

## Plasma hydrodynamics in view of quasilinear effects

K. V. Gamayunov,<sup>1</sup> G. V. Khazanov,<sup>2</sup> E. N. Krivorutsky,<sup>1</sup> T. I. Gombosi<sup>2</sup> and V. N. Oraevski<sup>3</sup>

<sup>1</sup>Altai State University, 656099 Barnaul, Russia

<sup>2</sup>Space Physics Research Laboratory, University of Michigan, Ann Arbor, MI 48104, U.S.A.

<sup>3</sup>IZMIRAN, 142092 Troitsk, Moscow Region, Russia

Received in final form 13 August 1992

**Abstract.** In this paper we have obtained expressions for the moments of the quasilinear integral of anisotropic plasma collisions. Resonant and adiabatic interactions of particles with an arbitrary wave mode are considered. Expressions for the moments describing the interactions of ion-cyclotron waves with ring current and magnetospheric helium ions have been derived as an example using obtained results.

### 1. Introduction

The hydrodynamical approach is widely used when describing the behavior of both cosmic and artificial plasmas. This approach makes it possible to analyze the behavior of multicomponent systems. In previous work, only interparticle collisions were considered (Oraevski *et al.*, 1985).

Early treatments which only accounted for interparticle collisions had a fairly restricted sphere of applicability, because they did not consider wave-particle interactions which are particularly important in the ring current region of the Earth's magnetosphere. The ring current's interaction with the plasmasphere leads to intensive generation of ion-cyclotron waves (Cornwall, 1964; Kennel and Petschek, 1966). These waves were used to explain the effects of precipitation of ring current protons into the Earth's atmosphere (Cornwall *et al.*, 1970), excitation of SAR arcs (Cornwall *et al.*, 1971), formation of the "hot zone" of the plasmasphere (Galeev, 1975) and many other phenomena.

The wave-particle interaction may be included if we add the corresponding moments of the quasilinear collision integral (Akhiezer, 1974) to the hydrodynamic equations. Similar attempts were made in recent articles (Konikov *et al.*, 1989; Gamayunov *et al.*, 1991; Gorbachev *et al.*, 1992). The authors of these papers examined the interaction of the ion-cyclotron waves generated by hot anisotropic protons of the ring current with electrons and protons of the background plasma. These results cannot be

employed to describe the interactions of other types of observed waves which interact with a multicomponent plasma.

In this paper, the system of hydrodynamic equations governing the behavior of a multicomponent anisotropic plasma will be supplemented by terms which take into account the quasilinear interactions of particles with arbitrary waves. Resonant quasilinear interaction of waves with particles (Konikov *et al.*, 1989; Gamayunov *et al.*, 1991) may be insufficient to describe a system. For example, both resonant and adiabatic interactions must be used to describe the interaction of Langmuir waves with electrons. Therefore, in the following, when obtaining the moments of the quasilinear collision integral, we will consider both resonant and adiabatic interactions.

### 2. Quasilinear interaction in hydrodynamics: general relationships

Let us consider the case of a weakly turbulent plasma. The expression for the quasilinear integral which considers both resonant and adiabatic interaction was obtained previously (Belikov *et al.*, 1974). Since it will subsequently be used in a somewhat different form, we will outline its calculation. The following equations for the regular ( $f^R$ ) and turbulent ( $f^{(1)}$ ) components of the distribution function are obtainable using a standard procedure (for more detail see Tsytovich, 1971):

$$\begin{aligned} \frac{\partial f_a^R}{\partial t} + \mathbf{v} \frac{\partial f_a^R}{\partial \mathbf{r}} + \frac{1}{m_a} \left[ \mathbf{F}_a + \frac{e_a}{c} (\mathbf{vB}_0) \right] \frac{\partial f_a^R}{\partial \mathbf{v}} \\ = - \frac{e_a}{m_a} \left[ \mathbf{E} + \frac{1}{c} (\mathbf{vB}) \right] \frac{\partial f_a^{(1)}}{\partial \mathbf{v}}, \quad (1) \end{aligned}$$

$$\begin{aligned} \frac{\partial f_x^{(1)}}{\partial t} + \mathbf{v} \frac{\partial f_x^{(1)}}{\partial \mathbf{r}} + \frac{1}{m_x} \left[ \mathbf{F}_x + \frac{e_x}{c} (\mathbf{vB}_0) \right] \frac{\partial f_x^{(1)}}{\partial \mathbf{v}} \\ = - \frac{e_x}{m_x} \left[ \mathbf{E} + \frac{1}{c} (\mathbf{vB}) \right] \frac{\partial f_x^R}{\partial \mathbf{v}}, \quad (2) \end{aligned}$$

where  $\mathbf{F}_x$  is the regular component of force and, for

brevity, the integrals of interparticle collisions are omitted on the right-hand side of equation (1). The regular force may include the gravity and electrical terms. We wrote the magnetic term in evident form. In the further description, the regular force is considered as constant on the time-space scale of the turbulent quantities.

Then, if we use an accompanying coordinate system moving with the mean velocity of particles of the type  $\alpha$  in considering a point in time and space

$$\left[ \mathbf{u}(\mathbf{r}, t) = \frac{1}{N} \int \mathbf{v} f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}, N = \int f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v} \right],$$

then equation (2) takes the form (we will omit index  $\alpha$ ):

$$\begin{aligned} \frac{\partial f^{(1)}(t', \mathbf{r}', \mathbf{v}')}{\partial t'} + \mathbf{v}' \frac{\partial f^{(1)}}{\partial \mathbf{r}'} + \frac{e}{m'c} [\mathbf{v}' \mathbf{B}_0] \frac{\partial f^{(1)}}{\partial \mathbf{v}'} \\ = - \frac{e}{m'} \left( \mathbf{E}' + \frac{1}{c} [\mathbf{v}' \mathbf{B}'] \right) \frac{\partial f^R}{\partial \mathbf{v}'}. \end{aligned} \quad (3)$$

