Journal of Mathematical Economics 22 (1993) 581–597. North-Holland

Price indeterminacy and bargaining in a market with indivisibilities

Benny Moldovanu*

University of Bonn, Bonn, Germany and University of Michigan, Ann Arbor MI, USA

Submitted November 1991, accepted February 1993

We consider an exchange market for an indivisible, heterogeneous good where pairs of buyers and sellers bargain over prices in different transactions. A stable outcome in one negotiation cannot be uniquely determined by the outcome of other negotiations, but the result of those transactions has influence through endogenously determined outside options. We prove existence of equilibria with the property that no agent wished to rebargain. These bargaining equilibria form a subset of the set of Walrasian equilibria. In a replicated market all Walrasian equilibria are already stable under rebargaining. A bargaining process is shown to converge to the set of equilibria.

1. Introduction

The main purpose of this paper is to study the process of negotiation and price formation in two-sided markets where the traded objects are heterogeneous and indivisible. Quoting from Shapely and Shubik (1971): 'Two-sided market models are important as Cournot, Edgeworth, Böhm-Bawerk, and others have observed, not only for the insights they may give into more general economic situations with many types of traders, but also for the simple reason that in real life many markets and most actual transactions are in fact bilateral – i.e., bring together a buyer and a seller of a single commodity'.

We consider here a market for a good which is available in indivisible units. The units of this good may differ in quality. There are two a-priori determined distinct groups: buyers and sellers. All agents may possess a perfectly divisible good called 'money', and each seller has also one unit of

Correspondence to: Benny Moldovanu, Department of Economics, University of Bonn, Adenauerallee 24-43, 5300 Bonn, Germany.

*I am indebted to an associate editor for valuable comments. I wish also to thank Ken Binmore, Mamoru Kaneko, Bezalel Peleg, Martine Quinzii, Al Roth, Ariel Rubinstein, Avner Shaked, Lloyd Shapley and Asher Wolinsky for some helpful discussions. I owe many thanks to Alexandra Hennemann who typed parts of the manuscript. Financial support from Deutsche Forschungs gemeinschaft, SFB 303, is gratefully acknowledged.

the indivisible good. Each agent has preferences on the possible combinations of money (which represents here the consumption of all 'other' goods) and units of the indivisible good. We assume that no agent wishes to own more than one unit of the indivisible good.

Shapley and Shubik (1971) studied such a market modeled as a game with transferable utility. Kaneko (1982) generalized their results to games without the transferable utility assumption. The main results are that the core of such a market is not empty, and that the set of competitive allocations coincides with the set of allocations in the core. For an excellent survey of two-sided markets see Roth and Sotomayor (1990).

We will use here Kaneko's model and our study begins with the following observation: Assume that a buyer and a seller contemplate a transaction, while regarding as fixed the prices paid for the units of the indivisible good in other transactions. The seller has then a minimum price for which he is willing to sell in the existing conditions. At lower prices he would find other buyers willing to pay more for his unit. Similarly, the buyer has a maximal price which she is prepared to pay for that unit given the existing conditions. If the price is higher, the buyer would prefer to buy other units, and she can afford it. The transaction can be mutually profitable if and only if the agreed price falls between these extremes, and, moreover, one can show that a certain system of transactions leads to a competitive allocation if and only if, for each pair in the system, the agreed price has the above described property. Note that the 'reservation prices' are not primitives of the model, but are endogenously arising at each specification of prices.

For a given pair, the environment (i.e. the prices in all other transactions) cannot uniquely determine the price in their deal. There is a whole range of agreements which could lead to a competitive allocation when combined with fixed outside prices. We are faced with the natural question of how, in each transaction, exactly one price is chosen by the matched agents? Any short examination of markets where few agents trade relatively valuable goods would reveal that the process of price formation involves some element of bargaining. The results of other negotiations can determine, and this is very important, the outside options available to the various agents.

We assume here a rather myopic and optimistic way of calculating outside options: the agents do not foresee the whole evolution of the market after possible breakdowns in present negotiations. They make only a rough assessment of their bargaining power at one point in time. We only require that the negotiated outcome depends on the outside options, and this dependence follows our intuition – an improvement in the terms of trade for one agent (due to price changes in other transactions) should not disadvantage this agent in his present negotiation.

By inter-relating the prices in different transactions through the influence of outside options the following problem arises: Negotiations in transaction

A determines the terms of trade in transaction B. Further, the resulting price in transaction B may change the terms of trade in A and at least one of the sides may have incentives to renegotiate. Renegotiation in A matters for B and so on ... The equilibrium concept must take care of this situation in a consistent manner.

The previous consideration suggests a sequential dynamic process of price formation between matched partners, and indeed we will present such a process which converges to equilibrium for some initial conditions. At equilibrium, the agents have no incentive to renegotiate (given the existing bargaining procedures).

This approach to exchange, negotiation, and the combination of the two suggests that the model is more suitable for situations where there is no one-shot exchange, but longer run relationships. Price contracts hold for a period of time and they may be renegotiated if the environment changed. Good examples would be the house-rental market, or the labor market for a given profession. Consider, for example, the following quotation from *The Economist*, August 11, 1990: '(Rock) Stars usually stick to their record labels but use outside offers to renegotiate better deals'.

For a correct application of our specific model to a labor market one needs to assume that employees care only about their pay and not about the identity of their employers, but more general models where the results hold can be easily analyzed.

