Nearly Commuting Projections

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ABSTRACT

It is well known that projection operators are typical elements in Boolean algebras, and a number of relevant theorems have been proved for commutative projections. We propose an extension of the concept of commutativity, which we call near-commutativity. We extend to this concept the main theorems on commutative projections, and in various ways we frame the class of nearly commutative projections in Boolean algebras.

1. INTRODUCTION

If p and q are two linear projections on a vector space V over $\mathbb{C}$, we say they *nearly commute* if

$$pqp = qp \quad \text{and} \quad qpq = pq. \tag{1}$$

We say they *antinearly commute* if

$$pqp = pq \quad \text{and} \quad qpq = qp. \tag{2}$$

If p and q commute, then they both nearly commute and antinearly commute. Also, p and q nearly commute if and only if their complements $I - p$ and $I - q$ antinearly commute.

Section 2 displays examples of these kinds of projections. Basic properties of nearly commuting projections appear in Section 3. Section 4 introduces

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two operators on sets of nearly commuting projections. Section 5 derives orthogonal projections from nearly commuting projections, and Section 6 does a decomposition of projections using orthogonal projections.

2. EXAMPLES

Let V be the vector space of functions $f : \mathbb{C}^3 \to \mathbb{C}$. Let

$$p(f)(z_1, z_2, z_3) = f(0, 0, z_3)$$

and

$$q(f)(z_1, z_2, z_3) = f(z_1, 0, 0).$$

Then p and q are commuting projections on V. Now let

$$p(f)(z_1, z_2, z_3) = f(0, 0, z_3)$$

and

$$q(f)(z_1, z_2, z_3) = f(z_1, 1, 1).$$

Then

$$pq(f)(z_1, z_2, z_3) = f(0, 1, 1),$$

$$qpq(f)(z_1, z_2, z_3) = f(0, 1, 1),$$

$$qp(f)(z_1, z_2, z_3) = f(0, 0, 1),$$

$$pqp(f)(z_1, z_2, z_3) = f(0, 0, 1).$$

Thus

$$pqp(f) = qp(f) \quad \text{and} \quad qpq(f) = pq(f).$$

In general, if V is the set of functions $f : \mathbb{C}^n \to \mathbb{C}$, the projections which substitute the same constant for different arguments all commute; whereas the projections which substitute different constants for arguments, in general, nearly commute.

Let p and q be linear projections on a vector space V over $\mathbb{C}$ and let a and b be two elements in V such that $a \in \operatorname{Ran}(I - p)$ and $b \in \operatorname{Ran}(I - q)$. Also, let $P(x) = a + p(x)$ and $Q(x) = b + q(x)$ for all $x \in V$. Then $P^2 = P$ and $Q^2 = Q$, i.e., P and Q are affine projections on V (see Wilde [1]). If p and q commute, then in general $PQP = PQ$ and $QPQ = QP$. Also, $PQP = PQ$ if and only if $pqp = pq$.

Our final example is a set of $(n+2) \times (n+2)$ matrices over $\mathbb{C}$. Let $a_1, a_2, \ldots, a_n \in \mathbb{C}$. Let E_{ij} be the $(n+2) \times (n+2)$ matrix with a 1 in the (i, j) spot and 0's elsewhere. Let $p_i = E_{11} + a_i E_{12}$ for $i = 1, 2, \ldots, n$, and let $q_j = E_{2+j,2+j}$ for $j = 1, 2, \ldots, n$. Then $p_i p_j = p_j$ and $p_j p_i = p_i$ for $i \neq j$; and $q_1, \ldots, q_n$ are pairwise orthogonal. Also, $p_i q_j = q_j p_i = 0$ for all i and j in $\{1, 2, \ldots, n\}$. All projections of the form "p_i plus sums of the q_j's" nearly commute. For instance, if $i \neq j$, then

$$(p_i + q_i)(p_j + q_i + q_j) = p_j + q_i,$$

$$(p_j + q_i + q_i)(p_i + q_i)(p_j + q_i + q_j) = p_j + q_i,$$

$$(p_j + q_i + q_j)(p_i + q_i) = p_i + q_i,$$

$$(p_i + q_i)(p_j + q_i + q_j)(p_i + q_i) = p_i + q_i,$$

i.e., $p_i + q_i$ and $p_j + q_i + q_j$ nearly commute.

