sigma)} = \prod_{a \in [n] \setminus S} [a-1]_q. \tag{36}$$

Proof. We prove (36) by constructing an appropriate bijection. Let $\sigma \in S_n$ be such that $S(\sigma) = S$ and let $a_1 \cdots a_n$ be the normal representation of σ. We define a function $f_\sigma : [n] \setminus S \to \mathbf{N}$ by letting, for $a \in [n] \setminus S$,

$$f_\sigma(a) \stackrel{\text{def}}{=} |\{i \in [n] : i > j, a_i < a\}|, \tag{37}$$

where $j \in [n]$ is such that $a_j = a$. Note that, since $a \in [n] \setminus S$, and $a_1 \cdots a_n$ is the normal representation of σ, $f_\sigma(a) \leqslant a - 2$. Conversely, let f: $[n] \setminus S \to \mathbf{N}$ be such that $f(a) \leqslant a - 2$ for all $a \in [n] \setminus S$, and let $\{b_1, ..., b_k\}_< =^{\text{def}} [n] \setminus S$. We construct a permutation σ_f inductively as follows. We first place b_1 in $\sigma^{(1)} \cdots \sigma^{(n-k)}$ so that there are exactly $f(b_1)$ elements to the right of b_1 that are smaller than b_1. Note that this is always possible because $f(b_1) \leqslant b_1 - 2$ and $\{1, 2, ..., b_1 - 1\} \subseteq \{\sigma^{(1)}, ..., \sigma^{(n-k)}\}$. Suppose now that we have already placed $b_1, ..., b_{j-1}$ in $\sigma^{(1)}, ..., \sigma^{(n-k)}$. We then place b_j in the leftmost position such that

$$\{1, 2, ..., b_j - 1\} \subseteq S \cup \{b_1, b_2, ..., b_{j-1}\}.$$

We then let σ_f be the unique permutation with $S(\sigma_f) = S$ that has as normal representation the word w obtained at the end of the preceding process (i.e., after the insertion of b_k). Note that σ_f always exists because, for each $a \in w$, the rightmost element of S that is to the left of a is $< a$. In fact, by our definition, when b_j is inserted the element to the left of it is $< b_j$ and this, by induction on j, proves our claim.

It is easy to see that the maps $\sigma \mapsto f_\sigma$ and $f \mapsto \sigma_f$ are inverses of each other. Furthermore, it follows from (37) that

$$\begin{aligned}\sum_{a \in [n] \setminus S} f_\sigma(a) &= |\{(i, j) \in [n] \times [n] : i > j, a_i < a_j, a_j \in [n] \setminus S\}| \\ &= \operatorname{inv}(\sigma),\end{aligned}$$

since $a_1 \cdots a_n$ is the normal representation of σ. Hence

$$\sum_{\{\sigma \in S_n : S(\sigma) = S\}} q^{\mathrm{inv}(\sigma)} = \sum_f \prod_{a \in [n] \setminus S} q^{f(a)}$$

$$= \prod_{a \in [n] \setminus S} (1 + q + q^2 + \cdots + q^{a-2})$$

(where f runs over all functions f: $[n] \setminus S \to \mathbf{N}$ such that $f(a) \leqslant a - 2$ for all $a \in [n] \setminus S$), as desired. ∎

We illustrate the preceding bijection with an example. Let $n = 9$ and $S = \{1, 2, 5, 7, 10\}$. If $\sigma = (14)(23)(56)(798)$ then $f_\sigma(3) = 0$, $f_\sigma(4) = 2$, $f_\sigma(6) = 0$, $f_\sigma(8) = 0$, and $f_\sigma(9) = 1$. Conversely, if $f(3) = 1$, $f(4) = 2$, $f(6) = 0$, $f(8) = 6$, and $f(9) = 4$ then successive insertions of 3, 4, 6, 8, and 9 into $\sigma^{(1)} \cdots \sigma^{(4)} = 1257$ yields, respectively,

13257,

143257,

1432567,

18432567,

184392567,

and hence $\sigma = (18439)(2)(56)(7)$.

