Nearly Commuting Projections

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Abstract

It is well known that projection operators are typical elements in Boolean algebras, and a number of relevant theorems have been proved for commutative projections. We propose an extension of the concept of commutativity, which we call near-commutativity. We extend to this concept the main theorems on commutative projections, and in various ways we frame the class of nearly commutative projections in Boolean algebras.

1. Introduction

If \( p \) and \( q \) are two linear projections on a vector space \( V \) over \( \mathbb{C} \), we say they nearly commute if

\[
pqq = qp \quad \text{and} \quad qpq = pq. \tag{1}
\]

We say they antinearly commute if

\[
pqq - pq \quad \text{and} \quad qpq - qp. \tag{2}
\]

If \( p \) and \( q \) commute, then they both nearly commute and antinearly commute. Also, \( p \) and \( q \) nearly commute if and only if their complements \( I - p \) and \( I - q \) antinearly commute.

Section 2 displays examples of these kinds of projections. Basic properties of nearly commuting projections appear in Section 3. Section 4 introduces
two operators on sets of nearly commuting projections. Section 5 derives orthogonal projections from nearly commuting projections, and Section 6 does a decomposition of projections using orthogonal projections.

2. EXAMPLES

Let $V$ be the vector space of functions $f : \mathbb{C}^3 \to \mathbb{C}$. Let

$$p(f)(z_1, z_2, z_3) = f(0, 0, z_3)$$

and

$$q(f)(z_1, z_2, z_3) = f(z_1, 0, 0).$$

Then $p$ and $q$ are commuting projections on $V$. Now let

$$p(f)(z_1, z_2, z_3) = f(0, 0, z_3)$$

and

$$q(f)(z_1, z_2, z_3) = f(z_1, 1, 1).$$

Then

$$pq(f)(z_1, z_2, z_3) = f(0, 1, 1),$$

$$qpq(f)(z_1, z_2, z_3) = f(0, 1, 1),$$

$$qp(f)(z_1, z_2, z_3) = f(0, 0, 1),$$

$$pqp(f)(z_1, z_2, z_3) = f(0, 0, 1).$$

Thus

$$pqp(f) = qp(f) \quad \text{and} \quad qpq(f) = pq(f).$$

In general, if $V$ is the set of functions $f : \mathbb{C}^n \to \mathbb{C}$, the projections which substitute the same constant for different arguments all commute; whereas the projections which substitute different constants for arguments, in general, nearly commute.
Let \( p \) and \( q \) be linear projections on a vector space \( V \) over \( \mathbb{C} \) and let \( a \) and \( b \) be two elements in \( V \) such that \( a \in \text{Ran}(I - p) \) and \( b \in \text{Ran}(I - q) \). Also, let \( P(x) = a + p(x) \) and \( Q(x) = b + q(x) \) for all \( x \in V \). Then \( P^2 = P \) and \( Q^2 = Q \), i.e., \( P \) and \( Q \) are affine projections on \( V \) (see Wilde [1]). If \( p \) and \( q \) commute, then in general \( PQP = PQ \) and \( QPQ = QP \). Also, \( PQP = PQ \) if and only if \( pqp = pq \).

Our final example is a set of \((n + 2) \times (n + 2)\) matrices over \( \mathbb{C} \). Let \( a_1, a_2, \ldots, a_n \in \mathbb{C} \). Let \( E_{ij} \) be the \((n + 2) \times (n + 2)\) matrix with a 1 in the \((i, j)\) spot and 0's elsewhere. Let \( p_i = E_{i1} + a_i E_{12} \) for \( i = 1, 2, \ldots, n \), and let \( q_j = E_{2+i, 2+j} \) for \( j = 1, 2, \ldots, n \). Then \( p_i p_j = p_j \) and \( p_j p_i = p_i \) for \( i \neq j \); and \( q_1, \ldots, q_n \) are pairwise orthogonal. Also, \( p_i q_j = q_j p_i = 0 \) for all \( i \) and \( j \) in \( \{1, 2, \ldots, n\} \). All projections of the form "\( p_i \) plus sums of the \( q_j \)'s" nearly commute. For instance, if \( i \neq j \), then

\[
(p_i + q_i)(p_j + q_i + q_j) = p_j + q_i,
\]

\[
(p_j + q_i + q_i)(p_i + q_i)(p_j + q_i + q_j) = p_j + q_i,
\]

\[
(p_j + q_i + q_j)(p_i + q_i) = p_i + q_i,
\]

\[
(p_i + q_i)(p_j + q_i + q_j)(p_i + q_i) = p_i + q_i,
\]

i.e., \( p_i + q_i \) and \( p_j + q_i + q_j \) nearly commute.