The primes indicate that all quantities are taken in a coordinate system moving with the plasma. If one neglects the background plasma's inhomogeneity and nonstationarity, equation (3) has the following solution (axis  $z \parallel \mathbf{B}_0$ ) (Akhiezer, 1974):

$$\begin{aligned} f_k^{(1)} = \exp \left( i \frac{k'_\perp v'_\perp}{\omega'_B} \sin(\varphi' - \chi') \right) \\ \times \sum_n \frac{\exp(-in\varphi')}{2\pi i (\omega' - k'_\perp v'_\perp - n\omega'_B)} \\ \times \int_0^{2\pi} d\varphi'' \exp \left( -i \frac{k'_\perp v'_\perp}{\omega'_B} \sin(\varphi'' - \chi') \right) \\ \times \exp(in\varphi'') \frac{e}{m'} \left[ \mathbf{E}'_k \left( 1 - \frac{\mathbf{k}' \mathbf{v}'}{\omega'} \right) + \frac{\mathbf{k}' (\mathbf{v}' \mathbf{E}'_k)}{\omega'} \right] \frac{\partial f^R}{\partial \mathbf{v}'}. \end{aligned} \quad (4)$$

where  $\mathbf{k}$  is the wavevector,  $\cos \chi' = k'_\perp / k'_\perp$ ,  $\cos \varphi' = v'_\perp / v'_\perp$ ,  $\omega'_B = eB'_0 / m'c$ .

Assuming that the mean velocities  $\mathbf{u}$  ( $|\mathbf{u}| \ll c$ ) on small scales are coordinate- and time-independent, the transformations (see e.g. Alexandrov *et al.*, 1988):

$$\begin{aligned} \omega' = \omega - \mathbf{k} \mathbf{u}, \quad \mathbf{k}' = \mathbf{k}, \quad \mathbf{E}'_k = \beta_{ij}(\mathbf{u}) \mathbf{E}_{k_i}, \\ \beta_{ij}(\mathbf{u}) = \frac{\omega - \mathbf{k} \mathbf{u}}{\omega} \delta_{ij} + \frac{\mathbf{k} \mathbf{u}_i}{\omega}, \end{aligned} \quad (5)$$

where  $\delta_{ij}$  is the Kronecker delta, permit equation (4) to be rewritten in a laboratory coordinate system:

$$\begin{aligned} f_k^{(1)} = \hat{\mathbf{L}} \exp \left( i \frac{k_\perp v'_\perp}{\omega_B} \sin(\varphi' - \chi) \right) \\ \times \sum_n \frac{\exp(in\varphi')}{2\pi i (\omega - \mathbf{k} \mathbf{u} - k v'_\perp - n\omega_B)} \\ \times \int_0^{2\pi} d\varphi'' \exp \left( -i \frac{k_\perp v'_\perp}{\omega_B} \sin(\varphi'' - \chi) \right) \\ \times \exp(in\varphi'') \frac{e}{m} \left[ - \frac{\mathbf{E}_k(\mathbf{k} \mathbf{u}) - \mathbf{k}(\mathbf{u} \mathbf{E}_k)}{\omega} \right] \end{aligned}$$

$$+ \mathbf{E}_k \left( 1 - \frac{\mathbf{k} \mathbf{v}'}{\omega} \right) + \frac{\mathbf{k} (\mathbf{v}' \mathbf{E}_k)}{\omega} \right] \frac{\partial f^R}{\partial \mathbf{v}'}. \quad (6)$$

Here  $\hat{\mathbf{L}} g(\mathbf{v}') \equiv g(\mathbf{v} - \mathbf{u})$ .

On substituting equation (6) into the right-hand side of equation (1), multiplying by  $mh(\mathbf{v} - \mathbf{u})$  and integrating over velocity, we obtain the following expression for the moments of the quasilinear integral:

$$\begin{aligned} M = \frac{e^2}{m} \int d\mathbf{k} I_k^\sigma(\mathbf{r}, t) \int d\mathbf{v} \left\{ \mathbf{d}_k^{\sigma*} - \frac{\mathbf{e}_k^{\sigma*}(\mathbf{k} \mathbf{v}) - \mathbf{k}(\mathbf{v} \mathbf{e}_k^{\sigma*})}{\omega_k^{\sigma*}} \right\} \frac{\partial h(\mathbf{v})}{\partial \mathbf{v}} \\ \times \sum_n \frac{\exp(-in\varphi)}{2\pi i (\omega_k^\sigma - \mathbf{k} \mathbf{u} - k_\perp v_\perp - n\omega_B)} \\ \times \int_0^{2\pi} d\varphi' \exp \left( -i \frac{k_\perp v'_\perp}{\omega_B} \sin(\varphi' - \chi) \right) \\ \times \exp(in\varphi') \left[ \mathbf{d}_k^\sigma - \frac{\mathbf{e}_k^\sigma(\mathbf{k} \mathbf{v}) - \mathbf{k}(\mathbf{v} \mathbf{e}_k^\sigma)}{\omega_k^\sigma} \right] \frac{\partial f^R}{\partial \mathbf{v}}, \end{aligned} \quad (7)$$

where primes were omitted for  $\mathbf{v}^\perp$  and the asterisk refers to complex conjugation.

We make the following definitions:

$$\mathbf{d}_k^\sigma = \frac{\mathbf{e}_k^\sigma (\omega_k^\sigma - \mathbf{k} \mathbf{u}) + \mathbf{k} (\mathbf{u} \mathbf{e}_k^\sigma)}{\omega_k^\sigma}; \quad I_k^\sigma(\mathbf{r}, t) = |E_k^\sigma(\mathbf{r}, t)|^2,$$

where  $\mathbf{e}_k^\sigma$  is a polarization vector of the wave oscillations of type  $\sigma$  and  $f^R$  refers to a distribution function in the reference frame moving with velocity  $\mathbf{u}$ .