Given an r-tuple of permutations $(\sigma_1, ..., \sigma_r)$ and a partition $\mu = (\mu_1, ..., \mu_r)$ we associate to them a shifted skew tabloid, denoted $ST_\mu(\sigma_1, ..., \sigma_r)$, by letting $\sigma_i^{(j)}$ be its $(i, i + \mu_i + j - 1)$ entry, for $i = 1, ..., r$, $j = 1, ..., k_i + 1$ (where k_i is the number of cycles of σ_i, for $i = 1, ..., r$). For example, if $\mu = (2, 1, 1)$, $\sigma_1 = (23)(1)(564)$, $\sigma_2 = (31)(2)$, and $\sigma_3 = (12)(78)(635)(4)$, then

$$ST_\mu(\sigma_1, \sigma_2, \sigma_3) = \begin{array}{ccccc} 7 & 4 & 2 & 1 & \\ 4 & 2 & 1 & & \\ & 9 & 7 & 4 & 3 \quad 1 \end{array}.$$

We usually omit the index μ when there is no danger of confusion.

LEMMA 7.4. *Let T be a row strict shifted skew plane partition of coshape $(n_1 + 1, ..., n_r + 1)$. Then*

$$\prod_{i \in RC(T)} [i]_q = \sum_{(\sigma_1, ..., \sigma_r)} \prod_{i=1}^{r} q^{\mathrm{inv}(\sigma_i)},$$

where the sum is over all r-tuples $(\sigma_1, ..., \sigma_r) \in S_{n_1+1} \times \cdots \times S_{n_r+1}$ *such that* $ST(\sigma_1, ..., \sigma_r) = \hat{T}$.

Proof. Let $T =^{\mathrm{def}} (T_{i,j})_{1 \leqslant i \leqslant r,\, c_i+i \leqslant j \leqslant d_i+i-1}$ be a row strict shifted skew plane partition of coshape $(n_1+1, ..., n_r+1)$. Then, by the definition of $RC(T)$, we have that

$$\begin{aligned} RC(T)_i + 1 &= ([n_i]) \backslash (\bar{T})_i) + 1 \\ &= [n_i+1] \backslash \{T_{i, c_i+i+1}+1, ..., T_{i, d_i+i-1}+1, 1\}, \end{aligned}$$

for $i = 1, ..., r$. Hence, by Lemma 7.3, we have that

$$\begin{aligned} \prod_{i \in RC(T)} [i]_q &= \prod_{i=1}^{r} \prod_{j \in RC(T)_i} [j]_q \\ &= \prod_{i=1}^{r} \prod_{j \in RC(T)_i+1} [j-1]_q \\ &= \prod_{i=1}^{r} \sum_{\sigma} q^{\mathrm{inv}(\sigma)}, \end{aligned}$$

where the sum is over all $\sigma \in S_{n_i+1}$ such that $S(\sigma) = \{n_i+2, T_{i, c_i+i+1}+1, ..., T_{i, d_i+i-1}+1, 1\}$, and the result follows. ∎

Since a shifted skew tabloid of the form $ST(\sigma_1, ..., \sigma_r)$ with $(\sigma_1, ..., \sigma_r) \in S_{n_1+1} \times \cdots \times S_{n_r+1}$ is necessarily row strict and of coshape $(n_1+2, ..., n_r+2)$ from the preceding lemma and Theorems 7.1 and 7.2 we deduce the following results.

THEOREM 7.5. *Let* $\{n_1, ..., n_r\}_>, \{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$C(q)\begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} = \sum_{(\sigma_1, ..., \sigma_r)} \prod_{i=1}^{r} q^{\mathrm{inv}(\sigma_i)},$$

where the sum is over all r-tuples $(\sigma_1, ..., \sigma_r) \in S_{n_1+1} \times \cdots \times S_{n_r+1}$ *such that* $ST(\sigma_1, ..., \sigma_r)$ *is a shifted plane partition of shape* $(k_1+2, ..., k_r+2)$ *(equivalently, such that* σ_i *has* k_i+1 *cycles and* $\sigma_i^{(j)} \geqslant \sigma_{i+1}^{(j-1)}$, *for* $1 \leqslant i \leqslant r-1$ *and* $2 \leqslant j \leqslant k_i+2$).

THEOREM 7.6. *Let* $\{m_1, ..., m_r\}_<, \{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ *and* $m, n \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$\{c[n+1, k+1]_q\}_{k \in \mathbf{N}}\begin{pmatrix} m_1, ..., m_r \\ k_1, ..., k_r \end{pmatrix} = \sum_{(\sigma_1, ..., \sigma_r)} \prod_{i=1}^{r} q^{\mathrm{inv}(\sigma_i)},$$

where the sum is over all r-tuples $(\sigma_1, \ldots, \sigma_r) \in (S_{n+1})^r$ *such that* $ST(\sigma_1, \ldots, \sigma_r)$ *is a skew plane partition of shape* $(m - m_1 + 3, \ldots, m - m_r + r + 2) \setminus (m - k_1 + 1, \ldots, m - k_r + r)$ *(equivalently, such that* σ_i *has* $k_i - m_i + 1$ *cycles and* $\sigma_i^{(k_i - j)} \geqslant \sigma_{i+1}^{(k_{i+1} - j - 1)}$ *for* $1 \leqslant i \leqslant r-1$ *and* $m_{i+1} - 3 \leqslant j \leqslant k_i - 1$*).*