3. MISCELLANEOUS PROPERTIES

We prove the following theorem.

**Theorem 1.** Let \( p, q, r \) be linear, pairwise nearly commuting projections on \( V \). Then

1. \( pq, p + q - pq, p + pq - qp, \text{ and } \frac{1}{2}(pq + qp) \) are linear projections on \( V \);
2. \( r \) nearly commutes with \( pq, p + q - pq, p + pq - qp, \text{ and } \frac{1}{2}(pq + qp) \);
3. \( \text{Ran} \ p \cap \text{Ran} \ q = \text{Ran} \ pq = \text{Ran} \ qp \); and
4. \( \text{Ran} \ p + \text{Ran} \ q = \text{Ran} (p + q - pq) = \text{Ran} (p + q - qp) \).

**Proof.** (1): Easy.
(2): $r$ nearly commutes with $pq$ because

$$r(pq)r = rp(qr) = rp(rqr) = (rpr)qr$$

$$= (pr)qr = p(rqr) = p(qr) = (pq)r$$

and

$$(pq)r(pq) = pq(rp)q = p(rp)q = (prp)q = (rp)q = r(pq).$$

This rest is just more calculation.

(3): Let $x \in \text{Ran } pq$. Then $x = pq(x)$, $p(x) = p(pq(x)) = pq(x) = x$, and $q(x) = q(pq(x)) = pq(x) = x$. Thus $\text{Ran } pq \subset \text{Ran } p \cap \text{Ran } q$. Let $x \in \text{Ran } p \cap \text{Ran } q$. Then $p(x) = x$ and $q(x) = x$, thus $pq(x) = p(x) = x$, and so $\text{Ran } p \cap \text{Ran } q \subset \text{Ran } pq$. By symmetry, $\text{Ran } p \cap \text{Ran } q = \text{Ran } qp$, although $pq$ does not always equal $qp$.

(4): Let $x \in \text{Ran } p$ and $y \in \text{Ran } q$; then $p(x) = x$ and $q(y) = y$, and

$$(p + q - pq)(x + y) = p(x) + q(x) - pq(x) + p(y) + q(y) - pq(y)$$

$$= x + qp(x) - pqp(x) + pq(y) + y - pq(y)$$

$$= x + qp(x) - qp(x) + y = x + y,$$

or $\text{Ran } p + \text{Ran } q \subset \text{Ran } (p + q - pq)$. Let $x \in \text{Ran } (p + q - pq)$. Then $x = (p + q - pq)(x) = p(x) + (I - p)q(x)$, where $p(x) \in \text{Ran } p$ and $(I - p)q(x) \in \text{Ran } q$, since $q((I - p)q(x)) = (I - p)q(x)$. Thus $\text{Ran } (p + q - pq) \subset \text{Ran } p + \text{Ran } q$. By symmetry, $\text{Ran } p + \text{Ran } q = \text{Ran } (p + q - qp)$.

Suppose $p, q, r$ are linear, pairwise nearly commuting projections on $V$. Let

$$E = \frac{1}{2}(pq + qp)$$

(3)

and

$$N = \frac{1}{2}(pq - qp).$$

(4)

Then we can prove the following.

**Theorem 2.**

1. $E^2 = E$, $N^2 = 0$;
2. $pE = E$, $qE = E$;
3. $pN = N$, $qN = N$;
4. $Ep = E - N$, $Eq = E + N$;
5. $Np = 0$, $Nq = 0$; and
6. $EN = N$, $NE = 0$. 

Also, $p + cN$, $E + cN$, and $p + q - E + cN$, for a scalar $c \in \mathbb{C}$, are linear projections, and $r$ nearly commutes with them. For this reason, we let $X$ be a maximal set of linear, pairwise nearly commuting projections on $V$, closed under the operations $p + cN$, $E + cN$, and $p + q - E + cN$. Note also the following theorem.