3. MISCELLANEOUS PROPERTIES

We prove the following theorem.

THEOREM 1. *Let p, q, r be linear, pairwise nearly commuting projections on V. Then*

(1) *pq, $p + q - pq$, $p + pq - qp$, and $\frac{1}{2}(pq + qp)$ are linear projections on V;*

(2) *r nearly commutes with pq, $p + q - pq$, $p + pq - qp$, and $\frac{1}{2}(pq + qp)$;*

(3) $\operatorname{Ran} p \cap \operatorname{Ran} q = \operatorname{Ran} pq = \operatorname{Ran} qp$; *and*

(4) $\operatorname{Ran} p + \operatorname{Ran} q = \operatorname{Ran}(p + q - pq) = \operatorname{Ran}(p + q - qp)$.

Proof. (1): Easy.

(2): r nearly commutes with pq because

$$\begin{aligned} r(pq)r &= rp(qr) = rp(rqr) = (rpr)qr \\ &= (pr)qr = p(rqr) = p(qr) = (pq)r \end{aligned}$$

and

$$(pq)r(pq) = pq(rp)q = p(rp)q = (prp)q = (rp)q = r(pq).$$

This rest is just more calculation.

(3): Let $x \in \operatorname{Ran} pq$. Then $x = pq(x)$, $p(x) = p(pq(x)) = pq(x) = x$, and $q(x) = q(pq(x)) = pq(x) = x$. Thus $\operatorname{Ran} pq \subset \operatorname{Ran} p \cap \operatorname{Ran} q$. Let $x \in \operatorname{Ran} p \cap \operatorname{Ran} q$. Then $p(x) = x$ and $q(x) = x$; thus $pq(x) = p(x) = x$, and so $\operatorname{Ran} p \cap \operatorname{Ran} q \subset \operatorname{Ran} pq$. By symmetry, $\operatorname{Ran} p \cap \operatorname{Ran} q = \operatorname{Ran} qp$, although pq does not always equal qp.

(4): Let $x \in \operatorname{Ran} p$ and $y \in \operatorname{Ran} q$; then $p(x) = x$ and $q(y) = y$, and

$$\begin{aligned} (p + q - pq)(x + y) &= p(x) + q(x) - pq(x) + p(y) + q(y) - pq(y) \\ &= x + qp(x) - pqp(x) + pq(y) + y - pq(y) \\ &= x + qp(x) - qp(x) + y = x + y, \end{aligned}$$

or $\operatorname{Ran} p + \operatorname{Ran} q \subset \operatorname{Ran}(p + q - pq)$. Let $x \in \operatorname{Ran}(p + q - pq)$. Then $x = (p + q - pq)(x) = p(x) + (I - p)q(x)$, where $p(x) \in \operatorname{Ran} p$ and $(I - p)q(x) \in \operatorname{Ran} q$, since $q((I - p)q(x)) = (I - p)q(x)$. Thus $\operatorname{Ran}(p + q - pq) \subset \operatorname{Ran} p + \operatorname{Ran} q$. By symmetry, $\operatorname{Ran} p + \operatorname{Ran} q = \operatorname{Ran}(p + q - qp)$. ■

Suppose p, q, r are linear, pairwise nearly commuting projections on V. Let

$$E = \tfrac{1}{2}(pq + qp) \tag{3}$$

and

$$N = \tfrac{1}{2}(pq - qp). \tag{4}$$

Then we can prove the following.

THEOREM 2.

(1) $E^2 = E$, $N^2 = 0$;
(2) $pE = E$, $qE = E$;
(3) $pN = N$, $qN = N$;
(4) $Ep = E - N$, $Eq = E + N$;
(5) $Np = 0$, $Nq = 0$; *and*
(6) $EN = N$, $NE = 0$.

Also, $p + cN$, $E + cN$, and $p + q - E + cN$, for a scalar $c \in \mathbb{C}$, are linear projections, and r nearly commutes with them. For this reason, we let X be a maximal set of linear, pairwise nearly commuting projections on V, closed under the operations $p + cN$, $E + cN$, and $p + q - E + cN$. Note also the following theorem.

THEOREM 3.

(1) $\operatorname{Ran} p = \operatorname{Ran}(p + cN)$;
(2) $\operatorname{Ran} pq = \operatorname{Ran}(E + cN)$; *and*
(3) $\operatorname{Ran}(p + q - pq) = \operatorname{Ran}(p + q - E + cN)$.