We illustrate Theorem 7.5, when $q = 1$, with an example. Suppose that $\{k_1, k_2, k_3\}_> =^{\text{def}} \{2, 1, 0\}$, and $\{n_1, n_2, n_3\}_> =^{\text{def}} \{5, 4, 3\}$, as in the example following Theorem 7.1. Then we are looking for triples $(\sigma_1, \sigma_2, \sigma_3) \in S_6 \times S_5 \times S_4$ such that $ST(\sigma_1, \sigma_2, \sigma_3)$ is a shifted plane partition of shape (4, 3, 2). Since $(\sigma_1, \sigma_2, \sigma_3) \in S_6 \times S_5 \times S_4$, $ST(\sigma_1, \sigma_2, \sigma_3)$ must be row strict and of the form

$$\begin{array}{cccc} 7 & \cdot & \cdot & 1 \\ & 6 & \cdot & 1 \\ & & 5 & 1 \end{array}$$

Hence the only possibility for $ST(\sigma_1, \sigma_2, \sigma_3)$ is

$$\begin{array}{cccc} 7 & 6 & 5 & 1 \\ & 6 & 5 & 1 \\ & & 5 & 1 \end{array}$$

This means that we are looking for triples $(\sigma_1, \sigma_2, \sigma_3) \in S_6 \times S_5 \times S_4$ such that $S(\sigma_1) = \{7, 6, 5, 1\}$, $S(\sigma_2) = \{6, 5, 1\}$, and $S(\sigma_3) = \{5, 1\}$. Hence σ_1, σ_2, and σ_3 must be of the form

$$\begin{aligned} \sigma_1 &= (1 x_1 x_2 x_3)(5)(6), \\ \sigma_2 &= (1 y_1 y_2 y_3)(5), \\ \sigma_3 &= (1 z_1 z_2 z_3), \end{aligned}$$

where $x_1 x_2 x_3$, $y_1 y_2 y_3$, and $z_1 z_2 z_3$ are any permutations of $\{2, 3, 4\}$. Hence there are 6 possibilities for each one of σ_1, σ_2, and σ_3 and therefore 216 possible triples $(\sigma_1, \sigma_2, \sigma_3)$, which is in accordance with (34) and (35) when $q = 1$.

8. q-Stirling Determinants of the Second Kind

In this section we consider determinants of q-Stirling numbers of the second kind. These were introduced by Gould in [10] and have been the subject of considerable interest recently (see, e.g., [16, 20, 23, 32]). They are defined inductively by letting

$$S[n, k]_q = S[n-1, k-1]_q + [k]_q \, S[n-1, k]_q$$

for $n \in \mathbf{P}$, and $S[0, k]_q =^{\text{def}} \delta_{0,k}$ for $k \in \mathbf{N}$ (with the convention that $S[n, k]_q =^{\text{def}} 0$ if either $n < 0$ or $k < 0$). We let

$$S(q) \stackrel{\text{def}}{=} (S[n+1, k+1]_q)_{n, k \in \mathbf{N}}.$$

All the results in this section are new even in the case $q = 1$.

THEOREM 8.1. *Let* $\{n_1, ..., n_r\}_>$, $\{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$S(q)\begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} = \sum_T \prod_{i \geqslant 1} [i]_q^{t_i(T) - t_{i+1}(T) - d_i(T)},$$

where the sum is over all row strict shifted plane partitions T *of shape* $(k_1+1, ..., k_r+1)$ *and coshape* $(n_1+1, ..., n_r+1)$.

