JOURNAL OF ALGEBRA **168**, 988–1026 (1994)

Homological Properties of (Graded) Noetherian PI Rings

J. T. STAFFORD* AND J. J. ZHANG†,‡

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Communicated by the Editors

Received April 29, 1993

Let R be a connected, graded, Noetherian PI ring. If $\mathrm{injdim}(R) = n < \infty$, then we prove that R is Auslander–Gorenstein and Cohen–Macaulay, with Gelfand–Kirillov dimension equal to n. If $\mathrm{gldim}(R) = n < \infty$, then R is a domain, finitely generated as a module over its centre and a maximal order in its quotient division ring. Similar results hold if R is assumed to be local rather than connected graded. Alternatively, suppose that R is a Noetherian PI ring with $\mathrm{gldim}(R) < \infty$ such that $\mathrm{hd}(R/M_1) = \mathrm{hd}(R/M_2)$ for any two maximal ideals M_i in the same clique. Then, R is a direct sum of prime rings, is integral over its centre, and is Auslander–Gorenstein. If R is a prime ring, then the centre $Z(R)$ of R is a Krull domain and R equals its trace ring TR. Moreover, $\mathrm{hd}(R/M) = \mathrm{height}(M)$, for every maximal ideal M of R. © 1994 Academic Press, Inc.

1. INTRODUCTION

For the purposes of this paper, a ring R is graded if it is $\mathbb{N}$-graded; that is $R = \oplus_{j \geq 0} R_j$ with $R_i R_j \subseteq R_{i+j}$ for all i and j. Of course, any ungraded ring can be regarded as a graded ring concentrated in degree zero. Throughout, R^+ will denote $\oplus_{j \geq 1} R_j$. A graded ring R is *connected graded* if $R_0 = k$ is a central subfield of R and each R_i is a finite dimensional k-vector space.

The study of connected graded rings has gained prominence recently, particularly through the study of the regular graded rings of [AS] and in the more general theory of quantum groups. The regularity condition used in [AS] is defined as follows: Let R be a connected graded ring. Then we define R to be Gor_0 or *Gorenstein in dimension zero* if R has finite (left and right) injective dimension, $\mathrm{injdim}(R) = n < \infty$, and (again on either side) $\mathrm{Ext}^m_R(R/R^+, R) = 0$ for all $m < n$. The regular rings of [AS], which we will term *AS regular rings*, are then the connected graded Gor_0 rings of finite global and Gelfand–Kirillov dimension. In [ATV1, ATV2] the AS

*The research of the first author was supported in part by an NSF grant. E-mail address: jts@math.lsa.umich.edu.

†E-mail address: jzhang@math.lsa.umich.edu.

‡Present address: Department of Mathematics, University of Washington, Seattle, WA 98195.

0021-8693/94 $6.00

regular rings of global dimension 3 are classified. In particular, they are always Noetherian. More significantly, the approach of [ATV] exhibits a fascinating interrelationship between these algebras and projective, elliptic curves. If the ring also satisfies a polynomial identity (PI), then there are interesting applications to the study of Brauer groups (see [Ar, Section 5]).

Of course, this would suggest that, even among connected graded PI rings of finite global dimension, the AS regular rings are very special. Yet, one of the main applications of this paper (Corollary 6.3) shows that this is not the case:

THEOREM 1.1. *Let R be any connected, graded, fully bounded Noetherian ring of finite injective dimension n. Then R is* Gor_0, *with* $GK\dim(R) = K\dim(R) = n$. *Indeed, R is even Auslander–Gorenstein and Cohen–Macaulay.*

The extra terms are defined as follows: Let R be a Noetherian graded ring with $\mathrm{injdim}(R) = n < \infty$. Then, R is *Auslander–Gorenstein* if, for every finitely generated R-module M and submodule $N \subseteq \mathrm{Ext}^j_R(M, R)$, one has $\mathrm{Ext}^i_R(N, R) = 0$ for all $i < j$. An Auslander–Gorenstein ring of finite global dimension will be called *Auslander–regular*. If $j(M) = \min\{j : \mathrm{Ext}^j_R(M, R) \neq 0\}$, then R is *Cohen–Macaulay* provided that $GK\dim(R) < \infty$ and $j(M) + GK\dim(M) = GK\dim(R)$ holds for every finitely generated R-module M.

Theorem 1.1 is false if R is not assumed to be Noetherian; a counterexample is provided by the PI ring $k\{x, y\}/(xy)$.

The Auslander–Gorenstein condition is one that has proved to be very useful elsewhere as it permits one to make effective use of homological techniques in non-commutative ring theory: see [Bj2] for a survey of some of these applications. If R is an AS regular ring of global dimension 3, then Levasseur has proved that R is both Auslander–Gorenstein and Cohen–Macaulay (see [Lv1]). However, Levasseur's proof uses the classification of these algebras given in [ATV1]. While the Auslander-Gorenstein condition is a very useful concept, it seems that the Cohen–Macaulay hypothesis is considerably stronger; for example, while many Artinian algebras are Auslander-regular, the only Artinian algebras that are Auslander-regular and Cohen–Macaulay are the semi-simple algebras. For connected graded rings, the Cohen–Macaulay condition is crucial in proving the following.

COROLLARY 1.2. *Let R be any connected graded, FBN ring of finite global dimension. Then R is a domain and a maximal order in its quotient division ring. If R is PI, then R is equal to its trace ring and is finite as a module over its centre Z(R). Consequently, Z(R) is a Noetherian, integrally closed domain.*

For applications of Theorem 1.1 and Corollary 1.2 to Skyanin algebras of dimension $n \geq 4$, see [TV].

As indicated in the abstract, analogues of Theorem 1.1 and Corollary 1.2 will also hold for local Noetherian PI rings. (If R is a ring, write $J(R)$ for the Jacobson radical of R. Then, R is *(semi)-local* provided that $R/J(R)$ is (semi)-simple Artinian.) The conclusion that R is a domain gives a partial answer to a question of Ramras (see [GW1, Question 5, p. 286] for a survey of the results on this question). In fact, the results of this paper are proved for a considerably more general class of Noetherian PI rings. Let R be a graded Noetherian PI ring with $\text{injdim}(R) = n < \infty$. Then R will be called *right (graded) injectively smooth* if $\text{Ext}^n_R(S, R) \neq 0$ for all simple, (graded) right R-modules. By Lemma 3.12, a graded Noetherian PI ring of finite injective dimension that is either connected graded or local will automatically be graded injectively smooth. The correct context for Theorem 1.1 is for the class of right graded injectively smooth Noetherian PI rings. However, in this generality the Gelfand–Kirillov dimension need not be defined and to circumvent this problem we use the (Rentschler–Gabriel) Krull dimension. Thus, an Auslander–Gorenstein ring R is called *(graded) Macaulay* if $j(M) + K\dim(M) = K\dim(R)$ holds for every finitely generated, (graded) R-module M. A more appropriate phrase for a Macaulay ring is "an equidimensional, Krull Cohen–Macaulay ring," since in the commutative case our definition of a Macaulay ring forces every maximal ideal to have the same height. We have used the phrase Macaulay since the definition of a (commutative) Macaulay ring in [Na] includes the equidimensionality assumption. If M is a finitely generated module over a connected graded, FBN ring R, then $GK\dim(M) = K\dim(M)$ (see Lemma 6.1) and so the Cohen–Macaulay and Macaulay concepts (with or without the prefix "graded") will all coincide for these rings. In general, however, these concepts are all distinct (see Section 4). The main theorem of the paper (Theorem 3.10) states the following.

THEOREM 1.3. *Let R be a graded, Noetherian PI ring that is right graded injectively smooth. Then, R is an Auslander–Gorenstein, graded Macaulay ring the* $K\dim(R) = \text{injdim}(R)$. *In particular, R is also left graded injectively smooth.*

Once again, Theorem 1.3 allows one to prove strong structure theorems for graded injectively smooth rings of finite global dimension and, by using localization techniques, the applications hold more generally. Write $\text{hd}(M)$ for the homological dimension of a module M. Generalizing the notions from [BwH] an (ungraded) PI ring R is called *homologically homogeneous* if $\text{gldim}(R) < \infty$ and $\text{hd}(R/M) = \text{hd}(R/M')$ whenever M and M' are maximal ideals of R lying in the same clique. The definition of homologically homogeneous rings in [BwH] is slightly different from that given here, since that paper is only concerned with rings integral over their centres. However, our extra generality is a mirage since the main result of Section 5 (see Theorem 5.6 and its corollaries) shows the following.

THEOREM 1.4. *Suppose that R is a Noetherian PI ring that is homologically homogeneous. Then*

(i) *R is a direct sum of prime rings, is integral over its centre, and is Auslander-regular.*

(ii) *Let Ω be a clique of maximal ideals of R. The Ω is finite and the localization R_Ω of R at Ω is Auslander-regular and Macaulay.*

(iii) *Assume that R is a prime ring. Then the centre $Z(R)$ of R is a Krull domain, R is integral over $Z(R)$, and R equals its trace ring.*

(iv) *If M is a maximal ideal of R, then* $j(M) = \mathrm{hd}(M) = \mathrm{height}(M)$.

Almost all of the conclusions of Theorem 1.4 will fail for Noetherian PI rings that are just Auslander-regular (see, in particular, Example 5.11 and [St3, Example 3.5]).

The definitions given above, particularly of smoothness and Macaulay, are forced on us by the fact that one cannot localize at maximal ideals of a non-commutative ring. In comparison with the commutative theory, one should think of an injectively smooth ring as being the analogue of a local commutative ring of finite injective dimension. Similarly, a Macaulay ring is the analogue of a local (or equidimensional) commutative Cohen–Macaulay ring. The definition of a homologically homogeneous ring is then the analogue of an arbitrary regular commutative ring. The results of this paper indicate that this analogy is appropriate, since they show that many of the standard commutative results do generalize. For example, parts (ii) or (iv) of Theorem 1.4 show that a homologically homogeneous PI ring is CM, in the commutative sense.

It is interesting to compare Theorem 1.4 with the standard methods for constructing "bad" Noetherian PI rings, notably of prime Noetherian PI rings that are not integral over their centres. Most, if not all, of these constructions can be modified so that the given ring R is also a semi-local ring of finite global dimension (see the discussion before Example 5.13). By Theorem 1.4, this forces the simple R-modules to have differing homological dimensions. These examples illustrate the problem with trying to use homological techniques in non-commutative ring theory—too many rings have finite global dimension—and this has led to the profusion of different possible definitions of a "noncommutative regular ring." However, the results of this paper show that if a PI ring satisfies reasonable local properties (thus, if the ring is local or connected graded or, in the global case, homologically homogeneous) then one is able to use homological techniques to prove strong results about the structure of the ring.

In outline, the paper is organized as follows. Some preliminary results and many of the basic definitions are collected in Section 2. Graded injectively smooth PI rings are studied in Section 3. In particular, Theorem 1.3 and some of its easier applications are proved there. In Section 5 the applications to homologically homogeneous rings and connected graded PI rings are given. Finally, some of the techniques of this paper apply to more than just PI rings. Applications to FBN rings are considered in Section 6 where, in particular, we prove Theorem 1.1. In Section 4 some easy applications to non-fully bounded rings are given.

2. Preparatory Results

In this section we give some of the basic definitions and results that will be needed in this paper. Unless specified otherwise, if a concept for a graded module M is prefixed by the word "graded" then that property is assumed to be defined in the category of graded modules, whereas if the word graded is omitted, then the property is assumed to hold in the category of all modules. Thus, for example, M is graded-uniform, if every two non-zero graded submodules have a non-zero intersection, but M is uniform if any arbitrary pair of non-zero submodules have a non-zero intersection. Usually, however, the graded and ungraded definitions are equivalent for a graded module and the proof of this fact can, in each case, be found in [NV]. In particular, this is true for the terms "prime ring," "Goldie ring," "Krull dimension," "uniform," and "projective." However, it is not true for "injective"; that is a graded-injective graded module need not be injective. Similarly, a graded Macaulay ring need not be Macaulay. For example, take $R = k[[x]][y]$, graded by degree in y. Then, as R has only one graded simple module, $R/(xR + yR)$, the commutative theory shows that R is graded Macaulay. However, R is not Macaulay, since the module $R/(1 - xy)R$ is simple of homological dimension one. The term FBN will always stand for a ring that is fully bounded and Noetherian as an ungraded ring.

Lemma 2.1. *Let M be a finitely generated, graded right module over an FBN graded ring R. Then there exists a chain of graded submodules $M = M_n \supseteq M_{n-1} \supseteq \cdots \supseteq M_0 = 0$ such that*

(i) *for each i, set $\overline{M}_i/M_{i-1}$ and $P_i = r\text{-ann}(\overline{M}_i)$. Then P_i is a prime, graded ideal and $\overline{M}_i$ is a torsion-free, uniform right R/P_i-module.*

(ii) *Each* $\overline{M}_i$ *is isomorphic to a uniform right ideal of* R/P_i. *There exists an integer* r *and a graded short exact sequence* $0 \to R/P_i \to \overline{M}_i^{(r)} \to K \to 0$, *where* K *is a torsion right* R/P_i*-module.*

(iii) *If* R *is prime and* M *is a torsion right* R*-module, then* $Mz = 0$, *for some regular homogeneous element* $z \in R$.

Proof. (i) This is very similar to the ungraded proof given, for example, in [GW1, Theorem 8.6]. Pick a graded-uniform submodule M_1 of M with $P_1 = r\text{-ann}(M_1)$ as large as possible. By [NV, Lemma A.II.5.9], M_1 is uniform. Clearly, P_1 is a (graded) prime ideal. Moreover, [NV, Lemma A.II.9.14] implies that the singular submodule T of M_1 is a graded submodule and so, as R/P_i is FBN, $r\text{-ann}_R(T) \supsetneq P_i$. Thus, $T = 0$ and M_1 is torsion-free. Now apply Noetherian induction.

(ii) Let U be a graded, uniform right ideal of R/P_i. By [GW1, Example 6M], there exists a non-zero (possibly ungraded) homomorphism $\theta : \overline{M}_i \to U$. By [NV, Corollary A.I.2.11], θ is a sum of graded homomorphisms, and so we may also assume that θ is graded. Since $\overline{M}_i$ is uniform, θ is an injection. If R has uniform dimension r, then R contains a direct sum of r graded, uniform right ideals ([NV, Corollary A.II.5.11]). Thus, $M^{(r)} \cong I$ (as graded modules), for some graded, essential right ideal I of R. Thus, R/I is a torsion module. Finally, by the graded version of Goldie's theorem ([NV, Corollary C.I.1.7 and Theorem A.I.5.8]), I contains a homogeneous, regular element of R.