The 'stability under renegotiation' idea used here goes back to Harsanyi. We quote from Harsanyi (1977, p. 196): A particular payoff vector 'will represent the equilibrium outcome of a bargaining among the n-players only if no pair of players has any incentive to redistribute their payoffs between them, as long as the other players' payoffs are kept constant'. This idea has been exactly formulated in a game theoretical framework by Davis and Maschler (1965) in their work on the kernel of a cooperative game. The work of Davis and Maschler has been elegantly used by Rochford (1984) in the framework of the assignment games with transferable utility due to Shapley and Shubik (1971). Abstact generalizations of Rochford's work can be found in Roth and Sotomayor (1988), and Moldovanu (1990).

Modeling the market situation as a game with transferable utility implies that the utility functions of the agents are separable, additive and linear in a commodity called 'money'. While this may be a reasonable modeling assumption in some cases, it has considerable drawbacks in a model where the value of the traded objects are high relative to income. The above mentioned utility functions do not allow for any income effect! We obtain several different pairings of buyers and sellers which are compatible with Walrasian prices, while with transferable utility there is generically one such pairing. Moreover, the set of utility vectors corresponding to competitive allocations for a given pairing may well be non-connected, as opposed to a convex set for transferable utility. This creates an additional difficulty when using topological fixed point arguments. For details see Moldovanu (1990).

Finally, we wish to mention the recent approach to bargaining in markets which uses explicit strategic models embedded in various market models. An excellent book on the subject is Osborne and Rubinstein (1990). While different in emphasis and treatment, the basic idea of a market as a network of interconnected bargainers is common to this literature and to our work. This holds also for the important role of outside options (although the calculation of these options differs), and for the comparison with the benchmark – the Walrasian equilibrium.

This paper is organized as follows: In section 2 we describe the exchange model and the competitive equilibria. In section 3 we describe the bargaining model, define the outside options of the agents, and relate the structure of the set of competitive equilibria with the bargaining range in bilateral transactions. The main results are gathered in section 4: We prove the existence of bargaining equilibria and compare this set with the set of competitive equilibria in usual and replicated markets. We also describe a dynamic process of bargaining which may converge to an equilibrium. In section 5 we discuss the results and some possible extensions. The proofs are gathered in section 6.

2. The exchange model

We consider a market for an indivisible good. The units of this good can be different in quality. We will call these items 'houses'.

There are two distinct groups of agents (or players), $B=\{b_1,\dots,b_t\}$ and $S=\{s_1,\dots,s_r\}$.

The members of B will be called 'buyers', and the members of S will be called 'sellers'.

Each player i in $N=B\cup S$ owns an amount of money w_i. Money is perfectly divisible and should be interpreted as a composite good.

Each seller owns, in addition to money, one unit of the indivisible good. The house of seller s_i will be denoted by h_i.

Thus, the initial endowment of seller s_i is $e_{s_i}=(w_{s_i},h_i)$ and the initial endowment of buyer b_j is $e_{b_j}=(w_{b_j},d)$, where owning d represents the situation of owning no house (or owning a 'dummy' house).

Let $H=\{h_1,h_2\dots h_r,h_{r+1}\}$, where $h_{r+1}=d$.

Each agent i has a utility function

$$U_i:\mathbb{R}\times H\to\mathbb{R}. \tag{2.1}$$

We make the following assumptions on the utility functions U_i:

$$\forall i\in N,\quad U_i \text{ is continuous and strictly increasing with respect to money.} \tag{2.2}$$

$$\forall i \in N, \quad \forall h \in H, \quad U_i(m,h) \geqq U_i(m,d). \tag{2.3}$$

$$\forall i \in N, \quad \forall h \in H, \quad U_i(m,d) \geqq U_i(0,h). \tag{2.4}$$

$$\forall s_j \in S, \quad \forall h \in H, \quad U_{sj}(m,d) = U_{sj}(m,h). \tag{2.5}$$

Condition (2.2) is clear. Condition (2.3) says that houses are desirable. Condition (2.4), which is slightly more unusual, has the following intuitive explanation [see also Kaneko (1982)]: Our model is a partial equilibrium model, where money plays the role of all commodities that are not explicitly considered. Therefore it is normal to have money but no house (in which case one may rent a hotel room), but it is not normal to own a house and not consume any other thing! Condition (2.5) is needed to ensure that sellers do not want to buy another house after selling the one they initially own.

We normalize the utility functions of the agents such that the utility of each agent from his initial endowment is exactly zero. The results will be independent of this normalization.

The above described economy will be denoted by $E = \{(e_i, U_i) \; i \in N\}$.

Due to the special properties of our markets (two types, indivisibility, satiation) one can consider w.l.o.g. only those allocations which are derived from exchanges which take place in coalitions of size no larger than two (one buyer and one seller). Indeed, only these coalitions will play a role when considering the core of the economy, and hence also for the competitive allocations. We now define special partitions of N which have as elements only mixed pairs (one buyer, one seller), and singletons:

A partition $\Pi = \{T_1, \dots, T_k\}$ of N will be called an E-partition if it satisfies:

$$\forall j, \quad 1 \leqq j \leqq k, \quad |T_j| \leqq 2. \tag{2.6}$$

$$\text{If} \quad |T_j| = 2 \quad \text{then} \quad T_j \cap B \neq \varnothing \text{ and } T_j \cap S \neq \varnothing. \tag{2.7}$$

An allocation x^i is feasible for i if it is just the initial endowment of this individual.

