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Fused Four-Dimensional Real Division Algebras

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Dickson's construction obtains the quaternions as pairs of complex numbers with a specific multiplication. We modify his construction to obtain four-dimensional division algebras from pairs of two-dimensional real division algebras and compare this class of algebras to other known classes. © 1994 Academic Press, Inc.

1. INTRODUCTION

Dickson's Construction. Dickson [8] obtained the complex numbers as pairs of real numbers, the quaternions as pairs of complex numbers, and the Cayley numbers as pairs of quaternions by defining addition and scalar multiplication componentwise and multiplication and conjugation by

$$(a,b)(c,d) = (ac - \bar{d}b, da + b\bar{c}) \qquad \text{and} \qquad \overline{(a,b)} = (\bar{a}, -b),$$

respectively, where $\bar{a} = a$ if a is a real number. (See also [5, 7, 10].)

We consider two modifications of Dickson's construction, each yielding four-dimensional real algebras from two-dimensional algebras. The first modification replaces the complex numbers with an arbitrary two-dimensional real algebra $\mathscr{A} = (\mathbb{R}^2, \cdot\,)$ and replaces conjugation by a nonsingular linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$. On the vector space $\mathbb{R}^2 \oplus \mathbb{R}^2$ we define multiplication by

$$(a,b)(c,d) = (a \cdot c - b \cdot \varphi(d), a \cdot d + b \cdot \varphi(c)).$$

Note that the order of the letters on the right is alphabetical and does not correspond to the order in Dickson's construction. This construction defines a four-dimensional algebra, which we call the *φ-algebra* $\mathscr{A}^{\varphi}$. The two-dimensional algebra $\mathscr{A}$ is called its *base algebra* and is isomorphic to the subalgebra of pairs $(a, 0)$, $a \in \mathscr{A}$.

The second modification involves two two-dimensional real algebras $\mathscr{A} = (\mathbb{R}^2, \cdot\,)$ and $\mathscr{B} = (\mathbb{R}^2, \times\,)$ with the following multiplication tables with

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respect to a basis $\{u, v\}$ for $\mathbb{R}^2$:

	$\cdot$	u	v
$\mathscr{A}$	u	$a_{11}u + b_{11}v$	$a_{12}u + b_{12}v$
	v	$a_{21}u + b_{21}v$	$a_{22}u + b_{22}v$

	$\times$	u	v
$\mathscr{B}$	u	$c_{11}u + d_{11}v$	$c_{12}u + d_{12}v$
	v	$c_{21}u + d_{21}v$	$c_{22}u + d_{22}v$

We define multiplication on the direct sum $\mathscr{A} \oplus \mathscr{B}$ by

$$(a, b)(c, d) = (a \cdot c - b \times d, a \cdot d + b \times c).$$

It is easy to check that this construction makes $\mathscr{A} \oplus \mathscr{B}$ into a real algebra. Setting $e_1 = (u, 0), e_2 = (v, 0), e_3 = (0, u), e_4 = (0, v)$, it is easy to verify that the algebra $\mathscr{A} \oplus \mathscr{B}$ has the multiplication table

	e_1	e_2	e_3	e_4
e_1	$a_{11}e_1 + b_{11}e_2$	$a_{12}e_1 + b_{12}e_2$	$a_{11}e_3 + b_{11}e_4$	$a_{12}e_3 + b_{12}e_4$
e_2	$a_{21}e_1 + b_{21}e_2$	$a_{22}e_1 + b_{22}e_2$	$a_{21}e_3 + b_{21}e_4$	$a_{22}e_3 + b_{22}e_4$
e_3	$c_{11}e_3 + d_{11}e_4$	$c_{12}e_3 + d_{12}e_4$	$-c_{11}e_1 - d_{11}e_2$	$-c_{12}e_1 - d_{12}e_2$
e_4	$c_{21}e_3 + d_{21}e_4$	$c_{22}e_3 + d_{22}e_4$	$-c_{21}e_1 - d_{21}e_2$	$-c_{22}e_1 - d_{22}e_2$

We call this table a *standard table* for the *$\mathscr{A}$-based fused algebra* $\mathscr{A} \oplus \mathscr{B}$. Again, $\mathscr{A}$ is isomorphic to the subalgebra of pairs $(a, 0)$.