(iii) As R is FBN, $r\text{-ann}(M)$ is a non-zero, graded ideal of R. Thus, as in the proof of part (ii), $r\text{-ann}(M)$ contains a regular, homogeneous element. ∎

We now turn to homological algebra. Let M be a finitely generated, graded right module over a graded Noetherian ring R. Then it will be useful to have the graded analogues of (co)homology groups. In the notation of [Mi], the category of graded right R-modules, gr-**mod**-R, is a *Grothendieck category*; that is, gr-**mod**-R is a cocomplete, C_3-category with a generator. Moreover, gr-**mod**-R has projective and injective objects. One consequence of this is that all the standard homological constructions work perfectly well in this category—see [Gr] and [Mi]. Thus, if M and N are graded (right) R-modules, define $\text{HOM}_R(M, N)$ to be the group of all graded R-module homomorphisms from M to N, which will be regarded as a graded group in the natural way:

$$\text{HOM}_R(M, N)_p = \{\theta \in \text{HOM}(M, N) : \theta(M_i) \subseteq N_{i+p} \text{ for all } i \in \mathbb{Z}\}.$$

We will write $\text{EXT}_R^n(M, N)$ for the corresponding homology group. The long exact sequences of EXT groups will be sequences of graded groups.

Suppose that N is now *a graded* $(S\text{-}R)$*-bimodule*; that is, R and S are graded rings and $N = \oplus_{i \in \mathbb{Z}} N_i$ satisfies $S_j N_i + N_i R_j \subseteq N_{i+j}$, for all i and j. Then we may regard $\mathrm{HOM}(\ , N)$ as taking values in gr-S-**mod**. Thus, $\mathrm{EXT}_R^n(M, N)$ will be a graded left S-module and the corresponding long exact sequences of EXT groups associated to a graded exact sequence $0 \to M_1 \to M \to M_2 \to 0$ will be sequences of graded left S-modules (this all follows from [Mi, Theorem VII.3.2] and the comments thereafter). Similar comments apply to other long exact sequences of (co)homology groups. We remark that if M is finitely generated, then [NV, Section A.1.2] implies that $\mathrm{Hom}_R(M, N) = \mathrm{HOM}_R(M, N)$ and $\mathrm{Ext}_R^n(M, N) = \mathrm{EXT}_R^n(M, N)$. Thus, in the definition of graded smoothness, etc., one could equally well have used EXT in place of Ext. However, if M is not finitely generated, then it is possible that $\mathrm{Ext}_R^n(M, N) \neq \mathrm{EXT}_R^n(M, N)$. No such complications arise for tensor products, but we still write $\mathrm{TOR}_n^R(M, N)$ for the corresponding homology groups to emphasize its graded structure.

The homological global dimension of a ring R will be written gldim(R) and the homological dimension of an R-module M will be written hd(M). The (right) injective dimension of R_R will be written r-injdim(R). Note that, by [Za, Lemma A], if R has finite right and left injective dimension, then r-injdim(R) = l-injdim(R), and we will write this common integer as injdim(R). In the graded case, the graded homological dimension of M equals hd(M). Similarly, by [Lv1, Lemma 3.3], r-injdim(R) equals the graded right injective dimension of R and so we will not need the graded analogue of either dimension.

Many of the results of this paper are proved by means of the following spectral sequence: Let A, B, S, and T be graded rings such that there exists a graded ring homomorphism from A to B. Let M be a graded $(T\text{-}A)$-bimodule and N a graded $(S\text{-}B)$-bimodule. Then there exists a convergent spectral sequence of graded $(S\text{-}T)$-bimodules:

$$\mathrm{EXT}_B^p\big(\mathrm{TOR}_q^A(M, B), N\big) \overset{p}{\Rightarrow} \mathrm{EXT}_A^n(M, N). \tag{2.2}$$

Thus, each object is a graded $(S\text{-}T)$-bimodule and the various homomorphism used to define the spectral sequence are graded $(S\text{-}T)$-bimodule homomorphisms. In particular, as a graded $(S\text{-}T)$-bimodule, $\mathrm{EXT}_A^n(M, N)$ is a subfactor of $\oplus_{p+q=n} \mathrm{EXT}_B^p(\mathrm{TOR}_q^A(M, B), N)$.

Unfortunately, we have been unable to find a reference for (2.2) in quite this generality although, at the level of a spectral sequence of homology groups, it is well-known (see, for example, [Ro, Theorem 11.65]). Thus, we should outline the modifications to the proof in [Ro] that are required to prove the above assertion.

Take a free $\mathbb{Z}$-algebra $\tilde{T}$ such that T is a homomorphic image of $\tilde{T}$. Thus, we may regard M as a graded left module over $\tilde{T}$. Equivalently, M is a graded right module over $A \otimes_{\mathbb{Z}} \tilde{T}^{\mathrm{op}}$. Take a projective resolution $F.$ of M as a right $A \otimes_{\mathbb{Z}} \tilde{T}^{\mathrm{op}}$-module. Thus, the complex $F. \otimes_A B$ is a complex of graded right modules over $B \otimes_{\mathbb{Z}} \tilde{T}^{\mathrm{op}}$ and, by the graded analogue of [Ro, Lemma 11.33] (and in the notation of the definition preceding that lemma), there exists a proper, graded resolution $\mathbf{M}_{pq}$ of that complex. The basic observation is that $\tilde{T}$ is a free $\mathbb{Z}$-module. Thus, by [CE, Chap. IX, Corollary 2.4, p. 166], the complex $F. \to M \to 0$ is also a projective resolution of M as a (graded) right A-module and $\mathbf{M}_{pq}$ is a proper resolution of graded right B-modules. However, the choice of our resolution implies that all the maps, and in particular the induced projective resolutions of the cycles, boundaries and homology of $F. \otimes_A B$, are graded left $\tilde{T}$-module homomorphisms. Now apply $\mathrm{HOM}_B(\_\_, N)$ to this complex and compute the spectral sequence as in [Ro, Proof of Theorem 11.38]. All the homomorphisms involved are therefore graded $(\tilde{T}\text{-}S)$-bimodule homomorphisms and the resulting spectral sequence (2.2) is a spectral sequence of graded $(\tilde{T}\text{-}S)$-bimodules. Finally, as the $\tilde{T}$-module structure of each object in (2.2) is induced from its T-module structure, this implies that (2.2) is indeed a spectral sequence of graded $(T\text{-}S)$-bimodules.

One can also prove this result in a more direct way by appealing to Grothendieck's original work. As above, it suffices to prove the result for $\tilde{T}$. Let $\mathscr{C}(\tilde{T}, A)$ denote the category of graded $(\tilde{T}\text{-}A)$-bimodules which, by the comments of the last paragraph, is a Grothendieck category with projective and injective objects. Then, $G = \_\_ \otimes_A B$ is a right exact functor $\mathscr{C}(\tilde{T}, A) \to \mathscr{C}(\tilde{T}, B)$ and $F = \mathrm{HOM}_B(\_\_, N)$ is a left exact functor $\mathscr{C}(\tilde{T}, B) \to \mathscr{C}(\tilde{T}, S)$. Thus, just as in the proof of [Gr, Theorem II.4.1], (2.2) is a spectral sequence with values in $\mathscr{C}(\tilde{T}, S)$, which is precisely our assertion.

COROLLARY 2.3. *Assume that B is a semi-simple Artinian ring. Then* (2.2) *collapses to an isomorphism* (*of graded bimodules*):

$$\mathrm{HOM}_B(\mathrm{TOR}_n^A(M, B), N) \cong \mathrm{EXT}_A^n(M, N). \quad \blacksquare$$

3. SMOOTH PI RINGS ARE GORENSTEIN

The aim of this section is to prove Theorem 1.3 of the Introduction; that is, to prove that any right graded injectively smooth, graded, Noetherian PI ring R is Auslander–Gorenstein and graded Macaulay. An intermediate step in the proof of this result is to prove that R satisfies the natural generalization of the Gor_0 condition defined in the Introduction. This is defined as follows. Let R be a graded ring with $\mathrm{injdim}(R) = n < \infty$. Then, R is gr-Gor_0 if $j(S) = n$, for every graded simple (left or right) R-module S. Note that this does not coincide with the concept of (ungraded) Gor_0 rings; the ring $k[[x]][y]$ considered in Section 2 is again a counterexample. However, we will never need to consider the ungraded condition.

If M is an $(R_1\text{-}R_2)$-bimodule for some rings R_i, then M will be said to have a property $\mathscr{P}$ if it has that property $\mathscr{P}$ as both a left R_1-module and a right R_2-module. Thus, for example, M is a *finitely generated bimodule*, if it is finitely generated as both a left R_1-module and a right R_2-module. Given that the rings A, S, T and the bimodules M and N in (2.2) are Noetherian, then $\mathrm{TOR}_n^A(M, N)$ is automatically a finitely generated bimodule. One of the basic ideas in this paper is to find situations in which (2.2) can then be used show that $\mathrm{EXT}_A^n(M, N)$ is also a Noetherian bimodule. Unfortunately, this is not true in general, even for FBN rings.

EXAMPLE 3.1. Pick division rings $T \subset B$ such that ${}_TB$ is finite but B_T is infinitely generated [Co, Theorem 5.6.1]. (In the notation of (2.2), we set $A = B = S$.) Then, $H = \mathrm{Hom}_B(B_B, B_B)$ is not finitely generated under its natural right T-module structure.

Proof. Pick $\{b_i\} \in B$ such that $\Sigma b_i T$ is an infinite direct sum and consider $\theta_i \in H$ defined by $\theta_i(1) = b_i$. If $\Sigma \theta_i T$ is finitely generated, then $\theta_m = \Sigma_1^{m-1} \theta_i a_i$, for some $a_i \in T$ and $b_m = \theta_m(1) = \Sigma \theta_i a_i(1) = \Sigma \theta_i(a_i 1) = \Sigma \theta_i(1) a_i = \Sigma b_i a_i$, a contradiction. ∎

A ring B will be called *central semi-simple* if B is a semi-simple Artinian ring that is finitely generated as a module over its center $Z(B)$. Examples of division rings satisfying the hypotheses of Example 3.1 are highly non-commutative; in particular, they cannot exist if B is central simple. This, in turn, can be used to show that the question raised before the example has a positive answer for PI rings.

LEMMA 3.2. *Let B be a prime, Noetherian PI ring and T a Noetherian ring. Let L be a finitely generated $(T\text{-}B)$-bimodule, torsion-free as a right B-module. Then $L^* = \mathrm{Hom}_B(L, B)$ is a finitely generated $(B\text{-}T)$-bimodule.*

Proof. (i) Set $E = \mathrm{End}_B(L)$. By replacing T by $T/l\text{-ann}(L)$, we may assume that L is a faithful left T-module. Hence, we may identify T with a subring of E, using the natural action of T on L. Since ${}_TL$ is faithful and Noetherian, so is ${}_EL$. Thus, [GW1, Theorem 8.9] implies that there is an E-module embedding $E \hookrightarrow L^{(r)}$ for some r. Hence, E is a left Noetherian T-module. Set $S = TZ(E) \subseteq E$ and note that $Z(E)$ is central in this ring. Thus, S is a left, and hence a right Noetherian T-module. Since L_B is a finitely generated, torsion-free module over a prime PI ring B, [MR, Proposition 3.1.15(ii)] implies that E is also a prime PI ring. Finally, by [MR, Theorem 13.6.10], there exist elements $z_1, \dots, z_t \in E$ and an E-module monomorphism

$$E \hookrightarrow \sum_{i=1}^{t} z_i Z(E) \subseteq \sum_{i=1}^{t} z_i S.$$

Hence E is a Noetherian right module over both S and T.

Clearly, L^* is a Noetherian left B-module. Moreover, L^* is a right E-module. The natural E-bimodule homomorphism $L \otimes_B L^* \to E$ induces an E-module injection $L^* \hookrightarrow L^\dagger = \mathrm{Hom}_E(L, E)$. Once again, as L is a finitely generated left E-module, $L^\dagger$ is a finitely generated right E-module. By the last paragraph, this implies that its submodule L^* is a Noetherian right T-module. ∎

The following well-known lemma will be used frequently.

LEMMA 3.3. *Let Z be any ring and $\mathscr{C}$ be an Ore set in a Noetherian ring R. Suppose that M is a $(Z$-$R)$-bimodule, finitely generated as a right R-module and that I is an ideal of R. Then, for any i, there is an isomorphism of $(R_{\mathscr{C}}$-$Z)$-bimodules:*

$$R_{\mathscr{C}} \otimes_R \mathrm{Ext}^i_R(M, I) \cong \mathrm{Ext}^i_{R_{\mathscr{C}}}(M \otimes_R R_{\mathscr{C}}, I \otimes_R R_{\mathscr{C}}).$$

Proof. See, for example, [BL, Proposition 1.6]. ∎

LEMMA 3.4. *Let R and T be Noetherian rings and suppose that M is a finitely generated $(T$-$R)$-bimodule. Let N be a right R-module and $x \in T$. Then, the homomorphism $x.: M \to M$, given by left multiplication by x, induces a group homomorphism: $.x: \mathrm{Ext}^n(M, N) \to \mathrm{Ext}^n(M, N)$ given by right multiplication by x. Similarly, it induces a homomorphism $x.: \mathrm{Ext}^n(N, M) \to \mathrm{Ext}^n(N, M)$ given by left multiplication by x.*

Remark. We always use this result for finitely generated graded modules over graded Noetherian rings, in which case $\mathrm{Ext}^n(N, M) = \mathrm{EXT}^n(N, M)$ and so the result also applies to $\mathrm{EXT}^n(N, M)$. However, the result is easier to prove for Ext.

Proof. In either case, pick a resolution of N and study the map induced by $x.$ on that resolution. The proof of the first assertion is given in detail in [Br, Lemma 2.1] and the second assertion is proved similarly. ∎

Let R_1 and R_2 be FBN rings and M a finitely generated $(R_1$-$R_2)$-bimodule. In the next theorem, and elsewhere, we use the fact that l-$K\dim_{R_1}(M) = r$-$K\dim_{R_2}(M)$ (see [GW1, Theorem 13.15]). In particular,

l-$K\dim(R_1) = r$-$K\dim(R_1)$ and, in each case, the prefix can, and usually will, be omitted.