Proof. $p(p + cN) = p + cN$, so $\operatorname{Ran}(p + cN) \subset \operatorname{Ran} p$. Also, $(p + cN)p = p$, so $\operatorname{Ran} p \subset \operatorname{Ran}(p + cN)$. Therefore, $\operatorname{Ran} p = \operatorname{Ran}(p + cN)$. The other identities follow analogously. ■

Now let

$$E_1 = E + cN, \tag{5.1}$$

$$E_2 = p - E + N, \tag{5.2}$$

$$E_3 = q - E - N, \tag{5.3}$$

$$E_4 = I - p - q + E - cN \tag{5.4}$$

for a scalar $c \in \mathbb{C}$. Then

$$E_i^2 = E_i \qquad (i = 1, 2, 3, 4), \tag{6.1}$$

$$E_i E_j = E_j E_i = 0 \qquad (i \neq j), \tag{6.2}$$

$$E_1 + E_2 + E_3 + E_4 = I, \tag{6.3}$$

i.e. E_1, E_2, E_3, E_4 are linear, idempotent, and orthogonal operators on V that add to I. They generate a set closed under the operations

$$x \vee y = x + y - xy, \qquad x \wedge y = xy, \quad \text{and} \quad x' = I - x.$$

Now we decompose p and q that are nearly commuting projections on V.

THEOREM 4. *p and q are two linear, nearly commuting projections on V if and only if p and q can be decomposed into sums*

$$p = p_1 + p_2,$$
$$q = q_1 + q_2,$$

where

(1) p_1, p_2, q_1, q_2 *are linear projections on* V;
(2) $p_1 p_2 = p_2 p_1 = 0,\ q_1 q_2 = q_2 q_1 = 0$;
(3) $p_1 q_2 = q_2 p_1 = 0,\ p_2 q_1 = q_1 p_2 = 0$;
(4) $p_1 q_1 = q_1,\ q_1 p_1 = p_1$; *and*
(5) $p_2 q_2 = q_2 p_2 = 0$.

Moreover, this decomposition is unique and is given by $p_1 = qp$, $p_2 = (I - q)p$, $q_1 = pq$, *and* $q_2 = (I - p)q$.

Proof. Let $p = p_1 + p_2$ and $q = q_1 + q_2$, where p_1, p_2, q_1, q_2 satisfy conditions (1)–(5). Then p and q are linear projections on V; and $qp = p_1$, $pqp = p_1$, $pq = q_1$, and $qpq = q_1$. Thus $pqp = qp$ and $qpq = pq$, making p and q nearly commute. Also, $p_1 = qp$, $p_2 = (I - q)p$, $q_1 = pq$, and $q_2 = (I - p)q$.

On the other hand, let p and q be any two linear, nearly commuting projections on V, and let $p_1 = qp$, $p_2 = (I - q)p$, $q_1 = pq$, and $q_2 = (I - p)q$. Then $p = p_1 + p_2$ and $q = q_1 + q_2$; and p_1, p_2, q_1, q_2 satisfy conditions (1)–(5). ∎

By methods similar to those used for Theorem 4, one can show that any two nearly commuting projections on any vector space V are given, after a suitable choice of basis for V, by matrices in the block form

$$\begin{bmatrix} I & I & & & & 0 \\ 0 & 0 & & & & \\ & & I & & & \\ & & & I & & \\ & & & & 0 & \\ 0 & & & & & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & -I & & & & 0 \\ 0 & 0 & & & & \\ & & I & & & \\ & & & 0 & & \\ & & & & I & \\ 0 & & & & & 0 \end{bmatrix}.$$

4. TWO OPERATORS

Let X be a maximal set of pairwise nearly commuting projections on a vector space V over $\mathbb{C}$, as before. Let H_p and F_p be two projection operators on X defined by

$$H_p(x) = p + px - xp \tag{7}$$

and

$$F_p(x) = x - px + xp \tag{8}$$

for $p, x \in X$. Note that $F_p(x) = H_x(p)$. Their basic properties are as follows.

THEOREM 5.