By making the following definitions [cf. equation (7)],

$$\begin{aligned} \sigma \begin{bmatrix} ijk \\ ij \\ i \\ 0 \end{bmatrix} = \frac{e^2}{m} \int d\mathbf{v} \exp \left( \frac{i}{v_\perp} (a_x v_y - a_y v_x) \right) \begin{bmatrix} v_x v_y \\ v_x \\ v_y \\ 1 \end{bmatrix} \\ \times \sum_n \frac{\exp(-in\varphi)}{2\pi i (\omega_k^\sigma - \mathbf{k} \mathbf{u} - k_\perp v_\perp - n\omega_B)} \\ \times \int_0^{2\pi} d\varphi' \exp \left( -i \frac{k_\perp v'_\perp}{\omega_B} \sin(\varphi' - \chi) + in\varphi' \right) \\ \times \left[ \mathbf{d}_k^\sigma - \frac{\mathbf{e}_k^\sigma(\mathbf{k} \mathbf{v}) - \mathbf{k}(\mathbf{v} \mathbf{e}_k^\sigma)}{\omega_k^\sigma} \right] \frac{\partial f^R}{\partial \mathbf{v}}; \\ v_{\perp i}^2 = \frac{T_\perp}{m}; \quad (i, j, k) = x, y, z \equiv 1, 2, 3; \end{aligned} \quad (8)$$

one may rewrite equation (7) in the form used in the hydrodynamical description (Orszag *et al.*, 1985), i.e.  $h = v_x, v_x v_y, v_x v_y v_z$ , etc.:

$$J_\alpha = \int d\mathbf{k} I_k^\sigma \left( d_{k\alpha}^{\sigma*} \sigma_0 - \frac{e_{k\alpha}^{\sigma*} k_i - k_\alpha e_{k_i}^{\sigma*}}{\omega_k^{\sigma*}} \sigma_i \right) \Big|_{a=k_\perp v_\perp / \omega_B}, \quad (9)$$

$$\begin{aligned} J_{\alpha\beta} = \int d\mathbf{k} I_k^\sigma \left[ \left( d_{k\alpha}^{\sigma*} \sigma_\beta - \frac{e_{k\alpha}^{\sigma*} k_i - k_\alpha e_{k_i}^{\sigma*}}{\omega_k^{\sigma*}} \sigma_{i\beta} \right) \right. \\ \left. + (\beta, \alpha) \right] \Big|_{a=k_\perp v_\perp / \omega_B}, \end{aligned} \quad (10)$$

$$J_{\alpha\beta\gamma} = \int dk I_k^\sigma \left[ \left( d_{k\alpha}^{\sigma*} \sigma_{\beta\gamma} - \frac{e_{k\alpha}^{\sigma*} k_i - k_\alpha e_{ki}^{\sigma*}}{\omega_k^{\sigma*}} \sigma_{i\beta\gamma} \right) + [\gamma, \alpha, \beta] + [\beta, \gamma, \alpha] \right] \Big|_{a=k_\perp v_\perp / \omega_B} \quad (11)$$

It is obvious that during the quasilinear interaction the number of particles is unchanged, i.e. for the  $h = 1$  moment,  $J_0 = 0$ . Here, the meaning of  $\alpha, \beta, \gamma$  are any  $x, y, z$ . The notations  $[\beta, \alpha], [\gamma, \alpha, \beta]$ , etc. mean that the first term must be copied with substitution  $\alpha \rightarrow \beta, \beta \rightarrow \alpha$  in equation (10);  $\alpha \rightarrow \gamma, \beta \rightarrow \alpha, \gamma \rightarrow \beta$  in the second term and  $\alpha \rightarrow \beta, \beta \rightarrow \gamma, \gamma \rightarrow \alpha$  in the third term in equation (11).

For receiving equations (9)–(11), different functions  $h$  ( $h = v_x, v_z v_\beta, v_x v_\beta v_\gamma$ , etc.) are replaced into equation (7). Then, the derivative  $\partial h / \partial v_i$  is multiplied by the bracket  $\{d_k^{\sigma*} - [e_{k\alpha}^{\sigma*}(\mathbf{k}\mathbf{v}) - \mathbf{k}(\mathbf{v}e_{k\alpha}^{\sigma*})] / \omega_k^{\sigma*}\}$ , and the definitions (8) are used. We introduced parameter  $a$  which is equal to  $k_x v_\perp / \omega_B, k_y v_\perp / \omega_B$  in the first exponential multiplier (8) in order to distinguish  $k_x, k_y$  in this term from the others. It will be used later for determining the connection between  $\sigma_{\alpha\beta}$  and  $\sigma_x, \sigma_{\alpha\beta\gamma}$  and  $\sigma_{\alpha\beta}$ , etc. [expressions (13)–(16)].

Equations (9)–(11) can be reduced to a simpler form if the relationships of continuity for the moments following from equation (3) are used. By introducing:

$$P \begin{bmatrix} v_y \\ v_x \\ 1 \end{bmatrix} = -\frac{e^2}{mk_\parallel 2\pi i} \int dv \exp\left(\frac{i}{v_\perp} (a_x v_y - a_y v_x)\right) \begin{bmatrix} v_y \\ v_x \\ 1 \end{bmatrix} \times \sum_n \exp(-in\varphi) \int_0^{2\pi} d\varphi' \exp\left(-i\frac{k_\perp v_\perp}{\omega_B} \sin(\varphi' - \chi)\right) \times \exp(in\varphi') \left\{ d_k^\sigma - \frac{e_k^\sigma(\mathbf{k}\mathbf{v}) - \mathbf{k}(\mathbf{v}e_k^\sigma)}{\omega_k^\sigma} \right\} \frac{\partial f^R}{\partial \mathbf{v}}, \quad (12)$$

we get:

$$J_x = \int dk I_k^\sigma \left\{ \frac{k_x}{\omega_k^{\sigma*}} [(\mathbf{u}e_k^{\sigma*}) \sigma_0 + e_{ki}^{\sigma*} \sigma_i] + e_{kx}^{\sigma*} \frac{\omega_k^{\sigma*} - \omega_k^\sigma}{\omega_k^{\sigma*}} \sigma_0 \right\} \Big|_{a=k_\perp v_\perp / \omega_B}, \quad (9')$$

$$J_{x\beta} = \int dk I_k^\sigma \left\{ \left[ \frac{k_x}{\omega_k^{\sigma*}} [(\mathbf{u}e_k^{\sigma*}) \sigma_\beta + e_{k\beta}^{\sigma*} \sigma_{i\beta}] + e_{kx}^{\sigma*} \frac{\omega_k^{\sigma*} - \omega_k^\sigma}{\omega_k^{\sigma*}} \sigma_\beta - \frac{k_\perp}{\omega_k^{\sigma*}} e_{kx}^{\sigma*} P_\beta - i \frac{\omega_B}{\omega_k^{\sigma*}} e_{kx}^{\sigma*} (\delta_{\beta 2} \sigma_1 - \delta_{\beta 1} \sigma_2) \right] + [\beta, \alpha] \right\} \Big|_{a=k_\perp v_\perp / \omega_B}, \quad (10')$$

