Subcube Fault-Tolerance in Hypercubes

Niall Graham*,+ and Frank Harary†

Computing Research Laboratory, Department of Computer Science, New Mexico State University, Las Cruces, New Mexico 88003-0001

AND

Marilynn Livingston‡,§ and Quentin F. Stout¶

Advanced Computer Architecture Laboratory, Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, Michigan 48109-2122

We consider the problem of determining the minimum number of faulty processors, $\kappa(n, m)$, and of faulty links, $\lambda(n, m)$, in an $n$-dimensional hypercube computer so that every $m$-dimensional subcube is faulty. Best known lower bounds for $\kappa(n, m)$ and $\lambda(n, m)$ are proved, several new recursive inequalities and new upper bounds are established, their asymptotic behavior for fixed $m$ and for fixed $n - m$ is analyzed, and their exact values are determined for small $n$ and $m$. Most of the methods employed show how to construct sets of faults attaining the bounds. An extensive survey of related work is also included, showing connections to resource allocation, $k$-independent sets, and exhaustive testing. © 1993 Academic Press, Inc.

1. Introduction

An $n$-dimensional hypercube computer, or $n$-cube, is a parallel computer with $2^n$ processors and network topology that of an $n$-dimensional binary cube. Each node of the cube is associated with a processor $P$ while each edge $(P_i, P_j)$ of the cube represents the direct communication link between processors $P_i$ and $P_j$. Hypercube computers have been studied since 1962 [35] and have recently become the focus of intense commercial and research activity [15–19].

* Present address: Los Alamos National Laboratory, Los Alamos, New Mexico 87545.
+ Research partially supported by ONR grant N00014-90-J-1860.
‡ On leave from Southern Illinois University, Edwardsville, Illinois 62026.
§ Research partially supported by NSF grant CCR-8808839.
¶ Research partially supported by NSF Grant DCR-8507851, NSF/DARPA Grant CCR-9004727, and an Incentives for Excellence award from Digital Equipment Corporation.
One of the attractive features of the \( n \)-cube topology is its behavior in the presence of faulty processors or links. Depending on the number and location of these faults it is possible that the network still contains large subcubes which are fault-free. Since most algorithms for the \( n \)-cube specify the dimension of the network as a parameter, these algorithms can still be used in the presence of faults, although with some degradation. Assuming some minimum acceptable level of degradation, it is natural to consider the following question:

In an \( n \)-dimensional hypercube, what is the minimum number of faulty processors (or faulty links) that cause all \( m \)-dimensional subcubes to be faulty?

This question can also be considered as part of the subcube allocation problem. In multitasking on an \( n \)-cube, the problem of dynamically assigning subcubes of a given dimension to a given task can be thought of as allocating subcubes in the presence of faults, where the busy processors and dedicated communication links can be considered "faulty".

The above question arises from problems in resource distribution [28] as well. To illustrate, suppose disks are to be attached to some of the processors of an \( n \)-cube in such a way that every \( m \)-dimensional subcube contains a processor with a disk. (We may, for example, be in a multiuser environment and want to ensure that each user has a disk in their allotted subcube.) For a given \( n \) and \( m \), the minimum number of disks necessary is the same as the minimum number of faulty processors needed to guarantee that every \( m \)-cube is faulty. A solution to this resource distribution problem, however, requires not only the number needed, but also a construction of a minimum set of nodes of \( Q_n \) that has a node in common with each \( m \)-dimensional subcube.

In order to facilitate our discussion we need to introduce some notation. Let \( Q_n \) denote a labeled \( n \)-dimensional binary cube, where the nodes of \( Q_n \) are all the \( n \)-bit strings and two nodes are adjacent if and only if their corresponding strings differ in exactly one position. Define \( \mathcal{I}(n, m) \) as the collection of all sets of nodes of \( Q_n \) whose removal leaves no \( Q_m \), and let \( \kappa(n, m) \) be the minimum size of a set in \( \mathcal{I}(n, m) \). Analogously, \( \mathcal{F}(n, m) \) denotes the collection of all sets of edges of \( Q_n \) whose removal from \( Q_n \) leaves no \( Q_m \) and \( \lambda(n, m) \) is the minimum size of a set in \( \mathcal{F}(n, m) \). When the context is clear, the informal term "fault set" will be used to mean a set in \( \mathcal{I}(n, m) \) or a set in \( \mathcal{F}(n, m) \). Figure 1 illustrates minimum node and edge fault sets for \( n = 4 \) and \( m = 2 \).

There are many alternative methods of fault tolerance not measured by the \( \kappa \) and \( \lambda \) functions. Two basic graph-theoretic approaches are to provide additional edges and/or nodes, or to weaken the notion of a subcube. In the former, hardware is added so that the system still has a \( Q_n \) as a sub-
system after a fault occurs [8, 33, 36]. This approach must be taken at the
time of hardware design, and can tolerate relatively few faults without inor-
dinate expense. In the latter approach, the notion of edge is weakened to
allow paths of length greater than one in order to route around faults. This
implementation is via software, perhaps together with hardware modifica-
tions to permit use of links to and from faulty processors. Generally, many
more faults can be tolerated with this approach and it is frequently possible
to provide a reconfigured subcube of the desired size in the presence of
several faults [14]. This solution suffers a performance penalty, however,
because each communication step in a reconfigured subcube takes longer
than a communication step in the original hypercube. Neither of these
approaches has yet been implemented in any commercial hypercube, and
we will not pursue these methods here.

The fault tolerance approach we analyze assumes no hardware modifica-
tion, incurs no communication penalties, and can be easily utilized on all

5 faulty nodes destroying every 2-cube

8 faulty edges destroying every 2-cube

FIG. 1. Minimum fault sets for n = 4, m = 2.
current commercial hypercubes. Further, the $\kappa$ and $\lambda$ functions are of interest for a variety of reasons beyond simple fault tolerance. This will be shown below, where ties are exhibited between these functions and problems in resource allocation, exhaustive testing, and $\kappa$-independent sets.

1.1. Prior Work

A family $F$ of sets is $k$-independent if for every pair of disjoint subsets $S_1$ and $S_2$ of $F$ such that $|S_1| + |S_2| = k$ there is at least one element common to all the sets in $S_1$ but which is in none of the sets in $S_2$. In Section 2 we show the direct relationship between $k$-independent sets and $\kappa$. The earliest published work relevant to evaluating $\kappa$ and $\lambda$ is apparently that of Schönheim [34] and Brace and Daykin [6], who determined the maximum size of a 2-independent family, and Kleitman and Spencer [26], who considered the general problem of determining the maximum size of families of $k$-independent sets. Kleitman and Spencer used a probabilistic argument to establish a lower bound for the maximum size of a family of $k$-independent subsets of a set, proved an upper bound for this maximum, and determined the maximum size of 2-independent sets by constructive means. These results yield the value of $\kappa(n, n-2)$ and bounds for $\kappa(n, n-k)$. Chandra et al. [7] studied the problem of finding the minimum number of boolean $n$-vectors such that every $k$-projection of them yields all possible $k$-vectors. In our notation this is $\kappa(n, n-k)$. They determined $\kappa(n, n-2)$, gave a construction for sets in $\mathcal{F}(n, n-3)$ of non-optimal size, and used essentially the same probabilistic argument as in [26] to obtain an upper bound for $\kappa(n, n-k)$. Becker and Simon [3], apparently unaware of the work in [7], repeated many of these results for $\kappa$, and used the same methods to establish bounds on $\lambda$. They also gave a construction, based on the work of Friedman [12], which yields an upper bound for $\kappa(n, n-k)$ that has the correct growth behavior for fixed $n-k$. In [27], Levitin and Karpovsky considered the problem of exhaustive testing of combinatorial devides with $n$ inputs, where each output is a boolean function of at most $k$ binary input variables. They used MDS codes to construct sets in $\mathcal{F}(n, m)$, although the sets were not of optimal size.

Several persons have worked on a problem complementary to determining $\kappa(n, m)$. Some time ago, Erdős asked for the maximum size of any set of nodes of $Q_n$ for which the induced subgraph contains no 4-cycle. Johnson and Entringer [24] found this maximum size and characterized the extremal graphs for this case. Let $f(n, m)$ denote the maximum size of any set of nodes of $Q_n$ for which the induced subgraph contains no $Q_m$, and $g(n, m)$ denote the corresponding number for edges. Note that $f(n, m) = 2^n - \kappa(n, m)$ and $g(n, m) = n 2^{n-1} - \lambda(n, m)$. Thus, the Johnson and Entringer result determines $\kappa(n, 2)$. In [20, 21, 23], Johnson has considered $f(n, m)$ and obtained bounds for the cases $m = 3, 4, 5$, and in
[22] has evaluated \( g(5, 2) \). Responding to a related question of Erdős [9] (see Section 3.4), F. Chung (personal communication, July 1988) established an upper bound for \( g(n, 2) \), thus providing a lower bound for \( \lambda(n, 2) \).

1.2. Organization

The following sections contain our new results on \( \kappa \) and \( \lambda \) as well as an extensive survey of related work. In view of the fault-tolerance applications on the one hand, and the exhaustive testing and resource distribution applications on the other, we address both the problem of determining the values of \( \kappa \) and \( \lambda \) and the problem of the construction of small fault sets. In Section 2 we derive several bounds for \( \kappa \). We establish new bounds for the maximum size of 3-independent families by using the non-constructive methods of Erdős et al. [10]. These bounds yield an improved upper bound for \( \kappa(n, n - 3) \).

We also give a construction for small sets in \( \mathcal{S}(n, n - 3) \) which yields a new recursive inequality for \( \kappa(n, n - 3) \), producing the best known upper bounds for it with \( n \) of any practical size. We make use of the results obtained by Kleitman and Spencer [26] for \( k \)-independent subsets to establish a new lower bound for \( \kappa(n, m) \).

Many of the techniques of Section 2 are easily modified to give corresponding results for \( \lambda \). These results are described in Section 3 and include an improved upper bound for \( \lambda(n, m) \) for \( m \) small relative to \( n \), a new lower bound for \( \lambda(n, m) \) that is the best known for \( n \) large, and a new lower bound for \( \lambda(n, m) \) that is the best when \( m \) is small. Here, as in Section 2, all but one of the bounds are established by constructive methods.

The asymptotic behavior of \( \kappa(n, m) \) and \( \lambda(n, m) \), discussed in Section 4, is not well understood for general \( n \) and \( m \). However, the new bounds we establish in Sections 2 and 3 do give new information for the cases when \( m \) is small relative to \( n \) and when \( n - m \) is small. In Section 5, we use a combination of the results of earlier sections together with computer programs to construct optimal or near optimal fault sets, thereby determining exact values or tight bounds for \( \kappa(n, m) \), for \( 0 \leq m \leq n \leq 10 \), and \( \lambda(n, m) \), for \( 1 \leq m \leq n \leq 7 \). In Section 6 we describe techniques for constructing fault sets when \( n \) is large. Section 7 contains a discussion of various related open problems and some generalizations.

Because of the large number of results and techniques in Section 2 and 3, the reader may prefer to initially skim these sections, proceeding to Sections 4, 5, and 6. These latter sections help to put the various inequalities into perspective. The reader may then return to the initial sections for a more careful reading.

Throughout, \( \lg \) denotes \( \log_2 \) and \( \ln \) denotes \( \log_e \).
2. The Values of $\kappa$

The theorems in this section are organized according to the methods employed in their proofs. Theorem 1 and 2 are proved by quite elementary means. A labeling technique is used to prove Theorem 3, whereas Theorem 4 is proved by the use of level sets, yielding a good upper bound for $\kappa(n, m)$ for fixed $m$. The results in Theorem 8 through 11 rely on the connection between $\kappa$ and independent sets mentioned in Section 1. A partitioning technique which can be viewed as an extension of the 2-independent set construction yields a recursive inequality for $\kappa(n, n - 3)$. The final theorem of this section uses a construction somewhat related to the partitioning method to establish a second recursive upper bound for $\kappa(n, n - 3)$. Whenever we establish a recursive inequality, the proof shows how to combine minimum fault sets for the larger side to get a fault set for the smaller side satisfying the inequality.