Every $\mathscr{A}$-based φ-algebra can be obtained as an $\mathscr{A}$-based fused algebra by defining $\times$ by $a \times b = a \cdot \varphi(b)$. We will see in Section 3 that not every fused algebra is a φ-algebra.

2. The Division Algebra Condition

Since fused algebras are more general than φ-algebras, we consider only the former. Our main theorem characterizing fused division algebras depends on two lemmas.

Lemma 1. *The quadratic form* $\Psi(x_1, x_2, x_3, x_4) = (Ax_1^2 + Bx_1x_2 + Cx_2^2 + Dx_3^2 + Ex_3x_4 + Fx_4^2)^2 + (Gx_1x_3 + Hx_1x_4 + Ix_2x_3 + Jx_2x_4)^2$ *is positive definite if and only if*

$$B^2 - 4AC < 0, \qquad E^2 - 4DF < 0, \qquad AD > 0.$$

Proof. (1) Suppose that $B^2 - 4AC \geq 0$. Then the following nontrivial substitutions make the form ψ zero:

(i) If $A \neq 0$, let $x_1 = -B \pm \sqrt{B^2 - 4AC}$, $x_2 = 2A$, $x_3 = x_4 = 0$.

(ii) If $A = 0$, let $x_1 = 1$, $x_2 = x_3 = x_4 = 0$.

Similarly, if $E^2 - 4DF \geq 0$, the form is not positive definite.

(2) Now suppose that $B^2 - 4AC < 0$ and $E^2 - 4DF < 0$ (so that neither A nor D is zero). Complete the squares on x_1, x_2 and x_3, x_4 to rewrite the form as

$$\gamma(y_1, y_2, y_3, y_4) = \left[(y_1^2 + y_2^2) + \varepsilon(y_3^2 + y_4^2)\right]^2 + (Ky_1y_3 + Ly_1y_4 + My_2y_3 + Ny_2y_4)^2,$$

where

$$y_1 = \sqrt{|A|}\left(x_1 + \frac{B}{2A}x_2\right), \qquad y_2 = \frac{\sqrt{4AC - B^2}}{2\sqrt{|A|}}x_2$$

$$y_3 = \sqrt{|D|}\left(x_3 + \frac{E}{2D}x_4\right), \qquad y_4 = \frac{\sqrt{4DF - E^2}}{2\sqrt{|D|}}x_4$$

and $\varepsilon = AD/|AD| = \pm 1$. Clearly ψ is positive definite if and only if γ is.

If $AD > 0$ so that $\varepsilon = 1$, then γ is positive definite. On the other hand, suppose that $AD < 0$, so that $\varepsilon = -1$. If $\begin{vmatrix} K & L \\ M & N \end{vmatrix} = 0$, then there exists $(y_1^0, y_2^0) \neq (0,0)$ such that

$$Ky_1^0 + My_2^0 = Ly_1^0 + Ny_2^0 = 0$$

so that

$$Ky_1^0y_3 + Ly_1^0y_4 + My_2^0y_3 + Ny_2^0y_4 = 0$$

for any (y_3, y_4) and, in particular, for $(y_3, y_4) = (y_1^0, y_2^0)$. Then $\gamma(y_1^0, y_2^0, y_1^0, y_2^0) = 0$, so γ and ψ are not positive definite.

If $\begin{vmatrix} K & L \\ M & N \end{vmatrix} \neq 0$, then for any pair $(y_1, y_2) \neq (0,0)$, we will have $(Ky_1 + My_2, Ly_1 + Ny_2) \neq (0,0)$. Then $Ky_1y_3 + Ly_1y_4 + My_2y_3 + Ny_2y_4 = (Ky_1 + My_2)y_3 + (Ly_1 + Ny_2)y_4 = 0$ if and only if $(y_3, y_4) = c(Ly_1 + Ny_2, -Ky_1 - My_2)$, for some $c \neq 0$.