Proof. It is well known (see, e.g., [22]) that

$$S[n+1, k+1]_q = h_{n-k}([1]_q, [2]_q, ..., [k+1]_q).$$

Hence, by Theorem 5.5,

$$S(q)\begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} = \sum_T \prod_{i \geqslant 1} [i]_q^{t_i(T) - t_{i+1}(T) - d_i(T)}$$

(since $(k_1+1, ..., k_r+1)' = (d_1(T), d_2(T), ...)$) where the sum is over all row strict shifted plane partitions T of shape $(k_1+1, ..., k_r+1)$ and coshape $(n_1+1, ..., n_r+1)$, as desired. ∎

We illustrate the above theorem with an example. Suppose that $\{k_1, k_2, k_3\}_> =^{\text{def}} \{2, 1, 0\}$, and $\{n_1, n_2, n_3\}_> =^{\text{def}} \{6, 5, 4\}$. Then there is only one row strict shifted plane partition T of shape $(3, 2, 1)$ and coshape $(7, 6, 5)$, namely,

$$\begin{matrix} 7 & 6 & 5 \\ & 6 & 5. \\ & & 5 \end{matrix}$$

Therefore

$$t(T) = (18, 11, 5, 0, ...),$$

$$\Delta(t(T)) = (7, 6, 5, 0, ...),$$

$$d(T) = (3, 2, 1, 0, ...),$$

hence

$$\prod_{i\geq 1}[i]_q^{t_i(T)-t_{i+1}(T)-d_i(T)}=[1]_q^4[2]_q^4[3]_q^4, \tag{38}$$

and

$$S(q)\begin{pmatrix}4 & 5 & 6\\ 0 & 1 & 2\end{pmatrix}=\det\begin{bmatrix}S[5,1]_q & S[5,2]_q & S[5,3]_q\\ S[6,1]_q & S[6,2]_q & S[6,3]_q\\ S[7,1]_q & S[7,2]_q & S[7,3]_q\end{bmatrix}$$
$$\begin{aligned}&=([1]_q[2]_q[3]_q)^4\\ &=1+8q+32q^2+84q^3+160q^4+232q^5\\ &\quad+262q^6+232q^7+160q^8\\ &\quad+84q^9+32q^{10}+8q^{11}+q^{12}.\end{aligned} \tag{39}$$

Reasoning as in the proof of the last theorem, but using Theorem 5.6, gives the following result.

THEOREM 8.2. *Let* $\{m_1, ..., m_r\}_<$, $\{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ *and* $m, k \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$\{S[n+1, k+1]_q\}_{n\in\mathbf{N}}\begin{pmatrix}m_1, ..., m_r\\ k_1, ..., k_r\end{pmatrix}=\sum_{\pi}\prod_{i=1}^{k+1}[i]_q^{t_i(\pi)-t_{i+1}(\pi)-r},$$

where the sum is over all row strict shifted plane partitions π *of shape* $((k+2)^r)$ *in which the* i*th row has largest part* $= m - m_i + 1$ *and smallest part* $= m - k_i$, *for* $i = 1, ..., r$.

Note that it is possible to use Corollary 5.4 instead of Theorem 5.6 to obtain an equivalent version of the preceding theorem. However, the version that we have chosen is much better suited for the applications that follow (see Lemmas 8.4 and 8.5).

We illustrate Theorem 8.2 with an example. Suppose that $k = 2$, $\{m_1, m_2, m_3\}_< =^{\text{def}} \{1, 2, 3\}$, $\{k_1, k_2, k_3\}_< =^{\text{def}} \{5, 6, 7\}$, and $m = 8$. Then we are looking for row strict shifted plane partitions of the form

$$\begin{matrix}8 & \cdot & \cdot & 3 & & \\ & 7 & \cdot & \cdot & 2 & \\ & & 6 & \cdot & \cdot & 1\end{matrix}$$

Clearly, there is only one such row strict shifted plane partition, namely

$$T=\begin{matrix}8 & 7 & 6 & 3 & & \\ & 7 & 6 & 3 & 2 & \\ & & 6 & 3 & 2 & 1\end{matrix}.$$

Therefore

$$\prod_{i=1}^{3} [i]_q^{t_i(T)-t_{i+1}(T)-3} = [1]_q^2 [2]_q^2 [3]_q^2, \tag{40}$$

and hence

$$\{S[n+1,3]_q\}_{n\in\mathbf{N}} \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} = \det \begin{bmatrix} S[5,3]_q & S[6,3]_q & S[7,3]_q \\ S[4,3]_q & S[5,3]_q & S[6,3]_q \\ S[3,3]_q & S[4,3]_q & S[5,3]_q \end{bmatrix}$$
$$= ([1]_q [2]_q [3]_q)^2$$
$$= 1 + 4q + 8q^2 + 10q^3 + 8q^4 + 4q^5 + q^6.$$

The same remark made after Theorem 7.2 applies to the last two results. In order to obtain " natural" versions of these results we need analogs of Lemmas 7.3 and 7.4 for (set) partitions.