For a node $q$ in $Q_n$, the weight of $q$ will denote the number of 1’s in its string. Extending our notation of $n$-bit strings for nodes, we will denote the subcubes of $Q_n$ by strings from $\{0, 1, \ast\}^n$, where the number of $\ast$'s in the string is the dimension of the subcube.

2.1. Elementary Bounds

The theorems in this section are proved by quite elementary and constructive means.

**Theorem 1.** For $n \geq 1$,

(i) $\kappa(n, n) = 1$

(ii) $\kappa(n, n - 1) = 2$

(iii) $\kappa(n, 0) = 2^n$

(iv) $\kappa(n, 1) = 2^{n - 1}$.

**Proof.** Parts (i) and (iii) follow directly from the definition of $\kappa(n, m)$.

For (ii), note that at least one node must be removed from each of two disjoint copies of $Q_{n-1}$ in $Q_n$. Moreover, if we remove any pair of antipodal nodes of $Q_n$, the remaining graph contains no $Q_{n-1}$. Thus (ii) holds.

For (iv), let $Q'$ and $Q''$ denote two disjoint copies of $Q_{n-1}$ in $Q_n$, and consider those edges with one node in $Q'$ and the other in $Q''$. Since at least one node of each of these edges must be removed in order to remove all the $Q'$'s from $Q_n$, we must have $\kappa(n, m) \geq 2^{n - 1}$. On the other hand, if we remove from $Q_n$ all nodes of even weight then no $Q_1$ can remain since every edge contains exactly one node of even weight. Part (iv) now follows. \qed
In the next theorem we give recursive upper and lower bounds for \( \kappa(n, m) \).

**Theorem 2.** For \( n, m \geq 1 \),

\[
\begin{align*}
(\text{i}) & \quad \kappa(n, m) \leq \kappa(n - 1, m - 1) + \kappa(n - 1, m), \\
(\text{ii}) & \quad \kappa(n, m) \geq \max \{2\kappa(n - 1, m), \kappa(n - 1, m - 1)\}.
\end{align*}
\]

**Proof.** Let \( Q' \) and \( Q'' \) be two node-disjoint copies of \( Q_{n - 1} \) in \( Q_n \).

For (i), let \( S_1 \subseteq Q' \), \( S_2 \subseteq Q'' \) be sets of size \( \kappa(n - 1, m - 1) \), \( \kappa(n - 1, m) \) in \( \mathcal{S}(n - 1, m - 1) \), \( \mathcal{S}(n - 1, m) \). Clearly \( S_1 \cup S_2 \) is in \( \mathcal{S}(n, m) \). For (ii), note that at least \( \kappa(n - 1, m) \) nodes must be removed from each of \( Q' \) and \( Q'' \) so that no \( Q_m \) remains in either \( (n - 1) \)-cube. Thus \( \kappa(n, m) \geq 2\kappa(n - 1, m) \).

To prove the second of the implied inequalities in (ii), let \( S \) be a set in \( \mathcal{S}(n, m) \) of size \( \kappa(n, m) \) and let \( S', S'' \) be the nodes of \( S \) in \( Q', Q'' \). Denote by \( T' \) the set of nodes of \( Q' \) that are adjacent to the nodes of \( S'' \). If \( Q' \) contains an \( (m - 1) \)-cube \( A' \) that is disjoint from \( S' \cup T' \), then \( Q'' \) contains a corresponding \( (m - 1) \)-cube \( A'' \) which combines with \( A' \) to form an \( m \)-cube disjoint from \( S \). Since this contradicts the choice of \( S \), we may conclude that \( S' \cup T' \) must contain at least \( \kappa(n - 1, m - 1) \) nodes and, therefore, \( \kappa(n, m) \geq \kappa(n - 1, m - 1) \).

Table 1 shows that sometimes the first term on the right side of the inequality in Theorem 2(ii) is the largest (for example, at \( n = 7 \) and \( m = 2 \)) and sometimes the second term is the largest (for example, at \( n = 6 \) and \( m = 4 \)). Part (ii) of Theorem 2 shows that \( \kappa(n, m) \) is strictly increasing in \( n \). Further, given any fault set \( S \) in \( \mathcal{S}(n, m) \), removal of any single node of \( S \) gives a fault set \( S' \) in \( \mathcal{S}(n, m + 1) \), since any \( (m + 1) \)-cube consists of two disjoint \( m \)-cubes, at least one of which is still faulty in \( S' \). Therefore \( \kappa(n, m) \) is strictly decreasing in \( m \).

The next theorem generalizes part (i) of Theorem 2. Consider the \( (n - 1) \)-dimensional subcubes \( A = 0 \cdots 0 \) and \( B = 1 \cdots 1 \) of \( Q_n \). We may visualize \( Q_n \) as a 1-cube with "supernodes" \( A \) and \( B \), where we label \( A \) with 0 and \( B \) with 1. Let \( S_1 \) be a subset of \( A \) whose removal from \( A \) leaves no \( m \)-cubes, and \( S_2 \) a subset of \( B \) whose removal from \( B \) leaves no \( (m - 1) \)-cubes. Part (i) of Theorem 2 was proved by observing that \( S_1 \cup S_2 \) is in \( \mathcal{S}(n, m) \). As a first step in generalizing this idea, visualize \( Q_n \) as a 2-cube with supernodes \( A_{00}, A_{01}, A_{10}, \) and \( A_{11} \), where \( A_{ij} = ij \cdots \) is an \( (n - 2) \)-cube of \( Q_n \) for \( i, j \in \{0, 1\} \). Assign label \( l_{ij} \) to supernode \( A_{ij} \) as follows: \( l_{00} = l_{11} = 0 \), \( l_{10} = 1 \), and \( l_{01} = 2 \). Next, for each \( i, j \in \{0, 1\} \), choose a minimum set \( S_{ij} \) of nodes of \( A_{ij} \) whose removal from \( A_{ij} \) leaves no \( (m - l_{ij}) \)-cube. We see that \( \bigcup_{i, j \in \{0, 1\}} S_{ij} \) is in \( \mathcal{S}(n, m) \) and so

\[
\kappa(n, m) \leq 2\kappa(n - 2, m) + \kappa(n - 2, m - 1) + \kappa(n - 2, m - 2).
\]

(1)
This result is not a consequence of iterating the inequality in part (i) of Theorem 2, for one iteration yields

\[
\kappa(n, m) \leq \kappa(n - 2, m) + 2\kappa(n - 2, m - 1) + \kappa(n - 2, m - 2)
\]

which is weaker than inequality (1). Figure 2 illustrates this labeling.

**Theorem 3.** Let \( r \) be a non-negative integer. Label the nodes of \( Q \), with integers in the interval \([0, r]\) such that for every \( j \) in \( 0 \ldots r \) each \( j \)-cube of \( Q \), has a node with label at least as large as \( j \). If \( l(q) \) is the label of node \( q \) in \( Q \), then

\[
\kappa(n, m) \leq \sum_{q \in Q} \kappa(n - r, m - l(q))
\]

for \( n \geq m \geq r \).

**Proof.** For each node \( a = a_1a_2\ldots a_r \) in \( Q\), let \( Q_n(a) \) be the \((n - r)\)-dimensional subcube of \( Q_n \) given by \( a_1\ldots a_r \ast \ast \ast \). Let \( S(a) \) be a set of \( \kappa(n - r, m - l(a)) \) nodes of \( Q_n(a) \) whose removal from \( Q_n(a) \) leaves no \((m - l(a))\)-cube. We claim that the removal of the set

\[
S = \bigcup_{a \in Q} S(a)
\]

Fig. 2. Labelings of small hypercubes.
from $Q_n$ leaves no $m$-cube. For suppose $T$ is an $m$-cube of $Q_n$, say $T = w_1 \cdots w_n$, where $w_{i(j)} = \bullet$ for $1 \leq i(1) < \cdots < i(m) \leq n$. Let $t = \max \{ j \mid i(j) \leq r \}$ and consider the $t$-dimensional subcube $T'$ of $Q_r$ given by $T' = w_1 \cdots w_r$. (Conceptually, $T$ can be thought of as a product of a $t$-dimensional subcube of $Q_r$ and an $(m-t)$-dimensional subcube of $Q_{n-r}$.) By our assumption on the labeling of $Q_r$, there is some node $v \in T'$ whose label $l(v)$ is at least $t$. Thus, the $(n-r)$-dimensional cube $Q_n(v)$ has no $(m-l(v))$-dimensional subcube after the removal of $S(v)$. Since $T \cap Q_n(v) = v_1 \cdots v_r, w_{r+1} \cdots w_n$ has dimension $m-t$, which is at least $m-l(v)$, this subcube must contain at least one element of $S$.

In the theorem just proved, if we take $r = 1$ and choose the labels 0 and 1, then inequality (3) reduces to the statement in part (i) of Theorem 2. An iteration of this inequality corresponds to selecting the labels 0, 1 (from 0) and 1, 2 (from 1) for $Q_2$ which yields inequality (2). However, with $r = 2$ and labels 0, 1, 1, 2 assigned to the appropriate nodes of $Q_2$, we obtain the stronger inequality (1). For each value of $r$, it is clear that there is a labeling of $Q$, which gives an inequality for $\kappa(n, m)$ which is stronger than that supplied by using a labeling obtained by iteration corresponding to a smaller value of $r$.

The results expressed in Theorem 3 are most useful in the construction of near optimum sets in $S(n, m)$ based on good constructions for near optimum sets in $S(n, m-j)$ for $0 \leq j \leq r$ for some $r \leq m$. In applying Theorem 3, the actual choice of $r$ will be determined by what is known about the optimum or near optimum sets in $S(n, m-j)$ for $0 \leq j < m$. In addition, since determining optimum labelings for $Q_r$ for large $r$ is a challenging combinatorial problem in itself, usually only near optimum labelings would be available. Consider, for example, the following construction. For each $j$ in $0, \ldots, r$, pick a set $S_j$ of $\kappa(r, j)$ nodes of $Q_r$ that contains 0 \cdots 0 and whose removal from $Q_r$ leaves no $j$-cube. Define a labeling $h$ as follows: for $q$ a node of $Q_r$, let $h(q) = \max \{ k \mid q \in S_k \}$. Clearly, for each collection of sets $S_1, S_2, \ldots, S_r$, the resulting labeling $h$ satisfies the requirements set forth in Theorem 3, but to obtain near optimum labelings by this method, one would want to choose the sets so that $S_j$ overlaps $S_j$, for $j < i$, as much as possible. Whatever the selection, 0 \cdots 0 will have label $r$, and those nodes with label 0 will not be in $S_1$. In the worst case, we would construct by this method a set $R$ in $S(n, m)$, where

$$|R| \leq \kappa(r, r) \kappa(n-r, m-r) + \left[ \kappa(r, 0) - \kappa(r, 1) \right] \kappa(n-r, m)$$

$$+ \sum_{j=1}^{r-1} \left[ \kappa(r, j) - 1 \right] \kappa(n-r, m-j).$$
2.2. Level Sets

Our next upper bound on \( \kappa \) is established by the simple device of removing all nodes at given distances from the origin \( 0 \cdots 0 \) of \( Q_n \). An example of such a fault set appears in Fig. 1.

**Theorem 4.** If \( n, m \geq 1 \) and \( a \) is any integer, then

\[
\kappa(n, m) \leq \sum_{k = a \mod m + 1}^n \binom{n}{k}.
\]

Further, this sum is minimized when \( a = \lfloor (n - m - 1)/2 \rfloor \).