Consider a pair $T = \{b_i, s_j\}$ with initial endowment (w_{b_i}, d) *and* (w_{s_j}, h_j) respectively. An allocation x^T is feasible for this pair if it has the form $\{(m_{b_i}, h_j), (m_{s_j}, d)\}$, or the form $\{(m_{b_i}, d), (m_{s_j}, h_j)\}$, where

$$m_{b_i} + m_{s_j} \leqq w_{b_i} + w_{s_j}. \tag{2.8}$$

An allocation x is feasible for the economy E if there exists an E-partition Π such that for each $T \in \Pi$, x^T is feasible for T.

An allocation x is individually-rational (IR) if the following holds:

$$\forall i \in N, \quad U_i(x^i) \geqq U_i(e_i) = 0. \tag{2.9}$$

A pair $T = \{b_i, s_j\}$ can improve upon an allocation x if there exists an allocation y^T feasible for T with $U_i(y^i) > U_i(x^i)$, $\forall i \in T$.

The core of the economy, $C(E)$, is the set of all individually-rational allocations which cannot be improved upon. This definition of the core coincides for the considered economies with the usual definition of the core where all coalitions are allowed to block. Note that in the core of our exchange economy there are no side-payments in the sense that, if a buyer gives up a certain amount of money in order to receive a house, this amount goes in full to the owner of this house and not, say, to 'bribe' another potential buyer. Also, it is easy to show that if $x \in C(E)$, then the allocation of a singleton (i.e. a trader not paired in the respective E-partition) is exactly her initial endowment, and her utility is hence zero.

A vector of prices is a vector $p \in \mathbb{R}_+^{r+1}$ where p_i represents the price of house h_i, for $1 \leqq i \leqq r$, and $p_{r+1} = 0$, this last price being the price of a dummy house.

An E-partition Π and a vector of prices p determine in a unique way an allocation $x = x(\Pi, p)$ as follows:

(1) For $i = T \in \Pi$,

$$x^i = e_i. \tag{2.10}$$

(2) Let $T = \{b_i, s_j\} \in \Pi$. Then we have

$$x^{b_i} = (w_{b_i} - p_j, h_j), \quad \text{and} \tag{2.11}$$

$$x^{s_j} = (w_{s_j} + p_j, d). \tag{2.12}$$

Such an allocation will be called price-generated.

Definition 2.1. A pair $(\bar{\Pi}, \bar{p})$ is a competitive equilibrium if the price-generated allocation $\bar{x} = x(\bar{\Pi}, \bar{p})$ has the following properties:

(1) $\forall b_i \in B$ if $\bar{x}^{b_i} = (\bar{m}_{b_i}, h_j)$ then

$$\bar{m}_{b_i} \leqq w_{b_i} - p_j, \tag{2.13}$$

$$\forall h_f \in H, \quad U_{b_i}(m_{b_i}, h_f) > U_{b_i}(\bar{m}_{b_i}, h_j) \Rightarrow m_{b_i} + \bar{p}_f > w_{b_i}. \tag{2.14}$$

(2) $\forall s_k \in S$ if $\bar{x}^{S_k} = (\bar{m}_{S_k}, h_i)$ then

$$\bar{m}_{S_k} \leqq w_{S_k} + p_k, \tag{2.15}$$

$$\forall h_f \in H, \quad U_{S_k}(m_{S_k}, h_f) > U_{S_k}(\bar{m}_{S_k}, h_i) \Rightarrow m_{S_k} + \bar{p}_f > w_{S_k} + \bar{p}_k. \tag{2.16}$$

The intuition behind this definition is the usual one: Conditions (2.13) and (2.15) are the budget constraints, and conditions (2.14) and (2.16) are the optimality requirements. The main results of Kaneko (1982) and Quinzii (1984) are:

Theorem 2.2 (Kaneko). Let E be a two-sided exchange economy with indivisibilities. Then the following hold:

(1) *The core of E is not empty.*
(2) *Every core allocation is price-generated by a pair (Π, p) which is a competitive equilibrium.*
(3) *Every competitive equilibrium (Π, p) price-generates an allocation in the core of the economy E.*

Note that point (3) above holds in general for exchange models, but point (2) is rather special.

3. The bargaining model

Consider a buyer and a seller which are about to make a transaction, and consider the situation where the prices of all the other (i.e. not involved in this transaction) houses are already determined. Assume that the buyer and the seller agree on a certain price. If the resulting vector of prices is not competitive there will be at least one buyer and one seller which will have the opportunity to achieve a better deal for both of them by leaving their actual partners and trade together. (This follows from the equivalence between core allocations and competitive allocations – Theorem 2.2). In this case the situation is fundamentally unstable, but even if we accept that final outcomes should be Walrasian, the question is whether the price of the considered house is uniquely determined. Note that we *do not* inquire the question of uniqueness of competitive equilibria, but we ask whether in a competitive equilibrium the result of all outside transactions already determines the result of the considered one. The answer to this question is negative: Usually there is a whole range of prices which could be agreed upon by our pair such that the resulting vector or prices is competitive.

