Primes in a Sparse Sequence

K. Soundararajan

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

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In this paper, we consider the sequence $1^1, + 1^1 + 2^2, 1^1 + 2^2 + 3^3, \ldots$ and prove some of its congruence properties. Surprisingly, this sequence is uniformly distributed in the residue classes (mod $m$) where $m \not\equiv 0 \pmod{4}$. Using these results and Selberg's sieve, we obtain an upper bound for the number of primes in the sequence which are $\leq x$.

1. Introduction

In [1], Hampel studied the sequence $n^n$ and proved some of its congruence properties. In this paper, we study the related sequence $1^1, 1^1 + 2^2, 1^1 + 2^2 + 3^3, \ldots$ with particular reference to the number of primes in this sequence.

The distribution of primes in some other sparse sequences has been dealt with in Chapter 7 of [2].

Our main result is

THEOREM. Let $a_1 = 1^1$ and $a_n = a_{n-1} + n^n$ for $n \geq 2$. Let

$$A = \{a_n : n \in \mathbb{N}\} \quad \text{and} \quad \pi_A(x) = \sum_{p \leq x \atop p \in A} 1.$$

Then, we have

$$\pi_A(x) \leq \frac{\log x}{(\log \log x)^2}.$$

Our proof of the theorem is based on an application of Selberg's upper bound sieve (see, for example, [3]). For this, it suffices to estimate

$$\sum_{n \in A \atop n \leq x \atop n \equiv 0 \pmod{d}} 1$$

for square free $d$. 
Our methods are, in fact, capable of yielding estimates for

\[ \sum_{\substack{n \in A \\ n \leq x \\ n \equiv a \pmod{d} }} 1 \]

for all \( a \) and \( d \).

2. PERIODICITIES OF \( n^a \) AND \( A \)

The following lemma was proved in a slightly more general form by Hampel [1].

**Lemma 1.** The sequence \( n^a \) is ultimately periodic \((\mod m)\) for all \( m \geq 2 \). Further if \( g(m) \) denotes the fundamental period of the sequence \((\mod m)\), then we have

\[ g(m) = \begin{cases} 
\text{l.c.m.} \{ p_1^{2^i}(p_1 - 1), \ldots, p_s^{2^i}(p_s - 1) \} \\
\text{if} \quad m = p_1^{2^i}p_2^{2^j} \cdots p_s^{2^j} \text{ is odd} \\
\text{l.c.m.} \{2, g(m/2)\} \quad \text{if} \quad m \equiv 2 \pmod{4}.
\end{cases} \]

It is quite easy to see that the phrase "ultimate periodicity of \( n^a \) \((\mod m)\)" may be replaced by the congruence

\[ l' \equiv (l + g(m))^{l + g(m)} \pmod{m} \quad \text{for all} \quad l > m. \]

We now consider the sequence \( A \) to the modulus \( p^x \), where \( p \) is an odd prime and \( x \in \mathbb{N} \). We begin with

**Lemma 2.** Let \( p \) be an odd prime, and \( x \geq 1 \) be a natural number. Then, we have

\[ a_{2p^{x}(p - 1)} - a_{p^{x}(p - 1)} \equiv p^{x - 1}(p - 1) \pmod{p^x}. \]

**Proof.** The proof proceeds by induction on \( x \). Now,

\[ a_{2p(p - 1)} - a_{p(p - 1)} = \sum_{p(p - 1) < r < 2p(p - 1)} r'. \]

We write \( r = q(p - 1) + s \), where \( 1 \leq s \leq (p - 1) \) and \( p \leq q \leq 2p - 1 \). Then, clearly
\[ a_{2^p(p-1)} - a_{p(p-1)} = \sum_{p(p-1) < r \leq 2p(p-1)} r^s = \sum_{p \leq q \leq 2p-1} (s-q)^s \]

\[ \equiv \sum_{0 \leq t \leq p-1} \sum_{1 \leq s \leq p-1} t^s \]

\[ \equiv 0 + (p-1) + \sum_{2 \leq i \leq p-1} \sum_{1 \leq s \leq p-1} t^s \]

\[ \equiv (p-1) + \sum_{2 \leq i \leq p-1} t(t^{p-1} - 1) \equiv (p-1) \pmod{p} \]

by Fermat's little theorem. Hence the lemma is true for the case \( x = 1 \).