$$J_{x\beta\gamma} = \int dk I_k^\sigma \left\{ \left[ \frac{k_x}{\omega_k^{\sigma*}} [(\mathbf{u}e_k^{\sigma*}) \sigma_{\beta\gamma} + e_{k\beta}^{\sigma*} \sigma_{i\beta\gamma}] + e_{kx}^{\sigma*} \frac{\omega_k^{\sigma*} - \omega_k^\sigma}{\omega_k^{\sigma*}} \sigma_{\beta\gamma} - \frac{k_\gamma}{\omega_k^{\sigma*}} e_{kx}^{\sigma*} P_{\beta\gamma} - i \frac{\omega_B}{\omega_k^{\sigma*}} e_{kx}^{\sigma*} (\delta_{\beta 2} \sigma_{1\gamma} - \delta_{\beta 1} \sigma_{2\gamma} + \delta_{\gamma 2} \sigma_{1\beta} - \delta_{\gamma 1} \sigma_{2\beta}) \right] \right\}$$

$$+ [\gamma, \alpha, \beta] + [\beta, \gamma, \alpha] \Big|_{a=k_\perp v_\perp / \omega_B} \quad (11')$$

Using equations (8) and (12) we have the following:

$$\sigma_1 = iv_\perp \frac{\partial \sigma_0}{\partial a_y}; \quad \sigma_2 = -iv_\perp \frac{\partial \sigma_0}{\partial a_x};$$

$$\sigma_3 = P_0 + \frac{\omega_k^{\sigma'}}{k_\parallel} \sigma_0 + i \frac{\omega_B}{k_\parallel} \left[ a_y \frac{\partial}{\partial a_x} - a_x \frac{\partial}{\partial a_y} \right] \sigma_0; \quad (13)$$

$$\sigma_{1x} = iv_\perp \frac{\partial \sigma_x}{\partial a_y}; \quad \sigma_{2x} = -iv_\perp \frac{\partial \sigma_x}{\partial a_x};$$

$$\sigma_{33} = P_3 + \frac{\omega_k^{\sigma'}}{k_\parallel} \sigma_3 + i \frac{\omega_B}{k_\parallel} \left[ a_y \frac{\partial}{\partial a_x} - a_x \frac{\partial}{\partial a_y} \right] \sigma_3; \quad (14)$$

$$\sigma_{1\beta\gamma} = iv_\perp \frac{\partial \sigma_{\beta\gamma}}{\partial a_y}; \quad \sigma_{2\beta\gamma} = -iv_\perp \frac{\partial \sigma_{\beta\gamma}}{\partial a_x};$$

$$\sigma_{333} = P_{33} + \frac{\omega_k^{\sigma'}}{k_\parallel} \sigma_{33} + i \frac{\omega_B}{k_\parallel} \left[ a_y \frac{\partial}{\partial a_x} - a_x \frac{\partial}{\partial a_y} \right] \sigma_{33}. \quad (15)$$

The relations (13)–(15) may be verified by the direct substitution of expressions (8) and (12) in these formulae.

If we use the following relationships:

$$P_1 = iv_\perp \frac{\partial P_0}{\partial a_y}; \quad P_2 = -iv_\perp \frac{\partial P_0}{\partial a_x};$$

$$P_{1x} = iv_\perp \frac{\partial P_x}{\partial a_y}; \quad P_{2x} = -iv_\perp \frac{\partial P_x}{\partial a_x}, \quad (16)$$

then, when calculating the functions  $P$ , one can put  $a = k_\perp v_\perp / \omega_B$ :

$$P_0|_{a=k_\perp v_\perp / \omega_B} = 0, \quad (17)$$

$$P_x|_{a=k_\perp v_\perp / \omega_B} = \frac{e^2}{imk_\parallel} \int dv f^R \left[ d_{kx}^\sigma - \frac{e_{kx}^\sigma(\mathbf{k}\mathbf{v}) - k_x(\mathbf{v}e_k^\sigma)}{\omega_k^\sigma} \right], \quad (18)$$

$$P_{x\beta}|_{a=k_\perp v_\perp / \omega_B} = \frac{e^2}{imk_\parallel} \int dv f^R \times \left\{ v_x \left[ d_{k\beta}^\sigma - \frac{e_{k\beta}^\sigma(\mathbf{k}\mathbf{v}) - k_\beta(\mathbf{v}e_k^\sigma)}{\omega_k^\sigma} \right] + [\beta, \alpha] \right\}. \quad (19)$$

Thus, we have written the expressions for the moments of the quasilinear integral of collisions in terms of the function  $\sigma_0$  (its physical meaning is clear from its relationship to the charge density of  $\alpha$  particles:  $\rho_\alpha = \sigma_0^z E_k|_{a=k_\perp v_\perp / \omega_B}$ ) and  $P$ . The hydrodynamic equations should be supplemented by an expression governing the energy density evolution of the plasma oscillations. The resulting system takes into account both the resonant and adiabatic quasilinear interactions.

### 3. Moments of the quasilinear integral of collisions for a bi-Maxwellian plasma

By virtue of the assumption made above [see equation (4)], the distribution function  $f^R$  can depend on coor-

dinates and on time [in expressions (8) and (12)] only as the parameters. If, moreover, interparticle collisions dominate and there is no need to consider plateau formation on  $f^R$ , the following expression (in the plasma frame) is often used:

$$f^R(\mathbf{v}) = \frac{N_\alpha}{(2\pi)^{3/2} v_{\perp 1}^2 v_{\parallel 1}} \exp\left(-\frac{v_{\parallel 1}^2}{2v_{\parallel 1}^2} - \frac{v_{\perp 1}^2}{2v_{\perp 1}^2}\right), \quad (20)$$

where  $N_\alpha$  is the density of  $\alpha$  particles and  $v_{\parallel 1}$ ,  $v_{\perp 1}$  are longitudinal (along the external magnetic field) and transverse thermal velocities, respectively. The density and temperatures in equation (20) depend parametrically on  $\mathbf{r}$  and  $t$ . Note that the calculation of moments of the quasilinear integral of collisions using equation (20) does not contradict the fact that, on scales in excess of the scale of averaging in equation (1), the distribution function can contain dissipative terms (for example, the Grad approximation).