**Theorem 3.**

1. $\text{Ran} \ p = \text{Ran}(p + cN)$,
2. $\text{Ran} \ pq = \text{Ran}(E + cN)$; and
3. $\text{Ran}(p + q - pq) = \text{Ran}(p + q - E + cN)$.

**Proof.** $p(p + cN) = p + cN$, so $\text{Ran}(p + cN) \subseteq \text{Ran} \ p$. Also, $(p + cN)p = p$, so $\text{Ran} \ p \subseteq \text{Ran}(p + cN)$. Therefore, $\text{Ran} \ p = \text{Ran}(p + cN)$. The other identities follow analogously.

Now let

\[ E_1 = E + cN, \]  
\[ E_2 = p - E + N, \]  
\[ E_3 = q - E - N, \]  
\[ E_4 = I - p - q + E - cN \]  

for a scalar $c \in \mathbb{C}$. Then

\[ E_i^2 = E_i \quad (i = 1, 2, 3, 4), \]  
\[ E_i E_j = E_j E_i = 0 \quad (i \neq j), \]  
\[ E_1 + E_2 + E_3 + E_4 = I, \]

i.e. $E_1, E_2, E_3, E_4$ are linear, idempotent, and orthogonal operators on $V$ that add to $I$. They generate a set closed under the operations

\[ x \vee y = x + y - xy, \quad x \wedge y = xy, \text{ and } x' = I - x. \]

Now we decompose $p$ and $q$ that are nearly commuting projections on $V$. 

Theorem 4. $p$ and $q$ are two linear, nearly commuting projections on $V$ if and only if $p$ and $q$ can be decomposed into sums

$$p = p_1 + p_2,$$

$$q = q_1 + q_2,$$

where

1. $p_1, p_2, q_1, q_2$ are linear projections on $V$;
2. $p_1p_2 = p_2p_1 = 0$, $q_1q_2 = q_2q_1 = 0$;
3. $p_1q_2 = q_2p_1 = 0$, $p_2q_1 = q_1p_2 = 0$;
4. $p_1q_1 = q_1p_1 = p_1$; and
5. $p_2q_2 = q_2p_2 = 0$.

Moreover, this decomposition is unique and is given by $p_1 = qp$, $p_2 = (1 - q)p$, $q_1 = pq$, and $q_2 = (1 - p)q$.

Proof. Let $p = p_1 + p_2$ and $q = q_1 + q_2$, where $p_1, p_2, q_1, q_2$ satisfy conditions (1)–(5). Then $p$ and $q$ are linear projections on $V$; and $qp = p_1$, $pq = p_1$, $pq = q_1$, and $qqpq = q_1$. Thus $pqp = qp$ and $qqpq = pq$, making $p$ and $q$ nearly commute. Also, $p_1 = qp$, $p_2 = (1 - q)p$, $q_1 = pq$, and $q_2 = (1 - p)q$.

On the other hand, let $p$ and $q$ be any two linear, nearly commuting projections on $V$, and let $p_1 = qp$, $p_2 = (1 - q)p$, $q_1 = pq$, and $q_2 = (1 - p)q$. Then $p = p_1 + p_2$ and $q = q_1 + q_2$; and $p_1, p_2, q_1, q_2$ satisfy conditions (1)–(5).

By methods similar to those used for Theorem 4, one can show that any two nearly commuting projections on any vector space $V$ are given, after a suitable choice of basis for $V$, by matrices in the block form

$$\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
I & -I \\
0 & I
\end{bmatrix}.$$ 

4. TWO OPERATORS

Let $X$ be a maximal set of pairwise nearly commuting projections on a vector space $V$ over $\mathbb{C}$, as before. Let $H_p$ and $F_p$ be two projection operators
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on \( X \) defined by

\[
H_p(x) = p + px - xp
\]

(7)

and

\[
F_p(x) = x - px + xp
\]

(8)

for \( p, x \in X \). Note that \( F_p(x) = H_x(p) \). Their basic properties are as follows.