We must now ask how is then a price chosen by a given pair, between the several consistent with the stability of the market. The answer offered here is one that involves bargaining between the agents. The prices in the outside transactions play an important role because they actually determine, at a given moment, the outside options available to the traders. Bargaining in one pair affects the options of other pairs, and some agents may wish to renegotiate. If this renegotiation takes place, again options changed for others, and so ad infinitum. We are confronted with a problem of stability and consistency when we observe not the isolated pair, but the entire system of bilateral interactions.

We first approach the question of indeterminacy, and define the outside options: Let Π be an E-partition, p a vector or prices and let $T=\{b_i, s_j\}\in\Pi$. Let $x=x(\Pi,p)$ be the price-generated allocation.

The outside option price q^{b_i} of buyer b_i is defined as the price such that

$$U_{b_i}(w_{b_i}-q^{b_i},h_j)=\max_{f\neq j} U_{b_i}(w_{b_i}-p_f,h_f). \tag{3.1}$$

The outside option price for the buyer in T is the maximal price b_i is prepared to pay for the house of s_j such that, by buying this house and paying this price, her utility will be no less than the utility achieved by buying any other house at current prices. Observe that the outside option price of the buyer is non-negative, because h_f can be a dummy house (this means that the buyer can decide not to buy at all), the price of dummy houses is always zero, and because we assumed desirability of houses [see (2.3)].

For a buyer b_k define p_j^k as the price such that

$$U_{b_k}(w_{b_k}-p_j^k,h_j)=U_{b_k}(x^{b_k}). \tag{3.2}$$

p_j^k is the price which buyer b_k is prepared to pay for the house of s_j such that her utility will be no less than his current one.

For a seller s_j define p_j^0 as the price such that

$$U_{s_j}(w_{s_j}+p_j^0,d)=U(e_{s_j}). \tag{3.3}$$

p_j^0 is the minimum price which s_j might accept for his house. At lower prices this seller prefers not to sell at all, and remain with his initial endowment. This price is non-negative by desirability of houses.

The outside option price q^{s_j} is defined then as

$$q^{s_j}=\max\left\{p_j^0,\max_{k\neq i} p_j^k\right\}. \tag{3.4}$$

The outside option price for the seller in T is the maximal price this seller could achieve by selling his house to other buyers, while keeping these buyers at their current levels of utility, provided that this is at least as large as the minimal acceptable price for this seller.

For agents which are not matched in Π we define the outside options prices in a similar way: because these agents are not paired we can remove the restriction $f\neq j$ in (3.1), and the restriction $k\neq i$ in (3.4).

The outside option prices depend, of course, on the pair (Π, p) but we will suppress this for the sake of simplicity of notation.

Given a vector of prices $\bar{p}=(\bar{p}_1, \bar{p}_2, \ldots \bar{p}_{r+1})$ we write $(\bar{p}_{-j}, p_j)$ for the vector that is identical to $\bar{p}$ in all but the jth coordinate, which is changed to p_j. We have the following:

Theorem 3.1. *(a) A pair $(\bar{\Pi}, \bar{p})$ is a competitive equilibrium if and only if the following conditions hold for each $T=\{b_i, s_j\} \in \bar{\Pi}$:*

$$q^{b_i} \geqq q^{s_j}, \tag{3.5}$$

$$\bar{p}_j \in [q^{s_j}, q^{b_i}]. \tag{3.6}$$

(b) Moreover, if $(\bar{\Pi}, \bar{p})$ is a competitive equilibrium and $T=\{b_i, s_j\} \in \bar{\Pi}$, then $(\bar{\Pi}, (\bar{p}_{-j}, p_j))$ remains a competitive equilibrium if and only if $p_j \in [q^{s_j}, q^{b_i}]$.

Theorem 3.1 describes exactly, for one pair, the bargaining range which is compatible with stability of the market. The outside option prices are then the extremities of this range. Note that Theorem 3.1 shows that the Walrasian paradigm in such a model (i.e. – with few agents) does not imply that agents are 'price-takers'. Prices can be negotiated in a whole range without affecting the stability of the market. Theorem 3.1 is a special case of a more general theorem about sections of the core found in Moldovanu (1989).

We now describe the bargaining model: Given the utility functions of the members in a pair, which are fixed throughout this exposition, bargaining should depend only on their outside option prices. Thus a bargaining function is a function $F: \mathbb{R}_+^2 \to \mathbb{R}_+$ where the argument is a pair of prices representing the outside options (first coordinate for seller, second for buyer) and the result is the negotiated price. We impose the following conditions on F:

$$\text{If} \quad p_1 \leqq p_2 \quad \text{then} \quad p_1 \leqq F(p_1, p_2) \leqq p_2. \tag{3.7}$$

F is continuous and monotonic non-decreasing in each coordinate. (3.8)

Condition (3.7) is a condition of individual rationality. Continuity is clear, and monotonicity represents our intuition about outside options: If p_1 (for the seller!) increases, then his situation improved and he should not be worse off than before. Similarly, if p_2 (for the buyer!) increases, then her situation *changed for the worse* (she is required to pay more outside) and therefore she should not be made better off – the result of the actual bargaining is also a price which is not lower than before.

All the main solutions to the bargaining problem (Nash, Kalai–Smorodinski, etc.) satisfy conditions (3.7) and (3.8). Moreover, noncooperative bargaining models with outside options [see Shaked and Sutton (1984)] also yield bargaining functions with these properties. We do not require that all pairs use the same bargaining procedure. The procedure used by a pair T, given the respective utility functions, will be denoted by F_T.