Take (y_3, y_4) of this form. Then $\gamma(y_1, y_2, y_3, y_4) = 0$ if and only if

$$\begin{aligned} 0 &= (y_1^2 + y_2^2) - (y_3^2 + y_4^2) \\ &= \left[1 - (K^2 + L^2)c^2\right] y_1^2 - 2(KM + LN)c^2 y_1 y_2 + \left[1 - (M^2 + N^2)c^2\right] y_2^2 \\ &\equiv \delta_c(y_1, y_2). \end{aligned}$$

This quadratic has discriminant

$$\rho(c) = 4\left[-(KN - LM)^2 c^4 + (K^2 + L^2 + M^2 + N^2)c^2 - 1\right].$$

Viewed as a quadratic in c^2, $\rho(c)$ has discriminant 16Δ, where

$$\begin{aligned} \Delta &= (K^2 + L^2 + M^2 + N^2)^2 - 4(KN - LM)^2 \\ &= \left[(K - N)^2 + (L + M)^2\right]\left[(K + N)^2 + (L - M)^2\right] > 0, \end{aligned}$$

and, hence, $\rho(c) = 0$ if and only if

$$c = \pm\sqrt{(K^2 + L^2 + M^2 + N^2 \pm \sqrt{\Delta})/2(KN - LM)^2} \neq 0.$$

But when $\rho(c) = 0$, there exists a pair $(y_1, y_2) \neq (0, 0)$ such that $\delta_c(y_1, y_2) = 0$ and hence $\gamma(y_1, y_2, y_3, y_4) = 0$. Thus, γ and ψ are not positive definite in this case.

The next lemma can be proved by expanding the left-hand side.

LEMMA 2.

$$\begin{vmatrix} p & q & w & x \\ r & s & y & z \\ -w & -x & p & q \\ -y & -z & r & s \end{vmatrix} = [(ps - qr) - (wz - xy)]^2 + (sw - qy - rx + pz)^2.$$

THEOREM 3. *A fused algebra $\mathscr{A} \oplus \mathscr{B}$ is a division algebra if and only if $\mathscr{A}$ and $\mathscr{B}$ are division algebras and in any standard table for $\mathscr{A} \oplus \mathscr{B}$*

$$(a_{11}b_{12} - b_{11}a_{12})(c_{11}d_{12} - d_{11}c_{12}) < 0.$$

Proof. Let L_α denote left translation by the element $\alpha = \sum x_i e_i$. Then the matrix of L_α with respect to the basis $\{e_1, e_2, e_3, e_4\}$ is

$$\begin{bmatrix} p & q & w & x \\ r & s & y & z \\ -w & -x & p & q \\ -y & -z & r & s \end{bmatrix},$$

where

$$\begin{aligned} p &= a_{11}x_1 + a_{21}x_2, & q &= a_{12}x_1 + a_{22}x_2 \\ r &= b_{11}x_1 + b_{21}x_2, & s &= b_{12}x_1 + b_{22}x_2 \\ w &= c_{11}x_3 + c_{21}x_4, & x &= c_{12}x_3 + c_{22}x_4 \\ y &= d_{11}x_3 + d_{21}x_4, & z &= d_{12}x_3 + d_{22}x_4 \end{aligned}$$

Let

$$A_1 = \begin{vmatrix} a_{11} & b_{11} \\ a_{12} & b_{12} \end{vmatrix}, \quad A_2 = \begin{vmatrix} a_{12} & b_{12} \\ a_{21} & b_{21} \end{vmatrix}, \quad A_3 = \begin{vmatrix} a_{21} & b_{21} \\ a_{22} & b_{22} \end{vmatrix}, \quad A_4 = \begin{vmatrix} a_{22} & b_{22} \\ a_{11} & b_{11} \end{vmatrix}$$

$$B_1 = \begin{vmatrix} c_{11} & d_{11} \\ c_{12} & d_{12} \end{vmatrix}, \quad B_2 = \begin{vmatrix} c_{12} & d_{12} \\ c_{21} & d_{21} \end{vmatrix}, \quad B_3 = \begin{vmatrix} c_{21} & d_{21} \\ c_{22} & d_{22} \end{vmatrix}, \quad B_4 = \begin{vmatrix} c_{22} & d_{22} \\ c_{11} & d_{11} \end{vmatrix}.$$