(1) $x \in \operatorname{Ran} H_p$ *if and only if* $px = x$ *and* $xp = p$.
(2) *The condition* "$pq = q$ *and* $qp = p$" *is that of an equivalence relation.*
(3) $x \in \operatorname{Ran} F_p$ *if and only if* $px = xp$.
(4) *If* $p, x, y \in X$, *then* $F_p(xy) = F_p(x)F_p(y)$.
(5) *If* $p, x, y \in X$, *then* $F_p(x + y - xy) = F_p(x) + F_p(y) - F_p(x)F_p(y)$.
(6) *If* $p, x, y \in X$, *then* $F_p(x + xy - yx) = F_p(x) + F_p(x)F_p(y) - F_p(y)F_p(x)$.

Proof. We need only prove (2). The relation is

(i) symmetric: $pp = p$ and $pp = p$;
(ii) reflexive: $pq = q$ and $qp = p$ implies $qp = p$ and $pq = q$; and
(iii) transitive: if $pq = q$ and $qp = p$, and if $qr = r$ and $rq = q$, then $pr = p(qr) = (pq)r = qr = r$ and $rp = r(qp) = (rq)p = qp = p$. ∎

Therefore, Ran H_p for each $p \in X$ is an equivalence class. Note that, for all $p, q \in X$, pq and qp are equivalent, and $p + q - pq$ and $p + q - qp$ are equivalent.

Let $p_1, p_2, \ldots, p_n, p, q, x, r$ be linear projections on V that nearly commute. Let $F_0(x) = x$, and let $F_n = F_{p_1}F_{p_2} \cdots F_{p_n}$. Now we prove a lemma.

LEMMA. $F_n(pq) = F_n(p)F_n(q)$.

Proof of lemma. By Theorem 5(4), $F_r(pq) = F_r(p)F_r(q)$. Note that q nearly commutes with $F_r(p)$ for any three projections $r, p, q \in X$. So we can apply $F_r(pq) = F_r(p)F_r(q)$ repeatedly with $p_n, p_{n-1}, \ldots, p_1$ as r. ∎

Let $p_i^* = F_{i-1}(p_i)$ for $i = 1, \ldots, n$. Now we prove a theorem.

THEOREM 6. $p_1^*, p_2^*, \ldots, p_n^*$ *pairwise commute.*

Proof. We want to show that $p_i^* p_n^* = p_n^* p_i^*$ for $i = 1, \ldots, n-1$. Note that $p_n^* = F_{i-1}F_{p_i}F_{p_{i+1}} \cdots F_{p_{n-1}}(p_n)$. Let $g_i = F_{p_i}F_{p_{i+1}} \cdots F_{p_{n-1}}(p_n)$. Now p_i commutes with g_i, and p_i and g_i each pairwise nearly commute with $p_1, p_2, \ldots, p_{i-1}$, which as a set of pairwise nearly commute. So, by our lemma,

$$
\begin{aligned}
p_i^* p_n^* &= F_{i-1}(p_i)F_{i-1}(g_i) \\
&= F_{i-1}(p_i g_i) \\
&= F_{i-1}(g_i p_i) \\
&= F_{i-1}(g_i)F_{i-1}(p_i) \\
&= p_n^* p_i^*.
\end{aligned}
$$

∎

Now we prove another theorem.

THEOREM 7. *Let $p_1, p_2, \ldots, p_n, x$ be linear projections on V that pairwise nearly commute. Then for each $n > 2$,*

$$F_{p^*_{n-1}}F_{p^*_{n-2}} \cdots F_{p^*_1}(x) = F_{p_1}F_{p_2} \cdots F_{p_{n-1}}(x). \tag{$*$}$$

Proof. Let $p_1 = p$, $p_2 = q$, and $x = r$. Then

$$
\begin{aligned}
F_{q^*}F_{p^*}(r) &= F_{F_p(q)}\big(F_p(r)\big) \\
&= F_p(r) - F_p(q)F_p(r) + F_p(r)F_p(q) \\
&= F_p(r - qr + rq) \\
&= F_pF_q(r).
\end{aligned}
$$

So

$$F_{F_p(q)}F_p(r) = F_pF_q(r), \tag{$**$}$$

and $(*)$ is true for $n = 3$. Assume it is true for n. $(*)$ can be written as

$$
\begin{aligned}
S_n &= F_{F_{p_1} \cdots F_{p_{n-2}}(p_{n-1})}F_{F_{p_1} \cdots F_{p_{n-3}}(p_{n-2})} \cdots F_{F_{p_1}(p_2)}F_{p_1}(x) \\
&= F_{p_1} \cdots F_{p_{n-1}}(x).
\end{aligned}
$$