We are led to a definition of bargaining equilibrium where no agent has the incentive to renegotiate.

Definition 3.2. A pair $(\bar{\Pi},\bar{p})$ will be called a bargaining equilibrium of the economy E if the resulting allocation is individually-rational, and if the following hold:

$$\forall s_j \in S, \quad \bar{q}^{s_j} \leqq \bar{p}_j, \tag{3.9}$$

$$\forall T=\{b_k, s_l\} \in \bar{\Pi}, \quad F_T(\bar{q}^{S_l}, \bar{q}^{b_k}) = \bar{p}_l, \tag{3.10}$$

where $\bar{q}$ is the vector of outside option prices based on $(\bar{\Pi},\bar{p})$.

Condition (3.9) is a stability condition: if some of the inequalities do not hold, then the respective sellers wish to change their status (i.e. not to sell at all, or to sell to another buyer). This condition, written in the language of outside options, is similar to the one in the definition of the core, but note that there are no explicit stability conditions for the buyers. The main novelty is condition (3.10): if agents bargain using the current outside options, the result is exactly the market price, so there is no incentive to renegotiate. Note that, by (3.7), this requirement is consistent with (3.9).

The next lemma says that in a bargaining equilibrium no buyer pays more than his outside option price [i.e. a stability condition similar to (3.9) is already implied also for buyers]. The reasoning behind this result is that, if a buyer pays more than her outside option, then condition (3.9) is violated for the seller with which that outside option is feasible.

Lemma 3.3. Let $(\bar{\Pi},\bar{p})$ be a bargaining equilibrium, and assume that buyer b_i buys house h_j in this equilibrium (h_j may be a dummy house if b_i remains single). Then $\bar{p}_j \leqq \bar{q}^{b_i}$.

4. The main results

Theorem 4.1. Every economy E has a bargaining equilibrium.

Theorem 4.2. Each bargaining equilibrium is a competitive equilibrium of E.

The set of bargaining equilibria is, generically, a strict subset of the set of competitive equilibria.

We obtain a converse to the previous theorem by regarding replicated markets. Let $\mathscr{E}$ be the m-fold replica of the economy E, where each of the original agents appears m times, and $m \geqq 2$.

Theorem 4.3. In any replicated market $\mathscr{E}$ each competitive equilibrium is a bargaining equilibrium.

The presence of identical agents – through the property of equal treatment – has the effect that in any negotiation the two outside option prices are equal, so there is really no scope for bargaining. This is exactly our intuition about truly competitive markets, where the price 'is taken as given' i.e. determined by the prices outside. It is quite interesting that one replication is enough for the previous result.

Next we describe a dynamic bargaining process which converges to a bargaining equilibrium for some special initial conditions. This process, which defines an operator on the set of feasible allocations, will also be used to prove the existence of bargaining equilibria without reference to the dynamics. The main tool will be then an algebraic fixed point theorem due to Tarski.

Let Π be an E-partition and let p be a vector or prices for which the allocation $x = x(\Pi, p)$ is individually-rational. Let $T_1, \dots, T_k$ be the matched pairs in Π, where the sellers are labeled such that $\forall i$ with $1 \leqq i \leqq k$, $s_i \in T_i$ (the houses $h_1 \dots h_k$ are those involved in actual transactions). Let now p^1 be the vector of prices resulting after the agents of T_1 bargained using F_{T_1} (and calculated their outside options prices based on $p = p^0$. Let p^2 be the price vector resulting after the agents in T_2 bargained using F_{T_2} (and calculated their outside options prices based on p^1). We proceed in a similar way for $p^3, \dots p^k$, and we define an operator $B_\pi(p) = p^k$. The allocation of singletons remains constant (as in x) throughout this process. Denote by B_π^m, $m \in \mathbb{N}$, the usual powers of the operator B_π (i.e. the procedure used to describe B_π is repeated m times). The operator B_π is not necessarily well-defined for every initial price vector p, but as we shall see in the proofs, it is well defined for all p such that (Π, p) is a competitive equilibrium. For two price vectors, p and p', we write $p \geqq p'$ $(p \leqq p')$, if $\forall j \in \mathbb{R}^{r+1}$, $p_j \geqq p'_j$ $(p_j \leqq p'_j)$. We have the following:

Theorem 4.4. Let (Π, p) be a competitive equilibrium and let p have the following property: There exsists $g \in \mathbb{N}$ such that either $B_\pi^g(p) \leqq B_\pi^{g+1}(p)$ or $B_\pi^g(p) \geqq B_\pi^{g+1}(p)$. Then the sequence $\{B_\pi^m(p)\}_{m \in \mathbb{N}}$ converges to a price vector p^* and (Π, p^*) is a bargaining equilibrium.

There are *many* initial price vectors with the property required for convergence in the previous theorem. These prices display an interesting polarization of interests for the members of one side of the market. Two simple and well-known examples are, for a given partition for which competitive prices exist, the vector of competitive prices with the property that *all* buyers obtain higher utility than the one they obtain in any other competitive allocation feasible for that partition, and the analog vector for the sellers. Such vectors always exist in our model, and this fact together with the previous theorem can supply an independent proof of existence of bargaining equilibria (see Theorem 4.1). Other price vectors with the property required for the convergence are also maximal (minimal) in the sense described above, but with respect to smaller sets – connected components of the set of competitive prices for a given partition.