Using Lemma 2 we find

$$\begin{aligned} \det L_\alpha = \{ & [A_1 x_1^2 - (A_2 + A_4) x_1 x_2 + A_3 x_2^2] \\ & - [B_1 x_3^2 - (B_2 + B_4) x_3 x_4 + B_3 x_3^2] \}^2 \\ & + \{ G x_1 x_3 + H x_1 x_4 + I x_2 x_3 + J x_2 x_4 \}^2. \end{aligned}$$

From [4] the division algebra conditions for the algebras $\mathscr{A}$ and $\mathscr{B}$ are, respectively,

$$(A_2 + A_4)^2 - 4A_1 A_3 < 0 \qquad \text{and} \qquad (B_2 + B_4)^2 - 4B_1 B_3 < 0.$$

Thus, by Lemma 1, $\mathscr{A} \oplus \mathscr{B}$ is a division algebra if and only if $\mathscr{A}$ and $\mathscr{B}$ are division algebras and $A_1 B_1 < 0$.

COROLLARY 4. *An $\mathscr{A}$-based φ-algebra is a division algebra if and only if $\mathscr{A}$ is a division algebra and* $\det \varphi < 0$.

Proof. Recall that a φ-algebra is a fused algebra with $a \times b = a \cdot \varphi(b)$. It is not hard to verify that for an $\mathscr{A}$-based φ-algebra the determinants in

Theorem 1 are related as

$$B_1 = (\det\varphi)A_1; \qquad B_3 = (\det\varphi)A_3; \qquad B_2 + B_4 = (\det\varphi)(A_2 + A_4),$$

where $\mathscr{B} = (\mathbb{R}^2, \times)$. It follows that if $\det\varphi \neq 0$, then $\mathscr{A}$ is a division algebra if and only if $\mathscr{B}$ is. The condition $A_1B_1 < 0$ is $(\det\varphi)A_1^2 < 0$ and the result follows.

3. Classes of Four-Dimensional Division Algebras

The first example in this section shows that the class of fused algebras is larger than the class of φ-algebras.

Example 5. Suppose a four-dimensional algebra is both a fused algebra $\mathscr{A} \oplus \mathscr{B}$ and a φ-algebra $\mathscr{C}^{\varphi}$. Let $\cdot$, $\times$, and $\circ$ denote the multiplications in $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$, respectively. Then for all a, b, c, and d in $\mathbb{R}^2$

$$\begin{aligned}(a,b)(c,d) &= (a \cdot c - b \times d, a \cdot d + b \times c)\\ &= (a \circ c - b \circ \varphi(d), a \circ d + b \circ \varphi(c)).\end{aligned}$$

Let $b = d = 0$. Then we have

$$(a \cdot c, 0) = (a \circ c, 0) \qquad \text{for all } a \text{ and } c.$$

Thus, $\cdot$ and $\circ$ coincide and $\mathscr{A} = \mathscr{C}$.