Suppose it is true for the case \( x = k \geq 1 \). Then,

\[ a_{2^{p^k+1}(p-1)} - a_{p^{k+1}(p-1)} = \sum_{p^{k+1}(p-1) < r \leq 2p^{k+1}(p-1)} r^s. \]

As before, we write \( r = qp^k(p-1) + s \) with \( p^k(p-1) + 1 \leq s \leq 2p^k(p-1) \) and \( 2p-2 \geq q \geq p-1 \). Then, clearly

\[ a_{2^{p^k+1}(p-1)} - a_{p^{k+1}(p-1)} \equiv \sum_{p^k(p-1) + 1 \leq s \leq 2p^k(p-1)} (qp^k(p-1) + s)^s \]

\[ \equiv \sum_{p^k(p-1) + 1 \leq s \leq 2p^k(p-1)} (s^s + s^s \cdot qp^k(p-1)) \]

(by the binomial theorem)

\[ \equiv \sum_{p^k(p-1) + 1 \leq s \leq 2p^k(p-1)} s^s \left( p + \frac{p(p-1)}{2} p^k(p-1) \right) \]

\[ \equiv p \sum_{p^k(p-1) + 1 \leq s \leq 2p^k(p-1)} s^s \]

\[ \equiv p^{k+1}(p-1) \pmod{p^{k+1}} \]

by the induction hypothesis.

This completes the proof.

An easy extension of Lemma 2 is

**Lemma 3.** Let \( l \) be any natural number. Then

\[ a_{l+2^{p^l}(p-1)} - a_{l+p^l(p-1)} = a_{2^{p^l}(p-1)} - a_{l+p^l(p-1)} \]

\[ \equiv p^{l-1}(p-1) \pmod{p^l}, \]

where, as before, \( p \) is an odd prime and \( x \geq 1 \) is an integer.
Our next lemma determines the fundamental period of $A \pmod{p^\alpha}$.

**Lemma 4.** Let $p$ be an odd prime and $\alpha \geq 1$ be an integer. Then, for all $l > p^\alpha(p - 1)$, we have

$$a_{l + p^\alpha + 1(p - 1)} \equiv a_l \pmod{p^\alpha}.$$  

Further $p^\alpha(p - 1)$ is the fundamental period.

**Proof.** It is easily seen, from Lemma 3, that

$$a_{l + p^\alpha + 1(p - 1)} \equiv a_l \pmod{p^\alpha}.$$  

Let $t$ be the fundamental period. Since $t$ is a period, clearly,

$$a_{l + t + 1} - a_{l + t} \equiv a_{l + 1} - a_l \pmod{p^\alpha} \quad \text{for all} \quad l > p^\alpha(p - 1),$$  

and so

$$(l + t + 1)^{t + 1} \equiv (l + 1)^{t + 1} \pmod{p^\alpha}.$$  

Hence by Lemma 1, $p^\alpha(p - 1) \mid t$ (as $g(p^\alpha) = p^\alpha(p - 1)$). Hence either $t = p^\alpha(p - 1)$ or $t = p^{\alpha + 1}(p - 1)$. However,

$$a_{l + p^\alpha(p - 1)} - a_l \equiv p^{\alpha - 1}(p - 1) \neq 0 \pmod{p^\alpha},$$  

where $l > p^\alpha(p - 1)$, by Lemma 3. Hence, $p^\alpha(p - 1)$ is the fundamental period which proves the result.

It is not hard to see that, if $\alpha \leq p$, then the sequence is periodic mod $p^\alpha$, from the beginning.

From Lemma 4, we deduce

**Lemma 5.** Let $m$ be an integer $\geq 1$, which is not divisible by 4. Then, the sequence $A$ is ultimately periodic $\pmod{m}$. Further, if $f(m)$ denotes the fundamental period of $A \pmod{m}$, then we have

$$f(m) = \begin{cases} 
\lcm\{p_1^{\alpha_1 + 1}(p_1 - 1), \ldots, p_s^{\alpha_s + 1}(p_s - 1)\} & \text{if } m = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s} \text{ is odd} \\
\lcm\{4, f(m/2)\} & \text{if } m \equiv 2 \pmod{4}.
\end{cases}$$

**Proof.** The case in which $m$ is odd is easily seen by Lemma 4. Further, as $A$ is periodic $\pmod{2}$ with fundamental period 4, we get the result for $m \equiv 2 \pmod{4}$.