**Proof.** The nodes of \( Q_n \) can be partitioned into levels, where level \( i \) consists of all nodes of weight \( i, 0 \leq w \leq n \). Any \( m \)-dimensional subcube of \( Q_n \) must include nodes from \( m + 1 \) consecutive levels. Consequently, if all the nodes are removed from at least one level in every set of \( m + 1 \) consecutive levels, then no \( Q_m \) will remain. This can be accomplished by removing all nodes whose weights are in a fixed congruence class \( a \) modulo \( m + 1 \). Furthermore, we can minimize the number of nodes removed in this way by judicious choice of \( a \). The level size is monotone decreasing away from the center level (or levels, for \( n \) odd). Selecting \( a = \lfloor (n - 1 - m)/2 \rfloor \) results in the removal of levels as far from the center level(s) as possible. A straightforward term-by-term comparison shows the optimality of this value of \( a \).

While many authors [3, 7, 20–22, 24] utilize the approach of the theorem just proved, most choose to express their result in the following simpler but weaker form.

**Corollary 4.1.** For \( n \geq m \geq 1 \),

\[
\kappa(n, m) \leq \frac{2^n}{m+1}.
\]

**Proof.** The desired result follows from the identity

\[
\sum_{a = 0}^{m} \sum_{k = a \mod m + 1}^n \binom{n}{k} = 2^n.
\]

The bound given by Theorem 4 in the case \( k = 2 \) is sharp according to the results of Johnson and Entringer [24], who used constructive methods to determine \( f(n, 2) \), the complement of \( \kappa(n, 2) \). We state their result in terms of \( \kappa \).
Theorem 5 [24]. For $n \geq 2$,

$$\kappa(n, 2) = \lfloor 2^n/3 \rfloor.$$

Before further discussion concerning the use of level sets, let us simplify notation by letting $C(n, m, a) = \sum_{k \equiv a \mod m + 1} \binom{n}{k}$, and setting $C^*(n, m) = \min\{C(n, m, a) : 0 \leq a \leq m\}$. In [21, 28] it was noted that, for fixed $m$, $C^*(n, m)$ satisfies a recursive equation, and this was later solved for $m = 3, 4, 5$ in [20, 21, 23]. These results yield upper bounds for $\kappa(n, m)$ for $m = 3, 4, 5$ which are improvements over those provided by Corollary 4.1. We summarize these in the following.

Theorem 6 [20, 21, 23]. For $n, m \geq 1$,

(i) \(\kappa(n, 3) \leq 2^n/4 - 2^{(n/2)^2}/2\).

(ii) \(\kappa(n, 4) \leq \begin{cases} 2^n/5 - (2/5) L_n & \text{n odd} \\ 2^n/5 - (1/5) L_{n+1} & \text{n even} \end{cases}\)

where $L_n$, the Lucas number, is $[(1 + \sqrt{5})^n + (1 - \sqrt{5})^n]/2^n$.

(iii) \(\kappa(n, 5) \leq \begin{cases} 2^n/6 - 3L_{n/2}/2 + 1/6 & \text{n odd} \\ 2^n/6 - 3L_{n/2}/3 + 1/3 & \text{n even} \end{cases}\).

Johnson [21] suggested that the bound $C^*(n, m)$ given by Theorem 4 may be sharp, and formally conjectured equality in that case $m = 4$. However, for any fixed $m > 2$, equality between $\kappa(n, m)$ and $C^*(n, m)$ cannot hold for all $n \geq m$. For $m = 3$ this follows from the fact that $\kappa(7, 3) = 24$, from Table 1, Section 5, whereas $C^*(7, 3) = 28$. For $m > 3$, we see that equality fails between $\kappa(m + 2, m)$, whose value is given by Theorem 9, and $C^*(m + 2, m)$, whose value is $m + 3$.

For fixed $m$ and large $n$, $C^*(n, m)$ is the best upper bound known for $\kappa(n, m)$, but it may still be far from optimal, for, as we shall see in Section 4, there is a large gap between $C^*(n, m)$ and the known lower bounds in these cases.

2.3. Independent Sets

We now turn to the theory of independent sets to help us in our study of $\kappa$. A family $F$ of sets is $k$-independent if for every pair of disjoint subsets $S_1$ and $S_2$ of $F$ such that $|S_1| + |S_2| = k$, there is at least one element common to all the sets in $S_1$, which is in none of the sets in $S_2$. The following lemma shows the close relationship between $k$-independent sets and sets in $\mathcal{F}(n, n-k)$. To state it, we first need some additional notation. Let $\mathcal{F}(r, k)$ denote all $k$-independent sets of subsets of $\{1, \ldots, r\}$. For any set $T$ of $i$ elements, by the orderings of $T$ we mean the set of $i!$ $i$-tuples which, when
viewed as unordered sets, are equal to $T$. Let $\mathcal{H}(r, k)$ denote the set of all orderings of all elements of $\mathfrak{F}(r, k)$, and let $\mathcal{H}(n, n-k)$ denote the set of all orderings of all elements of $\mathfrak{F}(n, n-k)$.

**Lemma 7.** Given positive integers $k, r, n$, there is a natural bijection between the $n$-tuples of $\mathcal{H}(r, k)$ and the $r$-tuples of $\mathcal{H}(n, n-k)$.

**Proof.** Let $F = (F_1, ..., F_n)$ be an $n$-tuple of $\mathcal{H}(r, k)$. $F$ can be used to construct an $r$-tuple in $\mathcal{H}(n, n-k)$ as follows. Let $M = (m_{ij})$ be the $r \times n$ matrix defined by

$$m_{ij} = \begin{cases} 1 & \text{if } i \in F_j \\ 0 & \text{otherwise} \end{cases}$$

Then, for each $1 \leq j \leq n$, the $j$th column of $M$ represents the characteristic function of the set $F_j$. Moreover, for each $1 \leq i \leq r$, the $i$th row of $M$ can be associated with the element $i$ of $\{1, ..., r\}$, and also represents a node of $Q_n$, where the $j$th entry of the row is the $j$th bit of the node's label. We denote by $S_M$ the $r$-tuple of nodes represented by the rows of $M$, and claim that $S_M$ is in $\mathcal{H}(n, n-k)$. To see why this is the case, let $A = a_1a_2\cdots a_n$ be an $(n-k)$-cube in $Q_n$ and define $S_1 = \{F_j; a_i = 1\}$ and $S_2 = \{F_j; a_i = 0\}$. Since $A$ is an $(n-k)$-cube, $|S_1| + |S_2| = k$, and since $F$ is $k$-independent there is at least one element, say $x$, that is in each set in $S_1$ and is in none of the sets in $S_2$. Thus the node represented by row $x$ is in both $A$ and $S_M$, proving that $S_M$ is in $\mathcal{H}(n, n-k)$.

It is clear that the above mapping from $n$-tuples of $\mathcal{H}(r, k)$ to $r$-tuples of $\mathcal{H}(n, n-k)$ is 1-1. To see that it is onto, let $S$ be an $r$-tuple in $\mathcal{H}(n, n-k)$. Create the $r \times n$ matrix $M$ by setting $m_{ij}$ equal to the $j$th bit of the $i$th element of $S$, and construct an $n$-tuple $F = (F_1, F_2, ..., F_n)$ of subsets of $\{1, ..., r\}$ by interpreting the $j$th column of $M$ as the characteristic function of the set $F_j$. We claim that $F$ is $k$-independent. To prove this, suppose $S_1$ and $S_2$ are disjoint subsets of $F$, where $|S_1| + |S_2| = k$, and $J_1, J_2$ are their index sets defined by $J_p = \{i: F_i \in S_p\}$ for $p = 1, 2$. Let $B = b_1b_2\cdots b_n$ be the $(n-k)$-dimensional subcube described by

$$b_i = \begin{cases} 1 & \text{if } i \in J_1 \\ 0 & \text{if } i \in J_2 \\ * & \text{otherwise} \end{cases}$$

Since $S$ is in $\mathcal{H}(n, n-k)$, $B$ must contain at least one element, say the $y$th element, of $S$. This means that $y \in F_i$ for each $i \in J_1$ and $y \notin F_i$ for $i \in J_2$, which allows us to conclude that $F$ is $k$-independent.

The correspondence established in the lemma, used in [3, 7], gives the following result.
THEOREM 8 [3, 7]. Let $F(r, k)$ denote the maximum size of a $k$-independent family of subsets of a set of $r$ elements. Then

$$\kappa(n, m) = \min \{ r \mid F(r, n-m) \geq n \}.$$  

Schönheim [34], Brace and Daykin [6], and Kleitman and Spencer [26] determined the maximum size of a family of $2$-independent sets. Kleitman and Spencer proved that $F(r, 2) = \binom{r-1}{\lfloor r/2 \rfloor - 1}$, observing that this maximum is attained by taking all subsets of size $\lfloor r/2 \rfloor$ that contain a fixed element of $X$. Using this result and the above theorem, one immediately obtains the following.

THEOREM 9. $\kappa(n, n-2)$ is the minimum positive integer $r$ such that $\binom{r-1}{\lfloor r/2 \rfloor - 1} \geq n$.

Chandra et al. [7] rediscovered this result and the following corollary, as did Becker and Simon [3].

COROLLARY 9.1 [3, 7]. $\kappa(n, n-2) = \log n + \frac{1}{2} \log \log n + O(1)$, where the $O(1)$ term is non-negative.

Kleitman and Spencer also obtained bounds for $F(r, k)$. They proved an upper bound for $F(r, k)$ for $k \geq 3$ [26, inequality (17)], from which we deduce the more convenient but slightly weaker form

$$F(r, k) \leq \frac{1}{2} \left( \frac{r}{(k-2)!} \left( \frac{r}{p} \right) \left( \frac{x}{p} \right) \right)^{1/(k-2)} + (k-3),$$  \hspace{1cm} (4)

where $x = \lfloor r/2^k \rfloor + 1$ and $p = \lfloor x/2 \rfloor + 1$. When we combine this result with Theorem 8, we obtain the following.

THEOREM 10. For $n \geq k \geq 3$,

$$\kappa(n, n-k) \geq \frac{k-2}{H(1/2^{k-1}) - 1/2^{k-2}} \log(n-k+3) - k \log k - 2 \log \log n,$$

where $H(x) = -[x \log x + (1-x) \log(1-x)]$.

At present, the lower bound just obtained is the best known for $\kappa(n, n-k)$ for $k$ fixed and large $n, k$. When it is rewritten in the slightly weaker form

$$\kappa(n, n-k) \geq 2^{k-1} \left( \frac{k-2}{k-3} + \log e \right) \log(n-k+3) - k \log k - 2 \log \log n,$$  \hspace{1cm} (5)
it can easily be compared with the improvement gained over the bound from [3]

$$\kappa(n, n-k) \geq 2^{k-2} \left[ \log(n-k+2) + 0.125 \log \log(n-k+2) \right],$$

which is the result of applying Theorems 2 and 9.

Now, in the other direction, Kleitman and Spencer [26] used a non-constructive probabilistic argument to prove that

$$F(r, k) \geq (1/2)(k!)^{1/k} (2^k/(2^k - 1))^{r/k}. \quad (6)$$

When this inequality is combined with Theorem 8, it is straightforward to show that

$$\kappa(n, n-k) \leq \frac{k}{\log(1 - 2^{-k})} \log n. \quad (7)$$

This inequality, first established in [7] and later in [3], provides the best known upper bound for fixed $k$ and large $n, k$. It will be discussed further in Section 4.

Using the non-constructive methods of Erdős et al. [10], we next derive a new upper bound for $\kappa(n, n-3)$ that, for $n$ large, is superior to any other known bounds. The best upper bound known previously, given by inequality (7) with $k = 3$, is $\kappa(n, n-3) \leq 15.571 \log n$.