Let $m, n \in \mathbf{P}$, $m < n$. Given a partition $\pi = \{B_1, ..., B_k\}$ of $[m, n]$ into k blocks we let $S(\pi) =^{\text{def}} \{\max(B_1), ..., \max(B_k), m-1\}$ and $\{\pi^{(1)}, ..., \pi^{(k+1)}\}_> =^{\text{def}} S(\pi)$. Let now π_i be the (unique) block of π containing $\pi^{(i)}$, for $i = 1, ..., k$. We define the *height* of π to be the number

$$ht(\pi) \overset{\text{def}}{=} \sum_{i=1}^{k} (i-1)(|\pi_i| - 1).$$

For example, if $\pi = 48/569/7$ then $S(\pi) = \{8, 9, 7, 3\}$, $\pi^{(1)} = 9$, $\pi^{(2)} = 8$, $\pi^{(3)} = 7$, $\pi^{(4)} = 3$, $\pi_1 = \{5, 6, 9\}$, $\pi_2 = \{4, 8\}$, $\pi_3 = \{7\}$, and

$$ht(\pi) = 1 \cdot 1 + 2 \cdot 0 = 1.$$

The verification of the next technical result is easy and is left to the reader.

LEMMA 8.3. *Let* $n \in \mathbf{P}$ *and* $S =^{\text{def}} \{s_1, ..., s_{k+1}\}_> \subseteq [n] \cup \{0\}$, $0, n \in S$. *Then*

$$\sum_{\{\pi \in \Pi([n]) : S(\pi) = S\}} q^{ht(\pi)} = [k]_q^{s_k - 1} \sum_{\{\pi \in \Pi([s_k+1, n]) : S(\pi) = S \setminus \{0\}\}} q^{ht(\pi)}.$$

LEMMA 8.4. *Let* $n \in \mathbf{P}$ *and* $S \overset{\text{def}}{=} \{s_1, ..., s_{k+1}\}_> \subseteq [n] \cup \{0\}$, 0, $n \in S$. *Then*

$$\sum_{\{\pi \in \Pi([n]) : S(\pi) = S\}} q^{ht(\pi)} = \prod_{i=1}^{k} [i]_q^{s_i - s_{i+1} - 1}. \tag{41}$$

Proof. We find it convenient to introduce the following notation. For $a \in [n] \setminus S$ we let $i(a)$ be the unique element of $[k]$ such that $s_{i(a)+1} < a < s_{i(a)}$.

We prove (41) by constructing an appropriate bijection. Let π be a partition of $[n]$ such that $S(\pi)=S$. For $a \in [n]\setminus S$ we let

$$f_\pi(a) \stackrel{\text{def}}{=} j-1 \tag{42}$$

if a and s_j are in the same block of π. Note that since $S(\pi)=S$, $f_\pi(a) \leqslant i(a)-1$, for all $a \in [n]\setminus S$. Conversely, given a function $f\colon [n]\setminus S \to \mathbf{N}$ such that $f(a) \leqslant i(a)-1$ for all $a \in [n]\setminus S$, we construct a partition π_f of $[n]$ by putting $a \in [n]\setminus S$ in the same block as $s_{f(a)+1}$ and putting $s_1, \ldots, s_k$ all in different blocks. Note that $a < s_{i(a)} \leqslant s_{f(a)+1}$ so that $S(\pi_f)=S$. It is easy to see that the maps $\pi \mapsto f_\pi$ and $f \mapsto \pi_f$ are inverses of each other. Furthermore, it follows from (42) that

$$ht(\pi) = \sum_{a \in [n]\setminus S} f_\pi(a).$$

Hence

$$\begin{aligned}
\sum_{\{\pi \in \Pi([n])\,:\, S(\pi)=S\}} q^{ht(\pi)} &= \sum_f \prod_{a \in [n]\setminus S} q^{f(a)} \\
&= \prod_{a \in [n]\setminus S} (1+q+q^2+\cdots+q^{i(a)-1}) \\
&= \prod_{i=1}^{k} (1+q+q^2+\cdots+q^{i-1})^{s_i - s_{i+1}-1}
\end{aligned}$$

(where f runs over all functions $f\colon [n]\setminus S \to \mathbf{N}$ such that $f(a) \leqslant i(a)-1$ for all $a \in [n]\setminus S$), and the thesis follows. ∎

We illustrate the above construction with an example. Let $n=9$ and $S \stackrel{\text{def}}{=} \{9, 7, 6, 2\}$, so that $s_1=9$, $s_2=7$, $s_3=6$, $s_4=2$ and hence $i(1)=4$, $i(3)=i(4)=i(5)=3$, and $i(8)=1$. If $\pi = 938/75/64/21$ then $f_\pi(3)=f_\pi(8)=0$, $f_\pi(5)=1$, $f_\pi(4)=2$, and $f_\pi(1)=3$. Conversely, if $f(1)=3$, $f(3)=2$, $f(4)=f(5)=1$, and $f(8)=0$ then $\pi_f = 98/745/63/21$.