THEOREM 3.5. *Let R and T be FBN graded rings. Let M be a finitely generated graded $(T$-$R)$-bimodule and N a finitely generated graded right R-module. Assume that*

$$(*)\ \textit{every graded prime factor ring } R/P \textit{ of } R/r\text{-ann}(N) \textit{ with } K\dim(R/P) \le K\dim(M) \textit{ satisfies a } \mathrm{PI}.$$

Then, $\mathrm{EXT}^n_R(M, N)$ *is a graded, Noetherian right T-module, for all $n \ge 0$.*

Proof. Given any graded submodule $N_1 \subseteq N$, then there exists an exact sequence of right T-modules

$$\mathrm{EXT}^n_R(M, N_1) \to \mathrm{EXT}^n_R(M, N) \to \mathrm{EXT}^n_R(M, N/N_1). \quad (3.5.1)$$

Let α be an ordinal. The theorem is vacuously true if $N = 0$, so, by induction on Krull dimension, assume that the theorem is true (for any rings satisfying the hypotheses of the theorem) if $K\dim(N) < \alpha$. Now assume that $K\dim(N) = \alpha$. By a Noetherian induction, we may assume that the theorem is true for any proper graded factor module of N and hence, by (3.5.1), we may replace N by any non-zero graded submodule of itself. Thus, by Lemma 2.1(i), we may assume that N is isomorphic to a graded uniform right ideal I/P of a graded prime factor ring of R. Since $\mathrm{EXT}(M, N^{(r)}) \cong \mathrm{EXT}(M, N)^{(r)}$, as graded right T-modules, we may replace N by a direct sum of copies of N. For some $r \ge 1$, Lemma 2.1(ii) provides a short exact sequence of graded modules $R/P \hookrightarrow N^{(r)} \twoheadrightarrow K$, where $K\dim(K) < \alpha$. Hence, by (3.5.1) and induction on the Krull dimension, we may replace N by $B = R/P$. Note that the condition $(*)$ still holds. Now consider the spectral sequence (2.2). Here, the modules $L_q = \mathrm{TOR}^R_q(M, B)$ are finitely generated, graded $(T$-$B)$-bimodules. Thus, if each $\mathrm{EXT}^p_B(L_q, N)$ is a finitely generated right T-module, then so is $\mathrm{EXT}^n_R(M, N)$. Set $L = L_q$, for some q. Then l-$\mathrm{ann}_T(M)L = 0$, and so, by [GW1, Theorem 13.15 and Lemma 7.1],

$$K\dim_B(L) = K\dim_T(L) \le K\dim_T(T/l\text{-ann}(M)) = K\dim_T(M) = K\dim_R(M).$$

Thus, hypothesis $(*)$ implies that, if L is not torsion as a right B-module, $K\dim(B) = K\dim(L)$ and so B is PI.

Thus, we have reduced the theorem to proving that $\mathrm{EXT}^n_B(L, B)$ is a finitely generated right T-module. Moreover, since $K\dim(B) \le \alpha$, the inductive hypothesis implies that $\mathrm{EXT}^i_B(L, \tilde{N})$ is a Noetherian right

T-module for every finite generated, graded, torsion right B-module $\tilde{N}$. Suppose, first, that $n = 0$ and let X denote the set of torsion elements of L considered as a right B-module. Since $\mathrm{HOM}_B(X, B) = 0$, clearly $\mathrm{HOM}_B(L, B) \cong \mathrm{HOM}_B(L/X, B)$, as $(T\text{-}B)$-bimodules. Thus, if $L/X = 0$ there is nothing to prove, while if $L/X \neq 0$ then, by the last paragraph, B is PI and the result follows from Lemma 3.2.

Thus, we may assume that $n > 0$. Let $Q = Q(B)$ denote the (ungraded) simple Artinian quotient ring of B. Then, by Lemma 3.3,

$$Q \otimes_B \mathrm{EXT}_B^n(L, B) = Q \otimes_B \mathrm{Ext}_B^n(L, B) \cong \mathrm{Ext}_Q^n(L \otimes_B Q, Q) = 0.$$

Thus, $\mathrm{EXT}_B^n(L, B)$ is a finitely generated, torsion left B-module. Since it is also graded, Lemma 2.1(iii) implies that $z\,\mathrm{EXT}_B^n(L, B) = 0$, for some homogeneous, regular element z. Consider the short exact sequence

$$0 \longrightarrow B \xrightarrow{z\cdot} B \longrightarrow B/zB \longrightarrow 0.$$

Applying $\mathrm{HOM}(L, \_\_)$ to this sequence and taking cohomology yields the following exact sequence of graded right T-modules:

$$\mathrm{EXT}_B^{n-1}(L, B/zB) \longrightarrow \mathrm{EXT}_B^n(L, B) \xrightarrow{\theta} \mathrm{EXT}_B^n(L, B) \longrightarrow \mathrm{EXT}_B^n(L, B/zB).$$

But, by Lemma 3.4, θ is given by left multiplication by z and hence is zero. Thus, there is an injection $\mathrm{EXT}_B^n(L, B) \hookrightarrow \mathrm{EXT}_B^n(L, B/zB)$ of right T-modules. Therefore, by the inductive hypothesis, $\mathrm{EXT}_B^n(L, B)$ is a Noetherian right T-module. ∎

The following condition is crucial to many of the results of this paper.

CONDITION 3.6. Let R be a Noetherian, graded ring. If M is a graded left, respectively right R-module of finite length, then $\mathrm{EXT}_R^n(M, R)$ is a right, respectively left, R-module of finite length.

This condition is also important when one studies the Proj of a connected graded ring (see [AZ]). It does not hold for arbitrary Noetherian rings, even connected graded ones (see [SZ]). However, Theorem 3.5 provides one situation where it holds.

COROLLARY 3.7. *Let R be a fully bounded, graded Noetherian ring such that R/Q is central simple for all graded maximal ideals Q of R. Then* (3.6) *holds.*

Proof. By induction on the length of M it suffices to consider the case when M is a simple right module. Let $P = \mathrm{ann}(M)$ and note that, as graded modules, $R/P \cong M^{(r)}$, for some r. Since EXT commutes with direct sums, we may replace M by R/P. Theorem 3.5 implies that $\mathrm{EXT}_R^n(M, R)$ is a Noetherian, and hence Artinian, right R/P-module. Since it is trivially a finitely generated left R-module, Lenagan's Lemma [MR, Theorem 4.1.6] implies that $\mathrm{EXT}_R^n(M, R)$ is also left Artinian. ∎

Notation. Given a finitely generated, graded (right) R-module M, set

$$E^j(M) = \mathrm{EXT}_R^j(M, R) \quad \text{and} \quad E^{ij}(M) = \mathrm{EXT}_R^i(\mathrm{EXT}_R^j(M, R), R).$$

Note that $E^j(M)$ is a finitely generated, graded left R-module and that $E^{ij}(M)$ is a finitely generated, graded right R-module.

THEOREM 3.8. *Suppose that R is a right graded injectively smooth, Noetherian PI ring or, more generally, a right graded injectively smooth Noetherian ring satisfying* (3.6). *Then R is* gr-Gor_0.

REMARK. Recall that a right graded injectively smooth PI ring is defined to have finite injective dimension. Thus, the theorem implies that a Noetherian PI ring R is right graded injectively smooth if and only if it is left graded injectively smooth and we can drop the prefix.

Proof. Let $\mathrm{injdim}(R) = n$. We need to prove that $E^j(S) = 0 \Leftrightarrow j < n$, for all simple, graded R-modules S. By definition, $E^n(S) \neq 0$ for any simple graded right R-module S while $E^j(M) = 0$ for every graded R-module M and $j > n$. Let

$$m = \min\{j(M) : M \text{ a graded } \textit{left } R\text{-module of finite length}\}.$$

By [BE, (1.3)] or (3.8.1) below $j(M) < \infty$ for each non-zero graded R-module M and hence $m \leq n$. We assume that $m < n$ and aim for a contradiction.

For a finitely generated graded left R-module M, consider the spectral sequence

$$^{I}E_2^{p,q} = \mathrm{EXT}_R^p(\mathrm{EXT}_R^q(M, R), R) \Rightarrow \mathbb{H}^{p-q}(M), \qquad (3.8.1)$$

where $\mathbb{H}^{p-q}(M) = 0$ if $p \neq q$ and $\mathbb{H}^0(M) = M$. For a proof of this, see [Lv1, (3.1)] and [Lv2]. We have used a non-standard indexing of $^{I}E_2^{p,q}$ in order to be consistent with the earlier definition of $E^{p,q}$. Let S_0 be an Artinian left R-module such that $j(S_0) = m$. Combined with the observations of the last paragraph, and following the notation of [Bj1, pp. 60–62], the only possible non-zero entries of the ${}^{I}E_2^{p,q}(S_0)$ table are

$$\begin{array}{cccc} E^{n,n}(S_0) & E^{n,n-1}(S_0) & \cdots & E^{n,m}(S_0) \\ E^{n-1,n}(S_0) & E^{n-1,n-1}(S_0) & \cdots & E^{n-1,m}(S_0) \\ \vdots & \vdots & \cdots & \vdots \\ E^{1,n}(S_0) & E^{1,n-1}(S_0) & \cdots & E^{1,m}(S_0) \\ E^{0,n}(S_0) & E^{0,n-1}(S_0) & \cdots & E^{0,m}(S_0) \end{array}.$$

In this table, the coboundary maps have bi-degree $(-1,2)$; that is, they map $E^{p,q}(S_0)$ to $E^{p+2,q+1}(S_0)$. At the mth stage, the coboundary maps have bi-degree $(-m, m+1)$.

As $m < n$ and $\mathbb{H}^{p-q}(S_0) = 0$ for $p \neq q$, the last paragraph implies that $E^{n,m}(S_0) = 0$. But, by (3.6), $E^m(S_0)$ is a non-zero, graded right R-module of finite length. Thus, $E^m(S_0)$ has a graded simple submodule L and one has the following exact sequence:

$$E^n(E^m(S_0)/L) \to E^{n,m}(S_0) \to E^n(L) \to E^{n+1}(E^m(S_0)/L).$$

Since $\operatorname{injdim}(R) = n$, the final term is zero, whereas $E^n(L) \neq 0$, since R is right graded injectively smooth. Therefore, $E^{n,m}(S_0) \neq 0$. This contradiction implies that $j(M) = n$ for every graded left R-module M of finite length. Thus, R is left gr-Gor_0 and hence left graded injectively smooth. By the left-handed analogue of the above argument, R is also right gr-Gor_0. ∎

REMARK 3.9. Suppose that R is a Noetherian gr-Gor_0 ring of injective dimension n. Then $E^n(S)$ is simple for every graded simple R-module S. Moreover, E^n provides a (contravariant) duality between the category of graded right R-modules of finite length and that of graded left R-modules of finite length. To prove this, use the arguments given in [Bj1, p. 76].

THEOREM 3.10. *Let R be an graded injectively smooth, Noetherian PI ring and set* $\operatorname{injdim}(R) = n$. *Then*

(i) *R is Auslander–Gorenstein graded Macaulay ring.*

(ii) $K\dim(R) = \operatorname{injdim}(R)$.

(iii) *Given any finitely generated, graded right or left R-module M, then*:

(a) $j(M) + K\dim(M) = n$.

(b) $K\dim(E^{j(M)}(M)) = K\dim(M)$.

(c) *For all* $m \le n$, $K\dim(E^m(M)) \le \min\{K\dim(M), n-m\}$.

REMARK 3.11. Parts (ii) and (iii) of the theorem follow easily from part (i). To see this, take a finitely generated, graded R-module M with $E^n(M) \neq 0$ (the existence of such a module is proved within the proof of Lemma 3.12, below). Thus, the Gorenstein property implies that $j(E^n(M)) = n$ and the graded Macaulay property implies that $K\dim(R) = n + K\dim(M) \geq n$. On the other hand, for any graded, simple R-module S, the graded Macaulay property implies that $K\dim(R) = j(S) \leq \operatorname{injdim}(R)$. Thus, part (ii) holds and part (iiia) becomes the definition of graded Macaulay. Part (iiib) now follows from [BE, Theorem 3.6] while part (iiic) follows from part (iiia) combined with the definition of Auslander–Gorenstein. We have included part (iii) as part of the theorem since it is by proving this assertion that we show that R is Auslander–Gorenstein.

Proof. Suppose that (iii) holds. Then part (iiia), with $M = R$, shows that (ii) holds. For any finitely generated, graded right R-module M and any graded submodule N of $E^p(M)$, part (iiic) implies that $K\dim(N) \leq n - p$ and hence part (iiia) implies that $E^q(N) = 0$ for $q < p$. By definition, this means that R is graded Auslander–Gorenstein which, by [Ek, Theorem 0.1], implies that R is Auslander–Gorenstein. Part (iiia) implies that R is graded Macaulay.

Thus, only part (iii) needs proof. Suppose first that M is a graded Artinian module. Then, (iii) is a restatement of Theorem 3.8 and Corollary 3.7. By induction, suppose that (iii) holds for all modules M' with $K\dim(M') < \alpha$, for some ordinal $\alpha \geq 1$. Let M be a graded right R-module with $K\dim(M) = \alpha$. Given a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$, then one has a long exact sequence of graded left R-modules:

$$E^{j-1}(M_1) \to E^j(M_2) \to E^j(M) \to E^j(M_1) \to E^{j+1}(M_2) \quad (3.10.1)$$

Suppose that part (iii) holds for M_1 and M_2. Then, $K\dim(M_k) \leq \alpha$ for each k and $K\dim(M_w) = \alpha$ for (at least) one w. Thus, if $j < n - \alpha$, then part (iiia) implies that $E^j(M_1) = E^j(M_2) = 0$ and hence that $E^j(M) = 0$. If $j = n - \alpha$, then $E^{j-1}(M_1) = 0$ while $K\dim(E^{j+1}(M_2)) < \alpha$. If w is chosen so $K\dim(M_w) = \alpha$, then part (iiib) implies that $K\dim(E^j(M_w)) = \alpha$ and hence that $K\dim(E^j(M)) = \alpha$. Thus $j = j(M)$. Finally, if $j > n - \alpha$, then parts (iiib, iiic) imply that

$$\begin{aligned} K\dim(E^j(M)) &\leq \max_k\{K\dim(E^j(M_k))\} \\ &\leq \min\left\{\max_k\{K\dim(M_k)\}, n - j\right\} \\ &\leq \min\{K\dim(M), n - j\}. \end{aligned}$$

Thus, it suffices to prove part (iii) for M_1 and M_2. By Lemma 2.1 and induction, it therefore suffices to prove the theorem in the case when $M = R/P$ is isomorphic to a graded prime factor ring of R, again with $K\dim(M) = \alpha \geq 1$. Pick a regular homogeneous element $z \in R/P$ and consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{z.} M \longrightarrow M/zM \longrightarrow 0,$$

where $z.$ denotes left multiplication by z. For each j, Lemma 3.4 implies that this induces the exact sequence of graded modules

$$E^j(M/zM) \to E^j(M) \xrightarrow{.z} E^j(M) \to E^{j+1}(M/zM), \quad (3.10.2)$$

where $.z$ denotes right multiplication by z. Note that, as z is regular, $M/zM = R/(zR + P)$ is a torsion right R/P-module and so $K\dim(M/zM) \leq \alpha - 1$.

Now consider (3.10.2) for various values of j and z. Suppose, first, that $j < n - \alpha$. Then $E^j(M/zM) = 0 = E^{j+1}(M/zM)$, by induction. Thus, by (3.10.2), $E^j(M)z = E^j(M)$, for all regular, homogeneous elements $z \in \mathscr{C}_{R/P}(0)$. By Theorem 3.5, $E^j(M)$ is a finitely generated, graded right R/P-module and so, if $E^j(M) \neq 0$, then $E^j(M)$ has a graded simple factor module $S = E^j(M)/L$. Since R/P is not Artinian, $r\text{-ann}_{R/P}(S) \neq 0$ and $Sz = 0$ for some regular, homogeneous element $z \in R\text{-ann}_{R/P}(S)$. Thus, $E^j(M)z \neq E^j(M)$, a contradiction. Therefore, $E^j(M) = 0$.