Now let $a = d = 0$. Then we have for all b and c:

$$(0, b \times c) = (0, b \circ \varphi(c)).$$

Thus, the $\times$ and $\circ$ multiplications are related by

$$b \times c = b \circ \varphi(c) \qquad \text{for all } b \text{ and } c.$$

Now consider an algebra $\mathscr{C} \oplus \mathscr{B} = \mathscr{C}^{\varphi}$ which has $\mathscr{C}$ as its only two-dimensional subalgebra, where $\mathscr{C}$ is isomorphic to $\mathbb{C}$. Then the only possible base for the algebra is $\mathscr{C}$ and the operation $\circ$ is ordinary complex multiplication. Thus, if φ has the matrix

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

with respect to the basis $\{\mathbf{1}, \mathbf{i}\}$ for $\mathbb{R}^2$, it is easy to check that $\mathscr{B}$ has multiplication table:

$\times$	$\mathbf{1}$	$\mathbf{i}$
$\mathbf{1}$	$p_{11}\mathbf{1} + p_{21}\mathbf{i}$	$p_{12}\mathbf{1} + p_{22}\mathbf{i}$
$\mathbf{i}$	$-p_{21}\mathbf{1} + p_{11}\mathbf{i}$	$-p_{22}\mathbf{1} + p_{12}\mathbf{i}$

On the other hand, consider the algebra $\mathscr{C} \oplus \mathscr{B}$ with $\mathscr{B} = (\mathbb{R}^2, \times)$ given by

$\times$	$\mathbf{1}$	$\mathbf{i}$
$\mathbf{1}$	$\mathbf{1} + \mathbf{i}$	$-\mathbf{i}$
$\mathbf{i}$	$-\mathbf{1} + 2\mathbf{i}$	$\mathbf{1}$

It is easy to check that $\mathscr{B}$ is a division algebra, as is the fused algebra $\mathscr{C} \oplus \mathscr{B}$. We now show that $\mathscr{C} \oplus \mathscr{B}$ has $\mathscr{C}$ as its only two-dimensional subalgebra.

Note that e_1 is a left identity. If α is idempotent, then the equation $(e_1 - \alpha)\alpha = 0$ implies that $e_1 = \alpha$. Thus, e_1 is the only idempotent. Since any two-dimensional subalgebra of $\mathbb{C} \oplus \mathscr{B}$ is a division algebra (and, hence, contains an idempotent), such a subalgebra must contain e_1. Suppose that $\{e_1, \alpha\}$ is a basis for such a subalgebra, where $\alpha = ae_1 + be_2 + ce_3 + de_4$. Then since $\alpha e_1 = ee_1 + f\alpha$ for some real numbers e and f, we have

$$ae_1 + be_2 + (c - d)e_3 + (c + 2d)e_4 = (e + fa)e_1 + fbe_2 + fce_3 + fde_4.$$

Equating the e_3 and e_4 coefficients, we find that $c = d = 0$, so α is an element of $\mathscr{C}$. Thus, $\mathscr{C}$ is the only two-dimensional subalgebra of $\mathscr{C} \oplus \mathscr{B}$.

Finally, by comparing the $\mathbf{1} \times \mathbf{1}$ and $\mathbf{i} \times \mathbf{1}$ entries of the two $\times$ multiplication tables, we see that $\mathscr{C} \oplus \mathscr{B}$ is not a φ-algebra.

There are several other known classes of four-dimensional real division algebras: quadratic algebras are treated in [9]; $\mathbb{C}$-associative algebras and algebras that satisfy two $\mathbb{C}$-associative conditions are classified in [1, 2], respectively; and rotational scaled quaternion algebras appear in [3]. We now turn to an examination of the intersections of these classes with the class of fused division algebras. The next example shows that there are four-dimensional fused algebras that lie in none of the other classes.

EXAMPLE 6. Let $\mathscr{A}$ and $\mathscr{B}$ be the following algebras:

	$\cdot$	u	v
$\mathscr{A}$	u	u	v
	v	$-v$	u

	$\times$	u	v
$\mathscr{B}$	u	u	$-v$
	v	$-v$	$-u-v$

It is not hard to check using Theorem 3 that the fused algebra $\mathscr{A} \oplus \mathscr{B}$ is a division algebra. A division algebra that is quadratic or $\mathbb{C}$-associative or satisfies two $\mathbb{C}$-associative conditions has an identity which is necessarily its only idempotent. Thus the algebra $\mathscr{A} \oplus \mathscr{B}$ is of none of these types, because e_1 is an idempotent but it is not the identity. We now verify that $\mathscr{A} \oplus \mathscr{B}$ is also not a rotational scaled quaternion algebra.