Given an r-tuple of partitions $(\pi_1, \ldots, \pi_r)$ we associate to it a shifted tabloid $ST(\pi_1, \ldots, \pi_r)$ by letting the elements of $S(\pi_i)$ (in decreasing order) be the ith row of it, for $i=1, \ldots, r$, and then shifting the resulting array. For example, if $\pi_1 = 82/7591/6/354$, $\pi_2 = 25/134$, and $\pi_3 = 71/52/3/46$ then

$$ST(\pi_1, \pi_2, \pi_3) = \begin{array}{cccccc} 9 & 8 & 6 & 5 & & \\ & 5 & 4 & & & \\ & & 7 & 6 & 5 & 3 \end{array}.$$

LEMMA 8.5. *Let T be a row strict shifted plane partition of coshape $(n_1+1, \ldots, n_r+1)$. Then*

$$\prod_{j \geqslant 1} [j]_q^{t_j(T) - t_{j+1}(T) - d_j(T)} = \sum_{(\pi_1, \ldots, \pi_r)} \prod_{i=1}^{r} q^{ht(\pi_i)}, \tag{43}$$

where the sum is over all r-tuples $(\pi_1, ..., \pi_r) \in \Pi([n_1+1]) \times \cdots \times \Pi([n_r+1])$ *such that* $ST(\pi_1, ..., \pi_r) = T$.

Proof. Let $T = (T_{i,j})_{1 \leqslant i \leqslant r,\, i \leqslant j \leqslant k_i + i}$ be a row strict shifted plane partition satisfying the hypotheses of the lemma. Then, for $1 \leqslant j \leqslant k_1 + 1$,

$$t_j(T) - t_{j+1}(T) - d_j(T) = \sum_{i=1}^{d_j(T)} (T_{i,i+j-1} - T_{i,i+j} - 1),$$

hence, by Lemma 8.4,

$$\begin{aligned}\prod_{j=1}^{k_1+1} [j]_q^{t_j(T) - t_{j+1}(T) - d_j(T)} &= \prod_{j=1}^{k_1+1} \prod_{i=1}^{d_j(T)} [j]_q^{T_{i,i+j-1} - T_{i,i+j} - 1} \\ &= \prod_{i=1}^{r} \prod_{j=1}^{k_i+1} [j]_q^{T_{i,i+j-1} - T_{i,i+j} - 1} \\ &= \prod_{i=1}^{r} \sum_{\{\pi \in \Pi([n_i+1]) \,:\, S(\pi) = T_i \cup \{0\}\}} q^{ht(\pi)},\end{aligned}$$

and (43) follows. ∎

Since a shifted tabloid of the form $ST(\pi_1, ..., \pi_r)$ with $(\pi_1, ..., \pi_r) \in \Pi([n_1+1]) \times \cdots \times \Pi([n_r+1])$ is necessarily row strict and of coshape $(n_1+1, ..., n_r+1)$ from Theorems 8.1 and 8.2 and the preceding lemma we deduce the following results.

THEOREM 8.6. *Let* $\{n_1, ..., n_r\}_>$, $\{k_1, ..., k_r\}_> \subseteq \mathbf{N}$. *Then*

$$S(q)\begin{pmatrix} n_r, ..., n_1 \\ k_r, ..., k_1 \end{pmatrix} = \sum_{(\pi_1, ..., \pi_r)} \prod_{i=1}^{r} q^{ht(\pi_i)},$$

where the sum is over all r-tuples $(\pi_1, ..., \pi_r) \in \Pi([n_1+1]) \times \cdots \times \Pi([n_r+1])$ *such that* $ST(\pi_1, ..., \pi_r)$ *is a shifted plane partition of shape* $(k_1+1, ..., k_r+1)$ *(equivalently, such that* π_i *has* k_i+1 *blocks and* $\pi_i^{(j)} \geqslant \pi_{i+1}^{(j-1)}$, *for* $1 \leqslant i \leqslant r-1$ *and* $2 \leqslant j \leqslant k_i+1$).