We note that if $\alpha \leq p$, for all $s \geq i \geq 1$, then the sequence is periodic $\pmod{m}$ from the beginning.
3. Uniform Distribution

In this section we show that the sequence $A$ is uniformly distributed in the residue classes $\mod m$ for all $m \not\equiv 0 \mod 4$. More precisely, we show

**Lemma 6.** Let $m = p_1^{a_1}p_2^{a_2}\cdots p_s^{a_s}$ be an odd number. Then among the numbers $a_{g(m)+1}, a_{g(m)+2}, \ldots, a_{g(m)+f(m)}$ exactly $f(m)/m$ numbers belong to each residue class $\mod m$.

**Proof.** The lemma is easily verified for the case $m = 3$. Suppose the lemma is true for all odd numbers $t$ satisfying $m > t \geq 1$. We now show it for $m$. Let

$$a_{2g(m)+i} - a_{g(m)+i} \equiv a_{2g(m)} - a_{g(m)} \equiv UV \mod m,$$

where $1 \leq UV \leq m$ and $(V, m) = 1$.

Let $2g(m) \geq i \geq g(m) + 1$ be an integer. Let $r$ run over all integers in the range $0 \leq r \leq (m/(m, U)) - 1$.

Then clearly

$$a_i + rUV \equiv a_i + (m, U)(rUV/(m, U)) \equiv a_i + s(m, U) \mod m,$$

where $s$ runs over all integers in the range $0 \leq s \leq (m/(m, U)) - 1$. (This is because $(U/(m, U), m) = 1$ and $(V, m) = 1$ and so $(UV/(m, U), m) = 1$.) Let there be $\lambda_b$ solutions to the congruence $a_x \equiv b \mod (m, U)$ with $g(m) + 1 \leq x \leq 2g(m)$. Then by the previous remark there are $\lambda_b$ solutions to each of the congruences $a_x \equiv b + s(m, U) \mod m$, where $0 \leq s \leq (m/(m, U)) - 1$ with $g(m) + 1 \leq x \leq g(m) + mg(m)/(m, U)$.

Thus if we show that the numbers $a_{g(m)+1}, \ldots, a_{2g(m)}$ are equally distributed in the residue classes $(\mod (m, U))$, it would follow that the numbers $a_{g(m)+1}, \ldots, a_{g(m)+mg(m)/(m, U)}$ are equally distributed in the residue classes $(\mod m)$.

Now as $a_i + mg(m)/(m, U) \equiv a_i + mUV/(m, U) \equiv a_i \mod (m, U)$ for all $i \geq g(m)$ it follows that $f(m) \| mg(m)/(m, U)$. Also, as

$$a_i + mg(m) \not\equiv a_i \mod m$$

for any $0 < \mu < m/(m, U)$, it follows that $f(m) = mg(m)/(m, U)$. This together with the earlier remark shows that (to prove the lemma) it suffices to prove that the numbers $a_{g(m)+1}, \ldots, a_{2g(m)}$ are equally distributed in the residue classes $(\mod (m, U))$.

As $g((m, U)) \mid g(m)$, it suffices to show that among the numbers $a_{g((m, U)) + 1}, \ldots, a_{g((m, U)) + g(m)}$ exactly $g(m)/(m, U)$ numbers belong to each residue class $\mod (m, U)$. Now as $f(m) \not\equiv g(m)$ clearly $(m, U) \not\equiv m$. Hence by the induction hypothesis it suffices to show $f((m, U)) \mid g(m)$.
Let $p_i^s \equiv (m, U)$, where $s \geq 1$. Then $p_i^{s+1} \mid g(m)$. (For, suppose $p_i^s \equiv g(m)$ with $1 \leq i < \beta_i + 1$. Also, clearly $p_i - 1 \mid g(m)$. Hence

$$a_{g(m)} - a_{g(m)} = a_{g(p_i^s)} - a_{p_i^s}(p_i - 1)$$

$$\equiv r(p_i - 1) (\mod p_i^s), \quad \text{where} \quad (r, p_i) = 1.$$

Hence $p_i^s X(m, U)$, which is false.) Hence $f((m, U)) \mid g(m)$, which, as noted before, yields the result.