In the case of Coulomb collisions, the question of the consistency of the approximations, in terms of which the cross-sections and moments of the integral of collisions are calculated, does not arise.

On substituting equation (20) into equation (8) and integrating over the velocity, we obtain:

$$\begin{aligned} \sigma_0 &= \frac{e^2 N}{imv_{\perp 1} \omega_k^\sigma} \sum_n \exp(in(\chi - \chi')) \left\{ \exp\left(-\frac{\alpha^2 + \lambda^2}{2}\right) I_n(a\lambda) \right. \\ &\times \left[ \frac{\omega_k^\sigma}{k_{\parallel} v_{\perp 1}} d_{k_{\parallel}}^\sigma (A+1)(1+i\sqrt{\pi}z_n w(z_n)) \right. \\ &+ \frac{n}{\lambda} \left[ \frac{d_{k_{\perp}}^\sigma \mathbf{k}_{\perp}}{k_{\perp}} i\sqrt{\pi}z_0 w(z_n) + A \left( \frac{e_{k_{\perp}}^\sigma \mathbf{k}_{\perp}}{k_{\perp}} - e_{k_{\parallel}}^\sigma \frac{k_{\perp}}{k_{\parallel}} \right) \right. \\ &\left. \left. \times (1+i\sqrt{\pi}z_n w(z_n)) \right] \right] \\ &+ i\lambda \exp\left(-\frac{\alpha^2 + \lambda^2}{2}\right) \left[ -I_n(a\lambda) + \frac{a}{\lambda} I_n'(a\lambda) \right] \\ &\left. \times \left[ \frac{(\mathbf{k}d_{k_{\perp}}^\sigma)_{\parallel}}{k_{\perp}} i\sqrt{\pi}z_0 w(z_n) + \frac{(\mathbf{k}e_{k_{\perp}}^\sigma)_{\parallel}}{k_{\perp}} A(1+i\sqrt{\pi}z_n w(z_n)) \right] \right\}. \quad (21) \end{aligned}$$

The designations used here are:

$$\begin{aligned} \tan \chi' &= \frac{a_y}{a_x}; \quad \lambda = \frac{k_{\perp} v_{\perp 1}}{\omega_B}; \quad A = \frac{T_{\perp}}{T_{\parallel}} - 1; \\ z_n &= \frac{\omega_k^\sigma - n\omega_B}{\sqrt{2}k_{\parallel} v_{\parallel 1}}; \quad z_0 = \frac{\omega_k^\sigma}{\sqrt{2}k_{\parallel} v_{\parallel 1}}; \end{aligned}$$

$I_n$  is the modified Bessel's function and  $w(z_n)$  is the probability integral of a complex argument:

$$w(z_n) = \exp(-z_n^2) \left[ 1 + \frac{2i}{\sqrt{\pi}} \int_0^{z_n} \exp(t^2) dt \right].$$

The expressions for the functions  $P$  [equation (12)] assume the form:

$$P_\alpha|_{a=k_{\perp} v_{\perp 1}/\omega_B} = \frac{\omega_p^2}{4\pi i k_{\parallel}} d_{k_{\alpha}}^\sigma;$$

$$P_{\alpha\beta}|_{a=k_{\perp} v_{\perp 1}/\omega_B} = \frac{\omega_p^2 (e_{k_{\alpha}}^\sigma k_{\beta} - e_{k_{\beta}}^\sigma k_{\alpha})}{4\pi i k_{\parallel} \omega_k^\sigma} \int d\mathbf{v} (v_\alpha^2 - v_\beta^2) f^R, \quad (22)$$

where  $\omega_p^2 = 4\pi N_\alpha e^2/m$  is the plasma frequency of the considered particles. It is most convenient to calculate the derivatives of  $\sigma_0$  in parameters  $a_x$  and  $a_y$  in a cylindrical coordinate system. In this case, as follows from equation (21),  $(\partial/\partial x')^l = (-in)^l$ , and the relationships:

$$\lambda \frac{\partial}{\partial a} Q_n|_{a=k_{\perp} v_{\perp 1}/\omega_B} = \lambda^2 [\exp(-\lambda^2) I_n(\lambda^2)]' \equiv x A_n'(x), \quad (23)$$

$$\lambda^2 \frac{\partial^2}{\partial a^2} Q_n|_{a=k_{\perp} v_{\perp 1}/\omega_B} = x^2 A_n''(x) - x A_n'(x), \quad (24)$$

$$\lambda^3 \frac{\partial^3}{\partial a^3} Q_n|_{a=k_{\perp} v_{\perp 1}/\omega_B} = x^3 A_n'''(x) - 3x^2 A_n''(x), \quad (25)$$

$$\lambda \frac{\partial}{\partial a} R_n|_{a=k_{\perp} v_{\perp 1}/\omega_B} = x A_n''(x) + \exp(-x) I_n'(x), \quad (26)$$

$$\begin{aligned} \lambda^2 \frac{\partial^2}{\partial a^2} R_n|_{a=k_{\perp} v_{\perp 1}/\omega_B} &= x^2 A_n'''(x) + x A_n''(x) + x [\exp(-x) I_n'(x)]' \\ &= x^2 A_n'''(x) + x A_n''(x) + x [\exp(-x) I_n'(x)]' \quad (27) \end{aligned}$$

and:

$$\begin{aligned} \lambda^3 \frac{\partial^3}{\partial a^3} R_n|_{a=k_{\perp} v_{\perp 1}/\omega_B} &= x^3 A_n^{IV}(x) + 3x^2 A_n'''(x) - 3x \exp(-x) I_n'(x), \quad (28) \end{aligned}$$

are valid, where:

$$Q_n = \exp\left(-\frac{a^2 + \lambda^2}{2}\right) I_n(a\lambda)$$

and

$$R_n = \exp\left(-\frac{a^2 + \lambda^2}{2}\right) \left[ -I_n(a\lambda) + \frac{a}{\lambda} I_n'(a\lambda) \right].$$

Using expressions (21)–(28), one can rewrite the moments of the quasilinear integral [equations (9)–(11)] for the case under consideration. The resulting expressions are rather unwieldy; however, for a number of particular situations (small Larmor radius, absence of transverse drift, resonant interaction, etc.), they simplify substantially.