So

$$
\begin{aligned}
S_{n+1} &= F_{F_{p_1}\cdots F_{p_{n-1}}(p_n)} F_{F_{p_1}\cdots F_{p_{n-2}}(p_{n-1})} \cdots F_{F_{p_1}(p_2)} F_{p_1}(x) \\
&= F_{F_{p_1}\cdots F_{p_{n-1}}(p_n)} S_n \\
&= F_{F_{p_1}(F_{p_2}\cdots F_{p_{n-1}})(p_n)} F_{p_1}\big(F_{p_2}\cdots F_{p_{n-1}}\big)(x).
\end{aligned}
$$

We can prove by induction on k, using $(**)$, that

$$
S_{n+1} = F_{p_1}\cdots F_{p_{k-1}} F_{F_{p_k}(F_{p_{k+1}}\cdots F_{p_{n-1}})(p_n)} F_{p_k}\big(F_{p_{k+1}}\cdots F_{p_{n-1}}\big)(x).
$$

Thus $S_{n+1} = F_{p_1}F_{p_2}\cdots F_{p_{n-1}}F_{p_n}(x)$. Thus $(*)$ is true by induction. ■

By equation $(**)$,

$$
\begin{aligned}
F_pF_qF_p(x) &= F_{F_p(q)}F_pF_p(x) \\
&= F_{F_p(q)}F_p(x) \\
&= F_pF_q(x),
\end{aligned}
$$

so F_p and F_q antinearly commute. Also, if p and q commute, p and x nearly commute, and q and x nearly commute, then $F_p(q) = q$ and $F_pF_q(x) = F_{F_p(q)}F_p(x) = F_qF_p(x)$, i.e., F_p and F_q commute. The projection operators $F_{p_1^*}, F_{p_2^*}, \ldots, F_{p_n^*}$ pairwise commute.

5. ORTHOGONAL PROJECTIONS

In Section 2, we displayed linear projections E_1, E_2, E_3, E_4 which were functions of p and q, and which were four orthogonal projections adding to I. Let $p_0, p_1, \ldots, p_{n-1}, p_n$ be $n+1$ linear projections on V that pairwise nearly commute. Suppose $E_1, E_2, \ldots, E_{2^n}$ are functions of $p_0, p_1, \ldots, p_{n-1}$ that are 2^n orthogonal projections that add to I. Then $p_nE_ip_n = E_ip_n$, and we have the following theorem.

THEOREM 8. *$\{E_ip_n | i = 1,2,\ldots,2^n\}$ and $\{(I-p_n)E_i | i = 1,2,\ldots,2^n\}$ are sets of 2^{n+1} orthogonal projections that add to I.*

Proof. If $i \neq j$, then

(1) $E_i p_n E_i p_n = E_i E_i p_n = E_i p_n$;

(2) $(I - p_n)E_i(I - p_n)E_i = E_i E_i - E_i p_n E_i - p_n E_i E_i + p_n E_i p_n E_i = E_i - E_i p_n E_i - p_n E_i + E_i p_n E_i = E_i - p_n E_i = (I - p_n)E_i$;

(3) $E_i p_n(I - p_n)E_i = 0$;

(4) $(I - p_n)E_i E_i p_n = E_i p_n - p_n E_i p_n = E_i p_n - E_i p_n = 0$;

(5) $E_i p_n(I - p_n)E_j = 0$;

(6) $(I - p_n)E_j E_i p_n = 0$;

(7) $E_i p_n E_j p_n = E_i E_j p_n = 0$;

(8) $(I - p_n)E_i(I - p_n)E_j = E_i E_j - E_i p_n E_j - p_n E_i E_j + p_n E_i p_n E_j = 0 - E_i p_n E_j - 0 + E_i p_n E_j = 0$;

(9) $\sum_{i=1}^{2^n} E_i p_n + \sum_{i=1}^{2^n}(I - p_n)E_i = Ip_n + (I - p_n)I = I$. ■