Suppose $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ were a basis with respect to which $\mathscr{A} \oplus \mathscr{B}$ has a table in standard form for a rotational scaled quaternion algebra [3]:

	α_1	α_2	α_3	α_4
α_1	$q\alpha_1$	$r\alpha_2$	$u\alpha_3$	$u\alpha_4$
α_2	$s\alpha_2$	$t\alpha_1$	$v\alpha_4$	$-v\alpha_3$
α_3	$w\alpha_3$	$x\alpha_4$	$y\alpha_1$	$z\alpha_2$
α_4	$w\alpha_4$	$-x\alpha_3$	$-z\alpha_2$	$y\alpha_1$

Let $\alpha_1 = ae_1 + be_2 + ce_3 + de_4$. The equation $\alpha_1^2 = q\alpha_1$ yields the system:

$$a^2 + b^2 - c^2 + d^2 = qa \tag{1}$$

$$2cd + d^2 = qb \tag{2}$$

$$2ac = qc \tag{3}$$

$$-2bc - bd = qd \tag{4}$$

Since $q \neq 0$ in a division algebra, Eqs. (2) and (4) imply that $b = d = 0$. If $c \neq 0$, then $q = 2a$ by Eq. (3) and Eq. (1) yields a contradiction. Thus, $c = 0$, $q = a$, and the only choices for α_1 are of the form ae_1.

The matrices of right translation by α_1 with respect to the $\{e_i\}$ and $\{\alpha_i\}$ bases are respectively

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{bmatrix}.$$

Consideration of eigenvalues reveals that $w = -a$, so that α_3 and α_4 are in the eigenspace of $-a$, which is spanned by e_2 and e_4. Then

$$\alpha_i = b_i e_2 + d_i e_4 \qquad \text{for } i = 3, 4$$

so

$$\alpha_i^2 = b_i^2 e_1 + d_i^2(e_1 + e_2) - b_i d_i e_4 = y\alpha_1 = yae_1 \qquad \text{for } i = 3, 4.$$

It follows that $d_3 = d_4 = 0$, whence α_3 and α_4 are dependent, a contradiction. Thus $\mathscr{A} \oplus \mathscr{B}$ is also not a rotational scaled quaternion algebra.

The algebra of quaternions is both a fused and a rotational scaled quaternion division algebra. The next example shows that it is not the only division algebra lying in the intersection of these classes.

Example 7. Let $\mathscr{A}$ and $\mathscr{B}$ be algebras given by the following multiplication tables:

	$\cdot$	u	v
$\mathscr{A}$	u	u	v
	v	$-v$	u
	$\times$	u	v
$\mathscr{B}$	u	u	$-v$
	v	$-v$	$-u-v$

It is not hard to check by Theorem 3 that the fused algebra $\mathscr{A} \oplus \mathscr{B}$ is a division algebra. It is not the algebra of quaternions, since it has an idempotent which is not its identity. The standard table for $\mathscr{A} \oplus \mathscr{B}$ is not a standard rotational scaled quaternion table. However, the basis change $\alpha_1 = e_1, \alpha_2 = e_3, \alpha_3 = e_2, \alpha_4 = e_4$, yields a rotational scaled quaternion table with $q = r = s = 1$, $t = -1$, $u = 1$, $v = w = x = -1$, and $y = z = 1$.

Note that a rotational scaled quaternion table with $q = u = r$, $s = -t = v$, and $w = -x = -y = z$ also represents a fused division algebra.

Our next example shows that there is also a nontrivial intersection for the class of fused division algebras and the class of division algebras satisfying two $\mathbb{C}$-associative conditions [2].