**Theorem 5.**

1. \( x \in \text{Ran } H_p \) if and only if \( px = x \) and \( xp = p \).
2. The condition "\( pq = q \) and \( qp = p \)" is that of an equivalence relation.
3. \( x \in \text{Ran } F_p \) if and only if \( px = xp \).
4. If \( p, x, y \in X \), then \( F_p(xy) = F_p(x)F_p(y) \).
5. If \( p, x, y \in X \), then \( F_p(x + y - xy) = F_p(x) + F_p(y) - F_p(x)F_p(y) \).
6. If \( p, x, y \in X \), then \( F_p(x + xy - yx) = F_p(x) + F_p(x)F_p(y) - F_p(y)F_p(x) \).

**Proof.** We need only prove (2). The relation is

(i) symmetric: \( pp = p \) and \( pp = p \);
(ii) reflexive: \( pq = q \) and \( qp = p \) implies \( qp = p \) and \( pq = q \); and
(iii) transitive: if \( pq = q \) and \( qp = p \), and if \( qr = r \) and \( rq = q \), then \( pr = p(qr) = (pq)r = qr = r \) and \( rp = r(qp) = (rq)p = qp = p \).

Therefore, \( \text{Ran } H_p \) for each \( p \in X \) is an equivalence class. Note that, for all \( p, q \in X \), \( pq \) and \( qp \) are equivalent, and \( p + q - pq \) and \( p + q - qp \) are equivalent.

Let \( p_1, p_2, \ldots, p_n, p, q, x, r \) be linear projections on \( V \) that nearly commute. Let \( F_0(x) = x \), and let \( F_n = F_{p_1}F_{p_2} \cdots F_{p_n} \). Now we prove a lemma.

**Lemma.** \( F_n(pq) = F_n(p)F_n(q) \).

**Proof of lemma.** By Theorem 5(4), \( F_r(pq) = F_r(p)F_r(q) \). Note that \( q \) nearly commutes with \( F_r(p) \) for any three projections \( r, p, q \in X \). So we can apply \( F_r(pq) = F_r(p)F_r(q) \) repeatedly with \( p_n, p_{n-1}, \ldots, p_1 \) as \( r \).

Let \( p_i^* = F_{i-1}(p_i) \) for \( i = 1, \ldots, n \). Now we prove a theorem.

**Theorem 6.** \( p_1^*, p_2^*, \ldots, p_n^* \) pairwise commute.
Proof. We want to show that $p_i^* p_n^* = p_n^* p_i^*$ for $i = 1, \ldots, n - 1$. Note that $p_n^* = F_{i-1} F_{i} F_{i+1} \cdots F_{n-1}(p_n)$. Let $g_i = F_{i+1} F_{i+2} \cdots F_{n-1}(p_n)$. Now $p_i$ commutes with $g_i$, and $p_i$ and $g_i$ each pairwise nearly commute with $p_1, p_2, \ldots, p_{i-1}$, which as a set of pairwise nearly commute. So, by our lemma,

$$p_i^* p_n^* = F_{i-1}(p_i) F_{i-1}(g_i) = F_{i-1}(p_i g_i) = F_{i-1}(g_i p_i) = F_{i-1}(g_i) F_{i-1}(p_i) = p_n^* p_i^*. $$

Now we prove another theorem.

**Theorem 7.** Let $p_1, p_2, \ldots, p_n, x$ be linear projections on $V$ that pairwise nearly commute. Then for each $n > 2$,

$$F_{p_n^*} F_{p_{n-2}} \cdots F_{p_1}(x) = F_{p_1} F_{p_2} \cdots F_{p_{n-1}}(x). \quad (\ast)$$

**Proof.** Let $p_1 = p$, $p_2 = q$, and $x = r$. Then

$$F_{q^*} F_{p^*}(r) = F_{F_{q^*}}(F_{p^*}(r)) = F_p(r) - F_p(q) F_p(r) + F_p(r) F_p(q) = F_p(r - qr + rq) = F_p F_q(r).$$

So

$$F_{F_{p}(q)} F_{p}(r) = F_p F_q(r), \quad (\ast\ast)$$

and $\ast$ is true for $n = 3$. Assume it is true for $n$. $\ast$ can be written as

$$S_n = F_{p_1} \cdots F_{p_{n-2}}(p_{n-1}) F_{p_{n-1}}(p_{n-2}) \cdots F_{p_1}(p_{n-2}) F_{p_1}(x) = F_{p_1} \cdots F_{p_{n-1}}(x).$$
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So

\[ S_{n+1} = F_{F_{p_1} - F_{p_{n-1}}(p_n)} F_{F_{p_1} - F_{p_{n-2}}(p_{n-1})} \cdots F_{F_{p_1}(p_2)} F_{p_1}(x) \]

\[ = F_{F_{p_1} - F_{p_{n-1}}(p_n)} S_n \]

\[ = F_{F_{p_1}(p_2)} \cdots F_{p_{n-1} x p_n} F_{p_1}(F_{p_2} \cdots F_{p_{n-1}})(x). \]