THEOREM 8.7. *Let* $\{m_1, ..., m_r\}_<$, $\{k_1, ..., k_r\}_< \subseteq \mathbf{N}$ *and* $m, k \in \mathbf{P}$, $m > \max\{m_r, k_r\}$. *Then*

$$\{S[n+1, k+1]_q\}_{n \in \mathbf{N}} \begin{pmatrix} m_1, ..., m_r \\ k_1, ..., k_r \end{pmatrix} = \sum_{(\pi_1, ..., \pi_r)} \prod_{i=1}^{r} q^{ht(\pi_i)},$$

where the sum is over all r-tuples $(\pi_1, ..., \pi_r) \in \Pi([m-k_1+1, m-m_1+1]) \times \cdots \times \Pi([m-k_r+1, m-m_r+1])$ *such that* $ST(\pi_1, ..., \pi_r)$ *is a shifted plane* *partition of shape* $((k+2)^r)$ (*equivalently, such that* π_i *has* $k+1$ *blocks, for* $i=1, ..., r$, *and* $\pi_i^{(j)} \geqslant \pi_{i+1}^{(j-1)}$ *for* $1 \leqslant i \leqslant r-1$ *and* $2 \leqslant j \leqslant k+2$).

Proof. Let T be a row strict shifted plane partition of shape $((k+2)^r)$ in which the ith row has largest part $= m - m_i + 1$ and smallest part $= m - k_i$, for $i = 1, ..., r$. Then by Lemmas 8.3 and 8.5 we obtain that

$$\begin{aligned}\prod_{j \geqslant 1} & [j]_q^{t_j(T) - t_{j+1}(T) - d_j(T)} \\ &= \prod_{i=1}^{r} \sum_{\{\pi \in \Pi([m - m_i + 1]) \,:\, S(\pi) = T_i \cup \{0\}\}} q^{ht(\pi)} \\ &= \prod_{i=1}^{r} [k+2]_q^{m - k_i - 1} \sum_{\{\pi \in \Pi([m - k_i + 1,\, m - m_i + 1]) \,:\, S(\pi) = T_i\}} q^{ht(\pi)} \\ &= [k+2]^{\sum_{i=1}^{r} (m - k_i - 1)} \sum_{(\pi_1, ..., \pi_r)} \prod_{i=1}^{r} q^{ht(\pi_i)},\end{aligned}$$

where the sum is over all r-tuples $(\pi_1, ..., \pi_r) \in \Pi([m - k_1 + 1, m - m_1 + 1]) \times \cdots \times \Pi([m - k_r + 1, m - m_r + 1])$ such that $ST(\pi_1, ..., \pi_r) = T$, and the thesis follows from Theorem 8.2. ∎

We illustrate Theorem 8.6, when $q = 1$, with an example. Suppose that $\{k_1, k_2, k_3\}_> =^{\mathrm{def}} \{2, 1, 0\}$, and $\{n_1, n_2, n_3\}_> =^{\mathrm{def}} \{6, 5, 4\}$, as in the example following Theorem 8.1. Then we are looking for triples $(\pi_1, \pi_2, \pi_3) \in \Pi([7]) \times \Pi([6]) \times \Pi([5])$ such that $ST(\pi_1, \pi_2, \pi_3)$ is a shifted plane partition of shape $(3, 2, 1)$. Since $(\pi_1, \pi_2, \pi_3) \in \Pi([7]) \times \Pi([6]) \times \Pi([5])$, $ST(\pi_1, \pi_2, \pi_3)$ must be row strict and of the form

$$\begin{array}{ccc} 7 & \cdot & \cdot \\ & 6 & \cdot \\ & & 5 \end{array}$$

Hence the only possibility for $ST(\pi_1, \pi_2, \pi_3)$ is

$$\begin{array}{ccc} 7 & 6 & 5 \\ & 6 & 5 \\ & & 5 \end{array}$$

This means that we are looking for triples $(\pi_1, \pi_2, \pi_3) \in \Pi([7]) \times \Pi([6]) \times \Pi([5])$ such that $S(\pi_1) = \{7, 6, 5\}$, $S(\pi_2) = \{6, 5\}$, and $S(\pi_3) = \{5\}$. Hence there are 3^4 possibilities for π_1, 2^4 possibilities for π_2, and one possibility for π_3. Therefore there are 1296 possible triples (π_1, π_2, π_3), which is in accordance with (38) and (39) when $q = 1$.