We observe that if $p_i \leq p_i$ for all $s \geq 1$ then the result is true from the beginning. We next state

**Lemma 7.** Let $m = 2p_1^{a_1}p_2^{a_2} \cdots p_s^{a_s} \equiv 2 (\mod 4)$. Then among the numbers $a_{g(m)+1}, a_{g(m)+2}, \ldots, a_{g(m)+f(m)}$, exactly $g(m)/m$ numbers lie in each residue class $\mod m$. Further if $p_i \leq p_i$ for all $s \geq 1$, the result is true from the beginning.

The proof of this runs along the same lines as that of Lemma 6 and is accordingly omitted.

4. **Proof of the Main Theorem**

Let $a$ and $b$ be any two positive integers such that $b \not\equiv 0 (\mod 4)$. Let $N$ be a natural number. Then thanks to Lemmas 6 and 7, we have

$$\sum_{n \leq N \atop n \equiv a (\mod b)} 1 = \frac{N}{b} + O \left( \frac{f(b)}{b} \right) = \frac{N}{b} + O(b^2).$$

In particular this result holds if $a = 0$ and $b = d$ is square free.

We now state a well-known version of Selberg's sieve.

**Lemma 8.** Let $B$ be a finite set of integers. Let $\wp$ be a set of primes and

$$p(z) = \prod_{p \leq z, p \in \wp} p,$$

where $z \geq 2$ is a real number. Let $B_d = \{b : b \in B \text{ and } b \equiv 0 (\mod d)\}$. Let $X$ be an approximation to $|B| = |B_1|$ and $w$ be a multiplicative function such that $w(p)X/p$ is an approximation to $|B_p|$. Let

$$R_d = |\hat{B}_d| - \frac{w(d)}{d} X.$$
Suppose \( 0 \leq w(p)/p \leq 1 - 1/A_1 \) for some constant \( A_1 \geq 1 \), \( w(p) = 0 \) for primes \( p \) not in \( \mathcal{O} \). If

\[
g(d) = \prod_{p \mid d \atop p \leq w(p)} \frac{w(p)}{p - w(p)}, \quad G(z) = \sum_{d < z} \mu^2(d) g(d),
\]

then we have

\[
S(B, \mathcal{O}, z) = \left| \{ b : b \in B, (b, p(z)) = 1 \} \right| 
\leq \frac{X}{G(z)} + \sum_{d_1, d_2 \leq z \atop d_1 \mid p(z) \atop d_2 \mid p(z)} |R_{l.c.m.(d_1, d_2)}|.
\]

This is Theorem 9.1 of [3].

By our earlier remark, we can choose \( \mathcal{O} \) to be all primes \( < z \), \( w(p) = 1 \) for these primes, and

\[
R_d = \sum_{n \leq A \atop n \leq A_N \atop n \equiv 0 \pmod{d}} 1 - \frac{N}{d} = O(d^2).
\]

Then \( g(d) = 1/\phi(d) \) for square free \( d \) and by standard arguments

\[
G(z) = \sum_{d \mid p(z) \atop d \leq z} \frac{1}{\phi(d)} \approx \log z.
\]

Hence, applying Lemma 8 to our sequence gives

\[
\pi_A(a_N) \ll \frac{N}{\log z} + \left( \sum_{d_1, d_2 \leq z \atop d_1 \mid p(z) \atop d_2 \mid p(z)} d_1^2 d_2^2 \right)
\]

\[
\ll \frac{N}{\log z} + O(z^{10})
\]

\[
\ll \frac{N}{\log N} \quad \text{by choosing } z = N^{1/11}, \text{ say.}
\]

We note that if

\[
N = \left[ \frac{\log x + 2 \log x \log \log \log x}{\log \log x + \left( \log \log x \right)^2} \right]
\]
then \( a_n \geq x \) and so

\[
\pi_A(x) \leq \pi_A(a_n) \leq \frac{\log x}{(\log \log x)^2},
\]

which completes the proof of our main theorem.

5. CONCLUDING REMARKS

We have only obtained an upper bound for the number of primes in \( A \) less than \( x \). We have not been able to show that there exist infinitely many primes in \( A \). We believe, however, that this is indeed the case. We also believe that

\[
\pi_A(x) \sim \frac{C \log x}{(\log \log x)^2} \quad \text{for large } x, \text{ where } C > 0 \text{ is a constant.}
\]

We have been able to derive results similar to Lemma 6, for the modulus \( m \equiv 0 \pmod{4} \). The details will appear elsewhere.

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