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An Improved Unifying Density Function

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Abstract—This paper presents an improved and simplified version of the unifying probability density function of [1]. The improved unifying density function is shown to be the parent of the Rayleigh distribution in addition to the Weibull-, gamma-, Erlang-, χ^2- and exponential distributions. The means and variances of the relating child distributions can be found easily by substitution using the established relationships.

Keywords—Probability distributions, Positive density functions, Probability densities.

1. INTRODUCTION

In [1] a unifying probability density function was presented. It was shown that the unifying density function is the parent of the Weibull-, gamma-, Erlang-, χ^2- and exponential distributions. This paper presents an improved version of that unifying probability density function.

The improved version establishes a relationship between the Weibull- and gamma distributions such that both distributions can be shown to be parents of the same exponential distribution. In addition, the Rayleigh distribution is shown to be a special case of the Weibull distribution. The means and variances of the gamma-, Erlang-, χ^2- and exponential distributions follow easily, as shown, e.g., in [2] from the known mean and variance of the gamma distribution. It is shown that the means and variances of the exponential and Rayleigh distributions follow similarly from the known mean and variance of the Weibull distribution. This paper assumes familiarity with the above-mentioned continuous positive distributions discussed in many introductory statistics and probability text books including [2–4].

2. THE IMPROVED UNIFYING DENSITY FUNCTION

The improvement in the presented form of the unifying density function, compared to the original unifying density function [1], is merely in the handling and simplification of the exponents of the original unifying density function. The changes, however, result into a significant improvement in explaining relationships between these positive distributions including their means and variances.

THEOREM 1. *The improved unifying density function $f(x)$ is the parent of the Weibull-, gamma-, Erlang-, χ^2-, exponential- and Rayleigh probability distributions:*

$$f(x) = \frac{\alpha}{\beta^{\alpha^{\varphi}}\Gamma(\alpha^{\varphi}+1)}\, x^{\alpha-1}\, e^{-\beta^{-1}x^{-(\alpha^{\varphi}-\alpha-1)}}, \qquad x > 0,$$
$$f(x) = 0, \qquad x \le 0,$$

where $\alpha > 0$, $\beta > 0$ and $\varphi = 0$ or $\varphi = 1$. Here α and β are the distribution shape parameters and φ is the unifying parameter.

PROOF.

1. Obtain the Weibull distribution by setting $\varphi = 0$.

$$f(x) = \frac{\alpha}{\beta^1 \Gamma(\alpha^0 + 1)} x^{\alpha-1} e^{-\beta^{-1} x^{-(\alpha^0 - \alpha - 1)}} \Rightarrow f(x) = \frac{\alpha}{\beta \Gamma(2)} x^{\alpha-1} e^{-x^\alpha/\beta}.$$

Using the property of the gamma function $\Gamma(n) = (n-1)!$, the Weibull distribution results:

$$f(x) = \frac{\alpha}{\beta} x^{\alpha-1} e^{-x^\alpha/\beta}.$$

Using $\lambda = 1/\beta$, the Weibull distribution may be written in its more commonly presented form:

$$f(x) = \alpha\lambda x^{\alpha-1} e^{-\lambda x^\alpha}.$$

2. Obtain the gamma distribution by setting $\varphi = 1$.

$$f(x) = \frac{\alpha}{\beta^\alpha \Gamma(\alpha + 1)} x^{\alpha-1} e^{-\beta^{-1} x^{-(\alpha - \alpha - 1)}} \Rightarrow f(x) = \frac{\alpha}{\beta^\alpha \Gamma(\alpha + 1)} x^{\alpha-1} e^{-x/\beta}.$$

Using the property of the gamma function that

$$\Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha - 1) \Rightarrow \Gamma(\alpha + 1) = \alpha\Gamma(\alpha) \Rightarrow \alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)},$$

the gamma distribution results:

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}.$$

3. Obtain the exponential distribution from the Weibull distribution by setting $\alpha = 1$:

$$f(x) = \frac{1}{\beta} x^{1-1} e^{-x/\beta} \Rightarrow f(x) = \frac{1}{\beta} e^{-x/\beta}.$$

4. Obtain the Rayleigh distribution from the Weibull distribution by setting $\alpha = 2$.

$$f(x) = \frac{2}{\beta} x^{2-1} e^{-x^2/\beta} \Rightarrow f(x) = \frac{2}{\beta} x\, e^{-x^2/\beta}.$$

Using $\lambda = 1/\beta$, the Rayleigh distribution may be written in its more commonly presented form:

$$f(x) = 2\lambda\, x\, e^{-\lambda x^2}.$$

As presented in, e.g., [2,3] the Erlang distribution is a gamma distribution with an integer valued shape parameter. The χ^2-distribution is obtained from the gamma distribution by setting $\alpha = \nu/2$ and $\beta = 2$. The exponential distribution is obtained, not only from the gamma distribution but also from the Weibull distribution by setting $\alpha = 1$. The above distributions are presented for $x > 0$, $\alpha > 0$, $\beta > 0$. For all above distributions, if $x \leq 0$ then $f(x) = 0$. ∎

Clearly, the Weibull- and gamma distributions are related. Both reduce to the exponential distribution for all values of β and $\alpha = 1$. Figure 1 summarizes the relationships between the above positive distributions and the unifying density function.

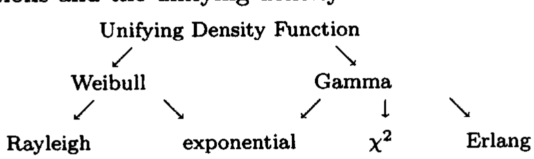

Figure 1. Relationships between the distributions and the unifying density.

2.1. Relationships between the Means and Variances

The means and variances of the above distributions, including the relationships between the means and variances of the gamma-, χ^2- and exponential distributions, are presented, e.g., in [2,3]. The means and variances of the Rayleigh and exponential distributions are obtained from the known mean and variance of the Weibull distribution using the presented relationships by substitution.

The mean and variance of the Weibull distribution [2]:

$$\mu = \lambda^{-1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right) = \beta^{1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right),$$

$$\sigma^2 = \lambda^{-2/\alpha}\left\{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right\} = \beta^{2/\alpha}\left\{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right\},$$

where $\lambda = 1/\beta$ as before.

If we set $\alpha = 1$, we obtain the mean and variance of the exponential distribution:

$$\mu = \lambda^{-1}\Gamma(2) = \lambda^{-1} = \beta,$$

$$\sigma^2 = \lambda^{-2}\left\{\Gamma(3) - [\Gamma(2)]^2\right\} = \lambda^{-2} = \beta^2.$$

If we set $\alpha = 2$, we obtain the mean and variance of the Rayleigh distribution:

$$\mu = \lambda^{-1/2}\Gamma\left(\frac{3}{2}\right) = \lambda^{-1/2}\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \lambda^{-1/2}\left(\frac{1}{2}\right)\sqrt{\pi} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}},$$

$$\sigma^2 = \lambda^{-1}\left\{\Gamma(2) - \left[\Gamma\left(\frac{3}{2}\right)\right]^2\right\} = \lambda^{-1}\left\{1 - \left[\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\right]^2\right\} = \lambda^{-1}\left\{1 - \frac{\pi}{4}\right\}.$$

3. CONCLUSION

This paper improves and extends the original unifying density function to serve as a parent of six positive continuous probability distributions. The improved handling and simplification of the exponents of the original unifying density function relate better the Weibull and gamma distributions. Using the means and variances of the Weibull and gamma distributions, the means and variances of the child distributions can be easily derived by substitution.

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