REFERENCES

1. A. B. BALANTEKIN AND I. BARS, Representations of supergroups, *J. Math. Phys.* **22** (1981), 1810–1818.
2. A. BERELE AND A. REGEV, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, *Adv. in Math.* **64** (1987), 118–175.
3. F. BRENTI, Combinatorial methods in the theory of total positivity, in preparation.
4. L. M. BUTLER, The q-log concavity of q-binomial coefficients, *J. Combin. Theory Ser. A* **54** (1990), 54–63.
5. L. COMTET, "Advanced Combinatorics," Reidel, Dordrecht/Boston, 1974.
6. P. H. DONDI AND P. D. JARVIS, Diagram and superfield techniques in the classical superalgebras, *J. Phys. A* **14** (1981), 547–563.
7. I. M. GELFAND AND M. L. TSEITLIN, Finite-dimensional representations of the group of unimodular matrices, *Dokl. Akad. Nauk USSR* **71** (1950), 825–828. [In Russian]
8. I. GESSEL AND G. VIENNOT, Binomial determinants, paths, and hook length formulae, *Adv. in Math.* **58** (1985), 300–321.
9. I. GESSEL AND G. VIENNOT, Determinants, paths, and plane partitions, preprint.
10. H. W. GOULD, The q-Stirling numbers of the first and second kinds, *Duke Math. J.* **28** (1961), 281–289.
11. T. JÓZEFIAK AND P. PRAGACZ, A determinantal formula for skew Schur Q-functions, *J. London Math. Soc.* **43** (1991), 76–90.
12. V. G. KAČ, Lie superalgebras, *Adv. in Math.* **26** (1977), 8–96.
13. V. G. KAČ, Representations of classical Lie superalgebras, *in* "Differential Geometry Methods in Mathematical Physics, II" (K. Bleuler, H. R. Petry, and A. Reetz, Eds.), pp. 597–626, Lecture Notes in Math., Vol. 676, Springer, New York, 1978.
14. S. KARLIN, "Total Positivity," Vol. 1, Stanford Univ. Press, Stanford, CA, 1968.
15. C. KRATTENTHALER, On the q-log-concavity of Gaussian binomial coefficients, preprint.
16. P. LEROUX, Reduced matrices and q-log concavity properties of q-Stirling numbers, *J. Combin. Theory Ser. A* **54** (1990), 64–84.
17. B. LINDSTRÖM, On the vector representations of induced matroids, *Bull. London Math. Soc.* **5** (1973), 85–90.
18. I. G. MACDONALD, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press, New York/London, 1979.
19. N. METROPOLIS, G. NICOLETTI, AND G. C. ROTA, A new class of symmetric functions, *Math. Anal. Appl. Part B Adv. in Math. Suppl. Stud.* **7** (1981), 563–575.
20. S. C. MILNE, Restricted growth functions, rank row matchings of partition lattices, and q-Stirling numbers, *Adv. in Math.* **43** (1982), 173–196.
21. B. E. SAGAN, Shifted tableaux, Schur Q-functions and a conjecture of R. Stanley, *J. Combin. Theory Ser. A* **45** (1987), 62–103.
22. B. E. SAGAN, Log concave sequences of symmetric functions and analogs of the Yacobi–Trudi determinants, *Trans. Amer. Math. Soc.* **329** (1992), 795–812.
23. B. E. SAGAN, A maj statistic for set partitions, *Europ. J. Comb.* **12** (1991), 69–79.
24. B. E. SAGAN, Inductive proofs of q-log concavity, *Disc. Math.* **99** (1992), 289–306.
25. I. SCHUR, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.* **139** (1911), 155–250.

607 98 1-5

26. R. Stanley, Theory and application of plane partitions, Parts 1, 2, *Stud. Appl. Math.* **50** (1971), 167–188, 259–279.
27. R. Stanley, Unimodality and Lie superalgebras, *Stud. Appl. Math.* **72** (1985), 263–281.
28. R. Stanley, "Enumerative Combinatorics," Vol. 1, Wadsworth and Brooks/Cole, Monterey, CA, 1986.
29. R. Stanley, A baker's dozen of conjectures concerning plane partitions, *in* "Combinatoire Enumerative" (G. Labelle and P. Leroux, Eds.), pp. 285–293, Lecture Notes in Mathematics, Vol. 1234, Springer-Verlag, Berlin/New York, 1986.
30. J. R. Stembridge, Shifted tableaux and the projective representations of symmetric groups, *Adv. in Math.* **74** (1989), 87–134.
31. J. R. Stembridge, Non-intersecting paths, pfaffians and plane partitions, *Adv. in Math.* **83** (1990), 96–131.
32. M. Wachs and D. White, p, q-Stirling numbers and set partition statistics, *J. Combin. Theory* **56** (1991), 27–46.
33. Iwahori, "Problem Session of the Conference in Combinatorics and Commutative Algebra, Nagoya, 1990."