Next, suppose that $n \geq j > n - \alpha$. Then (as left R-modules), $K\dim(E^j(M/zM)) \leq \min\{n - j, \alpha - 1\}$ while $K\dim(E^{j+1}(M/zM)) \leq \min\{n - j - 1, \alpha - 1\}$. Suppose that $K\dim(E^j(M)) = \gamma > \min\{n - j, \alpha\} \geq 0$. Recall that, as $E = E^j(M)$ is a Noetherian bimodule over the FBN ring R, [GW1, Theorem 13.15] implies that the Krull dimensions of E as a left or right R-module coincide. Moreover, by [NV, Proposition A.II.5.8], the graded and ungraded Krull dimensions of E also coincide. Thus, the definition of graded Krull dimension implies that there exists a graded *right* R/P-submodule $F \subset E$ such that $K\dim(E/F) = \gamma - 1$. Thus, as R is FBN, $I = r\text{-ann}_{R/P}(E/F) \neq 0$ and we may choose $z \in I$. Thus,

$$l\text{-}K\dim(E/Ez) \geq l\text{-}K\dim(E/EI) = r\text{-}K\dim(E/EI) \geq \gamma - 1.$$

However, from (3.10.2), $K\dim(E/Ez) \leq K\dim(E^{j+1}(M/zM)) \leq \gamma - 2$. This contradiction implies that $K\dim(E^j(M)) \leq \min\{n - j, \alpha\}$, and (iiic) holds for $j > n - \alpha$.

Finally, suppose that $j = n - \alpha$. In this case, as $E^j(M)$ is a finitely generated right R/P-module, certainly $K\dim(E^j(M)) \leq \alpha$. If $K\dim(E^j(M)) < \alpha$, then $K\dim(E^k(M)) < \alpha$ holds *for all* k. But now the inductive hypothesis (applied to left modules) implies that $K\dim(E^{l,k}(M)) < \alpha$ for all l and k. The spectral sequence (3.8.1) therefore implies that $K\dim(M) < \alpha$, a contradiction. ▮

We end this section with some easy consequences of the results, or more precisely the techniques, of this section. Suppose that R is a Noetherian PI ring with $\text{injdim}(R) = n < \infty$. To have any hope of proving that R is graded injectively smooth, one must first prove that R has at least one simple module M with $\text{EXT}_R^n(M, R) \neq 0$. If R is flat over some central subring, then this follows from [Br, Theorem D]. However, as we show next, Theorem 3.5 can be used to give a second proof of this result, without the assumption on the central subring.

LEMMA 3.12. *Let $R = \oplus R_i$ by an FBN, graded ring with* r-injdim$(R) = n < \infty$. *Assume that either R is PI or that R_0 is Artinian. Then there exists a simple, graded right R-module S with* $\text{EXT}^n(S, R) \neq 0$.

Proof. By [Rw, Lemma 9.11 and Theorem 9.7] there exists a cyclic module $M = R/L$ such that $\text{EXT}^n(R/L, R) \neq 0$. Now, regard R as a filtered ring by using the natural filtration $\Gamma_j(R) = \bigoplus_{i=0}^{j} R_i$ and take the induced filtration on M. Thus, $\text{gr}(M) = \text{gr}(R)/\text{gr}(L) = R/\text{gr}(L)$. Then, [Bj2, Corollary 3.12] implies that $\text{EXT}^n(R/\text{gr}(L), R) \neq 0$. Thus, we may choose a graded, finitely generated module M, with $K\dim(M)$ as small as possible, such that $\text{EXT}^n(M, R) \neq 0$.

As in the proof of Theorem 3.10, the exact sequence (3.10.1) and induction allows one to replace M by R/P, for some graded prime ideal P satisfying $K\dim(R/P) \leq K\dim(M)$. If P is maximal, then the lemma follows for S a simple summand of R/P. So, suppose that P is not maximal and pick a homogeneous, regular element $z \in R/P$. Then, by the inductive hypothesis,

$$\text{EXT}^n(R/(P+zR), R) = \text{EXT}^{n+1}(R/(P+zR), R) = 0.$$

Thus, from the exact sequence (3.10.2),

$$\text{EXT}^n(R/P, R) = \text{EXT}^n(R/P, R)z.$$

We now consider the two cases separately. If R is PI, then $\text{EXT}^n(R/P, R)$ is a Noetherian right R/P-module, by Theorem 3.5. As in the argument after (3.10.2), this implies that $\text{EXT}^n(R/P, R) = 0$, giving the required contradiction.

Suppose that R_0 is Artinian. Since $\text{EXT}^n(R/P, R)$ is a Noetherian left R-module, there exists an integer r_0 such that $\text{EXT}^n(R/P, R)_r = 0$, for all $r < r_0$. Choose r_0 maximal with respect to this property. Since R/P is not Artinian, $R^+ \subsetneq P$ and so we may pick $z \in R^+$. Now, $\text{EXT}^n(R/P, R)$ is a graded R-bimodule and so

$$\mathrm{EXT}^n(R/P, R) = \mathrm{EXT}^n(R/P, R)z \subseteq \mathrm{EXT}^n(R/P, R)\cdot R^+ \subseteq \bigoplus_{r=r_0+1}^{\infty} \mathrm{EXT}^n(R/P, R)_r.$$

Thus, $\mathrm{EXT}^n(R/P, R)_{r_0} = 0$, contradicting the choice of r_0. ∎

The *graded Jacobson radical*, written gr-J(R), of a graded ring R is defined to be the intersection of the right graded-primitive ideals of R. (Note that the right graded-primitive ideals are just the right primitive ideals containing R^+.) A graded ring R is called *graded-local*, respectively *graded-semilocal*, if $R/$gr-J(R) is simple Artinian, respectively semisimple Artinian.

COROLLARY 3.13. *Suppose that R is a graded-local, Noetherian PI ring and assume that* injdim(R) $< \infty$. *Then R is graded injectively smooth. Consequently, R is Auslander–Gorenstein and graded Macaulay.*

Proof. Combine Theorem 3.10 with Lemma 3.12. ∎

COROLLARY 3.14. *Let R be a graded injectively smooth, Noetherian PI ring. Suppose that x is a homogeneous element with $x \in$* gr-J(R). *Let M be a finitely generated, graded R-bimodule, such that $xm \neq 0$ for any non-zero $m \in M$. Then* $K\dim(M/xM) = K\dim(M) - 1$. *In particular, for every factor ring $\bar{R}$ of R and every regular homogeneous element $x \in$* gr-J($\bar{R}$), *one has* $K\dim(\bar{R}/x\bar{R}) = K\dim(\bar{R}) - 1$.

Proof. By [GW1, Lemma 13.6], $K\dim(M/xM) \leq K\dim(M) - 1$. Let $\alpha = j(M)$ and injdim(R) $= n$. As R is graded Macaulay, $j(M/xM) = n - K\dim(M/xM) \geq \alpha + 1$. Thus, by Lemma 3.4, the short exact sequence $0 \to M \to M \to M/xM \to 0$ induces an exact sequence

$$0 \longrightarrow E^{\alpha}(M) \xrightarrow{\cdot z} E^{\alpha}(M) \longrightarrow E^{\alpha+1}(M/xM).$$

By Theorem 3.5, $E^{\alpha}(M)$ is a finitely generated, graded right R-module. As $x \in$ gr-J(R), this implies that $E^{\alpha+1}(M/xM) = E^{\alpha}(M)/E^{\alpha}(M)x \neq 0$. Thus, $j(M/xM) = \alpha + 1$ and, by Theorem 3.10, $K\dim(M/xM) = K\dim(M) - 1$. ∎

If R is a semi-local, Noetherian PI ring, then the conclusion of Corollary 3.14 still holds, but our proof of this fact is rather indirect.

Finally, we note that Theorem 3.10 provides strong information about the injective resolution of R. Given a prime ideal P of a ring R, take a uniform right ideal M/P in R/P and write I_P for the injective hull of $(M/P)_R$.

COROLLARY 3.15. *Suppose that R is an (ungraded) injectively smooth, Noetherian PI ring. Let $0 \to R \to I^{\cdot}$ be the minimal injective resolution of R_R. Then, for each s, $I^s = \oplus\{(I_P)\}$, where the sum is taken over all prime ideals P such that $K\dim(R/P) + s = K\dim(R)$. This sum has only finitely many copies of I_P, for each P.*

Proof. By [GW, Proposition 8.13], each I^s is a direct sum of I_P, for some P. By [Br, Lemma 2.3], I_P is a summand of I^s if and only if $\mathrm{Ext}^s(R/P_R, R)$ is not torsion as a right R/P-module. But, by Theorem 3.5, $\mathrm{Ext}^s(R/P_R, R)$ is a Noetherian right R/P-module. Thus, $\mathrm{Ext}^s(R/P_R, R)$ is not torsion as a right R/P-module if and only if $r\text{-}K\dim(\mathrm{Ext}^s(R/P, R)) = l\text{-}K\dim(\mathrm{Ext}^s(R/P, R)) = K\dim(R/P)$. By parts (iiib) and (iiic) of Theorem 3.10 this happens if and only if $s = j(R/P) = K\dim(R) - K\dim(R/P)$.

Let $C = \mathrm{Coker}(I^{s-1} \to I^s)$ and P be a prime ideal of R with $s = j(R/P)$. Then the last paragraph implies that $\mathrm{Hom}(R/P, I^{s-1}) = 0$ and hence that $\mathrm{Ext}^s(R/P, R) = \mathrm{Hom}_R(R/P, C)$. Suppose that I^s contains an infinite direct sum of copies of I_P. Then I^s is the injective hull of C and so C contains an infinite direct sum, say X, of copies of a uniform right ideal of R/P. But this would imply that $X = \mathrm{Hom}_{R/P}(R/P, X)$ is a non-Noetherian *right* R/P-module. Since $\mathrm{Hom}_{R/P}(R/P, X) \subseteq \mathrm{Hom}_R(R/P, C) = \mathrm{Ext}^s(R/P, R)$, this contradicts Theorem 3.5. ∎

4. OTHER DEFINITIONS OF MACAULAY RINGS

Recall that, by [Ek, Theorem 0.1], a graded Noetherian ring is graded Auslander–Gorenstein if and only if R is Auslander–Gorenstein. Moreover, by [Bj2, Theorem 4.1], if a filtered ring S has a Noetherian, Auslander–Gorenstein associated graded ring then S is also Auslander–Gorenstein. These sorts of results do not hold for the Macaulay (or Cohen–Macaulay) condition, largely because there is in general no connection between the Krull or Gelfand–Kirillov dimensions of a filtered module M and its associated graded module $\mathrm{gr}(M)$. In this short section we discuss the various types of Macaulay conditions that can be imposed on a ring and the relationship between them.

If a graded, Noetherian PI ring R is Auslander–Gorenstein and graded Macaulay, then R need not be (ungraded) Macaulay; the ring $k[[x]][y]$ mentioned in Section 2 being the obvious counterexample. However, as the next result shows, if one assume some finiteness assumptions, then R will be Macaulay. Let C be an arbitrary commutative ring and S a

C-algebra that is also a filtered ring; $S = \bigcup_{i \geq 0} \Gamma_i(S)$. Then S is called a *filtered C-algebra* if the image of C in S is contained in $\Gamma_0(S)$. Similar comments apply to graded rings. The ring S is called *C-affine* if S is finitely generated as a C-algebra. The ring C is called *Jacobson* if $J(C/I) = 0$, for every prime ideal I of C.

PROPOSITION 4.1. *Let C be any commutative Jacobson ring and suppose that* $R = \bigcup_{i \geq 0} \Gamma_i(R)$ *is a filtered C-algebra that satisfies a PI. Assume that the associated graded ring* $\mathrm{gr}(R) = \oplus \Gamma_i(R)/\Gamma_{i+1}(R)$ *is a PI Noetherian, C-affine algebra that is Auslander–Gorenstein and graded Macaulay. Then R is Auslander–Gorenstein and Macaulay. If* $\mathrm{gr}(R)$ *is Auslander-regular, then so is R.*

Proof. The fact that $\mathrm{gr}(R)$ is an affine C-algebra forces R to be an affine C-algebra. Thus, if S is any simple R-module, then [MR, Theorem 13.10.4] implies that S is a C-module of finite length. Therefore, $\mathrm{gr}(S)$ is also of finite length both as a C-module and as a $\mathrm{gr}(R)$-module. Thus, by [Bj2, Theorem 4.3] and Remark 3.11, $j(S) = j(\mathrm{gr}(S)) = K\dim(\mathrm{gr}(R)) = \mathrm{injdim}(\mathrm{gr}(R))$. By [Bj2, Corollary 3.12], $\mathrm{injdim}(R) \leq \mathrm{injdim}(\mathrm{gr}(R))$ and so $j(S) = \mathrm{injdim}(R)$. In particular, as an ungraded ring, R is injectively smooth and so Theorem 3.10 implies that R is Auslander–Gorenstein and Macaulay. The final assertion follows from the fact that $\mathrm{gldim}(R) \leq \mathrm{gldim}(\mathrm{gr}(R))$ [Bj2, Corollary 3.12]. ∎

COROLLARY 4.2. *Let C be any commutative, Jacobson ring and suppose that R is a Noetherian, C-affine, graded C-algebra that satisfies a PI. If R is graded injectively smooth then R is injectively smooth (as an ungraded ring) and is Auslander–Gorenstein and Macaulay.* ∎

This corollary has a curious consequence. Keep the assumptions of the corollary and assume that $\mathrm{gldim}(R) < \infty$. Then, the fact that R is graded injectively smooth is equivalent to demanding that $\mathrm{hd}(S) = \mathrm{gldim}(R)$ for every graded simple module. Yet the conclusion of the corollary implies that $\mathrm{hd}(S) = \mathrm{gldim}(R)$ for every simple module. We do not know of a direct proof of this observation.

Proposition 4.1 fails badly if R is not assumed to be PI; for example, use the existence of non-holonomic modules [St2] over the Weyl algebra A_2. The usual way around this problem is to use the Gelfand–Kirillov dimension in place of the Krull dimension. More generally, if R is an Auslander–Gorenstein graded ring and δ is a dimension function in the sense of [MR, Section 6.8.4], then R will be called *(graded) δ-Macaulay* if $j(M) + \delta(M) = \delta(R)$, for all finitely generated (graded) R-modules M. Thus, the Macaulay rings of Section 3 are $K\dim$-Macaulay while the connected graded, Cohen–Macaulay rings of the Introduction are graded $GK\dim$-Macaulay rings. Let R be an Auslander–Gorenstein, Noetherian ring.

Then the function δ_0, defined by $\delta_0(M) = \text{injdim}(R) - j(M)$ for a finitely generated module M, is always an exact, partitive dimension function (see [Lv1, Proposition 4.5]). Thus, R is automatically δ_0-Macaulay. At first glance, this might suggest that demanding that an Auslander–Gorenstein ring be K dim-Macaulay is a fairly weak assumption. However, this is far from the case; indeed, almost all of the results of Section 5 will fail for Auslander-regular, Noetherian PI rings.