5. Discussion

The bargaining model we use is general, the only limitation being the special way in which the outside options are calculated. This mode of calculation should not be interpreted literally: it is meant to be just one possible alternative of how agents take into account the existing market conditions. This method requires (apart from complete information) less sophistication than other models where the agents are required to compute the whole future evolution of a market after a possible breakdown in present negotiations. Note that traders use here optimistic assessments, because several agents may have the same outside option in mind. In a sense our traders exhibit a kind of bounded rationality when they assess their bargaining power.

The dynamic procedure exhibited to show convergence to equilibria may give some insights to a type of situations where contracts remain valid for a period of time. At the end of the period renegotiation takes place, based on the new environment and so on (recall the quotation about rock-stars!).

Most strategic models, where the interaction between agents is more finely modeled, are based on simple exchange markets (for example, all sellers have the same type of object and reservation price, all buyers have the same reservation price). It is not entirely clear how these models perform in the presence of finely modeled markets [for possible 'failures' see Osborne and Rubinstein (1990, p. 130, p. 60). For a full market model combined with strategic bargaining see Gale (1986a, b)].

Agents may be uncertain about the possible profits when they write contracts depending on future events, and they might be also uncertain about the conditions outside. This is translated to uncertainty about the set of feasible agreements and about the outside options, respectively. The use of a particular bargaining method may create a divergence of interests: one agent may prefer to sign a contract now, the other may prefer to wait till uncertainty is resolved. Bargaining methods exist which ensure that all agents agree to contract *before* the uncertainty is resolved, thus ensuring Pareto-optimality. For a discussion of this, see Chun and Thomson (1990). The methods described there satisfy the conditions imposed in our model.

If the utility functions are separable, additive and linear in 'money' we can show that bargaining equilibria of the type analyzed here are defined, and exist, also for markets where one agent may be both seller and buyer of the indivisible good [see Shapley and Scarf (1974), Quinzii (1984)]. Unfortunately, results in Moldovanu (1990) imply that this may not hold for general utility functions. The two-sided markets are exceptional in this aspect. For another exceptional role played by games of pairs, in the context of the bargaining set, see Peleg (1963).

6. Proofs

Proof of Theorem 3.1. Let $(\bar{\Pi},p)$ be a competitive equilibrium, and let $T=\{b_i,s_j\}\in\bar{\Pi}$. We must have $\bar{p}_j\leqq q^{b_i}$. Otherwise, by the definition of outside options prices [see (3.1)] buyer b_i can afford to buy another house at current prices (or remain single) while increasing his utility. This is a contradiction to $(\bar{\Pi},\bar{p})$ being a competitive equilibrium. Similarly we must have $\bar{p}_j\geqq q^{s_j}$, otherwise seller s_j can find another buyer willing to pay q^{s_j} (or remain single) and this would increase his utility, again a contradiction. The last two inequalities prove the 'only if' parts of (a) and (b).

For the 'if' part of (a) let $(\bar{\Pi},\bar{p})$ be such that conditions (3.5) and (3.6) are satisfied. Using Kaneko's theorem (Theorem 2.2) it is enough to show that the allocation $x=x(\bar{\Pi},\bar{p})$ is in the core of the economy. Because x is price generated and by condition (3.5) we obtain that x is individually-rational. Assume, on the contrary, that x is not in the core. Then, by the definition of the core, we must have a pair R which can improve upon x. Because x is price generated and individually rational it cannot be the case that $R\in\bar{\Pi}$. Therefore assume w.l.o.g. that $R=[s_j,b_k]$ with $b_k\neq b_i$, where b_i is the partner of s_j in $\bar{\Pi}$ (if s_j is single in $\bar{\Pi}$ the proof is completely analogous). Then there exists a price p_j' such that $p_j'>\bar{p}_j$ and $U_{b_k}(w_{b_k}-p_j',h_j)>U_{b_k}(x^{b_k})$. This implies that $p_j'<p_j^k$ [see (3.2)] and, by the definition of the outside option price for sellers (3.4), we obtain $\bar{p}_j<q^{s_j}$, which is a contradiction to condition (3.6).

For the 'if' part of (b) let $(\bar{\Pi},\bar{p})$ be a competitive equilibrium and let $T=\{b_i,s_j\}\in\bar{\Pi}$. Varying the price in that transaction may change the outside options of other agents. We have to prove that $(\bar{\Pi},(\bar{p}_{-j},p_j))$ remains an equilibrium if $p_j\in[q^{s_j},q^{b_i}]$. Again, it is enough to prove that the price generated allocation $y=y(\bar{\Pi},(\bar{p}_{-j},p_j))$ is in the core. Assume, on the contrary, that a pair R can improve upon y. Obviously $R\neq T$, and if $R\subset N\setminus T$ we obtain a contradiction to the competitiveness of $(\bar{\Pi},\bar{p})$. Therefore assume w.l.o.g. that $R=(s_j,b_k)$ where $b_k\neq b_i$. The proof now continues exactly as at point (a). Q.E.D.