#### 4. Moments of the quasilinear integral of collisions in the absence of a transverse drift

If  $\mathbf{u} = 0$ , then neither  $I_k^\sigma$ , nor  $\sigma_0$ , nor the derivatives on  $x'$  and  $a$  (after identifying  $\mathbf{a} = \mathbf{k}_{\perp} v_{\perp 1}/\omega_B$ ), depend on  $x$ , and this allows us to simplify expressions (9)–(11). We choose the polarization vector in the form (Akhiezer, 1974):

$$\mathbf{e}_k^\sigma = \frac{1}{|\mathbf{g}^\sigma|} \{ \cos \chi + iT^\sigma \sin \chi, \sin \chi - iT^\sigma \cos \chi, e_3^\sigma |\mathbf{g}^\sigma| \}, \quad (29)$$

where  $T^\sigma$  is a combination of the components of the tensor of dielectric permeability  $\varepsilon_{ij}(\omega_k^\sigma, \mathbf{k})$ ;  $\varepsilon_3 \equiv \varepsilon_{zz}(\omega_k^\sigma, \mathbf{k})$  and  $|q^\sigma|$  is the normalization. Now we can integrate expressions (9')–(11') over  $\chi$ . Then:

$$J_x = \delta_{\alpha 3} \int d\mathbf{k} I_k \frac{1}{\omega_k^*} \sum_n \left[ e_{k\parallel}^* (\omega_k^* - n\omega_B) + \frac{k_\parallel \omega_B}{k_\perp |g|} \left( T^* \lambda \frac{\partial}{\partial a} + n \right) \right] \sigma_0^n \Big|_{a=k_\perp v_{\perp i} / \omega_B}, \quad (30)$$

$$J_{x\beta} = \int d\mathbf{k} I_k \frac{1}{\omega_k^* k_\parallel} \left\{ (\delta_{\alpha 1} \delta_{\beta 1} + \delta_{\alpha 2} \delta_{\beta 2} - 2\delta_{\alpha 3} \delta_{\beta 3}) \frac{\omega_p^2}{4\pi i} \times [(\mathbf{k} \mathbf{e}_k^*) \mathbf{d}_k]_\parallel + (\delta_{\alpha 1} \delta_{\beta 1} + \delta_{\alpha 2} \delta_{\beta 2}) \sum_n \left[ n\omega_B e_{k\parallel}^* (\omega_k - n\omega_B) + (n\omega_B + \omega_k^* - \omega_k) \frac{\omega_B k_\parallel}{k_\perp |g|} \left( T^* \lambda \frac{\partial}{\partial a} + n \right) \right] \sigma_0^n + 2\delta_{\alpha 3} \delta_{\beta 3} \times \sum_n (\omega_k' - n\omega_B) \left[ e_{k\parallel}^* (\omega_k^* - n\omega_B) + \frac{\omega_B k_\parallel}{k_\perp |g|} \left( T^* \lambda \frac{\partial}{\partial a} + n \right) \right] \sigma_0^n \right\} \Big|_{a=k_\perp v_{\perp i} / \omega_B}, \quad (31)$$

$$J_{\alpha\beta\gamma} = \int d\mathbf{k} I_k \frac{1}{\omega_k^* k_\parallel^2} \left\langle - [\delta_{\alpha 3} (\delta_{\beta 1} \delta_{\gamma 1} + \delta_{\beta 2} \delta_{\gamma 2}) + \delta_{\beta 3} (\delta_{\gamma 1} \delta_{\alpha 1} + \delta_{\gamma 2} \delta_{\alpha 2}) + \delta_{\gamma 3} (\delta_{\alpha 1} \delta_{\beta 1} + \delta_{\alpha 2} \delta_{\beta 2}) - 3\delta_{\alpha 3} \delta_{\beta 3} \delta_{\gamma 3}] \times \frac{\omega_p^2}{4\pi i} \left[ \frac{k_\parallel v_{\parallel i}}{\omega_k} A |(\mathbf{e}_k^* \mathbf{k})_\perp|^2 + i\omega_B (\mathbf{e}_k^* \mathbf{k})_\perp \mathbf{d}_k \right] - 3\delta_{\alpha 3} \delta_{\beta 3} \delta_{\gamma 3} \frac{\omega_p^2}{4\pi i} \{ \omega_k' [(\mathbf{k} \mathbf{e}_k^*) \mathbf{d}_k]_\parallel - \omega_k^* k_\parallel |e_{k\parallel}|^2 + \omega_k^* e_{k\parallel}^* \mathbf{k}_\perp \mathbf{d}_k \} + 3\delta_{\alpha 3} \delta_{\beta 3} \delta_{\gamma 3} \sum_n (\omega_k' - n\omega_B)^2 \times \left[ (\omega_k^* - n\omega_B) e_{k\parallel}^* + \frac{\omega_B k_\parallel}{k_\perp |g|} \left( T^* \lambda \frac{\partial}{\partial a} + n \right) \right] \sigma_0^n + [\delta_{\alpha 3} (\delta_{\beta 1} \delta_{\gamma 1} + \delta_{\beta 2} \delta_{\gamma 2}) + \delta_{\beta 3} (\delta_{\gamma 1} \delta_{\alpha 1} + \delta_{\gamma 2} \delta_{\alpha 2}) + \delta_{\gamma 3} (\delta_{\alpha 1} \delta_{\beta 1} + \delta_{\alpha 2} \delta_{\beta 2})] \sum_n (\omega_k' - n\omega_B) \left[ n\omega_B e_{k\parallel}^* (\omega_k - n\omega_B) + (n\omega_B + \omega_k^* - \omega_k) \frac{\omega_B k_\parallel}{k_\perp |g|} \left( T^* \lambda \frac{\partial}{\partial a} + n \right) \right] \sigma_0^n - [\delta_{\alpha 3} (\delta_{\beta 1} \delta_{\gamma 1} + \delta_{\beta 2} \delta_{\gamma 2}) + \delta_{\beta 3} (\delta_{\gamma 1} \delta_{\alpha 1} + \delta_{\gamma 2} \delta_{\alpha 2}) + \delta_{\gamma 3} (\delta_{\alpha 1} \delta_{\beta 1} + \delta_{\alpha 2} \delta_{\beta 2})] \times \frac{\omega_B^2 k_\parallel^2}{2k_\perp^2} \sum_n \left\{ e_{k\parallel}^* (\omega_k^* - n\omega_B) \left( \lambda^2 \frac{\partial^2}{\partial \alpha^2} + \lambda \frac{\partial}{\partial \alpha} - n^2 \right) \right.$$