**Theorem 11.** For $n$ sufficiently large, $\kappa(n, n-3) < 7.57 \log n$.

**Proof.** Let $r$ be an even positive integer and let $X$ be a set of $r$ elements. We will prove that there is a 3-independent family of subsets of $X$ that contains at least $(1.0959)^r$ elements when $r$ is sufficiently large. From this we will be able to conclude that $\kappa(n, n-3) < \log n/\log 1.0959$ for $n$ sufficiently large, which will complete the proof of the theorem.

Let $X'$ be the set of all subsets of $X$ of size $r/2$, and let $p$ be a real number, $0 < p < 1$, whose value will be determined later. Denote by $S$ a random collection of subsets obtained by choosing independently and with probability $p$ each of the subsets in $X'$. Using $S$, we form a 3-independent family by successively deleting any set $A$ from $S$ for which there are sets $B$ and $C$ in $S$ that satisfy either

1. $A \subseteq B \cup C$, or
2. $B \cap C \subseteq A$.

For a fixed $A \in X'$, let $b_1(A, r)$ denote the number of pairs $(B, C) \in X' \times X'$ for which (1) holds and let $b_2(A, r)$ be defined analogously for (2). Setting $b(r) = \sum_{A \in X'} [b_1(A, r) + b_2(A, r)]$, we see that the expected number of members deleted from $S$ is at most $p^3(r/2) b(r)$. By choosing $p = (2b(r))^{1/2}$,
the existence of a 3-independent set with at least \((1/2)(2b(r))^{-1/2}(r')^2\) members can be guaranteed.

We need an upper bound for \(b(r)\), but it will suffice to determine an upper bound for \(b_1(A, r)\) because \(b_1(A, r) = b_2(X' - A, r)\) for \(A \in X'\) and \(b(r) = 2\sum_{A \in X'} b_1(A, r)\). To this end, suppose \(A\) is a given set in \(X'\). The pairs \((B, C) \in X' \times X'\) for which condition (1) holds can be put in one-to-one correspondence with the four-tuples of sets \((U_1, U_2, V_1, V_2)\) which satisfy the set of restrictions

\[
R: \quad V_1 \subseteq U_1 \subseteq A, \\
U_2, V_2 \subseteq X - A, \\
|U_1| + |U_2| = r/2, \\
|V_1| + |V_2| = |U_1|.
\]

To illustrate the intended correspondence, if we are given the pair \((B, C)\) for which condition (1) holds, take \(U_1 = A \cap B, U_2 = B - A, V_1 = U_1 \cap C,\) and \(V_2 = C - A\). It is straightforward to check that \((U_1, U_2, V_1, V_2)\) does satisfy each condition of \(R\). Conversely, if the four-tuple \((U_1, U_2, V_1, V_2)\) satisfies all of the conditions listed in \(R\), then with \(B = U_1 \cup U_2\) and \(C = V_1 \cup (A - U_1) \cup V_2\), the pair \((B, C)\) satisfies condition (1). It follows that \(b_1(A, r)\) is the number of such four-tuples satisfying the conditions in \(R\). Hence,

\[
b_1(A, r) = \sum_{0 \leq x \leq r/2} \binom{r/2}{x} \binom{r/2}{r/2 - x} \sum_{0 \leq y \leq x} \binom{x}{y} \binom{r/2}{x - y}.
\]

Since the ratio of consecutive terms in the sum for \(b_1(A, r)\) is monotone decreasing, the maximum term occurs where this ratio is approximately 1, namely for \(x \sim 0.309r\). Using Stirling's approximation, \(n! \sim (n/e)^n (2\pi n)^{1/2}\), we find that \(b(r) < (3.3302)^r\) and so \((2b(r))^{-1/2}(r')^2 > 2(1.0959)^r\) for \(r\) sufficiently large. 

2.4. Partitions

We now introduce another technique for obtaining upper bounds for \(\kappa(n, m)\). While the results of this section are asymptotically weaker than those of the previous section, they provide good recursive constructions for small values of \(n\) which are not available from the probabilistic arguments employed.

Consider a collection \(P_1, P_2, ..., P_r\) of partitions of \(\{1, 2, ..., n\}\) with the following property.

Property \(\mathcal{P}(n, k)\): for every pair of disjoint subsets \(U\) and \(V\) of \(\{1, 2, ..., n\}\) for which \(|U \cup V| = k\), there is some partition \(P_i\)
such that no cell of $P_i$ contains both an element of $U$ and an element of $V$.

A collection of $r$ partitions satisfying property $\mathcal{P}(n, k)$ can be used to construct a $k$-independent family of size $n$ as well as a set in $\mathcal{I}(n, n-k)$. In view of the correspondence described at the beginning of Section 2.3 between $k$-independent sets and sets in $\mathcal{I}(n, m)$, it suffices to show how a collection of partitions satisfying property $\mathcal{P}(n, k)$ can be used to construct a set in $\mathcal{I}(n, n-k)$.

Let $P$ be a partition of $\{1, 2, ..., n\}$ with non-empty cells $A_1, A_2, ..., A_p$, and for each $i$, $1 \leq i \leq n$, let $\phi(i)$ be the unique integer $j$ for which $i \in A_j$. Further, if $J \subseteq \{1, 2, ..., n\}$, let $\phi(J) = \{\phi(i) : i \in J\}$. We use $P$, and therefore $\phi$, to construct a function $\tau$ from $\{0, 1\}^p$ to $\{0, 1\}^n$ as follows:

$$\tau(a_1, a_2, ..., a_p) = (a_{\phi(1)}, a_{\phi(2)}, ..., a_{\phi(n)}).$$

That is, for each subset $W$ of $\{1, 2, ..., p\}$, $\tau$ maps the characteristic function of $W$ to the characteristic function of $\bigcup_{i \in W} A_i$ as a subset of $\{1, 2, ..., n\}$. Now, suppose $P_1, P_2, ..., P_r$ is a collection of partitions of $\{1, 2, ..., n\}$ satisfying property $\mathcal{P}(n, k)$, where $P_j$ has $c_i$ non-empty cells $A_{i1}, A_{i2}, ..., A_{ic_i}$. Further, let $\tau_j$ and $\phi_j$ be the functions obtained from $P_j$ as described above. For each $i$, $1 \leq i \leq r$, choose a minimum size set $S_j \in \mathcal{I}(c_i, c_i - k)$ and let $X_i = \{\tau_i(s) : s \in S_i\}$. Since we can always choose a minimum set in $\mathcal{I}(n, m)$ that contains $(0, ..., 0)$, we will do so, and then modify each $X_i$ for $i > 1$ by removing the $n$-tuple $(0, ..., 0)$. We claim that the resulting set $X = \bigcup_{i=1}^r X_i$ is in $\mathcal{I}(n, n-k)$. To prove this, suppose $U = u_1 u_2 \cdots u_n$ is a subcube of $Q_n$ of dimension $n-k$, and let $J_1$ and $J_2$ be the index sets determined by

$$u_i = \begin{cases} 1 & \text{if } i \in J_1 \\ 0 & \text{if } i \in J_2 \\ * & \text{otherwise} \end{cases}$$

We want to show that there is some element of $X$ that is in $U$. Since property $\mathcal{P}(n, k)$ is satisfied by the collection $P_1, P_2, ..., P_r$, at least one of these partitions, say $P_\ell$, is such that none of its cells contains an element of $J_1$ and an element of $J_2$. Thus, we can define the subcube $V = v_1 v_2 \cdots v_n$ of $Q_n$ by

$$v_i = \begin{cases} 1 & \text{if } i \in \phi_\ell(J_1) \\ 0 & \text{if } i \in \phi_\ell(J_2) \\ * & \text{otherwise} \end{cases}$$
and conclude that $V$ is of dimension $c_u - k$. Furthermore, since $S_u \in \mathcal{S}(c_u, c_u - k)$, there is an element $x = x_1 x_2 \cdots x_{c_u} \in (V \cap S_u)$, which shows that $\tau_u(x) \in (X \cap U)$. We formalize this result in terms of $\kappa$ in the following.

**Theorem 12.** Let $n \geq k \geq 1$. Suppose $P_1, P_2, \ldots, P_r$ is a collection of partitions of $\{1, 2, \ldots, n\}$ satisfying property $\mathcal{P}(n, k)$, where $P_i$ has $c_i$ non-empty cells for $1 \leq i \leq r$. Then

$$\kappa(n, n - k) \leq \sum_{i=1}^{r} \kappa(c_i, c_i - k) - r + 1.$$ 

This inequality is very useful when a small collection of partitions satisfying property $\mathcal{P}(n, k)$ can be found, as illustrated in the following for $k = 3$.

**Corollary 12.1.** For all integers $s$ and $t$ such that $st \geq n \geq s \geq t \geq 3$,

$$\kappa(n, n - 3) \leq 2\kappa(s, s - 3) + \kappa(t, t - 3) - 2.$$ 

**Proof.** Choose the integers $s$ and $t$ in the given range. Let $P_1$ denote the partitions of $\{1, 2, \ldots, n\}$ with cells

$$A_{1j} = \{m: 1 \leq m \leq n, m \equiv j \mod s\},$$

for $0 \leq j \leq s - 1$; let $P_2$ denote the partition with cells

$$A_{2j} = \left\{m: 1 \leq m \leq n, \left\lfloor \frac{m}{s} \right\rfloor = j\right\},$$

for $0 \leq j \leq t$; and let $P_3$ denote the partition with cells

$$A_{3j} = \left\{m: 1 \leq m \leq n, \left\lfloor \frac{m}{s} \right\rfloor + m \equiv j \mod s\right\},$$

for $0 \leq j \leq s - 1$. It is relatively easy to check that this collection of partitions $P_1, P_2, P_3$ does indeed have property $P(n, 3)$.

Although Corollary 12.1 gives an upper bound for $\kappa(n, n - 3)$ which is $O((\lg n)^{g_3})$, its results, when combined with Theorem 13 and the exact values of $\kappa$ in Table 1, actually give a better upper bound than that provided by Theorem 11 for $n \leq 1600$.

Friedman [12] showed how to construct, for any fixed $k$ and $n$, a collection of $O(\lg n)$ partitions of $\{1, 2, \ldots, n\}$ such that for any subset $T$ of $\{1, 2, \ldots, n\}$ of size $k$, there is a partition in which each of its cells contain at most one element of $T$. Becker and Simon [3] used Friedman's result
to construct sets in $\mathcal{S}(n, n - k)$ of size at most $\lg n (k^4/\lg k) 2^{2k \lg k + 3k}$. While this construction yields sets of the right order of magnitude, namely $O(\lg n)$, for small $k$ they are impractically large. For example, if $k = 3$ and $n = 40$, they are of size $2^{20} \lg 40$, whereas the above corollary with $s = 9$ and $t = 5$ yields a set of size 32.

The next theorem contains an upper bound for $\kappa(n, n - 3)$ which is also established by constructive means. Although its methods yield sets of size $O((\lg n)^2)$, as pointed out earlier, when it is combined with Corollary 12.1 and Table 1, it yields superior bounds for $\kappa(n, n - 3)$ for $n \leq 1600$. A similar construction was used by Chandra et al. [7] to construct sets in $\mathcal{S}(n, n - k)$ of size $O((\lg n)^{k-1})$, but for specific $n$ and $k$, their sets are somewhat larger than ours because they could not utilize the results in Table 1.

**Theorem 13.** For $n \geq 5$,

$$\kappa(n, n - 3) \leq \kappa(\lceil n/2 \rceil, \lceil n/2 \rceil - 3) + \kappa(\lceil n/2 \rceil, \lceil n/2 \rceil - 2).$$

**Proof.** If $n$ is odd then the bound given for $\kappa(n, n - 3)$ is the same as the one for $\kappa(n + 1, n - 2)$. Since part (ii) of Theorem 2 shows that $\kappa(n, n - 3) \leq \kappa(n + 1, n - 2)$, it suffices to consider only the case when $n$ is even.