We can prove by induction on \( k \), using \((**1)\), that

\[ S_{n+1} = F_{p_1} \cdots F_{p_{k-1}} F_{p_k} F_{p_k+1} \cdots F_{p_{n-1}} F_{p_n}(x). \]

Thus \( S_{n+1} = F_{p_1} F_{p_2} \cdots F_{p_{n-1}} F_{p_n}(x) \). Thus \((*)\) is true by induction. \( \blacksquare \)

By equation \((**2)\),

\[ F_{p} F_{q} F_{p}(x) = F_{p_{q}} F_{p} F_{p}(x) \]

\[ = F_{p_{p}(q)} F_{p}(x) \]

\[ = F_{p_{q}} (x), \]

so \( F_{p} \) and \( F_{q} \) antilinearly commute. Also, if \( p \) and \( q \) commute, \( p \) and \( x \) nearly commute, and \( q \) and \( x \) nearly commute, then \( F_{p}(q) = q \) and \( F_{p} F_{q}(x) = F_{p_{q}} F_{p}(x) = F_{q} F_{p}(x) \), i.e., \( F_{p} \) and \( F_{q} \) commute. The projection operators \( F_{p_1}, F_{p_2}, \ldots, F_{p_n} \) pairwise commute.

5. ORTHOGONAL PROJECTIONS

In Section 2, we displayed linear projections \( E_1, E_2, E_3, E_4 \) which were functions of \( p \) and \( q \), and which were four orthogonal projections adding to \( I \). Let \( p_0, p_1, \ldots, p_{n-1}, p_n \) be \( n + 1 \) linear projections on \( V \) that pairwise nearly commute. Suppose \( E_1, E_2, \ldots, E_2^* \) are functions of \( p_0, p_1, \ldots, p_{n-1} \) that are \( 2^n \) orthogonal projections that add to \( I \). Then \( p_n E_i p_n = E_i p_n \), and we have the following theorem.

**Theorem 8.** \( \{ E_i p_n | i = 1, 2, \ldots, 2^n \} \) and \( \{(I - p_n)E_i | i = 1, 2, \ldots, 2^n \} \) are sets of \( 2^{n+1} \) orthogonal projections that add to \( I \).
Proof. If \( i \neq j \), then

\[
\begin{align*}
(1) & \quad E_i p_n E_i p_n = E_i E_i p_n = E_i p_n; \\
(2) & \quad (I - p_n)E_i(I - p_n)E_i = E_i E_i - E_i p_n E_i - p_n E_i E_i + p_n E_i p_n E_i = E_i \\
& \quad - E_i p_n E_i - p_n E_i + E_i p_n E_i = E_i - p_n E_i = (I - p_n)E_i; \\
(3) & \quad E_i p_n (I - p_n)E_i = 0; \\
(4) & \quad (I - p_n)E_i E_i p_n = E_i p_n - p_n E_i p_n = E_i p_n - E_i p_n = 0; \\
(5) & \quad E_i p_n (I - p_n)E_i = 0; \\
(6) & \quad (I - p_n)E_j E_i p_n = 0; \\
(7) & \quad E_i p_n E_j p_n = E_i E_j p_n = 0; \\
(8) & \quad (I - p_n)E_i(I - p_n)E_j = E_i E_j - E_i p_n E_j - p_n E_i E_j + p_n E_i p_n E_j = 0 \\
& \quad - E_i p_n E_j - 0 + E_i p_n E_j = 0; \\
(9) & \quad \sum_{i=1}^{2^n} E_i p_n + \sum_{i=1}^{2^n} (I - p_n)E_i = \sum_{i=1}^{2^n} E_i + (I - p_n)I = I.
\end{align*}
\]