On the other hand, one would prefer to consider GK dim-Macaulay rings rather than K dim-Macaulay rings, simply because Gelfand–Kirillov dimension is usually more pleasant than Krull dimension. (Throughout this discussion, whenever the GK dimension of a module over a ring R is mentioned, we will assume that R is a k-algebra over a fixed field k and the GK dimension is defined over that field.) If one restricts one's attention to k-affine PI algebras, then the distinction disappears:

LEMMA 4.3. *Let R be a k-affine, Noetherian PI algebra. Then*:

(i) $GK\,\dim(M) = K\,\dim(M)$, *for any finitely generated R-module M.*

(ii) *Assume that R is also an Auslander–Gorenstein graded ring. Then the terms "graded Macaulay," "Macaulay," "graded GK dim-Macaulay" and "GK dim-Macaulay" are all equivalent.*

Proof. (i) Since $R/\text{ann}(M)$ embeds into a finite direct sum of copies of M, we may replace M by $R/\text{ann}(M)$. Now [KL, Corollary 10.16] implies that $\text{GK}\,\dim(R/\text{ann}(M)) = K\,\dim(R/\text{ann}(M))$.

(ii) This follows from part (i) and Proposition 4.1. ∎

In Section 6, we will show that $K\,\dim(M) = Gk\,\dim(M)$ also holds every finitely generated, graded module M over a connected graded, FBN ring. One advantage of working with the GK dimension is that it works well when passing from associated graded rings.

LEMMA 4.4. *Let $R = \bigcup_{i \geq 0} \Gamma_i(R)$ be a filtered k-algebra such that $\Gamma_0(R)$ is a finite dimensional k-vector space. Assume that the associated graded ring $\text{gr}(R)$ is a Noetherian, Auslander–Gorenstein, graded GK dim-Macaulay ring. Then R is a Noetherian, Auslander–Gorenstein, GK dim-Macaulay ring.*

Proof. Since $\text{gr}(R)$ is Noetherian, $\text{gr}(R)^+$ is a finitely generated $\text{gr}(R)$-module. Since $\Gamma_0(R)$ is finite dimensional, this implies that every $\Gamma_i(R)$ is finite dimensional and that $\text{gr}(R)$ is k-affine. Thus, R is Auslander–Gorenstein and Noetherian by [Bj2, Theorem 4.1]. Moreover, for any finitely generated R-module M, [MR, Proposition 8.6.5] implies that

$GK\dim(M) = GK\dim(\mathrm{gr}(M))$ while [Bj2, Theorem 4.3] implies that $j(M) = j(\mathrm{gr}(M))$. Thus, $j(M) + GK\dim(M) = GK\dim(\mathrm{gr}(R)) = GK\dim(R)$, as required. ∎

Corollary 4.5. *Let $R = \bigcup_{i\geq 0}\Gamma_i(R)$ be a filtered k-algebra, such that $\Gamma_0(R) = k$. If $\mathrm{gr}(R)$ is a Noetherian, PI ring of finite injective dimension, then R is an Auslander–Gorenstein, GK dim-Macaulay ring.*

Proof. By the hypothesis on $\Gamma_0(R)$, $\mathrm{gr}(R)$ is a connected graded ring. Thus, by Corollary 3.13 and Lemma 4.3, $\mathrm{gr}(R)$ is Auslander–Gorenstein and graded GK dim–Macaulay. Now Lemma 4.4 implies that R is an Auslander–Gorenstein, GK dim-Macaulay ring. ∎

If one works with filtrations that are not finite dimensional, then $GK\dim(M)$ need not equal $GK\dim(\mathrm{gr}(M))$ (see [MS]) and so the analogue of Corollary 4.5 will presumably fail.

5. Structure of PI Auslander Regular Rings

The results of Sections 3 and 4 show that a large number of Noetherian PI rings are Auslander-regular and Macaulay. The aim of this section is to show that these rings have very pleasant properties; in particular, we prove Theorem 1.4 of the Introduction. For simplicity, we will only consider ungraded rings in this section, unless we explicitly state otherwise. However, by Section 4, these results will apply to many graded rings—see Remark 5.5.

An injectively smooth, Noetherian ring of finite global dimension will be called *smooth*; equivalently, a Noetherian ring R is smooth if $\mathrm{hd}(M) = \mathrm{gldim}(R) < \infty$, for all simple R-modules M. Let S be a Noetherian PI ring. Then we will let Λ_j denote the set of prime ideals Q of S such that $K\dim(S/Q) = K\dim(S) - j$. We will use the standard terminology of localization theory, as defined in [GW1] or [Ja]. Note that if P and Q are linked prime ideals of S, then $P \in \Lambda_j \Leftrightarrow Q \in \Lambda_j$, by [GW1, Corollary 12.6 and Theorem 13.13]. If Ω is a clique of prime ideals, set $\mathscr{C}(\Omega) = \bigcap\{\mathscr{C}(P) : P \in \Omega\}$. Let δ be an exact dimension function on a ring R. Then the nilradical $N(R)$ is called *right* δ-wii if $\delta(M \otimes N(R)) < \delta(R)$, whenever M is a finitely generated, right R-module with $\delta(M) < \delta(R)$.

Let R be an Auslander-regular Macaulay ring and M a finitely generated R-module. Then we will again write $E^i(M)$ or $E^i_R(M)$ for $\mathrm{Ext}^i_R(M, R)$ and $E^0(M)$ will usually be denoted by M^*. Note that, by the Macaulay and Auslander properties,

$$K\dim(E^i(M)) = K\dim(R) - j(E^i(M)) \leq K\dim(R) - i, \qquad \text{for all } i. \tag{(5.1)}$$

PROPOSITION 5.2. *Let S be an Auslander–Gorenstein, Macaulay, FBN ring. Then*

(i) *Λ_0 is precisely the set of minimal prime ideals of S. Moreover, S is Krull-homogeneous; that is $K\dim(I) = K\dim(S)$ for every non-zero (right or left) ideal I of S. Also, S has a quasi-Frobenius, classical quotient ring.*

(ii) *If, in addition, S is Auslander-regular, then S is semiprime.*

Proof. If I is a non-zero (right) ideal of S, then $\operatorname{Hom}_S(I, S) \neq 0$ and hence $K\dim(I) = K\dim(S) - j(I) = K\dim(S)$. Thus S is Krull-homogeneous. By [MR, Proposition 6.4.16] $N(S)$ is right Krull-wii and so, by [MR, Theorem 6.8.15], S has an Artinian quotient ring $Q(S)$. Suppose that P is a prime ideal of S such that $K\dim(S/P) < K\dim(S)$. Then [MR, Proposition 6.8.14(ii) and Theorem 6.8.15] imply that P contains a regular element of R. Thus, P cannot be minimal. Consequently, Λ_0 is the set of minimal prime ideals of S.

Fix $i \geq 1$ and let P be a minimal prime of S. Then, by [GW1, Theorem 8.9] and (5.1), $K\dim(S/l\text{-ann}(E^i_S(S/P))) = K\dim(E^i_S(S/P)) < n = K\dim(S)$. Thus, by [MR, Proposition 6.8.14], $xE^i(S/P) = 0$, for some $x \in S$ with $K\dim(S/Sx) < n$. But, by [MR, Theorem 6.8.15], $x \in \mathscr{C}(0)$. Therefore, by Lemma 3.3,

$$\operatorname{Ext}^i_{Q(S)}(Q(S)/PQ(S), Q(S)) \cong Q(S) \otimes_S \operatorname{Ext}^i_S(S/P, S) = 0, \qquad \text{for all } i \geq 1.$$

Thus, $Q(S)$ is quasi-Frobenius. Finally, if $\operatorname{gldim}(R) < \infty$, then $\operatorname{gldim}(Q(R)) < \infty$ and so S is semi-prime. ∎

Suppose that S is any Auslander–Gorenstein, δ-Macaulay, Noetherian ring, where δ is an exact dimension function. If $N(S)$ is δ-wii, then the proof of the proposition will still work and so the conclusions of the proposition will hold for S. Of course, by the comments in Section 4, any Auslander–Gorenstein, Noetherian ring is δ_0-Macaulay and δ_0 is exact, by [Lv1, Proposition 4.5]. Unfortunately, all this shows is that δ_0 is rarely wii. The reason is that there exist many Artinian, Auslander-regular rings that are not semi-simple; for example, take $S = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$.

Suppose that S is a semi-prime, Noetherian ring, with semi-simple quotient ring $Q(S)$. If I is a finitely generated S-submodule of $Q(S)$, then we identify I^* with its natural image $I^* \subseteq \operatorname{Hom}_{Q(S)}(IQ(S), Q(S)) \subseteq Q(S)$. Thus, $I \subset I^{**}$ and $I^* = I^{***}$.

LEMMA 5.3. *Let S be a Auslander-regular, Macaulay Noetherian PI ring that is not Artinian. Then*

(i) *If Ω is a clique of prime ideals in Λ_1, then Ω is finite and the localization S_Ω is an hereditary ring and a direct sum of prime rings.*

(ii) *If I is a right ideal of S then $I^* = S \Leftrightarrow j(S/I) \geq 2 \Leftrightarrow I_\Omega = S_\Omega$ for all cliques Ω of prime ideals in Λ_1.*

Proof. (i) By Proposition 5.2(ii), S is semiprime. Pick $Q \in \Lambda_1$ and let Ω be the clique of Q. Then, by Proposition 5.2 and the remarks before that result, Q is a height one prime ideal of S and Ω is entirely contained in Λ_1. By [Rw, Lemma 1.7.21] there exists a central, regular element $z \in Q$. By [GW1, Lemma 11.7], $z \in P$, for every $P \in \Omega$. But, as z is regular, $K\dim(S/zS) \leq K\dim(S) - 1$ and hence each $P \in \Omega$ must be minimal over zS. Thus, Ω is finite and hence localizable, by [MR, (4.3.14) and Theorem 4.3.16]. Form the localization S_Ω and note that the maximal ideals of S_Ω are simply the $\{P_\Omega : P \in \Omega\}$. Hence, $K\dim(S_\Omega) = 1$. If $i \geq 2$, then $K\dim(E_S^i(S/Q)) \leq K\dim(S) - 2$, by (5.1). Hence $LE^i(S/Q) = 0$ for some ideal L of S with $K\dim(S/L) \leq K\dim(S) - 2$. In particular, $S_\Omega L = S_\Omega$. By Lemma 3.3 this implies that

$$\operatorname{Ext}^i_{S_\Omega}(S_\Omega/Q_\Omega, S_\Omega) \cong S_\Omega \otimes_S \operatorname{Ext}^i_S(S/Q, S) = 0, \qquad \text{for all } i \geq 2.$$

Thus, every simple S_Ω-module has homological dimension ≤ 1 and so $\operatorname{gldim}(S_\Omega) \leq 1$. By [MR, Theorem 5.4.6], this implies that S_Ω is a direct sum of prime rings and Artinian rings. However, if S_Ω has an Artinian summand, then it would have a maximal prime ideal that is also minimal. In other words, some $P \in \Lambda_1$ would be minimal, contradicting Proposition 5.2.

(ii) Set $\alpha = K\dim(S)$. Then,

$$I^* = S \Leftrightarrow \operatorname{Ext}^0_S(S/I, S) = \operatorname{Ext}^1_S(S/I, S) = 0 \Leftrightarrow j(S/I) \geq 2$$
$$\Leftrightarrow K\dim(S/I) \leq \alpha - 2.$$

If $K\dim(S/I) \leq \alpha - 2$, then $I \cap \mathscr{C}(P) \neq \varnothing$ for each $P \in \Lambda_1$ and so $I_\Omega = S_\Omega$ for each clique $\Omega \subseteq \Lambda_1$. Conversely, if $K\dim(S/I) \geq \alpha - 1$, then there exists a right ideal $J \supseteq I$ such that S/J is $(\alpha - 1)$-critical. But, if $L = \operatorname{ann}(S/J)$, then [St1, Proposition 3.9] implies that L is a prime ideal with $K\dim(S/L) = \alpha - 1$. In particular, $J \cap \mathscr{C}(L) = \varnothing$. Thus, if Ω is the clique of L, then $I_\Omega \subseteq J_\Omega \neq S_\Omega$. ∎

THEOREM 5.4. *Let R be a Auslander-regular, Macaulay, Noetherian PI ring. Then*

(i) *R is a direct sum of prime rings, each of which is Auslander-regular and Macaulay.*

(ii) *Assume that R is prime, but not Artinian. Then, for any clique of prime ideals $\Omega \subseteq \Lambda_1$, R_Ω is an hereditary Noetherian prime ring. Moreover, $R = \bigcap R_\Omega$, where the intersection is taken over all cliques $\Omega \subseteq \Lambda_1$.*

(iii) *If R is prime, then R is equal to its trace ring* TR. *Moreover, the centre $Z(R)$ of R is a Krull domain and R is integral over $Z(R)$.*

REMARK 5.5. By Theorem 3.10, this theorem applies to any smooth, Noetherian PI ring R. In particular, it applies to any local, Noetherian PI ring of finite global dimension. By the results of Section 4, it also applies to many graded smooth rings. For example, by Corollary 4.5, it applies to any connected graded, Noetherian PI ring of finite global dimension.

Proof. (i) Suppose that $R = R_1 \oplus \cdots \oplus R_m$ is a direct sum of rings and that M is any R_1-module. Then, by taking a projective resolution of M as an R_1-module, it follows that $\operatorname{Ext}^j_R(M, R) = \operatorname{Ext}^j_{R_1}(M, R_1)$. Thus, each R_i is Auslander-regular and Macaulay. Therefore, it suffices to prove that R is a direct sum of prime rings. If R is Artinian, then R is semisimple, by Proposition 5.2(ii), and so we may assume that R is not Artinian.

Let the minimal prime ideals of R be $\{P_1, \ldots, P_r\}$ and set $L = P_2 \cap \cdots \cap P_r$. Given that $P = P_1$ is arbitrary and R is semi-prime, $L \cap P = 0$ and so it suffices to prove that $R = P + L$. Let Ω be an arbitrary clique in Λ_1. Thus, by Lemma 5.3, Ω is finite and localizable. By [MR, Proposition 2.1.16(vii)], the minimal primes of R_Ω are precisely the $(P_i)_\Omega$, for which $(P_i)_\Omega \neq R_\Omega$. Thus, L_Ω is just the intersection of the minimal prime ideals of R_Ω other than P_Ω. By Lemma 5.3(i), this implies that $(P + L)_\Omega = P_\Omega + L_\Omega = R_\Omega$. By Lemma 5.3(ii), this implies that $R = (P + L)^* \cong P^* \oplus L^*$ (as left R-modules). Thinking of P^* as $\operatorname{Hom}(P, R)$ gives $P^*(P)L = P^*(PL) = 0$, and so $P^*(P) \subseteq l\text{-ann}(L) = P$. Hence, $LP^*(P) = 0$ and $LP^* = 0$. Thus, the identification $R = P^* \oplus L^*$, as left R-modules, implies that $P^* \subseteq r\text{-ann}(L) = P$ and similarly $L^* \subseteq L$. Hence, $R = P \oplus L$, as required.

(ii) Lemma 5.3(i) implies that each R_Ω is an hereditary ring. As R is prime, $R \subseteq R_\Omega \subseteq Q(R)$. Set $\tilde{R} = \bigcap R_\Omega$, where the intersection is over all cliques $\Omega \subseteq \Lambda_1$. If $t \in \tilde{R}$ then, for all cliques $\Omega \subseteq \Lambda_1$, there exists $c \in \mathscr{C}(\Omega)$ such that $tc \in R$. Therefore, if $I = \{r \in R : \operatorname{tr} \in R\}$, the $I_\Omega = R_\Omega$ for each $\Omega \subseteq \Lambda_1$. Thus, Lemma 5.3(ii) implies that $t \in I^* = R$.