For an arbitrary positive integer $m$, let $x$ and $y$ be two binary $m$-bit strings and denote by $xy$ the $2m$-bit string formed by the concatenation of $x$ and $y$. The complement of $x$ is denoted by $x'$, that is, $x'$ is the string $x'_1 \cdots x'_m$ where $x'_i = 0$ if $x_i = 1$ and $x'_i = 1$ otherwise, $1 \leq i \leq m$.

Now, let $S_1$, $S_2$ denote sets in $\mathcal{S}(n/2, n/2 - 3)$, $\mathcal{S}(n/2, n/2 - 2)$, respectively, of minimum size. Define the set $S$ by

$$S = \{xx \mid x \in S_1\} \cup \{yy' \mid y \in S_2\}.$$  

To complete the proof, we show that $S$ is in $\mathcal{S}(n, n - 3)$.

Suppose $i$, $j$, and $k$ are integers in $\{1, \ldots, r\}$ and $a, b, c \in \{0, 1\}$. Let $T = T(r; i, j, k; a, b, c)$ denote the $(r - 3)$-cube of $Q$, given by $u_1 \cdots u_r$, where $u_i = a$, $u_j = b$, $u_k = c$, and $u_h = \ast$ for all $h \neq i, j, k, 1 \leq h \leq r$. Analogously, we denote by $W(r; i, j; a, b)$ the $(r - 2)$-cube of $Q$, given by $u_1 \cdots u_r$, where $u_i = a$, $u_j = b$, and $u_h = \ast$ for all $h \neq i, j, k, 1 \leq h \leq r$.

Let $T$ be an $(n - 3)$-cube of $Q$, say $T = T(n; i, j, k; a, b, c)$, where we may suppose, without loss of generality, that $i < j < k$. We divide the proof into cases and show, in each case, that $T$ contains an element of $S$. The details of the proof in the case where both $i \leq n/2$ and $n/2 < j$ are omitted since they are handled almost identically to those listed below.
• $k \leq n/2$ or $n/2 < i$. If $k \leq n/2$, then $T(n/2; i, j, k; a, b, c) \cap S_1 \neq \emptyset$, whereas if $n/2 < i$, then $T(n/2; i, n/2, k; a, b, c) \cap S_1 \neq \emptyset$. In either case, if $x$ is an element of this intersection, then $xx \in T \cap S$.

• $j \leq n/2$, $n/2 < k$, and $k - n/2 \neq 0, j$. Since $T(n/2; i, j, k; a, b, c) \cap S_1 \neq \emptyset$, it follows, as in the previous case, that $T \cap S \neq \emptyset$.

• $j \leq n/2$, $k - n/2 = i$, and $c = a$ or if $j \leq n/2$, $k - n/2 = j$, and $c = b$. Choose an integer $k_1$ in $1 \cdots n/2$ other than $i, j$. Since $T(n/2; i, j, k_1; a, b, 0) \cap S_1 \neq \emptyset$, we have $T \cap S \neq \emptyset$.

• $j \leq n/2$, $k - n/2 = i$, and $c \neq a$ or if $j \leq n/2$, $k - n/2 = j$, and $c \neq b$. Since $W(n/2; i, j; a, b) \cap S_2 \neq \emptyset$, if $y$ is an element of this intersection, then $yy' \in T \cap S$.

The proof of the above theorem can be extended to show that for all $n \geq 2h \geq 2$,

$$\kappa(n, n - h) \leq \kappa\left(\left[\frac{n}{2}\right], \left[\frac{n}{2}\right] - h\right) + \sum_{i=2}^{h-2} \kappa\left(\left[\frac{n}{2}\right], \left[\frac{n}{2}\right] - h + i\right)\left(\kappa\left(\left[\frac{n}{2}\right], \left[\frac{n}{2}\right] - i\right) - 1\right).$$

A slightly weaker result appears in [7], where the factor $\kappa\left(\left[\frac{n}{2}\right], \left[\frac{n}{2}\right] - i\right) - 1$ in the above summation is replaced by $\kappa\left(\left[\frac{n}{2}\right], \left[\frac{n}{2}\right] - i\right)$.

3. The Values of $\lambda$

Turning to the corresponding questions involving edge faults instead of node faults, we find that many of the results and proof techniques for $\kappa$ have their analogs for $\lambda$. Once again, proofs of recursive bounds show how to construct small fault sets. By a slight abuse of notation, we will use $a_1 \cdots a_1 a_{i+2} \cdots a_n$ to denote the edge of the 1-cube as well as the 1-cube itself. When we speak of removing the edge $a_1 \cdots a_1 a_{i+2} \cdots a_n$ from $Q_n$, we remove the edge but not the nodes to which it is incident.

3.1. Elementary Bounds

**Theorem 14.** For $n \geq 1$,

(i) $\lambda(n, n) = 1$

(ii) $\lambda(n, n - 1) = 3$ for $n \geq 3$

(iii) $\lambda(n, 1) = n2^{n-1}$. 
**Proof.** Parts (i) and (iii) are immediate. To establish (ii), let $Q'$ and $Q''$ denote two disjoint copies of an $(n-1)$-cube in $Q_n$. Clearly, at least one edge must be removed from each of $Q'$ and $Q''$. Moreover, at least one edge with one endpoint in $Q'$ and the other in $Q''$ must be removed to prevent an $(n-1)$-cube made up of corresponding $(n-2)$-cubes in $Q'$ and $Q''$. Thus, $\lambda(n, n-1) \geq 3$. To realize this bound for $n \geq 3$, take the set of edges of $Q_n$ given by $T = \{\ast0 \cdots 0, 11\ast1 \cdots 1, 0\ast10 \cdots 0\}$. It is easy to check that $T$ is in $\mathcal{S}(n, n-1)$.

Recursive upper and lower bounds corresponding to Theorem 2 are now established for $\lambda$.

**Theorem 15.** For $n \geq m$,

(i) $\lambda(n, m) \geq \max \{\lfloor (n/m) 2^{n-m} \rfloor, \lceil 2\lambda(n-1, m)n/(n-1) \rceil, \kappa(n, m)\}$

(ii) $\lambda(n, m) \leq \min \{\lambda(n-1, m-1) + \lambda(n-1, m), (n-m+1)\kappa(n, m)\}$

(iii) If $\lambda(n+1, m+1) < n+1$ then $\lambda(n, m) \leq \lambda(n+1, m+1)$.

**Proof.** For (i), first note that there are $\binom{n}{m} 2^{n-m}$ copies of $Q_m$ in $Q_n$ and each edge is contained in $\binom{n-1}{m-1}$ of the $Q_m$'s. Thus $\lambda(n, m) \geq (n/m) 2^{n-m}$.

To show that $\lambda(n, m) \geq 2\lambda(n-1, m) + \lfloor \lambda(n, m)/n \rfloor$, from which the second implied inequality of part (i) follows, let $T$ denote a set of minimum size in $\mathcal{S}(n, m)$. There exist at least $\lceil \lambda(n, m)/n \rceil$ parallel edges in $T$, and without loss of generality, we may suppose these are parallel to $\ast0 \cdots 0$. The desired inequality now follows from the observation that the two node-disjoint cubes of dimension $(n-1)$ given by $0\ast \cdots \ast$ and $1\ast \cdots \ast$ must each contain at least $\lambda(n-1, m)$ edges of $T$.

To see that $\lambda(n, m) \geq \kappa(n, m)$, observe that if $T$ is a set of size $\lambda(n, m)$ in $\mathcal{S}(n, m)$ then the set $S = \{ v \mid \{v, w\} \in T \text{ and weight}(v) < \text{weight}(w) \}$ is in $\mathcal{S}(n, m)$.

In order to show the first of the implied inequalities in (ii), we construct a set $T$ in $\mathcal{S}(n, m)$ as follows. Let $Q', Q''$ be node-disjoint $(n-1)$-cubes of $Q_n$. Choose sets $T_1, T_2$ each of $\lambda(n-1, m)$ edges from $Q', Q''$, respectively, whose removal from $Q', Q''$ leaves no $Q_m$. Further, choose a set of $S$ of $\kappa(n-1, m-1)$ nodes of $Q'$ whose removal from $Q'$ leaves no $Q_{m-1}$, and let $T_3$ be the set of edges of $Q_n$ with one endpoint in $S$ and the other in $Q''$. It is straightforward to verify that $T = T_1 \cup T_2 \cup T_3$ is in $\mathcal{S}(n, m)$. Thus, $\lambda(n, m) \leq 2\lambda(n-1, m) + \kappa(n-1, m-1)$.

The inequality $\lambda(n, m) \leq \lambda(n-1, m-1) + \lambda(n-1, m)$ can be proved in the same way as the corresponding inequality for $\kappa$ in Theorem 2.
To prove the last of the implied inequalities in (ii), choose a set of nodes $S$ of size $\kappa(n, m)$ in $\mathcal{S}(n, m)$ and let

$$ T = \{ \{u, v\} \mid \{u, v\} \text{ is an edge of } Q_n, u \in S, \\ v \text{ has the same first } m - 1 \text{ components as } u \}. $$

Since any $m$-cube of $Q_n$ contains at least one node in $S$, it will contain at least one edge of $T$. Thus, $T \in \mathcal{T}(n, m)$ and $|T| \leq (n - m + 1) \kappa(n, m)$.

For the proof of (iii), suppose for some $n$ that $\lambda(n + 1, m + 1) < n + 1$, and let $T$ be a set of minimum size in $\mathcal{T}(n + 1, m + 1)$. By our assumption on $n$, there is some $j, 1 \leq j \leq n + 1$, for which no element of $T$ has a $*$ for its $j$th component. If we project $T$ on this component we see that the resulting set is in $\mathcal{T}(n, m)$.

In part (i) of Theorem 15, each of the first two terms providing a lower bounds for $\lambda(n, m)$ is larger than the remaining two for certain values of $n$ and $m$. For $m = 1$, $(n/m)2^{n-m} = \lambda(n, m)$, and for $n = 7$ and $m = 4$, the term $\lceil 2\lambda(n - 1, m) n/(n - 1) \rceil$ gives the best bound. We have not found an example for which the third term, $\kappa(n, m)$, exceeds the other two, but neither have we been able to prove that it is always at most the maximum of the other two. In the inequality occurring in part (ii) of the above theorem, we find that for $m = 1$, the first and third terms are equal to $\lambda(n, 1)$, whereas for $n = 7$ and $m = 5$, the second term is less than the other two. We have not found an example for which the third term, $(n - m + 1) \kappa(n, m)$, is less than the other two, nor have we been able to show that it is always at least as the minimum of the other two. The example $8 = \lambda(4, 2) < \lambda(3, 1) = 12$ shows that, unlike the corresponding inequality for $\kappa$, the conclusion of part (iii) of our theorem does not hold for all $n \geq m$. Figure 1 illustrates $\lambda(4, 2)$.

We state two straightforward consequences of Theorem 15 which were also observed in [3].

**Corollary 15.1.** For $n \geq 3$,

(i) $\kappa(n, n - 2) \leq \lambda(n, n - 2) \leq \kappa(n - 1, n - 3) + 6$

(ii) $\lambda(n, n - 2) = \lg n + \frac{1}{2} \lg \lg n + O(1)$.

The labeling technique used in Theorem 3 has an analog for $\lambda(n, m)$.

**Theorem 16.** Let $r$ be a fixed integer, $0 \leq r \leq m$, and let $r' = \min\{r, m - 1\}$. Label the nodes of $Q$, with integers in $[0, r]$ and label some subset $E$, of the edges of $Q$, with integers in $[0, r']$ in such a way that every $l$-cube of $Q$, has either a node or an edge whose label is at least $l$. 