Proof of Lemma 3.3. Denote by $\bar{x}$ the resulting allocation in this equilibrium. If h_j is a dummy house the result is clear because the price of dummy houses is zero, and the outside option prices are always non-negative [see remark after (3.1)]. Assume then that b_i buys the house owned by s_j, and the inequality does not hold. Then we have

$$U_{b_i}(w_{b_i}-\bar{p}_j,h_j)<U_{b_i}(w_{b_i}-\bar{q}^{b_i},h_j)=\max_{f\neq j}U_{b_i}(w_{b_i}-\bar{p}_f,h_f).$$

Assume that the maximum in the last term is achieved for $h_l\neq h_j$. Then there exists $\varepsilon>0$ such that

$$U_{b_i}(w_{b_i}-(\bar{p}_1+\varepsilon),h_l)=U_{b_i}(w_{b_i}-\bar{p}_j,h_j)=U_{b_i}(\bar{x}^{b_i}).$$

Then $p_l^i=\bar{p}_l+\varepsilon$ [see (3.2)], and this is a contradiction because, by the definition of bargaining equilibria, (3.9), and by the definition of outside options, (3.4), we know that $\bar{p}_l\geqq\bar{q}^{s_l}\geqq p_l^i$. Q.E.D.

For the proof of Theorem 4.1 we will use the following elegant fixed point theorem due to Tarski:

Theorem 6.1 [Tarski (1955)]. *Let $(L,\succ)$ be a complete lattice and let A be a monotonic operator on L (i.e. $X\succ Y$ implies $A(X)\succ A(Y)$). Then A has a fixed point.*

Let Π be an E-partition, and define

$$P_\pi=\{p\,|\,(\Pi,p)\text{ is a competitive equilibrium}\}. \tag{6.1}$$

We define an order relation on the set P_π: $p'\succ p$ if and only if $p'\geqq p$.

For two vectors p, p' define two new vectors $p\wedge p'$ and $p\vee p'$, where

$$(p\wedge p')_i=\min(p_i,p_i'), \tag{6.2}$$

$$(p\vee p')_i=\max(p_i,p_i'). \tag{6.3}$$

The next three lemmata will enable the use Tarski's Theorem in our framework. The first one displays a property that is well known in two-sided models [see Demange and Gale (1985), Roth and Sotomayor (1990)]:

Lemma 6.2. Let $P_\pi\neq\varnothing$. Then $(P_\pi,\succ)$ is a complete lattice.

Proof. We have to prove that for each $p,p'\in P_\pi$ also $p\wedge p'$ and $p\vee p'$ belong to P_π. We start by showing that $(\Pi,p\wedge p')$ is indeed a competitive equilibrium. Obviously the resulting allocation x is feasible and individually-rational. It is enough to show that $x \in C(E)$. Assume, on the contrary that this is not the case. Then, there is a pair $T=\{b_i, s_j\}$ which can improve upon x. Assume w.l.o.g. that $(p \wedge p')_j = p_j$. This yields a contradiction to (Π, p) being a competitive equilibrium. Similarly, if $(p \vee p')_j = p'_j$. In the same way we get the result for $(p \wedge p')$. Q.E.D.

Lemma 6.3. Let Π be an E-partition such that $P_\pi \neq \varnothing$. Then, for each $p \in P_\pi$, the operator B_π is well defined (see section 4), and $B_\pi(p) \in P_\pi$.

Proof. Let $T_1 \ldots, T_k$ be the ordered pairs in Π, and let p^1 the vector or prices resulting after the members of T_1 have bargained. Then $p_1^1 = F_{T_1}(q^{s_1}, q^{b_1})$ where the outside options are calculated on the basis of p. By Theorem 3.1 and by the definition of bargaining functions we know that $p_1^1 \in [q^{s_1}, q^{b_1}]$. The other coordinates of p^1 are identical to those in p and therefore we can conclude, by Theorem 3.1, that (Π, p^1) is a competitive equilibrium and $p^1 \in P_\pi$. In a similar way we continue for $p^2, p^3 \ldots$ till we obtain $p^k = B_\pi(p) \in P_\pi$. Q.E.D.

Lemma 6.4. The operator B_π is monotonic on P_π.

Proof. Let $p, p' \in P_\pi$ and assume $p \leqq p'$. The prices of all houses are lower in p than in p', and hence all buyers' outside option prices based on p are lower than those based on p'. (All buyers are advantaged by the transition from p' to p). Because of the monotonicity property of the bargaining functions [see (3.7)] we obtain $B_\pi(p) \leqq B_\pi(p')$. Q.E.D.

Proof of Theorem 4.1. Let Π be an E-partition for which P_π is not empty. Such partitions always exist by Kaneko's Theorem (Theorem 2.2). By Lemma 6.2, P_π is a complete lattice. By Lemma 6.3 we know that $B_\pi(P_\pi) \subset P_\pi$, and by Lemma 6.4 we know that B_π is monotonic. By Tarski's Theorem (Theorem 6.1) we can conclude that B_π has a fixed point. By definition of the operator B_π, for any p a fixed point of B_π, (Π, p) is a bargaining equilibrium. Q.E.D.

Proof of Theorem 4.2. Let $(\bar{\Pi}, \bar{p})$ be a bargaining equilibrium. By the definition of bargaining equilibria, (3.9), by Lemma 3.3 and by definition of the bargaining functions, (3.7), we know that for each pair $T=\{b_i, s_j\} \in \Pi$ we have $\bar{p}_j \in [\bar{q}^{s_j}, \bar{q}^{b_i}]$. By Theorem 3.1 $(\bar{\Pi}, \bar{p})$ is also a competitive equilibrium. Q.E.D.