(iii) By part (ii), $Z(R) = \bigcap Z(R_\Omega)$ where, as usual, the intersection is over all cliques $\Omega \subseteq \Lambda_1$. Also, $Z(R) \subseteq Z(R_\Omega) \subseteq Z(Q(R)) = Q(Z(R))$. By [MR, Theorem 13.9.16] each $Z(R_\Omega)$ is a Dedekind domain, and hence a maximal order in $Q(Z(R))$. Thus, $Z(R)$ is also a maximal order in $Q(Z(R))$. The proof of [MR, Proposition 5.1.10(ii)] implies that $Z(R)$ is a Krull domain. By [MR, Proposition 13.9.6] this implies that $Z(R) = T$, the ring of traces, and so [MR, Proposition 13.9.5] implies that R is integral over $Z(R)$. ∎

In [BwH] the authors define a ring R to be homologically homogeneous if R is a Noetherian ring of finite homological dimension such that R is integral over $Z(R)$ and $\mathrm{hd}(R/M_1) = \mathrm{hd}(R/M_2)$ whenever M_1 and M_2 are maximal ideals of R with $M_1 \cap Z(R) = M_2 \cap Z(R)$. Since we are interested in rings that need not integral over their centres, we will define a ring R to be *homologically homogeneous*, or *hom-hom*, if R is a Noetherian ring of finite global dimension such that $\mathrm{hd}(R/M_1) = \mathrm{hd}(R/M_2)$ whenever M_1 and M_2 are maximal ideals of R lying in the same clique. When R is integral over its centre, this is formally weaker than the definition in [BwH], but this is likely to illusory; the two definitions do coincide for PI rings integral over their centres (see Remark 5.7). Hom-hom rings need not be smooth—just consider the commutative rings $k[x] \oplus k[y, z]$ and $k[x, y]_{(x-1)\cap(x,y)}$—although, as in the commutative case, they are locally smooth.

THEOREM 5.6. *Let R be a hom-hom, PI ring. Then*

(i) *R is a direct sum of prime rings.*

(ii) *R is integral over its centre and is Auslander-regular. Moreover, if R is prime, then Z(R) is a Krull domain and R equals its trace ring.*

(iii) *Let Ω be a clique of maximal ideals of R. Then Ω is finite and localizable. The localization R_Ω is smooth and hence is both Auslander-regular and Macaulay.*

(iv) *Suppose that R contains a field k. If either* $\mathrm{char}(k) = 0$ *or R is localization of a finitely generated k-algebra, then R is a finite module over its centre.*

Proof. (i) The idea of the proof is to take a clique Ω of maximal ideals of R and pass to the localization R_Ω. This will be smooth, after which the proposition follows easily from Theorem 5.4. The catch with this argument is that it is an open question as to whether infinite cliques are necessarily localizable. To get around this problem we change rings.

Let $S = R((x)) = \sum_{j=n}^{\infty}\{r_j x^j : r_j \in R\}$ denote the ring of Laurent power series in a commuting indeterminate x. By [GS], S is a Noetherian, PI ring with $\mathrm{gldim}(S) = \mathrm{gldim}(R)$. Fix a finitely generated R-module M. Write

$M((x)) = M \otimes_R S$ and identify M with the natural R-submodule of $M((x))$. Note that M is an R-direct summand of $M((x))$ and that S is a flat R-module. Hence, if $\mathrm{wd}_R(X)$ stands for the weak homological dimension of an R-module X, then

$$\mathrm{hd}_R(M) = \mathrm{wd}_R(M) \leq \mathrm{wd}_R M((x)) \leq \mathrm{wd}_S M((x)) \leq \mathrm{wd}_R M$$

(use, for example, [MR, Proposition 7.2.2]). Thus, $\mathrm{hd}_R(M) = \mathrm{hd}_S M((x))$. Next, if Ω is a clique of maximal ideals of R, then [St4] implies that $\Omega((x)) = \{P((x)) : P \in \Omega\}$ is a clique of ideals of S. By [Wa, Theorem 8], $\Omega((x))$ is a classically localizable clique. Here, a clique Γ of prime ideals of a ring A is classically localizable if (among other things) (i) $\mathscr{C}(\Gamma) = \bigcap\{\mathscr{C}(P) : P \in \Gamma\}$ is an Ore set and (ii) the maximal ideals of $A_\Gamma = A_{\mathscr{C}(\Gamma)}$ are precisely the $\{P_{\Gamma} : P \in \Gamma\}$. For notational reasons, we write the localization $M((x))_{\Omega((x))}$ as $M((x))_\Omega$. If M is a simple right R-module then $r\text{-ann}(M) \in \Omega$ for some clique of maximal ideals Ω of R and so $M((x))_\Omega \neq 0$. Moreover, [GS, Lemma 1] implies that $M((x))$ is a simple S-module. Thus, $M((x))_\Omega \cong M((x))$, as S-modules. Therefore, by [MR, Proposition 7.4.2] and the last displayed equation,

$$\mathrm{hd}_{S_\Omega}(M((x))_\Omega) = \mathrm{hd}_S(M((x))) = \mathrm{hd}_R(M).$$

We now return to the proof. Let Ω be a clique of maximal ideals of R and form $S_\Omega = R((x))_{\Omega((x))}$, as above. Since $\mathrm{hd}_R(R/P_1) = \mathrm{hd}_R(R/P_2)$ for any $P_j \in \Omega$, the comments of the last paragraph imply that $\mathrm{hd}_{S_\Omega}(M_1) = \mathrm{hd}_{S_\Omega}(M_2)$ for any simple S_Ω-modules M_j. In other words, S_Ω is a smooth, Noetherian PI ring. By Theorems 3.10 and 5.4, S_Ω is a direct sum of prime rings. Of course, the direct sum of the S_Ω, as Ω ranges over the cliques of maximal ideals of R, is also a faithfully flat extension of R.

Suppose that the minimal primes of R are $\{P_1, \ldots, P_r\}$ and let $L = P_2 \cap \cdots \cap P_r$. Once again, in order to prove that R is a direct sum of prime rings, it suffices to show that $R = P + L$ and $P \cap L = 0$. Since the minimal prime ideals of S are just the $P_i((x))$, it follows from [MR, Proposition 2.1.16(vii)] that the minimal primes of S_Ω are just those $P_i((x))_\Omega$ for which $P_i((x))_\Omega \neq S_\Omega$. Thus, the fact that S_Ω is a direct sum of prime rings implies that $(P + L)((x))_\Omega = S_\Omega$ and $(P \cap L)((x))_\Omega = 0$. Since $\oplus_\Omega S_\Omega$ is a faithfully flat extension of R, this implies that $P + L = R$ and $P \cap L = 0$. Thus, $R \cong \oplus R/RPdi$ is indeed a direct sum of prime rings.

(iii) Let $\{P_i\}$ be the minimal prime ideals of R and note that, by part (i), each P_i is generated by a central idempotent of R. Thus, if Ω is a clique of maximal ideals of R, then [GW1, Lemma 11.7] implies that there exists i such that $P_i \subseteq Q$, for all $Q \in \Omega$. Write $P = P_i$. For any $Q_1, Q_2 \in \Omega$, one has $Q_1 Q_2 \supseteq P^2 = P$. Thus, $\overline{\Omega} = \{Q/P : Q \in \Omega\}$ is a clique of prime ideals in $\bar{R} = R/P$ and $R_\Omega = \bar{R}_{\bar{\Omega}}$. It is also routine to see that $\bar{R}$ is a hom-hom ring. Thus, we may replace R by $\bar{R}$ and assume that R is prime.

In this case, if Ω is any clique of R, then $S \subseteq S_\Omega$ and so the maximal ideals of S_Ω form a single clique. By part (i) of the proof, S_Ω is a smooth, Noetherian PI ring and so, by Theorems 3.10 and 5.4, S_Ω is integral over its centre. By the proof of [BW, Proposition 3], this implies that every clique of prime ideals in S_Ω is finite. In particular, Ω is finite. Thus the localization R_Ω exists by [MR, Theorem 4.3.16]. By the argument used in the first part of this proof, R_Ω is smooth and hence Auslander-regular, Macaulay, and integral over its centre.

(ii) We may again assume that R is prime. Thus, if Ω is a clique of maximal ideals of R, then $R_\Omega \subseteq Q(R)$, the simple Artinian quotient ring of R. Moreover, $R = \bigcap_\Omega R_\Omega$, where the intersection runs over all cliques of maximal ideals of R. By Theorems 5.4, each $Z(R_\Omega)$ is a maximal order. Thus, just as in the proof of Theorem 5.4(iii), $Z(R) = \bigcap Z(R_\Omega)$ is a Krull domain and R is integral over $Z(R)$.

It remains to prove that R is Auslander-regular. Let M be a finitely generated right R-module and $N \subseteq \operatorname{Ext}^j_R(M, R)$, for some j. If $i < j$, and Ω is a clique of maximal ideals of R, then Lemma 3.3 implies that $R_\Omega \otimes_R \operatorname{Ext}^i_R(N, R) \cong \operatorname{Ext}^i_{R_\Omega}(N_\Omega, R_\Omega)$. However,

$$N_\Omega \subseteq R_\Omega \otimes_R \operatorname{Ext}^j_R(M, R) \cong \operatorname{Ext}^j_{R_\Omega}(M_\Omega, R_\Omega).$$

Since R_Ω is Auslander-regular, this implies that $R_\Omega \otimes_R \operatorname{Ext}^i_R(N, R) = 0$. As Ω is arbitrary, this implies that $\operatorname{Ext}^i_R(N, R) = 0$ and hence that R is Auslander-regular.

(iv) Once again, we may assume that R is a prime ring, integral over its centre and equal to its trace ring TR. Suppose, first, that $\mathbb{Q} \subseteq Z(R)$. Then it is apparently folk-lore that our hypotheses force R to be a finite $Z(R)$-module, and we thank L. W. Small for bringing this result to our attention. The argument is as follows. Consider the reduced trace as a map from R to the ring of traces, $T \subseteq Z(R)$. Since the PI degree of R is invertible in $Z(R)$, this map is surjective and so $R \cong Z(R) \oplus N$, as $Z(R)$-modules. In particular, $IR \cap Z(R) = I$, for any ideal I of $Z(R)$. Since R is Noetherian, this implies that $Z(R)$ is Noetherian. Finally, [Rw, Corollary 5.1.4] implies that R is a finitely generated $Z(R)$-module.

Alternatively, suppose that $R = S_{\mathscr{C}}$ is a localization of a finitely generated k-algebra S. Changing notation slightly, write $T(A)$ for the trace ring of a PI ring A. Then [BS, Lemma 2] implies that $R = T(R) = T(S)_{\mathscr{C}}$ and [MR, Proposition 13.9.11] implies that $T(S)$ is an affine k-algebra and a finitely generated $Z(T(S))$-module. Thus, we may replace S by $T(S)$. By

[BS, Lemma 1], we may assume that $\mathscr{C}$ consists of central elements. Thus, $R = S_{\mathscr{C}}$ is a finitely generated $Z(S)_{\mathscr{C}}$-module. ∎

REMARKS 5.7. (i) If R is a local Noetherian PI ring of finite global dimension, then R need not be a finite module over its centre [JJ].

(ii) Let R be a prime, Noetherian PI ring that is hom-hom. Then, R is integral over its centre, by Theorem 5.6. Let M_1 and M_2 be two maximal ideals of R. Then, by [BW, Theorem A], M_1 and M_2 are in the same clique if and only if $M_1 \cap Z(R) = M_2 \cap Z(R)$. Thus, for Noetherian PI rings, the definition of hom-hom in this paper coincides with that in [BwH].

The structure of hom-hom rings integral over their centres has been studied in [BwH] and a number of results reminiscent of the commutative theory are proved there. Of course, by the above remark, these results hold for all hom-hom rings.

COROLLARY 5.8. *Let R be a Noetherian, PI hom-hom ring. Then*

(i) *for any prime ideal P of R one has* $j(R/P) = \text{height}(P)$. *If P is maximal, then* $j(R/P) = \text{hd}(R/P) = \text{height}(P)$.

(ii) *If* Ω *is any clique of prime ideals of R, then* Ω *is finite and the localization* R_Ω *is Auslander-regular and Macaulay.*

(iii) *R (and Z(R)) satisfy the saturated chain condition; that is, given prime ideals* $P \subset Q$ *of R, then the length of any two chains of prime ideals from P to Q are equal.*

Proof. (i) Combine Remark 5.7 with [BwH, Theorem 3.6].

(ii) As in the proof of Theorem 5.6, we may assume that R is prime. Given any $P \in \Omega$, then [BW, Theorem A] implies that Ω is precisely the set of prime ideals Q of R with $P \cap Z(R) = Q \cap Z(R)$. Thus, [BwH, Theorem 3.4] implies that R_Ω is hom-hom. Theorem 5.6(iii) now implies that R_Ω is Auslander-regular and Macaulay.

(iii) This follows from [BwH, (2.6)]. ∎

A number of results concerned with r-sequences in a hom-hom ring are also given in [BwH]. However, since these sequences necessarily consist of central elements, the results are not as strong as those for commutative rings.

Suppose that R is a local, Noetherian PI ring of finite global dimension. Then, Theorem 5.6 and Corollary 3.13 imply that R is integral over its centre. It follows from [BHM] and [Gy] that $R \cong M_n(D)$, for some domain D and that R is a maximal order in its quotient simple Artinian ring. The next corollary generalizes those results.

COROLLARY 5.9. *Let R be a smooth, Noetherian PI ring. Then*

(i) *if R is stably free, then R is a domain.*

(ii) *If R has a trivial state space, then R is a prime ring and a maximal order in its simple Artinian quotient ring.*

Proof. The proof we give is the one outlined in [St3, Section 3]. The state space St(R) is defined in [GW2] as the set of additive functions $s : K_0(R) \to \mathbb{R}$ such that $s([R]) = 1$ and $s([P]) \geq 0$, for all finitely generated projective modules P. It is called trivial if there exists a unique state. This is the case, for example, if $K_0(R) \cong \mathbb{Z}$ or R is a commutative domain. By Theorems 3.10 and 5.4, R is a direct sum of prime rings. Clearly, each summand of R contributes a non-trivial summand to St(R), and so R is prime. Part (i) now follows from a standard argument; use, for example, the proof of [Lv1, Theorem 4.8]. If $\mathscr{C}$ is an Ore set of regular elements of R then, since gldim(R) $< \infty$, the proof of [GW2, Proposition 7.2] shows that St(R) maps onto St($R_{\mathscr{C}}$). Thus, if Ω is a clique of height one prime ideals of R, then St(R_Ω) is still trivial. However, by Theorems 3.10 and 5.4, R_Ω is an hereditary Noetherian prime ring and so [GW2, Corollary 6.6] implies that R_Ω is a Dedekind domain. The fact that R is a maximal order now follows from Lemma 5.3(ii). ∎

If R is a commutative, Noetherian domain of finite global dimension, then R is automatically a maximal order (this even follows from Corollary 5.9). However, this is not true in the non-commutative case. For example, any hereditary, Noretherian prime PI ring is automatically Auslander–Gorenstein and Macaulay: a typical example is $\begin{pmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Let R be a Noetherian PI ring. Then the results of this section may be summarized as saying that, if R is Auslander-regular and Macaulay (and hence if R is a ring of finite global dimension that is either local, or connected graded, or smooth, or hom-hom), then R has very pleasant properties. We end the section with a number of examples that show that these pleasant properties will not hold if one deletes any of the hypotheses on R. The first two examples show that the finite global dimension and Macaulay hypotheses are necessary.