0 \leq l \leq r. If l(q) is the label of node q in Q, and l'(e) is the label of edge e in E, then

$$\lambda(n, m) \leq \sum_{q \in E} \lambda(n - r, m - l(q)) + \sum_{e \in E} \kappa(n - r, m - l'(e)).$$

Proof. We construct a set T of edges of Q as follows. For each q = q_1 \cdots q_{r} \in Q, choose a set T_q of \lambda(n - r, m - l(q)) edges of the (n - r)-cube Q_{s}(q) = q_1 \cdots q_{r} \cdots q_{r} whose removal from Q_{s}(q) leaves no (m - l(q))-cube. Further, if e = u_1 \cdots u_{i} \cdots u_{i+2} \cdots u_r is an edge in E, choose a set S_e of \kappa(n - r, m - l'(e)) nodes of the (n - r)-cube Q_{s}(u_1 \cdots u_{i} \cdots u_{i+2} \cdots u_r) = u_1 \cdots u_{i} \cdots u_{i+2} \cdots u_r whose removal from Q_{s}(u_1 \cdots u_{i} \cdots u_{i+2} \cdots u_r) leaves no cube of dimension (m - l'(e)). Now, let T_e be the set of edges with one endpoint in S_e and the other in Q_{s}(u_1 \cdots u_{i} \cdots u_{i+2} \cdots u_r) = u_1 \cdots u_{i} \cdots u_{i+2} \cdots u_r. It is straightforward to verify that the set T = (\bigcup_{q \in Q} T_q) \cup (\bigcup_{e \in E} T_e) is in \mathcal{F}(n, m). \square.

3.2. Level Sets

We now construct sets in \mathcal{F}(n, m) by removing edges from Q, whose nodes are at a specified distance from the origin. The size of the sets constructed by this technique are, for fixed m and large n the smallest yielded by any of the known constructions.

Theorem 17. If n \geq m \geq 1 and a is any integer, then

$$\lambda(n, m) \leq (n - m + 1) \left[ \sum_{k \equiv a \mod m \atop k \leq n/2} \left( \begin{array}{c} n - 1 \\ k \end{array} \right) + \sum_{k \equiv a + 1 \mod m \atop k > (n + 1)/2} \left( \begin{array}{c} n - 1 \\ k - 1 \end{array} \right) \right].$$

Further, this sum is minimized when a = \lfloor (n - m)/2 \rfloor.

Proof. Consider, as in the proof of Theorem 4, the nodes of Q partitioned into levels in which all nodes of weight i compose level i, 0 \leq i \leq n. Suppose that in every set of m + 1 consecutive levels there are two consecutive levels, say level i and i + 1, in the set from which we have removed each edge that joins a node in level i to a node in level i + 1. Clearly, no Q_m can remain. We can improve upon this, for if we fix some n - m + 1 dimensions, we need only remove those edges that join a node in level i to a node in level i + 1 along these dimensions. Equivalently, we could have removed the edges that join nodes in level i and i + 1 along these dimensions. To be more explicit, let N_0(i, j), N_1(i, j) denote the set of nodes of Q at level i with jth component equal to 0, 1, respectively.

If 0 \leq i < n/2 and 0 \leq j \leq n - m + 1, let T_{ij} denote the set of edges of Q with one endpoint in N_0(i, j) and the other in N_1(i + 1, j); if (n + 1)/2 < i \leq n and 0 \leq j \leq n - m + 1, let T_{ij} be the set of edges of Q with
one endpoint in $N_r(i,j)$ and the other in $N_r(i-1,j)$. Further, setting $T_i = \bigcup_{0 \leq j \leq n - m + 1} T_{ij}$, for $0 \leq i \leq n$, we see that $|T_i| = (n - m + 1)(n^{-1})$. Now, fix some integer $a$. If we remove the edges in $T_i$ for $i < n/2$ and $i \equiv a \mod m$ together with the edges in $T_k$ for $k > (n+1)/2$ and $k \equiv a + 1 \mod m$ then no set of $m+1$ consecutive levels can contain an $m$-dimensional subcube of $Q_n$. Again, as in Theorem 4, we choose $a$ in order to minimize the number of edges removed in this manner. The value $a = \lfloor (n - m)/2 \rfloor$ ensures that, where possible, we avoid removing edges incident with level $n/2$ when $n$ is even and levels $(n - 1)/2$ and $(n + 1)/2$ when $n$ is odd.

**Corollary 17.1.** For $n \geq m \geq 1$,
\[ \lambda(n, m) \leq (n - m + 1) 2^{n-1}/m. \]

**Proof.** The desired result is a consequence of the identity
\[ \sum_{a=0}^{m-1} \left[ \sum_{k \equiv a \mod m} \binom{n-1}{k} + \sum_{k \equiv a+1 \mod m} \binom{n-1}{k-1} \right] \leq 2^{n-1}. \]

3.3. **Partitions**

Using techniques similar to those in Section 2.4, a collection of $r$ partitions satisfying property $\mathcal{P}(n, k)$ can be used to construct sets in $\mathcal{S}(n, k)$. We merely need to modify the method used to construct sets in $S(n, m)$ by changing $\tau$ as follows. If $P$ is a partition of $\{1, 2, ..., n\}$ with non-empty cells $A_1, A_2, ..., A_r$, and if $t = (t_1, t_2, ..., t_r)$ is an edge of $Q_r$, where $t_e = \ast$, say, then let $\tau(t_1, t_2, ..., t_r) = (d_1, d_2, ..., d_n)$, where $d_i = t_i^{-1}(i)$ if $i$ is not in $\phi^{-1}(e)$, $d_i = \ast$ if $i$ is the least element in $\phi^{-1}(e)$, and $d_i = 0$ otherwise. The same methods as those used to establish Theorem 12 and Corollary 12.1 can be used to prove the following.

**Theorem 18.** Let $n \geq k \geq 1$. Suppose $P_1, P_2, ..., P_r$ is a collection of partitions of $\{1, 2, ..., n\}$ satisfying property $\mathcal{P}(n, k)$, where $P_i$ has $c_i$ non-empty cells for $1 \leq i \leq r$. Then
\[ \lambda(n, n-k) \leq \sum_{i=1}^{r} \lambda(c_i, c_i-k) - r + 1. \]

This theorem is very useful when a small collection of partitions satisfying property $\mathcal{P}(n, k)$ can be found, as illustrated in the following for $k = 3$.

**Corollary 18.1.** For all integers $s$ and $t$ such that $s \geq n \geq s \geq t \geq 3$,
\[ \lambda(n, n-3) \leq 2\lambda(s, s-3) + \lambda(t, t-3) - 2. \]
The next result is an analog of Theorem 13.

**Theorem 19.** For \( n \geq 6 \),
\[
\lambda(n, n - 3) \leq \lambda(\lceil n/2 \rceil, \lceil n/2 \rceil - 3) + \lambda(\lceil n/2 \rceil, \lceil n/2 \rceil - 2).
\]

**Proof.** We use an argument similar to that in the proof of Theorem 13 except that special consideration is needed for the case \( n \) odd.

First we introduce some notation to show how two edges of \( Q_n \) will be used to form an edge in \( Q_{2k} \). If \( x = x_1 \cdots x_k * x_{i+2} \cdots x_k \) is a 1-cube in \( Q_k \), let
\[
x \tilde{x} = x_1 \cdots x_k x_1 \cdots x_i 0 x_{i+2} \cdots x_k
\]
and
\[
x \tilde{x}' = x_1 \cdots x_k x'_1 \cdots x'_i 0 x'_{i+2} \cdots x'_k
\]
Thus, \( x \tilde{x} \) and \( x \tilde{x}' \) are 1-cubes in \( Q_{2k} \).

Suppose \( n = 2p \). Choose sets \( T_1, T_2 \) of minimum size in \( \mathcal{F}(p, p - 3), \mathcal{F}(p, p - 2) \), respectively and define
\[
T = \{ x \tilde{x} \mid x \in T_1 \} \cup \{ y \tilde{y}' \mid y \in T_2 \}.
\]

The proof that \( T \in \mathcal{F}(n, n - 3) \) is almost identical to that used for Theorem 13 and so we suppress the details.

Now, suppose \( n = 2p - 1 \). Let \( W_1, W_2 \) be sets in \( \mathcal{F}(p, p - 3), \mathcal{F}(p, p - 2) \), respectively, each of minimum size. We form a set \( W \) in \( \mathcal{F}(n, n - 3) \) in the same way as we formed \( T \) in the \( n \) even case, except that we project the \( (n + 1) \)-tuples on the last component. That is, if \( x = x_1 \cdots x_{n+1} \in Q_{n+1} \) and \( P_{n+1}(x) = x_1 \cdots x_n \) we take \( W = \{ P_{n+1}(x \tilde{x}) \mid x \in W_1 \} \cup \{ P_{n+1}(y \tilde{y}') \mid y \in W_2 \} \). We suppress the details of the proof that \( W \) is in \( \mathcal{F}(n, n - 3) \) as they are straightforward.

3.4. **Lower Bounds for \( \lambda \)**

Since \( \lambda(n, m) \geq \kappa(n, m) \), from Theorem 15, various lower bounds for \( \lambda(n, m) \) can be derived from the lower bounds for \( \kappa(n, m) \). In particular, the new lower bound proved in Theorem 10 gives us the improved lower bound for \( \lambda(n, n - k) \) for fixed \( k \) and large \( n \) which we state below.

**Theorem 20.** For \( n \geq k \geq 3 \),
\[
\lambda(n, n - k) \geq \frac{k - 2}{H(1/2^{k-1})} - 1/2^{k-2} \lg(n - k + 3) - k \lg k - 2 \lg \lg n,
\]
where \( H(x) = -[x \lg x + (1 - x) \lg(1 - x)] \).
At present, the lower bound just obtained is the best known for $\lambda(n, n-k)$ for $k$ fixed and large $n, k$. We write it in the following slightly weaker form to make it easier to see the size of the bound:

$$\lambda(n, n-k) \geq 2^{k-1} \left( \frac{k-2}{k-3} + \log e \right) \log(n - k + 3) - k \log k - 2 \log \log n. \quad (8)$$

The next theorem gives lower bounds for $\lambda(n, m)$ which, for small $m$, are better than those available from the inequality in Theorem 20. Its proof is an extension and generalization of an argument used by Johnson [22], who proved that $g(5, 2) \leq 56$. Our extension establishes that $g(n, 2) \leq k 2^{n-1} + b/2$, where $k$ is the integer such that $4^k \leq (\binom{n}{2}) \leq 4^{k+1}$, and $b$ is the largest integer such that $(2^n - b)(\binom{k}{2}) + b(k+1) \leq (\binom{n}{2}) 2^{n-2}$, and further generalizes this to arbitrary $g(n, m)$. F. Chung (personal communication, July 1988) independently proved a result which is essentially the same as our result for $g(n, 2)$, namely that the edge density $x = g(n, 2)/(n 2^{n-1})$ must satisfy $(n-1)(n-2) \geq 4x(n-1)(n-2).$ Thus, for large $n$, the edge density is bounded above by $(1/4)^{1/3}$. In terms of $\lambda(n, 2)$, we see that, for $n$ large, at least 0.37 of the edges must be faulty in order that every $Q_2$ is faulty. By Theorems 1 and 15(ii), at most $\frac{1}{2}$ of the edges need be faulty to ensure that every $Q_2$ is faulty. Some time ago, Erdős [9] conjectured that, for every $\varepsilon < 0$, there is an $n_\varepsilon$ such that, for all $n > n_\varepsilon$, $g(n, 2) < (\frac{1}{2} + \varepsilon)n 2^{n-1}$, i.e., that the edge density becomes arbitrarily close to $\frac{1}{2}$. He also conjectured that $g(n, k) < cn^{a_k} 2^n$, where $a_k < 1$ and $a_k \to 0$ as $k \to \infty$.