$$\left. + \frac{\omega_B k_\parallel}{k_\perp |g|} \left[ T^* \left( \lambda^3 \frac{\partial^3}{\partial \alpha^3} + \lambda^2 \frac{\partial^2}{\partial \alpha^2} - (1+n^2) \lambda \frac{\partial}{\partial \alpha} + 2n^2 \right) + n \left( \lambda^2 \frac{\partial^2}{\partial \alpha^2} + \lambda \frac{\partial}{\partial \alpha} - n^2 \right) \right] \right\} \sigma_0 + \frac{\omega_p^2}{4\pi i} [\delta_{\alpha 3} (\delta_{\beta 1} \delta_{\gamma 1} + \delta_{\beta 2} \delta_{\gamma 2}) + \delta_{\beta 3} (\delta_{\gamma 1} \delta_{\alpha 1} + \delta_{\gamma 2} \delta_{\alpha 2}) + \delta_{\gamma 3} (\delta_{\alpha 1} \delta_{\beta 1} + \delta_{\alpha 2} \delta_{\beta 2})] \times \{ k_\parallel \omega_k^* e_{k\perp}^* \mathbf{d}_k \perp + \omega_k [(\mathbf{k} \mathbf{e}_k^*) \mathbf{d}_k]_\parallel \} \Big|_{a=k_\perp v_{\perp i} / \omega_B}, \quad (32)$$

where  $\sigma_0$  has the form:

$$\sigma_0 = \sum_n \sigma_0^n = \frac{\omega_p^2}{4\pi i v_{\perp i} \omega_k} \times \sum_n \left\langle \exp \left( -\frac{a^2 + \lambda^2}{2} \right) I_n(a\lambda) e_{k\parallel} \times \left\{ \frac{\omega_k}{k_\parallel v_{\perp i}} \left[ 1 + A \left( 1 - \frac{n\omega_B}{\omega_k^\sigma} \right) \right] [1 + i\sqrt{\pi} z_n w(z_n)] + \frac{n\omega_B}{\omega_k} \frac{u_\parallel}{v_{\perp i}} i\sqrt{\pi} z_n w(z_n) \right\} + \frac{\lambda}{|g|} \left\{ i\sqrt{\pi} z_0 w(z_n) + A [1 + i\sqrt{\pi} z_n w(z_n)] \exp \left( -\frac{a^2 + \lambda^2}{2} \right) \times \left\{ I_n(a\lambda) \frac{n}{\lambda^2} + T \left[ -I_n(a\lambda) + \frac{a}{\lambda} I_n'(a\lambda) \right] \right\} \right\}. \quad (33)$$

When substituting the derivatives  $(\partial/\partial a)^i$  into equations (30)–(32), it is more convenient to use, not equations (23)–(28), but their combinations. Expressions for these combinations are given in the Appendix.

Thus, we have obtained the moments of the quasilinear collision integral which take into account the resonant and adiabatic interactions of arbitrary types of oscillations with arbitrary plasma particles. Expressions (9')–(11'), (13)–(17), (21) and (22) are suitable for describing the plasma in the field of arbitrarily directed regular force  $\mathbf{F}_\alpha$ , or equations (30)–(33) in the absence of a transverse drift.

As mentioned above, the system of hydrodynamic equations, taking into account the contributions from the quasilinear interaction, should be solved together with the equation describing the evolution of the spectral energy of waves  $I_k^a$ . However, as a first approximation, we can use the energy spectra of the given type of waves registered on satellites in near-terrestrial space.

In conclusion we will derive the expressions for the moment of the quasilinear integral of collisions which can be used when modelling the interactions of ion-cyclotron waves with ions in the ring current region.

As we mentioned in the Introduction, the anisotropic protons of the ring current generate intensive left-hand polarized ion-cyclotron waves in the region of the plasmasphere. Moreover, the generation takes place during the cyclotron interaction. Using equations (30)–(33) and equations (A1)–(A6) for  $n = 1$ , assuming  $\lambda \ll 1$  and neglecting  $e_{k\parallel}$ , we get fairly simple expressions which take

into account the interaction of ring current protons and ion-cyclotron waves:

$$J_{\alpha} = \delta_{\alpha 3} \frac{\omega_p^2}{8\sqrt{\pi}} \int dk I_k \frac{k_{\parallel}}{\omega_R^2 |g|^2} |1+T|^2 \times (z_0 + Az_1) \exp(-z_1^2) \text{sign } k_{\parallel}, \quad (34)$$

$$J_{\alpha\beta} = \frac{\omega_p^2 \omega_B}{8\sqrt{\pi}} \int dk I_k \frac{|1+T|^2}{\omega_R^2 |g|^2} \times \left[ \delta_{\alpha 1} \delta_{\beta 1} + \delta_{\alpha 2} \delta_{\beta 2} + 2\delta_{\alpha 3} \delta_{\beta 3} \left( \frac{\omega'_R}{\omega_B} - 1 \right) \right] \times (z_0 + Az_1) \exp(-z_1^2) \text{sign } k_{\parallel}, \quad (35)$$

$$J_{\alpha\beta\gamma} = \frac{\omega_p^2 \omega_B^2}{8\sqrt{\pi}} \int dk I_k \frac{|1+T|^2}{\omega_R^2 k_{\parallel} |g|^2} \times \left\{ [\delta_{\alpha 3} (\delta_{\beta 1} \delta_{\gamma 1} + \delta_{\beta 2} \delta_{\gamma 2}) + \delta_{\beta 3} (\delta_{\gamma 1} \delta_{\alpha 1} + \delta_{\gamma 2} \delta_{\alpha 2}) + \delta_{\gamma 3} (\delta_{\alpha 1} \delta_{\beta 1} + \delta_{\alpha 2} \delta_{\beta 2})] \left[ \frac{\omega'_R}{\omega_B} - 1 + \frac{2k_{\parallel}^2 v_{\perp 1}^2}{\omega_B^2} \right] + 3\delta_{\alpha 3} \delta_{\beta 3} \delta_{\gamma 3} \left( \frac{\omega'_R}{\omega_B} - 1 \right)^2 \right\} \times (z_0 + Az_1) \exp(-z_1^2) \text{sign } k_{\parallel}, \quad (36)$$