6. A FURTHER DECOMPOSITION

Suppose \( p, q, r, x \) are linear, pairwise nearly commuting projections on \( V \). Then \( F_p(x) = xp + p'x \) where \( p' = I - p \). Let \( q' = I - q \) and \( r' = I - r \) also. Let \( P = p, Q = F_p(q) \), and \( R = F_p F_q(r) \). Then, by Theorem 6, \( P, Q, \) and \( R \) pairwise commute. Also,

\[
P = p = qp + q'p \\
= (rqp + r'qp) + (q'rqp + q'r'p),
\]

\[
Q = F_p(q) = qp + p'q \\
= (rqp + r'qp) + (p'rqp + p'r'q),
\]

\[
R = F_p F_q(r) = F_q(r)p + p'F_q(r) \\
= (rq + q'r)p + p'(rq + q'r) \\
= rqp + q'rp + p'rq + p'q'r.
\]

By Theorem 8, these triples of \( p, q, r, p', q' \), and \( r' \) are orthogonal.

We generalize these formulas to \( n \) projections. Let \( p_1, p_2, \ldots, p_n \) be \( n \) linear, pairwise nearly commuting projections on \( V \), let \( p_i^{(1)} = p_i \), and let
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For \( k = 1, \ldots, n \), let \( E_k^n(i_k, \ldots, i_n) \) be a function from \( \{0, 1\}^{n-k+1} \) into the set of linear projections on \( V \), defined recursively by

(i) \( E_k^n(i_n) = p_n^{(i_n)} \),

(ii) \( E_k^{n-1}(1, i_k, \ldots, i_n) = E_k^n(i_k, \ldots, i_n) p_{k-1} \) and \( E_k^{n-1}(0, i_k, \ldots, i_n) = p_{k-1} E_k^n(i_k, \ldots, i_n) \)

for \( n > k > 2 \). Then \( E_k^n(i_1, \ldots, i_n) \) is in general a product of \( n \) projections such that the first few are primed \( p_i \)'s in numerical order followed by the rest unprimed in reverse numerical order. Moreover, the products \( E_k^n(i_1, \ldots, i_n) \) for \( i_1 = 0, 1; \ldots; i_n = 0, 1 \) are (by Theorem 8) \( 2^n \) orthogonal projections that add to \( I \).

Taking \( k \) such that \( k = 1, \ldots, n \), note that \( p_{k+1} \) is in the same position in \( E_{k+1}^n(i_1, \ldots, i_n, 1) \) that \( p_k \) is in \( E_k^n(i_1, \ldots, 1, 0) \). Removing \( p_{k+1} \) or \( p_k \) from their positions gives us \( E_k^n(i_1, \ldots, i_n) \). Since \( p_{k+1} + p_k = I \),

\[
E_k^n(i_1, \ldots, i_n) = E_k^{n+1}(i_1, \ldots, i_k, 1) + E_k^{n+1}(i_1, \ldots, i_k, 0).
\]

By induction,

\[
E_k^n(i_1, \ldots, i_n) = \sum_{i_{k+1}=0}^{1} \cdots \sum_{i_n=0}^{1} E^n_1(i_1, \ldots, i_{k-1}, 1, i_{k+1}, \ldots, i_n). \tag{12}
\]

Let \( P_1 = p_1 \) and \( P_k = F_{p_1} \cdots F_{p_k} \) for \( k = 2, \ldots, n \). Then by (12) and \( F_p(x) = xp + p'x \),

\[
P_k^{(1)} = P_k = \sum E_k^n(i_1, \ldots, i_{k-1}, 1)
= \sum E_1^n(i_1, \ldots, i_{k-1}, 1, i_{k+1}, \ldots, i_n), \tag{13}
\]

where \( \sum \) denotes the sum over all indices \( i_j \) without substituted values. Since \( P_k^{(0)} = I - P_k^{(1)} \),

\[
P_k^{(0)} = P_k' = \sum E_1^n(i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_n), \tag{14}
\]

where \( \sum \) denotes the same type of sum. By Theorem 6, \( P_1, \ldots, P_n \) pairwise commute. So the product

\[
P_1^{(i_1)} \cdots P_n^{(i_n)} = E_1^n(i_1, \ldots, i_n) \tag{15}
\]
follows by Equations (13) and (14) and the fact that all products of the right-hand side of (15) are orthogonal.

REFERENCES


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