**Theorem 21.** For $n \geq m \geq 1$, let $g(n, m)$ be the largest number of edges in a subgraph of $Q_n$ that contains no $Q_m$. Then

$$g(n, m) \leq k 2^{m-1} + b/2,$$

where $k$ is the integer determined by

$$2^{m+1} \left( \frac{k}{m+1} \right) \leq (2^{m+1} - 6) \left( \frac{n}{m+1} \right) < 2^{m+1} \left( \frac{k+1}{m+1} \right)$$

and

$$b = \left[ \left( 2^{m+1} - 6 \right) \left( \frac{n}{m+1} \right) 2^n \left( \frac{k}{m+1} \right) \right] / \left( \frac{k}{m} \right).$$

**Proof.** We call a node in $Q_n$ together with its $n$ incident edges an $n$-star, and refer to the node as its center. We first note that any induced subgraph of $Q_{m+1}$ with at least $2^{m+1} - 5$ of the $(m+1)$-stars must contain a $Q_m$. For suppose $H$ is an induced subgraph of $Q_{m+1}$ that is lacking only five
(\(m+1\))-stars. Split \(H\) into two node-disjoint subgraphs \(H_0\) and \(H_1\), where the nodes of \(H_i\) have first coordinate \(i\), \(i = 0, 1\). If either \(H_0\) or \(H_1\) lacks only one \((m+1)\)-star, then it must be a \(Q_m\). Without loss of generality, suppose \(H_0\) lacks only two \((m+1)\)-stars, centered at nodes \(p_0\) and \(q_0\). Then \(p_0\) and \(q_0\) must be adjacent, for otherwise \(H_0\) would be a \(Q_m\). Let \(A_0\) denote an \((m-1)\)-cube of \(H_0\) that does not contain nodes \(p_0\) and \(q_0\) and let \(A_1\) be its neighbor in \(H_1\). Since \(A_1\) must be missing at least one edge, it must contain at least two of the centers, say \(r\) and \(s\), of the missing \((m+1)\)-stars of \(H_1\) and these must be adjacent. Now, there are at least two node-disjoint \((m-1)\)-cubes of \(H_0\), say \(B_0\) and \(C_0\), with \(p_0\) a node of \(B_0\) and \(q_0\) a node of \(C_0\). At least one of their neighbors in \(H_1\), \(B_1\) or \(C_1\), contains all of its possible \((m+1)\)-stars and so will form an \(m\)-cube with its neighbor in \(H_0\).

Let \(E\) denote a set of edges of \(Q_n\), let \(G\) denote the subgraph of \(Q_n\) induced by \(E\), and suppose \(G\) contains no \(Q_{m+1}\). Since there are \(\binom{n}{m+1}2^{n-m-1}\) cubes of dimension \(m+1\) in \(Q_n\), \(G\) can contain at most \((2^{m+1}-6)\binom{n}{m+1}2^{n-m-1}\) of the \((m+1)\)-stars. Let \(x_i\) denote the number of nodes of \(G\) of degree \(i\), for \(1 \leq i \leq n\), then \(|E| = \frac{1}{2} \sum_{i=1}^{n} ix_i\). We must have

\[
\sum_{k=m+1}^{n} \binom{k}{m+1} x_k \leq (2^{m+1}-6) \binom{n}{m+1} 2^{n-m-1}.
\]

Now, let \(M(z) = M(z_1, ..., z_n) = \frac{1}{2} \sum_{i=1}^{n} iz_i\), and consider the problem of maximizing \(M\) subject to the following three constraints:

\[
\begin{align*}
C1: & \quad z_i \text{ is an integer in } [0, 2^n] \text{ for } 1 \leq i \leq n, \\
C2: & \quad \sum_{i=1}^{n} z_i \leq 2^n, \\
C3: & \quad \sum_{i=m+1}^{n} \binom{n}{m+1} z_i \leq (2^{m+1}-6)\binom{n}{m+1} 2^{n-m-1}.
\end{align*}
\]

Note that if \(y = (y_1, ..., y_n)\) satisfies these constraints and if, say, \(y_i, y_{i+1}, \text{ and } y_{i+2}\) are all non-zero, then the \(n\)-tuple \(y' = (y'_1, ..., y'_n)\) also satisfies these constraints, where \(y'_i = y_i - 1, y'_{i+1} = y_{i+1} + 2, y'_{i+2} = y_{i+2} - 1\), and \(y'_j = y_j\) otherwise. Moreover, \(M(y') = M(y)\). In view of this property, if \(x = (x_1, ..., x_n)\) yields a maximum value for \(M\) subject to these constraints, then we may assume without loss of generality that all but at most two of the \(x_i\)'s are 0, and that these two are consecutive. We see that the integer \(k\) given in the statement of the theorem is the integer for which \(x_k \neq 0\), and \(x_i = 0\) for all \(i \neq k, k+1\). The theorem now follows from the simpler problem of maximizing \(M = (kx_k + (k+1)x_{k+1})/2\) subject to the modified constraints:

\[
\begin{align*}
C1': & \quad x_k \text{ and } x_{k+1} \text{ are non-negative integers,} \\
C2': & \quad x_k + x_{k+1} = 2^n, \\
C3': & \quad \binom{k}{m+1} x_k + \binom{k+1}{m+1} x_{k+1} \leq (2^{m+1}-6)\binom{n}{m+1} 2^{n-m-1}.
\end{align*}
\]
Corollary 21.1. For $n \geq m \geq 1$,

$$\lambda(n, m) \geq (n-k)2^{n-1} - b/2,$$

where $k$ is the integer determined by

$$2^{m+1} \binom{k}{m+1} \leq (2^{m} + 6) \binom{n}{m+1} < 2^{m+1} \binom{k+1}{m+1}$$

and

$$b = \left[ \frac{(2^m + 6) \binom{n}{m+1} 2^{n-m} - (2^m + 1) 2^n}{\binom{k}{m}} \right].$$

4. Asymptotics

What is known about the behavior of $\kappa$ and $\lambda$ for large values of $n$ and $m$ falls roughly into two categories: results for $n-m$ fixed and those for $m$ fixed. The successful techniques for studying the case $n-m$ fixed are quite different from those that succeed in the case of $m$ fixed. Moreover, the bounds obtained for fixed $n-m$ are not useful for fixed $m$, and conversely. In this section we describe the best known bounds for each of these cases and mention several open problems concerning the relative sizes of $\kappa$ and $\lambda$.

Kleitman and Spencer [26] used probabilistic methods to determine bounds for the maximum size of families of $k$-independent sets. Chandra et al. [7] used a probabilistic argument equivalent to that in [26] to prove the following bound on $\kappa$. Becker and Simon [3] rediscovered this result, and used similar arguments to establish an upper bound for $\lambda$. These bounds are stated in the following.

Theorem 22. [3, 7]. For all $n \geq m \geq 1$,

$$\kappa(n, m) \leq (\ln 2)(n-m) 2^{n-m} \lg n$$

$$\lambda(n, m) \leq (\ln 2)(n-m) 2^{n-m} (n/m) \lg n.$$ 

Combining these bounds with those given by Theorems 10 and 20, we see that both $\kappa(n, m)$ and $\lambda(n, m)$ are $\Theta(\log n)$ for $n-m$ fixed, but there are significant gaps between these bounds.

Question. For fixed $n-m$, does the limit $\lim_{n \to \infty} \kappa(n, n-m)/\lg n$ exist?

This limit exists for $n-m=2$ by the result on 2-independent sets as does the corresponding limit with $\lambda$ in place of $\kappa$. Another question suggested by the slowly increasing nature of $\kappa$ along the diagonals is:
Question. For fixed $n-m$, is it true that $\kappa(n+1, m+1) - \kappa(n, m) \leq 1$ for all sufficiently large $n$?

The same question could, of course, be asked for $\lambda$ as well.

When $m$ is fixed, and $m$ and $n$ are large, Theorem 10 and Corollary 4.1 combine to show that $\kappa(n, m) = \Theta(2^n)$. Analogously, Theorem 20 and Corollary 17.1 show that $\kappa(n, m) = \Theta(n2^n)$, but here, too, there are significant gaps between the upper and lower bounds for both $\kappa$ and $\lambda$. Let

$$x_m = \lim_{n \to \infty} \frac{\kappa(n, m)}{2^n}.$$

We see that $x_m$ exists for all $m \geq 0$, for Theorem 2 implies that $\kappa(n, m)/2^n$ is non-decreasing for fixed $m$. Moreover, the inequalities $2x_{m+1} \geq x_m \geq x_{m+1}$ follow from this same theorem. The definition of $x_m$ and the results of Theorems 1 and 5 show that

$$x_0 = 1, \quad x_1 = \frac{1}{2}, \quad x_2 = \frac{1}{3},$$

and that $x_m$ satisfies the inequality

$$\frac{\kappa(n, m)}{2^n} \leq x_m \leq \frac{1}{m+1}$$

for every $n$. It would be interesting to know the exact values of $x_m$ for $m \geq 3$.

In the case of edges, there are many analogies to the above. Let

$$\beta_m = \lim_{n \to \infty} \lambda(n, m)/(n2^{n-1}).$$

The fact that $\beta_m$ exists for all $m \geq 1$ is a consequence of part (i) of Theorem 15 which shows that the sequence $\lambda(n, m)/n2^{n-1}$ is nondecreasing for fixed $m$. Theorem 14 shows that $\beta_1 = 1$, and Corollary 17.1 together with the definition of $\beta_m$ shows that

$$\frac{\lambda(n, m)}{n2^{n-1}} \leq \beta_m \leq \frac{1}{m}$$

for $m \geq 1$. Using Theorem 21, we find $\beta_2 \geq 0.37$ and $\beta_3 \geq 0.112$. Table 2 gives $\lambda(7, 4) \geq 19$, which yields $\beta_4 \geq 0.042$. The bound $\beta_5 \geq 0.016$ is from Theorem 21. Stronger lower bound results and extensions of the tables of values of $\kappa$ and $\lambda$ would be of considerable interest as they can improve our asymptotic estimates as well as yield more information about $x_m$ and $\beta_m$.

Considering the relative sizes of $\kappa$ and $\lambda$, we see that $\lambda(n, m)/\kappa(n, m)$ is $\Theta(n)$ when $m$ is fixed. Another question concerning the behavior of these functions along the diagonal is:
**Question.** If $n - m$ is constant, is it true that $\lambda(n, m)/\kappa(n, m)$ is $\Theta(1)$?

Along the same lines, we have seen that, for $n - m \leq 2$, the difference $\lambda(n, m) - \kappa(n, m)$ is bounded, which prompts us to ask the following:

**Question.** For $n - m$ fixed, is it true that $\lambda(n, m) - \kappa(n, m) = O(1)$?

## 5. Exact Values

In application to hypercubes, the behavior of $\kappa$ and $\lambda$ for relatively small values of $n$ and $m$ is more important than their asymptotic values. We must keep in mind that $n$ represents the dimension of the hypercube and so values of $n \geq 50$, say, represent a hypercube with more than a quadrillion processors! Consequently, in most applications, the exact values and constructive bounds that yield good approximations for $n < 50$ are most useful.

Values of $\kappa(n, m)$ for $0 \leq m \leq n \leq 10$ are represented in Table 1, where exact values are given if known, and otherwise lower and upper bounds are given in the form lower-upper. The exact values for $m = 0, 1, n - 1$, and $n$ follow from Theorem 1, the values of $\kappa(n, n - 2)$ are from Theorem 9, and the $\kappa(n, 2)$ values are obtained from Theorem 5. A computer program using a greedy heuristic was developed to construct small sets $S$ in $\mathcal{F}(n, m)$. To a partially constructed set $S$, the program randomly adds a node to $S$ that is in the largest number of remaining fault-free $m$-cubes. This program found sets that resulted in the upper bounds for $\kappa(9, 4)$, $\kappa(10, 5)$, $\kappa(10, 6)$, and $\kappa(10, 7)$ shown in the table. In the remaining cases, the upper and

<table>
<thead>
<tr>
<th>$n$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$9$</th>
<th>$10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>8</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>16</td>
<td>10</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>32</td>
<td>21</td>
<td>12</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>128</td>
<td>64</td>
<td>42</td>
<td>24</td>
<td>12</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>256</td>
<td>128</td>
<td>85</td>
<td>48-56</td>
<td>24</td>
<td>12</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>512</td>
<td>256</td>
<td>170</td>
<td>96-120</td>
<td>48-64</td>
<td>24</td>
<td>12</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1024</td>
<td>512</td>
<td>341</td>
<td>192-240</td>
<td>96-165</td>
<td>48-68</td>
<td>24-25</td>
<td>12-13</td>
<td>6</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
lower bounds for \( \kappa(n, m) \) are determined from part (ii) of Theorem 2 and Theorem 4.