EXAMPLE 8. An algebra is both middle and right $\mathbb{C}$-associative if and only if it has a basis $\{1, i, J, iJ\}$ yielding a multiplication table of the form:

	1	i	J	iJ
1	1	i	J	iJ
i	i	-1	iJ	$-J$
J	J	$-iJ$	$p + qi + J$	$q - pi + iJ$
iJ	iJ	J	$-q + pi - iJ$	$p + qi + J$

and is a division algebra if and only if $4p + 1 < 4q^2$. Let $\mathscr{B}$ denote the algebra

$\mathscr{B}$

$\times$	u	v
u	u	$-u - v$
v	v	$u - v$

In the algebra $\mathbb{C} \oplus \mathscr{B}$, set $\mathbf{1} = e_1$, $\mathbf{i} = e_3$, and $\mathbf{J} = \frac{1}{2}e_1 + e_2$. Then it is easy to check that the basis $\{\mathbf{1}, \mathbf{i}, \mathbf{J}, \mathbf{iJ}\}$ yields a middle and right $\mathbb{C}$-associative table in which $p = -\frac{5}{4}$ and $q = 0$, so $\mathbb{C} \oplus \mathscr{B}$ is a division algebra. Finally, the algebra is not the quaternions since it is not associative: for example, $e_3(e_2e_2) = -e_3$ but $(e_3e_2)e_2 = 2e_4$.

Our last results deal with the intersection of the class of fused division algebras with the classes of quadratic and $\mathbb{C}$-associative algebras.

THEOREM 9. *The quaternion algebra is the only quadratic fused division algebra.*

Proof. Let $\mathscr{A} \oplus \mathscr{B}$ be a quadratic fused division algebra. Then $\mathscr{A} \oplus \mathscr{B}$ has an identity, which is its unique idempotent. Viewed as a subalgebra of $\mathscr{A} \oplus \mathscr{B}$, $\mathscr{A}$ is a division algebra and hence contains an idempotent, which must be the identity of $\mathscr{A} \oplus \mathscr{B}$. Since $\mathbb{C}$ is the only two-dimensional real division algebra with identity, $\mathscr{A}$ must be isomorphic to $\mathbb{C}$, so $\mathscr{A} \oplus \mathscr{B}$ has a table of the form

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	e_4	$-e_3$
e_3	e_3	$c_{12}e_3 + d_{12}e_4$	$-e_1$	$-c_{12}e_1 - d_{12}e_2$
e_4	e_4	$c_{22}e_3 + d_{22}e_4$	$-e_2$	$-c_{22}e_1 - d_{22}e_2$

In a quadratic algebra, every element α satisfies a quadratic

$$\alpha^2 = A\alpha + Be_1$$

for real numbers A and B. Thus, in particular,

$$A(e_2 + e_3) + Be_1 = (e_2 + e_3)^2 = -2e_1 + c_{12}e_3 + (1 + d_{12})e_4,$$

whence $c_{12} = 1 + d_{12} = 0$.

Similarly, a consideration of $(e_2 + e_4)^2$ yields $c_{22} - 1 = d_{22} = 0$. Thus, $\mathscr{A} \oplus \mathscr{B}$ is the quaternion algebra.

COROLLARY 10. *The quaternion algebra is the only* $\mathbb{C}$*-associative fused division algebra.*

Proof. Let $\mathscr{A} \oplus \mathscr{B}$ be a $\mathbb{C}$-associative fused division algebra. Then $\mathscr{A} \oplus \mathscr{B}$ has a table of the form

	1	i	J	iJ
1	1	i	J	iJ
i	i	-1	iJ	$-J$
J	J	$-iJ$	$p + qi$	$q - pi$
iJ	iJ	J	$-q + pi$	$p + qi$

As a $\mathbb{C}$-associative algebra, $\mathscr{A} \oplus \mathscr{B}$ has an identity, so it has a fused table of the form in Theorem 9. Note that $\mathscr{A} \oplus \mathscr{B}$ contains at least two distinct copies of $\mathbb{C}$: namely, the ones with bases $\{e_1, e_2\}$ and $\{e_1, e_3\}$. By Lemma 13 of [1], the presence of two copies of $\mathbb{C}$ forces q to be zero. Then by Theorem 18 of [1], the algebra is quadratic and the result follows.

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