EXAMPLE 5.10. There exists a connected graded, Noetherian, PI ring R that is Auslander–Gorenstein and Macaulay, but such that R is neither a semiprime ring nor integral over its centre.

Proof. Let k be a field and $\lambda \in k$ be an element transcendental over the prime subfield. Set $S = k\{x, y\}/(xy - \lambda yx)$, graded by total degree in x and y. Since $x \in S_1$ is a regular normal element, with $S/xS \cong k[y]$, [Lv1, Theorem 5.10] implies that S is Auslander–Gorenstein and

GK dim-Macaulay. The same result implies that $R = S/x^2S$ is also Auslander–Gorenstein and GK dim-Macaulay. Since R is now a Noetherian PI ring, Lemma 4.3 implies that R is also Macaulay. As λ is transcendental, $Z(R) = k$ and so R is not integral over its centre. ▌

EXAMPLE 5.11. There exists an Auslander-regular, Artinian (and hence semi-local), PI ring R that is not integral over its centre.

REMARK. We thank E. Kirkman and B. Zimmermann Huisgen for their helpful comments concerning this example.

Proof. Let $k = \mathbb{Q}(x)$ be a transcendental extension of $\mathbb{Q}$ and σ the automorphism of k defined by $x \mapsto (x+1)$. The example is

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & a^{\sigma} \end{pmatrix} : a, b, c, d, e \in k \right\} \subset M_3(k).$$

Let $\{e_{ij}\}$ denote the standard matrix units of $M_3(k)$. Write $Q_1 = (e_{11} + e_{33})R$ and $Q_2 = e_{22}R$ for the two indecomposable projective right R-modules and $S_1 = e_{23}R$, respectively $S_2 = Q_2/S_1$, for their simple factor modules. Then, there exist short exact sequences

$$0 \to S_1 \to Q_2 \to S_2 \to 0 \quad \text{and} \quad 0 \to Q_2 \to Q_1 \to S_1 \to 0.$$

Since Q_1 is indecomposable, S_1 is not projective and so $\text{hd}(S_i) = i$ for $i = 1, 2$. In particular, $\text{gldim}(R) = 2$. Also, $Z(R) = k^{\sigma} = \mathbb{Q}$ which ensures that R is not an integral $Z(R)$-module.

We next compute some Ext groups. Clearly, $E^2(S_2) \cong E^1(S_1)$ while $E^2(S_1) = 0$. By inspection, $E^0(S_2) = 0$ while both $E^0(Q_2)$ and $E^0(S_1)$ are two dimensional k-vector spaces. Thus, from the exact sequence

$$0 \to E^0(S_2) \to E^0(Q_2) \to E^0(S_1) \to E^1(S_2) \to 0,$$

one finds that $E^1(S_2) = 0$. Now consider the exact sequence

$$0 \longrightarrow E^0(S_1) \longrightarrow E^0(Q_1) \xrightarrow{\phi} E^0(Q_2) \longrightarrow E^1(S_1) \longrightarrow 0.$$

Here, $\phi \neq 0$ since the identity map on Q_1 induces a non-zero map $Q_2 \to R$. Since $\dim_k(E^0(Q_2)) = 2$, this implies that $\dim_k(E^1(S_1)) \leq 1$. Since $\text{hd}(S_1) = 1$, certainly $E^1(S_1) \neq 0$ and so $E^2(S_2) \cong E^1(S_1)$ is simple.

In order to prove that R is Auslander regular, an easy induction on the length of a module shows that it suffices to prove that, for all *simple* R-modules M and all submodules N of $E^i(M)$, one has $E^j(N) = 0$ for $j < i$. As R is anti-isomorphic to itself, it suffices to prove this for right modules. From the computations of the last paragraph, it therefore suffices to prove that $E^{02}(S_2) = 0 = E^{12}(S_2)$. This follows either from direct computations or from the spectral sequence (3.8.1). Indeed, recall (from the proof of Theorem 3.8) that the coboundary maps for this spectral sequence have bidegree $(-m, m+1)$ at the mth stage. Since $\text{gldim}(R) = 2$, this implies that there are no (non-zero) coboundary maps involving $E^{02}(S_2)$ or $E^{12}(S_2)$. Since $\mathbb{H}^{p-q}(S_2) = 0$ if $p \neq q$, this forces $E^{02}(S_2) = 0 = E^{12}(S_2)$. ∎

We now turn our attention to prime rings. It is well-known that prime Noetherian PI rings need not be integral over their centres and there exist a great many methods for constructing such examples (see, for example, [Rw, Chapter 5]). Most, if not all, of these constructions can be modified so that the ring becomes a semi-local ring of finite global dimension. The idea (which has also been used to effect in [BuH]) is as follows. Let R be such an example, with trace ring TR. By [MR, Propositions 13.9.6, 13.9.11, and 13.9.5], R and TR have a non-zero ideal, say I, in common, TR is a finite R-module, and TR is integral over its centre. Since trace rings can be inconvenient to work with, let T be any overring of R having these three properties. The first observation is that T and R/I have—or can be arranged to have—very pleasant properties, which for our purposes means that they should be semi-local and have finite global dimension. Now, replace T by the 2×2 matrix ring $\tilde{T} = M_2(T)$ and R by

$$S = \begin{pmatrix} R & I \\ T & T \end{pmatrix} \supseteq J = \begin{pmatrix} I & I \\ T & T \end{pmatrix}.$$

Then, S is a finitely generated R-module, containing an ideal $M_2(I)$ of $\tilde{T}$. Thus, S is a prime Noetherian ring, with $Z(S) = Z(R)$. Consequently, S is not integral over its centre. However, J is now an ideal of S that is a right ideal of $\tilde{T}$ satisfying $\tilde{T}J = \tilde{T}$. In these circumstances, [MR, Theorem 7.5.13] implies that

$$\text{gldim}(S) \leq \text{gldim}(\tilde{T}) + \text{gldim}(R/I) + 1. \qquad ((5.12))$$

Thus, the properties of S and R are very similar except that if T and R/I have finite global dimension, then so does S. We illustrate this procedure with an example of Wadsworth and Small [Rw, Example 5.1.1].

EXAMPLE 5.13. There exists a semilocal, Noetherian, prime PI ring S, of global dimension 2, such that S is not integral over its centre.

Proof. Set $L = \mathbb{Q}(x)$, for an indeterminate x. As in [Rw, Example 5.1.1], pick subfields L_1 and L_2 of L such that $[L : L_i] = 2$, for each i, but

$L_1 \cap L_2 = \mathbb{Q}$. Now take

$$I = \begin{pmatrix} yL[[y]] & yL[[y]] \\ yL[[y]] & yL[[y]] \end{pmatrix} \subset R$$
$$= \begin{pmatrix} L_1 + yL[[y]] & yL[[y]] \\ yL[[y]] & L_2 + yL[[y]] \end{pmatrix} \subset T = M_2(L[[y]]).$$

It is routine to check that R is a Noetherian, prime PI ring of Krull dimension one. Moreover, $Z(R) = \mathbb{Q} + y\mathbb{Q}(x)[[y]]$, over which R is certainly not integral. In this case T is not the trace ring of R, but certainly TR is a finite module both over its centre $L[[y]]$ and over R, while I is a common ideal of R and TR.

Now construct S as above. Then $\operatorname{gldim}(\tilde{T}) = \operatorname{gldim}(T) = 1$ and, since $R/I \cong L_1 \oplus L_2$, $\operatorname{gldim}(R/I) = 0$. Thus, (5.12) implies that $\operatorname{gldim}(S) \leq 2$. By [MR, Theorem 13.9.16], $\operatorname{gldim}(S) \geq 2$. Finally, we need to check that S is semilocal. But, $M_2(I) = yM_4(L[[y]])$ is an ideal of S that is also a quasiregular ideal of $\tilde{T} \cong M_4(L[[y]])$. Thus, $0 \neq M_2(I) \subseteq J(S)$. Since $K\dim(S) = 1$, this implies that S is semilocal. ∎

A somewhat more complicated version of this example appears in [BuH, Example 14]. Further examples of Noetherian PI rings R of finite global dimension for which $Z(R)$ has various bad properties can also be found in [BuH]. The examples given there are not semi-local, but that hypothesis can easily be arranged. All these examples can equally well be modified so that, rather than assuming that the ring is semi-local, one assumes that the ring is graded, say $R = \oplus_{i \geq 0} R_i$, with each R_i finite dimensional over the base field. For an example of an Auslander-regular, Noetherian, prime PI ring R that is a finite module over its centre, but such that $Z(R)$ is not integrally closed, see [St3, Example 3.5].

Finally, we note that the Auslander condition does not always hold for PI rings of finite global dimension.

Example 5.14. There exists a semilocal, Noetherian, prime PI ring S, of finite global dimension, such that S is not Auslander-regular.

Proof. Let $K = (x, y) \subset C = \mathbb{Q}[[x, y]]$, and set

$$J = \begin{pmatrix} K & K \\ C & C \end{pmatrix} \subset S = \begin{pmatrix} C & K \\ C & C \end{pmatrix} \subset \tilde{T} = M_2(C).$$

In this case, $Z(S) \cong C$ and S is a finite $Z(S)$-module. As in the previous example, it is readily checked that S is a semilocal, prime Noetherian ring. Also, J is an ideal of S that is a right ideal of $\tilde{T}$ with $\tilde{T}J = \tilde{T}$. Thus, (5.12) implies that

$$\text{gldim}(S) \leq \text{gldim}(\tilde{T}) + \text{gldim}(S/J) + 1 = 3.$$

Note that J is a maximal ideal of S with $({}_SJ)^* \supseteq \tilde{T}$. Thus, by the dual basis lemma, ${}_SJ$ is projective and hence reflexive. However, $J \supsetneq Sx$ and $S/Sx \cong \begin{pmatrix} \mathbb{Q}[[y]] & y\mathbb{Q}[[y]] \\ \mathbb{Q}[[y]] & \mathbb{Q}[[y]] \end{pmatrix}$. It follows that Sx is prime ideal that is again reflexive as a left module. Thus, by [Lv1, Proposition 4.5(iii)], S cannot be Auslander-regular. ∎

6. Fully Bounded, Noetherian Rings

In this section we discuss the extent to which the results of Section 3 can be generalized to work for FBN rings. (See also the note added in proof.) There are two obstacles to such a generalization: First, one needs to find circumstances in which (3.6) holds. Second, one needs to find ways of proving Theorem 3.10 without the assumption that $\text{EXT}^i_R(R/P_R, R_R)$ is a finitely generated right R/P-module. For both these questions, we have partial answers.

We begin by considering FBN connected graded rings. Consider a function $f(n): \mathbb{Z} \to \mathbb{N}$. If there exists a positive integer t and polynomial functions $p_1(n), \ldots, p_t(n) \in \mathbb{Q}[n]$ such that $f(n) = p_s(n)$ for all $n \equiv s(\text{mod } t)$, then $f(n)$ is called a *multi-polynomial function*. Define $\deg(f) = \max\{\deg(p_s): s = 1, \ldots, t\}$. Given a graded ring R and a finitely generated, graded R-module $M = \oplus M_i$, set $f_M(n) = \dim_k(M_n)$.

Lemma 6.1. *Assume that $R = \oplus R_i$ is an FBN, graded k-algebra such that R_0 is a finite dimensional k-vector space and let M be a finitely generated, graded right R-module. Then*

(i) *the function $f_M(n)$ is a multi-polynomial function for $n \gg 0$.*

(ii) $GK\dim(M) = K\dim(M) \in \mathbb{N}$. *In particular,* $GK\dim(R) < \infty$.

Proof. (i) If M is Artinian, then M is finite dimensional over k and the result is obvious. Thus, for some $\alpha > 0$, assume that $f_N(n)$ is a multi-polynomial function for any finitely generated, graded module N with $K\dim(N) < \alpha$ and every $n \gg 0$. Suppose that $K\dim(M) = \alpha$. Now, $f_M(n) = f_K(n) + f_{M/K}(n)$, for any submodule K of M. Thus, by induction and Lemma 2.1, we may assume that $M \cong \bar{I}$, where $\bar{I}$ is a graded-uniform right ideal of a graded prime factor ring $\bar{R} = R/P$ of R. Pick a homogeneous element $x \in \bar{I}$ of positive degree such that $x\bar{I} \neq 0$. Since $\bar{I}$ is uniform, $x\bar{I} \cong \bar{I}$ and $L = \bar{I}/x\bar{I}$ satisfies $K\dim(L) < \alpha$. Now compute dimensions in the graded short exact sequence

$$0 \longrightarrow M \xrightarrow{x.} M \longrightarrow L \longrightarrow 0. \tag{6.1.1}$$

If $x \in \bar{R}_m$, then $f_M(n) - f_M(n-m) = f_L(n)$ for all n. For all $n \gg 0$, the inductive hypothesis implies that $f_L(n)$ is a multi-polynomial function, say of degree r, and so $f_M(n)$ is a multi-polynomial function of dgree $r+1$.

(ii) It follows immediately from part (i) that $GK\dim(M) = 1 + \deg(f_M) \in \mathbb{N}$. Following [MR, Section 8.3.17], $GK\dim$ is called finitely right partitive if the following condition holds for every finitely generated, graded right R-module M: There exists an integer v such that, for any descending chain $M = M^0 \supset M^1 \supset \cdots \supset M^m$ with $GK\dim(M^i/M^{i-1}) = GK\dim(M)$ for each i, one has $m \le v$. Let $f_M(n) = p_s(n)$, for $n \equiv s(\operatorname{mod} t)$ and some polynomial functions $p_s(x) = \sum_{i=1}^{r(s)} \lambda_{s,i} x^i$, as defined before the statement of the lemma. Then the fact that $f_M(n)$ is always a non-negative integer implies that $t(s) = r(s)! \cdot \lambda_{s,r(s)} \in \mathbb{N}$, for each s. Since $f_M(n) = f_L(n) + f_{M/L}(n)$, for any graded submodule L of M, this implies that $GK\dim$ is finitely right partitive; just take $v = \max\{t(s)\}$. Thus, [MR, Proposition 8.3.18] implies that $GK\dim(M) \ge K\dim(M)$, for any finitely generated right R-module M.