Proof of Theorem 4.3. Let $(\bar{\Pi}, \bar{p})$ be a competitive equilibrium of a *replica* economy $\mathscr{E}$. It is easy to see that the resulting allocation has the equal treatment property. Let $T=\{b_i, s_j\}$ be any pair in $\bar{\Pi}$. Because of equal treatment the price of all houses of type h_j must be $\bar{p}_j$. Therefore $\bar{q}^{s_j} \geqq \bar{p}_j$. Because in a competitive equilibrium we must have $\bar{q}^{s_j} \leqq \bar{p}_j$ (see Theorem 3.1), we obtain $\bar{q}^{s_j} = \bar{p}_j$. In a similar way we obtain $\bar{q}^{b_i} = \bar{p}_j$. The bargaining range of the pair T is reduced to a single point and $F_T(\bar{q}^{s_j}, \bar{p}^{b_i}) = \bar{p}_j$. Q.E.D.

Proof of Theorem 4.4. Let $B_\pi : P_\pi \to P_\pi$ and let p have the stated property. Then, by Lemma 6.4, the sequence $B_\pi^{g+i}(p), i \in \mathbb{N}$, is a monotonic sequence. P_π is a compact set, and therefore this sequence converges to a limit p^*. The operator B_π is continuous because it is a composition of continuous bargaining functions [see (3.8)], and because the outside options prices are continuous functions of p [see (2.2), (3.1)–(3.4)]. Therefore we have

$$B_\pi(p^*) = B_\pi\left(\lim_{i\to\infty} B_\pi^{g+i}(p)\right) = \lim_{i\to\infty} B_\pi(B_\pi^{g+i}(p)) = \lim_{i\to\infty} B_\pi^{g+i+1}(p) = p^*.$$

Thus p^* is a fixed point of B_π, and (Π, p^*) is a bargaining equilibrium. Q.E.D.

It remains to exhibit price vectors with the monotonicity property used in the previous theorem. As we have seen, for a non-empty P_π the range of the operator B_π is again P_π. Apart from the fact that there may be *generically* several Π with $P_\pi \neq \emptyset$, (this does not happen with linear utility functions!), the structure of the set P_π is quite complicated – these sets may be not connected. Each connected component of a set P_π is by itself a complete lattice (with the same operations as before). The operator B_π preserves connected sets in B_π. This can be seen with the help of Theorem 3.1, where any point in an entire connected interval can be substituted while preserving the stability implied by the competitive equilibrium. We can conclude that in every connected component there are minimal and maximal elements (relative to the order defined on P_π) and, their image under B_π being in the same component, they must exhibit the property required by Theorem 4.4, with $g=1$.

References

Chun, Y., and W. Thomson, 1990, Bargaining with uncertain disagreement points, Econometrica 58, 951–960.

Crawford, V.P. and S.C. Rochford, 1986, Bargaining and competition in matching markets, International Economic Review 27, 329–348.

Davis, M. and M. Maschler, 1965, The kernel of a cooperative game, Naval Research Logistics Quarterly 12, 223–259.

Demange, G. and David Gale, 1985, The strategy structure of two-sided matching markets, Econometrica 53, 873–888.

Gale, Douglas, 1986a, Bargaining and competition Part I: Characterization, Econometrica 54, 785–806.

Gale, Douglas, 1986b, Bargaining and competition Part II: Existence, Econometrica 54, 806–818.
Gale, David and L.S. Shapley, 1962, College admission and the stability of marriage, American Mathematical Monthly 69, 9–15.
Harsanyi, J.C., 1977, Rational behaviour and bargaining equilibrium in games and social situations (Cambridge University Press, Cambridge).
Kaneko, M., 1982, The central assignment game and the assignment markets, Journal of Mathematical Economics 10, 205–232.
Moldovanu, B., 1989, A note on a theorem of Aumann and Drèze, International Journal of Game Theory 18, 471–477.
Moldovanu, B., 1990, Stable bargained equilibria for assignment games without side payments, International Journal of Game Theory 19, 171–190.
Peleg, B., 1963, Bargaining sets of cooperative games without side payments, Israel Journal of Mathematics 1, 197–200.
Osborne, M.J. and M. Rubinstein, 1990, Bargaining and markets (Academic Press, San Diego, CA).
Quinzii, M., 1984, Core and competitive equilibria with indivisibilities, International Journal of Game Theory 13, 41–60.
Rochford, S.C., 1984, Symmetrical pairwise bargained allocations in an assignment market, Journal of Economic Theory 34, 262–281.
Roth, A.E. and M.O. Sotomayor, 1988, Interior points in the core of two-sided matching markets, Journal of Economic Theory 45, 85–101.
Roth, A.E. and O. Sotomayor, 1990, Two sided matching (Cambridge University Press, Cambridge).
Shaked, A. and J. Sutton, 1984, The semi-Walrasian economy, Discussion paper 84/98 (London School of Economics, London).
Shapley, L.S. and M. Shubik, 1971, The assignment game: The core, International Journal of Game Theory 1, 111–130.
Shapley, L.S. and H. Scarf, 1974, On cores and indivisibilities, Journal of Mathematical Economics 1, 23–37.
Tarski, A., 1955, A lattice-theoretical fixpoint theorem and its applications, Pacific Journal of Mathematics 5, 285–309.