6. A FURTHER DECOMPOSITION

Suppose p, q, r, x are linear, pairwise nearly commuting projections on V. Then $F_p(x) = xp + p'x$ where $p' = I - p$. Let $q' = I - q$ and $r' = I - r$ also. Let $P = p$, $Q = F_p(q)$, and $R = F_p F_q(r)$. Then, by Theorem 6, P, Q, and R pairwise commute. Also,

$$P = p = qp + q'p$$

$$= (rqp + r'qp) + (q'rp + q'r'p), \tag{9}$$

$$Q = F_p(q) = qp + p'q$$

$$= (rqp + r'qp) + (p'rq + p'r'q), \tag{10}$$

$$R = F_p F_q(r) = F_q(r)p + p'F_q(r)$$

$$= (rq + q'r)p + p'(rq + q'r)$$

$$= rqp + q'rp + p'rq + p'q'r. \tag{11}$$

By Theorem 8, these triples of p, q, r, p', q', and r' are orthogonal.

We generalize these formulas to n projections. Let $p_1, p_2, \ldots, p_n$ be n linear, pairwise nearly commuting projections on V, let $p_i^{(1)} = p_i$, and let

$p_i^{(0)} = p_i'$ for $i = 1, \ldots, n$. For $k = 1, \ldots, n$, let $E_k^n(i_k, \ldots, i_n)$ be a function from $\{0, 1\}^{n-k+1}$ into the set of linear projections on V, defined recursively by

(i) $E_n^n(i_n) = p_n^{(i_n)}$,

(ii) $E_{k-1}^n(1, i_k, \ldots, i_n) = E_k^n(i_k, \ldots, i_n)p_{k-1}$ and $E_{k-1}^n(0, i_k, \ldots, i_n) = p_{k-1}' E_k^n(i_k, \ldots, i_n)$

for $n \geqslant k \geqslant 2$. Then $E_1^n(i_1, \ldots, i_n)$ is in general a product of n projections such that the first few are primed p_i's in numerical order followed by the rest unprimed in reverse numerical order. Moreover, the products $E_1^n(i_1, \ldots, i_n)$ for $i_1 = 0, 1; \ldots; i_n = 0, 1$ are (by Theorem 8) 2^n orthogonal projections that add to I.

Taking k such that $k = 1, \ldots, n$, note that p_{k+1} is in the same position in $E_1^{k+1}(i_1, \ldots, i_k, 1)$ that p_{k+1}' is in $E_1^{k+1}(i_1, \ldots, i_k, 0)$. Removing p_{k+1} or p_{k+1}' from their positions gives us $E_1^k(i_1, \ldots, i_k)$. Since $p_{k+1} + p_{k+1}' = I$,

$$E_1^k(i_1, \ldots, i_k) = E_1^{k+1}(i_1, \ldots, i_k, 1) + E_1^{k+1}(i_1, \ldots, i_k, 0).$$

By induction,

$$E_1^k(i_1, \ldots, i_k) = \sum_{i_{k+1}=0}^{1} \cdots \sum_{i_n=0}^{1} E_1^n(i_1, \ldots, i_k, i_{k+1}, \ldots, i_n). \tag{12}$$

Let $P_1 = p_1$ and $P_k = F_{p_1} \cdots F_{p_{k-1}}(p_k)$ for $k = 2, \ldots, n$. Then by (12) and $F_p(x) = xp + p'x$,

$$\begin{aligned} P_k^{(1)} = P_k &= \sum E_1^k(i_1, \ldots, i_{k-1}, 1) \\ &= \sum E_1^n(i_1, \ldots, i_{k-1}, 1, i_{k+1}, \ldots, i_n), \end{aligned} \tag{13}$$

where Σ denotes the sum over all indices i_j without substituted values. Since $P_k^{(0)} = I - P_k^{(1)}$,

$$P_k^{(0)} = P_k' = \sum E_1^n(i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_n), \tag{14}$$

where Σ denotes the same type of sum. By Theorem 6, $P_1, \ldots, P_n$ pairwise commute. So the product

$$P_1^{(i_1)} \cdots P_n^{(i_n)} = E_1^n(i_1, \ldots, i_n) \tag{15}$$

follows by Equations (13) and (14) and the fact that all products of the right-hand side of (15) are orthogonal.

REFERENCES

1 Alan C. Wilde, Properties of affine projections, *Rev. Mat. Univ. Parma Ser. 4* 14:223–229 (1989).
2 George Boole, *An Investigation of the Laws of Thought*, first printing, 1854; Dover, New York, 1951.

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