Much less has been determined about the exact values of \( \lambda(n, m) \). Table 2 displays values of \( \lambda(n, m) \) for \( 1 \leq m \leq n \leq 7 \), showing lower and upper bounds when exact values are not known. The values for \( m = 1, n - 1 \), and \( n \) follow from Theorem 14. All remaining lower bounds can be obtained from Theorems 21.1 and 15. Upper bounds for \( \lambda(n, 2) \) are from part (ii) of Theorem 15, while the remaining upper bounds in the table were found by construction. A computer program analogous to the one for \( \kappa \) was designed to construct small sets in \( \mathcal{S}(n, m) \). A separate program was developed to determine that \( \lambda(7, 5) = 7 \).

(A copy of the sets constructed for \( \kappa \) and \( \lambda \) may be obtained by writing to Quentin F. Stout.)

It would be very useful to extend the table of values of \( \kappa \) and \( \lambda \) both for practical instances and because it would, in turn, yield improvements in known bounds for \( \kappa(n, m) \) and \( \lambda(n, m) \) not included in the table. However, finding small sets in \( \mathcal{S}(n, m) \) and \( \mathcal{F}(n, m) \) is computationally very difficult.

6. CONSTRUCTIONS

The construction of fault sets that are of nearly minimum size is of interest to saboteurs, to computer architects solving resource allocation problems such as those described in [28], and to persons needing to construct \( k \)-independent sets for testing purposes [27]. Unfortunately, finding such sets is a very difficult problem in general. Arguments in [26, 7] show that nondeterministic methods have a high probability of success for \( n \) large and \( n - m \) fixed. Probabilistic arguments similar to those in [26] were used in [3, 7] to prove that, with high probability, a randomly
chosen set of \((\ln 2)(n - m) 2^{n - m} \log n\) nodes of \(Q_n\) is in \(\mathcal{S}(n, m)\). An analogous argument shows that, with high probability, a randomly chosen set of \((\ln 2)(n - m) 2^{n - m} (n/m) \log n\) edges of \(Q_n\) is in \(\mathcal{I}(n, m)\) [3].

Levithin and Karpovsky [27] developed constructive methods for a problem equivalent to the study of \(\kappa(n, m)\). The problem involves the exhaustive testing of devices with \(n\) inputs where each output is a Boolean function of at most \(k\) binary input variables. Using MDS codes, they constructed an \(r \times n\) binary matrix such that all \(2^k\) possible binary \(k\)-vectors appear in each of the \(k\) columns, where \(r = O(\log^w n)\) and \(w\) can be chosen arbitrarily close to 1. Their results give a construction of a set in \(\mathcal{S}(n, m)\) of size \(O(\log^w n)\), where \(w\) can arbitrarily close to 1. Alon [1] has given a construction of a family of \(k\)-independent subsets of a set of size \(r\). Through the correspondence between independent sets and elements of \(\mathcal{S}(n, m)\), shown in the proof of Theorem 8, this yields a construction of a set in \(\mathcal{S}(n, m)\) of size \(O((n - m)^{cn - m^2} \log n)\) for some constant \(c\) about 24. For fixed \(n - m\), this size is the same order of magnitude as a minimum set in \(\mathcal{S}(n, m)\), but even for \(n - m = 3\), say, it is more than \(3^{216} \log n\).

As discussed in Section 2.4, Becker and Simon [3] used results of Friedman [12] to construct sets in \(\mathcal{S}(n, n - k)\) of size at most \(\log n (k^2/\log k) 2^{2k} 2^{3k} \log k + 3k\). When \(n - m = k\) is fixed, this has the right order of growth, but for small values such as \(n - m = 3\) and \(n = 20\), say, this bound is more than \(2^{20} \log 20\). On the other hand, the construction in Theorem 13 yields a set in \(\mathcal{I}(20, 17)\) of size 19. Even the construction using level sets, Theorem 4, yields a set of size 40 in this case.

When \(m\) is fixed, the constructions in [12] and [3] give sets whose sizes are far from the same order of magnitude as \(\kappa(n, m)\). In this case the best constructions for near minimum fault sets are given by the level sets in Theorems 4 and 17. It would certainly be of interest to find constructions of sets in \(\mathcal{I}(n, m)\) of size \(\sim \kappa(n, m)\) and corresponding sets in \(\mathcal{I}(n, m)\) of size \(\sim \lambda(n, m)\) for \(m\) fixed.

To construct small fault sets for practical sizes of \(n\) and \(m\), the best strategy is to employ the constructive methods that led to the recursive inequalities of Sections 2 and 3 coupled with the computational results that led to Tables 1 and 2.

7. CONCLUDING REMARKS

Our analysis of subcube fault-tolerance assumes that it is sufficient to find an arbitrary fault-free \(m\)-dimensional subcube. However, the problem of determining a fault-free subcube of a given dimension is computationally intensive, so in practice the allocation routines examine the availability of only a certain subset of the subcubes of a given dimension. Most allocation
schemes use some variant of the "buddy system" allocating only m-cubes of the form $a_1 \cdots a_{n-m} \cdots \cdots \cdots$ [32].

Under a given allocation scheme $\mathcal{A}$, let $\mathcal{A}Q_n$ denote the set of all subcubes of $Q_n$ that are recognized by $\mathcal{A}$. A natural extension of $\kappa(n, m)$ is to $\kappa(\mathcal{A}; n, m)$, which we define as the least number of nodes that need to be removed from $Q_n$ so that the resulting graph contains no m-cube in $\mathcal{A}Q_n$. We define $\lambda(\mathcal{A}; n, m)$ in an analogous way. As an example, if $\mathcal{B}$ denotes the buddy allocation scheme then $\mathcal{B}Q_n = \{a_1 \cdots a_r \cdots \cdots \cdots | r = 0, ..., n\}$, and it is easy to check that $\kappa(\mathcal{B}; n, m) = \lambda(\mathcal{B}; n, m) = 2^n - m$ for $n \geq m \geq 1$. While the buddy system is the only allocation scheme used on hypercube computers thus far, we see it is not particularly fault-tolerant. For some specific allocation schemes of interest, Livingston and Stout [29] determined $\kappa(\mathcal{A}; n, m)$.

For arbitrary allocation scheme $\mathcal{A}$, Becker and Simon [3] showed that the problem of determining $\kappa(\mathcal{A}; n, n-2)$ is equivalent to a graph-coloring problem. The general problem of determining $\kappa(\mathcal{A}; n, m)$ and $\lambda(\mathcal{A}; n, m)$ is open.

The fault-intolerance questions considered here can be generalized to arbitrary architectures and arbitrary graph properties. That is, given a graph $G$ which represents the connectivity of the processors, how tolerant is $G$ to retaining some specific graph property $P$ under the removal of successive copies of a subgraph $H$? Here, we define the quantity $\kappa(P, H; G)$ as the minimum number of copies of $H$ whose deletion from $G$ leaves the resulting graph without property $P$. For example, suppose $G$ is $Q_n$, $P$ is the property of being connected, and $H$ is a single edge; then $\kappa(P, H; Q_n) = n$. If $G$ is $Q_n$, $P$ is the property of containing an $m$-cube, and $H$ is a single $m$-cube, then $\kappa(Q_m, Q_m; Q_n)$ is the mispacking number mispack$(Q_m, Q_n)$ discussed in [13]. As a final example along these lines, consider the problem, described in [25], due to Yuzvinski: How many nodes of the $n$-cube must be removed in order that no connected component of the rest contains an antipodal pair of nodes? Kleitman [25] solved this problem by establishing the more general result that at least $\binom{n}{\lceil n/2 \rceil}$ nodes must be removed from $Q_n$ if no connected component of the remaining graph is to contain more than $2^{n-1}$ nodes.

In the generalized problem considered above, asking for the minimum number of copies of $H$ whose removal from $G$ destroys $P$ is appropriate in an adversarial situation, in certain resource allocation problems [28], in designing efficient test [27], or in constructing $k$-independent sets [26]. However, suppose each copy of $H$ to be removed is selected uniformly and at random from the set of all copies of $H$ in $G$. A natural question that arises is What is the expected number of copies of $H$ that must be removed from $G$ so that the resulting graph fails to have property $P$? Consider, for example, the case in which $G$ is $Q_n$, $H$ is a single node, and $P$ is the property of containing an $(n-1)$-cube. In contrast to $\kappa(n, n-1) = 2$ we
find that its expected value, denoted \( \kappa_E(n, n-1) \), is \( \Theta(\log n) \). Some of the properties of \( \kappa_E(n, m) \) and and \( \lambda_E(n, m) \) for arbitrary \( n \) and \( m \), and of \( \kappa_E(\mathcal{A}; n, m) \) and \( \lambda_E(\mathcal{A}; n, m) \) for certain allocation schemes \( \mathcal{A} \), are studied in [29, 30]. A related but somewhat different situation arises if we are only concerned that, with high probability, \( G \) fails to have property \( P \). What is the expected number of copies of \( H \) that must be removed in this case? Becker and Simon [3] considered an instance of this question in which \( G \) is \( Q_n \), \( H \) is a single node, and \( P \) denotes the property of containing an \( m \)-cube. They showed that if at least \( (n - m) 2^{n-m} \lg n \) nodes are removed from \( Q_n \), the probability that there are no remaining \( m \)-cubes approaches 1 as \( n \) tends to infinity.

A variation of these questions appears in the work of Burton [2], and Erdős and Spencer [11]. Using a probabilistic model of \( Q_n \) in which each edge is deleted independently and with fixed probability \( p \), they showed that if \( P_1(Q_n, p) \) denotes the probability that the resulting subgraph of \( Q_n \) is connected then

\[
\lim_{n \to \infty} P_1(Q_n, p) = \\
\begin{cases} 
1 & \text{if } p < 1/2; \\
1/e & \text{if } p = 1/2; \\
0 & \text{otherwise.}
\end{cases}
\]

When \( p \) is allowed to vary with \( n \), Bollobás [4, 5] proved that if \( \mu > 0 \) and

\[
p = p(n) = 1 - \frac{1}{2} \{ \mu + o(1) \}^{1/n}
\]

then

\[
\lim_{n \to \infty} P_1(Q_n, p) = e^{-\mu}.
\]

Suppose that instead of deleting edges from \( Q_n \) we delete nodes, together with their incident edges, with fixed probability \( p \) and define \( P_0(Q_n, p) \) as the probability that the resulting subgraph of \( Q_n \) is connected. Najjar and Gaudiot [31], investing the reliability of the hypercube network in the presence of node faults, used Monte Carlo simulation to estimate \( P_0(Q_n, p) \) for small \( n \). In [30], and analog of the above results for \( P_1(Q_n, p) \) is proved for \( P_0(Q_n, p) \), namely

\[
\lim_{n \to \infty} P_0(Q_n, p) = \\
\begin{cases} 
1 & \text{if } p < 1/2; \\
1/2 & \text{if } p = 1/2; \\
0 & \text{otherwise.}
\end{cases}
\]
ACKNOWLEDGMENTS

We thank the referees for their useful and incisive comments, and Karen Johnson and Fan Chung for providing comments and access to their unpublished work.

RECEIVED August 1, 1988; FINAL MANUSCRIPT RECEIVED February 28, 1991

REFERENCES