where:

$$\omega_R = \text{Re}(\omega_k); \quad \omega'_R = \omega_R - k_{\parallel} u; \\ z_1 = \frac{\omega'_R - \omega_B}{\sqrt{2k_{\parallel} v_{\parallel 1}}}; \quad T = \frac{i\epsilon_{\alpha 1}}{\epsilon_{\alpha 1} - N^2}; \quad N^2 = \frac{k^2 c^2}{\omega_k^2}.$$

The tensor of dielectric permeability of the cold magnetized plasma in previous expressions may be used.

If there is a small addition of helium ions to the ring current region, then the dispersive dependence of ion-cyclotron waves is greatly modified (Young *et al.*, 1981). The presence of helium leads to three branches in the dispersion relation for oblique propagation of waves. The first branch (class I) is left-hand polarized and lies below the gyrofrequency of helium ions  $\text{He}^+$  (Korth *et al.*, 1984). The interaction of such waves with  $\text{He}^+$  will also be described by expressions (34)–(36) with substitution of the anisotropic proton parameters for the parameters of  $\text{He}^+$ .

In conclusion, we should like to stress that the moments of the quasilinear collision integrals obtained in this paper take into account both resonant and adiabatic interactions of arbitrary waves and particles. It gives us an opportunity to use the system of the hydrodynamical equations for investigation of multicomponent plasma in which the developed wave activity takes place.

The obtained results include, as a particular case, the outcome of previous papers (Konikov *et al.*, 1989; Gamayunov *et al.*, 1991; Gorbachev *et al.*, 1992) and may be useful for a hydrodynamical description of the solar wind, plasmas of the other planets and so on.

*Acknowledgement*—This work was supported by the NASA Grant Nos. NAGW-1619 and NAG5-1500, as well as NSF Grant ATM-9114409. We thank the referee for his assistance in evaluating this paper.

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## Appendix

$$\hat{D}_1 Q_n |_{a=k_{\perp 1}, \omega_B} = x \left[ -A_n T^* + \frac{1+T^*}{2} A_{n-1} - \frac{1-T^*}{2} A_{n+1} \right], \quad (A1)$$

$$\hat{D}_2 Q_n |_{a=k_{\perp 1}, \omega_B} = x [2(x-1)A_n - x(A_{n-1} + A_{n+1})], \quad (A2)$$

$$\hat{D}_3 Q_n |_{a=k_{\perp 1}, \omega_B} = x^2 \left\{ T^*(4-3x)A_n + [2(1+T^*) - (2T^*+1)x]A_{n-1} + [2(1-T^*) - (2T^*-1)x]A_{n+1} - \frac{1+T^*}{2} x A_{n-2} - \frac{1-T^*}{2} x A_{n+2} \right\}; \quad (A3)$$

$$\begin{aligned} \hat{D}_1 \left( Q_n \frac{n}{x} + TR_n \right) \Big|_{a=k_\perp v_\perp / \omega_B} &= \frac{x}{2} \left\{ (3|T|^2 - 1) A_n \right. \\ &+ \frac{1+T^*}{2} (1+T-2xT) A_{n-1} - \frac{1-T^*}{2} (1-T+2xT) A_{n+1} \\ &\left. + \frac{|1+T|^2}{4} x A_{n-2} + \frac{|1-T|^2}{4} x A_{n+2} \right\}, \quad (\text{A4}) \end{aligned}$$

$$\begin{aligned} \hat{D}_2 \left( Q_n \frac{n}{x} + TR_n \right) \Big|_{a=k_\perp v_\perp / \omega_B} &= x \left\{ T(4-3x) A_n \right. \\ &+ [2T(x-1) + x-2] A_{n-1} + [2T(x-1) - x+2] A_{n+1} \\ &\left. - \frac{1+T}{2} x A_{n-2} + \frac{1-T}{2} x A_{n+2} \right\}, \quad (\text{A5}) \end{aligned}$$

$$\begin{aligned} \hat{D}_3 \left( Q_n \frac{n}{x} + TR_n \right) \Big|_{a=k_\perp v_\perp / \omega_B} &= x \left\{ (2-x)(1-5|T|^2) x A_n \right. \\ &+ \left[ -2|1+T|^2 + x(1+4T^*+4T+7|T|^2) \right. \\ &\left. + \frac{x^2}{4} (1-5T^*-5T-15|T|^2) \right] A_{n-1} \end{aligned}$$

$$\begin{aligned} &+ \left[ -2|1-T|^2 + x(1-4T^*-4T+7|T|^2) \right. \\ &\left. + \frac{x^2}{4} (1+5T^*+5T-15|T|^2) \right] A_{n+1} \\ &+ x \left[ -2|1+T|^2 + \frac{x^2}{2} (1+2T^*+2T+3|T|^2) \right] A_{n-2} \\ &+ x \left[ -2|1-T|^2 + \frac{x^2}{2} (1-2T^*-2T+3|T|^2) \right] A_{n+2} \\ &\left. - \frac{x^2}{4} |1+T|^2 A_{n-3} - \frac{x^2}{4} |1-T|^2 A_{n+3} \right\}, \quad (\text{A6}) \end{aligned}$$

where :

$$\hat{D}_1 = T^* \lambda \frac{\partial}{\partial a} + n; \quad \hat{D}_2 = \lambda^2 \frac{\partial^2}{\partial a^2} + \lambda \frac{\partial}{\partial a} - n^2$$

and :

$$\hat{D}_3 = T^* \left( \lambda^3 \frac{\partial^3}{\partial a^3} + \lambda^2 \frac{\partial^2}{\partial a^2} - (1+n^2) \lambda \frac{\partial}{\partial a} + 2n^2 \right) + n \hat{D}_2.$$