It remains to prove that $K\dim(M) \ge GK\dim(M)$. As in the proof of part (i), we can reduce to the case when $M \cong \bar{I}$ and there exists a short exact sequence (6.1.1). Moreover, by induction on Krull dimension, we may assume that $K\dim(L) = GK\dim(L)$. By part (i), $GK\dim(M) = 1 + GK\dim(L)$. Thus, as M is critical, $K\dim(M) \ge K\dim(L) + 1 = GK\dim(L) + 1 = GK\dim(M)$. ∎

It is a reasonable conjecture that Lemma 6.1 will hold for any connected graded Noetherian ring R, at least when $GK\dim(R) < \infty$. For other special cases where it is known to hold see [ATV2, Proposition 2.21] and [Lo]. This lemma also provides slight evidence for an old conjecture about FBN rings; that the set of prime ideals satisfies DCC (see [GW1, Question 3, p. 285]).

THEOREM 6.2. *Let R be a graded injectively smooth, FBN, graded k-algebra, such that R_0 is a finite dimensional k-module. Then R is Auslander–Gorenstein and Macaulay (as a graded or ungraded ring). In particular, $GK\dim(R) = K\dim(R) = \operatorname{injdim}(R)$.*

Proof. Suppose that R is Auslander–Gorenstein and GK dim-Macaulay as a graded ring. As R is Noetherian, certainly each R_i is finite dimensional over k. Hence, by Lemma 4.4, R is Auslander–Gorenstein and GK dim-Macaulay as an ungraded ring and so, by Lemma 6.1, R is Macaulay. Thus, we need only prove the graded result.

The proof is similar to that of Theorem 3.10, so some of the details will be left to the reader. By Corollary 3.7, R is a gr-Gor$_0$ ring. As in the proof of Theorem 3.10, it suffices to prove that the following conditions hold for any finitely generated, graded right R-module M.

(a) $j(M) + GK\dim(M) = n$.

(b) $GK\dim(E^{j(M)}(M)) = GK\dim(M)$.

(c) *For all* $m \leq n, GK\dim(E^m(M)) \leq \min\{GK\dim(M), n - m\}$.

Since R is a gr-Gor$_0$ ring, these conditions do hold if $GK\dim(M) = 0$. Thus, for some integer $\alpha \geq 1$, assume that (a), (b), and (c) hold for every finitely generated, graded R-module M' with $GK\dim(M') < \alpha$. Let $GK\dim(M) = \alpha$. As in the proof of Theorem 3.10, we may assume that $M = R/P$ for some prime ideal P of R. Pick a homogeneous, regular element $z \in (R/P)_r$, for some $r > 0$, and consider the exact sequence

$$E^j(M/zM) \longrightarrow E^j(M) \xrightarrow{\cdot z} E^j(M) \longrightarrow E^{j+1}(M/zM). \quad (6.2.1)$$

In this case, $E^j(M) = \oplus_{k \geq k_0} E^j(M)_k$ is a graded $(R\text{-}R/P)$-bimodule and so $E^j(M)_k z \subseteq E^j(M)_{r+k}$ for each k. Therefore, $E^j(M) \neq E^j(M)z$, unless $E^j(M) = 0$, and so $E^j(M) = 0$ if $j < n - \alpha$. In order to complete the proof it suffices to show that, for $j \geq n - \alpha$, one has $GK\dim(E^j(M)) \leq \beta = \min\{GK\dim(M), n - j\}$. Let X be the largest graded left submodule of $E^j(M)$ that satisfies $GK\dim(X) \leq \beta$. By induction, $GK\dim(E^j(M/zM)) \leq \beta$ and $GK\dim(E^{j+1}(M/zM)) \leq \beta - 1$. Thus, (6.2.1) induces an exact sequence

$$0 \longrightarrow E^j(M)/X \xrightarrow{\cdot z} E^j(M)/X \longrightarrow K \longrightarrow 0.$$

Here, K is a subfactor of $E^{j+1}(M/zM)$ and so $GK\dim(K) \leq \beta - 1$. Therefore, by Lemma 6.1, $GK\dim(E^j(M)) \leq GK\dim(K) + 1 \leq \beta$. ∎

COROLLARY 6.3. (i) *Let R be an FBN, connected graded ring with* injdim$(R) < \infty$. *Then R is Auslander–Gorenstein and Macaulay.*

(ii) *Let R be an FBN, connected graded ring with* gldim$(R) < \infty$. *Then, R is a domain and a maximal order in its quotient division ring.*

Proof. (i) By Lemma 3.12, R is graded injectively smooth. Thus, the result follows from Theorem 6.2.

(ii) This follows from part (i) combined with [St3, Theorem 2.10]. ∎

One can generalize Theorem 6.2 in several directions. First, the result actually holds for any FBN graded ring R of finite injective dimension for which R_0 is an Artinian ring. To prove this, one needs to make two changes. First, the Gelfand–Kirillov dimension may not be available, but in its place one may use the dimension function defined in the same manner as the GK dimension, except that one replaces the dimension of k-vector spaces by the length of R_0-modules. Second, one has to prove that R is a gr-Gor_0 ring; just as graded arguments were used in place of Theorem 3.5 in the proof of Theorem 6.2, one proves this by using graded arguments in place of Theorem 3.5 in the proof of Theorem 3.8.

The theorem and its corollary also hold for any connected, graded, Neotherian ring R of finite injective dimension that satisfies the following condition:

($*$) In every prime factor ring $\bar{R}$ of R, there exists a non-zero, central, homogeneous element $x \in \bar{R}^+$.

The key point in the proof is that, for any critical R-module M with $\mathrm{ass}(M) = \mathrm{ann}(M) = P$, there exists a short exact sequence $0 \longrightarrow M \xrightarrow{\phi} M \longrightarrow L \longrightarrow 0$, where ϕ is now given by *right* multiplication by a central element $x \in R/P$. Now mimic the proof of Theorem 6.2. If one assumes that x is normal rather than central in ($*$), then one can still show that $GK\,\mathrm{dim}(R) = K\,\mathrm{dim}(R) = \mathrm{injdim}(R)$ and, if $\mathrm{gldim}(R) < \infty$, that R is a domain. (The same proof works, except that one assumes that M is an R-bimodule and that the homomorphism ϕ is given by left multiplication by a normal element z from the appropriate factor ring of R.)

Finally, recall that, by Theorem 3.8, a graded injectively smooth FBN ring will be Gor_0 provided that every simple Artinian factor ring is central simple. As the next result shows, these rings will frequently be Auslander–Gorenstein. Given a prime ideal P of a ring R, pick a uniform right ideal K/P in R/P and write I_P for the injective hull of $(K/P)_R$.

THEOREM 6.4. *Suppose that R is an (ungraded)* Gor_0, FBN *ring such that R contains an uncountable, central subfield. Then*

(i) *R is Auslander–Gorenstein and Macaulay. Thus,* $K\,\mathrm{dim}(R) = \mathrm{injdim}(R)$.

(ii) *Let $0 \longrightarrow R \longrightarrow I^{\cdot}$ be the minimal injective resolution of R_R and fix an integer $s \geq 0$. Then $I^s = \oplus\{I_P\}$, where the sum is taken over all prime ideals P such that $K\,\mathrm{dim}(R/P) + s = K\,\mathrm{dim}(R)$ (possibly with repetitions).*

We will not prove this theorem, but the basic idea is as follows. The hypothesis that R contains an uncountable field allows one to apply the results of [Br]. In particular, [Br, Lemma 2.3] implies that $I^s = \oplus\{I_P\}$, where the sum is taken over those prime ideals P such that $E^s(R/P)$ is not torsion as a right R/P-module. By [Br, Theorem C], this forces $K\,\mathrm{dim}(R/P) \leq \mathrm{injdim}(R) - s$. Thus, this proves "half" of part (ii). In order to prove the theorem, one first proves part (ii) and then uses that to prove part (i).

Note added in proof: In his thesis [Te], Kok-Ming Teo has proved that every smooth FBN ring R is Auslander–Gorenstein and Macaulay. It follows, for example, that Corollary 5.9 holds for smooth FBN rings.

References

[AR] M. Artin, Geometry of quantum planes, *in* "Azumaya Algebras, Actions and Groups" (D. Haile and J. Osterberg, Eds.), Contemp. Math., Vol. 124, Amer. Math. Soc., Providence, RI, 1992.

[AS] M. Artin and W. Schelter, Graded algebras of global dimension 3, *Adv. in Math.* **66** (1987), 171–216.

[ATV1] M. Artin, J. Tate and M. Van Der Bergh, Some algebras associated to automorphisms of curves, *in* "The Grothendieck Festschrift" (P. Cartier, *et al.*, Eds.), Birkhauser, Basel, 1990.

[ATV2] M. Artin, J. Tate and M. Van Den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.* **106** (1991), 335–389.

[AZ] M. Artin and J. J. Zhang, Noncommutative projective schemes, *Adv. Math.*, to appear.

[Bj1] J. E. Björk, "Rings of Differential Operators," North Holland, Amsterdam, 1979.

[Bj2] J. E. Björk, Filtered Noetherian rings, *in* "Noetherian Rings and their Applications" (L. W. Small, Ed.), Math. Surveys and Monographs, Vol. 24, Amer. Math. Sco., Providence, RI, 1987.

[BE] J. E. Björk and E. K. Ekström, Filtered Auslander–Gorenstein rings, *in* "Colloque en l'honneur de J. Dixmier," Birkhauser, Bosel, 1990.

[BuH] A. Braun and C. R. Hajarnavis, Finitely generated PI rings of global dimension two, *J. Algebra*, to appear.

[BS] A. Braun and L. W. Small, Localization in prime, Noetherian PI rings, *Math. Z.* **193** (1986), 323–330.

[BW] A. Braun and R. B. Warfield, Jr., Symmetry and localization in Noetherian prime PI rings, *J. Algebra* **118** (1988), 322–335.

[Br] K. A. Brown, Fully bounded rings of finite injective dimension, *Quart. J. Math. Oxford Ser.* **41** (1990), 1–13.

[BwH] K. A. Brown and C. R. Hajarnavis, Homologically homogeneous rings, *Trans. Amer. Math. Soc.* **281** (1984), 197–208.

[BHM] K. A. Brown, C. R. Hajarnavis and A. B. MacEacharn, Rings of finite global dimension integral over their centres *Comm. in Algebra* **11** (1983), 67–93.

[BL] K. A. Brown and T. Levasseur, Cohomology of bimodules over enveloping algebras, *Math. Z.* **189** (1985), 393–413.

[CE] H. Cartan and S. Eilenberg, "Homological Algebra," Princeton Uinv. Press, Princeton, 1956.

[Co] P. M. Cohn, "Skew Field Constructions," London Math. Soc. Lecture Note Series, Vol. 27, Cambridge Univ. Press, 1977.

[Ek] E. K. Ekström, The Auslander condition on graded and filtered Noetherian rings, "Séminaire Dubreil-Malliavin, 1987–8," Lecture Notes in Math. Vol. 1404, Springer-Verlag, Berlin, 1989.

[GS] K. R. Goodearl and L. W. Small, Krull verses global dimension in Noetherian PI rings, *Proc. Amer. Math. Soc.* **92** (1984), 175–178.

[GW1] K. R. GOODEARL AND R. B. WARFIELD, JR., "An Introduction to Noncommutative Noetherian Rings," London Math. Soc. Student Texts, Vol. 16, Cambridge Univ. Press Cambridge, 1989.

[GW2] K. R. GOODEARL AND R. B. WARFIELD, JR., State spaces of K_0 of Noetherian rings, *J. Algebra* **71** (1981), 322–378.

[Gy] A. J. GRAY, A class of maximal orders integral over their centres, *Glasgow Math. J.* **24** (1983), 177–180.

[Gr] A. GROTHENDIECK, Sur quelques points d'algèbre homologique, *Tôhoku Math. J.* **9** (1957), 119–221.

[Ja] A. V. JATEGAONKAR, "Localization in Noetherian Rings," London Math. Soc. Lecture Note Series, Vol. 98, Cambridge Univ. Press, Cambridge, 1986.

[JJ] C. JENSEN AND S. JONDRUP, Centres and fixed points of Artinian rings, *Math. Z.* **130** (1973), 189–197.

[KL] G. KRAUSE AND T. H. LENAGAN, "Growth of Algebras and Gelfand–Kirillov Dimension, Pitman, London, 1985.

[Lv1] T. LEVASSEUR, Some properties of non-commutative regular rings, *Glasgow J. Math.*, **34** (1992), 277–300.

[Lv2] T. LEVASSEUR, Complexe bidualisant en algèbre non-commutative, *in* "Séminaire Malliavin," Lecture Notes in Math., Vol. 1146, Springer-Verlag, Berlin, 1985.

[Lo] M. LORENZ, "Gelfand–Kirillov Dimension and Poincaré Series," Cuarderos de Algebra, Vol. 7, Granada, 1988.

[MR] J. C. MCCONNELL AND J. C. ROBSON, "Non-commutative Noetherian Rings," Wiley–Interscience, Chichester, 1987.

[MS] J. C. MCCONNELL AND J. T. STAFFORD, Gelfand–Kirillov dimension and associated graded modules, *J. Algebra* **125** (1989), 197–214.

[Mi] B. MITCHELL, "Theory of Categories," Academic Press, New York, 1965.

[Na] M. NAGATA, "Local Rings," Wiley, New York, 1962.

[NV] C. NASTACESCU AND F. VAN OYSTAEYEN, "Graded Ring Theory," North Holland, Amsterdam, 1982.

[PR] I. PALMÉR AND J.-E. ROOS, Explicit formulæ for the global homological dimensions of trivial extensions of rings, *J. Algebra* **27** (1973), 380–413.

[Ro] J. J. ROTMAN, "An Introduction to Homological Algebra," Academic Press, New York, 1979.

[Rw] L. H. ROWEN, "Polynomial Identities in Ring Theory," Academic Press, New York, 1980.

[St1] J. T. STAFFORD, Stable structure of noncommutative Noetherian rings, II, *J. Algebra* **52** (1978), 218–235.

[St2] J. T. STAFFORD, Non-holonomic modules over Weyl algebras and enveloping algebras, *Invent. Math.* **79** (1985), 619–638.

[St3] J. T. STAFFORD, Auslander regular algebras and maximal orders, *J. London Math. Soc.*, to appear.

[St4] J. T. STAFFORD, Infinite cliques in Noetherian rings, *preprint*.

[SZ] J. T. STAFFORD, AND J. J. ZHANG, Examples of graded rings, *Math. Proc. Camb. Phil. Soc.*, to appear.

[Te] K.-M. TEO Homological properties of fully bounded Noetherian rings and Sklyanin algebras, Thesis, University of Michigan, 1994.

[TV] J. TATE AND M. VAN DEN BERGH, Homological properties of Sklyanin algebras, *to appear*.

[Va] V. W. VASCONCELOS, On quasi-local regular algebras, *in* "Sympos. Math.," Vol. 11, pp. 11–22, Academic Press, London, 1973.

[Wa] R. B. WARFIELD, JR., Non-commutative localized rings, *in* "Séminaire Malliavin," Lecture Notes in Math., Vol. 1220, Springer-Verlag, Berlin, 1986.

[Za] A. ZAKS, Injective dimension of semi-primary rings, *J. Algebra* **13** (1969), 73–89.