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Note

R-sequenceability and R^*-sequenceability of abelian 2-groups

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Abstract

A group of order n is said to be *R-sequenceable* if the nonidentity elements of the group can be listed in a sequence $a_1, a_2, \ldots, a_{n-1}$ such that the quotients $a_1^{-1}a_2, a_2^{-1}a_3, \ldots, a_{n-2}^{-1}a_{n-1}, a_{n-1}^{-1}a_1$ are distinct. An abelian group is R^*-sequenceable if it has an R-sequencing $a_1, a_2, \ldots, a_{n-1}$ such that $a_{i-1}a_{i+1} = a_i$ for some i (subscripts are read modulo $n-1$). Friedlander, Gordon and Miller (1978) showed that an R^*-sequenceable Sylow 2-subgroup is a sufficient condition for a group to be R-sequenceable. In this paper we also show that all noncyclic abelian 2-groups are R^*-sequenceable except for $\mathscr{Z}_2 \times \mathscr{Z}_4$ and $\mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_2$.

A group of order n is said to be *R-sequenceable* if the nonidentity elements of the group can be listed in a sequence $a_1, a_2, \ldots, a_{n-1}$ such that the quotients $a_1^{-1}a_2, a_2^{-1}a_3, \ldots, a_{n-2}^{-1}a_{n-1}, a_{n-1}^{-1}a_1$ are distinct. The concept of R-sequenceability has been around for more than 40 years in one form or another. In 1951 Paige observed that it is a sufficient condition for a group to have a complete mapping. In 1955 Hall and Paige [3] showed that a solvable group has a complete mapping if and only if its Sylow 2-subgroup is either trivial or noncyclic. In 1974 Ringel [5] was led to the concept of R-sequenceability in his solution of the map coloring problem for all compact two-dimensional manifolds except the sphere. In their book [1] Dénes and Keedwell used an alternative definition of R-sequenceable and discussed the topic in great depth. They also showed that an abelian group is a super P-group if and only if it is either R-sequenceable or sequenceable. Friedlander et al. [2] showed that the following types of abelian groups are R-sequenceable: cyclic groups of odd order greater than 1; groups of odd order whose Sylow 3-subgroup is cyclic; groups whose orders are relatively prime to 6; elementary abelian p-groups, except the group of order 2; groups of type $\mathscr{Z}_2 \times \mathscr{Z}_{4k}$, $k \geq 1$; groups whose Sylow p-subgroup has the form $\mathscr{Z}_2^m$, $m > 1$ but $m \neq 3$; groups G whose Sylow p-subgroup has the form $S = \mathscr{Z}_2 \times \mathscr{Z}_n$

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where $n=2^k$ and either k is odd or $k \geqslant 2$ is even and G/S has a direct cyclic factor of order congruent to 2 modulo 3. Ringel [1] has claimed that abelian groups of the form $\mathscr{Z}_2 \times \mathscr{Z}_{6k+2}$ are R-sequenceable.

Friedlander et al. [2] conjectured that an abelian group is R-sequenceable if and only if its Sylow 2-subgroup is either trivial or noncyclic. This paper proves the conjecture for abelian 2-groups.

The following types of nonabelian groups are known to be R-sequenceable: groups of order pq where p and q are odd primes, $p<q$, and p has 2 as a primitive root [4]; dihedral groups of order $2n$ where n is even [4]; dicyclic groups of order $4n$ where n is divisible by 4 [6].

An abelian group is R^*-sequenceable if it has an R-sequencing $a_1, a_2, \ldots, a_{n-1}$ such that $a_{i-1}a_{i+1}=a_i$ for some i (subscripts are read modulo $n-1$). The term was introduced by Friedlander et al. [2], who showed that the existence of an R^*-sequenceable Sylow 2-subgroup is a sufficient condition for a group to be R-sequenceable. In this paper we also show that all noncyclic abelian 2-groups are R^*-sequenceable except for $\mathscr{Z}_2 \times \mathscr{Z}_4$ and $\mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_2$.

We begin with two results of Friedlander et al. concerning abelian 2-groups.

Lemma 1 (Friedlander et al. [2]). *$(\mathscr{Z}_2)^m$ is R^*-sequenceable for $m>1$, $m \neq 3$, $\mathscr{Z}_2 \times \mathscr{Z}_{2^k}$ is R^*-sequenceable for k odd, and R-sequenceable for all k.*

Lemma 2 (Friedlander et al. [2]). *$\mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_2$ and $\mathscr{Z}_2 \times \mathscr{Z}_4$ are R-sequenceable but not R^*-sequenceable.*

Lemma 3. *If an abelian group G is an extension of $\mathscr{Z}_2 \times \mathscr{Z}_2$ by an R^*-sequenceable group H, then G is R^*-sequenceable.*

Proof of Lemma 3. Let $n=|H|$. Since H is R^*-sequenceable, the cosets of $\mathscr{Z}_2 \times \mathscr{Z}_2$, excluding $\mathscr{Z}_2 \times \mathscr{Z}_2$ itself, have an ordering $K_1, \ldots, K_{n-1}$ that is an R-sequence with $K_{n-1}K_2=K_1$. Choose k_i, $1 \leqslant i \leqslant n-1$, such that $k_i \in K_i$ and $k_{n-1}k_2=k_1$. Then any element in G can be uniquely expressed as a product of an element in $\mathscr{Z}_2 \times \mathscr{Z}_2$ and an element in $\{k_1, \ldots, k_{n-1}, e\}$. Let $\{y_i\}_{i=1}^{4n-1}$ be the sequence $k_1, k_2, \ldots, k_{n-1}$, e, $k_2, k_3, \ldots, k_{n-1}$, $k_1, k_1, k_1, k_2, \ldots, k_{n-1}$, e, e, $k_2, k_3, \ldots, k_{n-1}$. Let a and b be generators of the $\mathscr{Z}_2 \times \mathscr{Z}_2$ subgroup of G. Define $\{x_i\}_{i=1}^{4n-1}$ as follows.

Case 1: $|H| \bmod 3 \equiv 0$. Let $3k=|H|$, $\{x_i\}$ is given by the successive rows of the $4 \times n$ matrix

$$\begin{pmatrix} e & e & \cdots & & & b & a \\ ab & k-2 \text{ copies of } \{a,b,ab\} & a & ab & ab & b & a \\ ab & b & k-2 \text{ copies of } \{ab,a,b\} & ab & b & a & ab \\ b & a & k-2 \text{ copies of } \{b,ab,a\} & b & a & e & \end{pmatrix}.$$

If $k=1$, then $H=\mathscr{Z}_3$, so $G=\mathscr{Z}_2\times\mathscr{Z}_6$, which is R^*-sequenceable since its Sylow 2-subgroup is R^*-sequenceable.

Case 2: $|H| \bmod 3\equiv 1$. Let $3k+1=|H|$. $\{x_i\}$ is read from the successive rows of the $4\times n$ matrix.

$$\begin{pmatrix} e & e & \cdots & b & a \\ ab & k-1 \text{ copies of } \{b,a,ab\} & ab & b & a \\ ab & b & k-1 \text{ copies of } \{a,ab,b\} & a & ab \\ b & a & k-1 \text{ copies of } \{ab,b,a\} & e & \end{pmatrix}.$$

Case 3: $|H| \bmod 3\equiv 2$. Let $3k+2=|H|$. $\{x_i\}$ is read from the successive rows of the $4\times n$ matrix

$$\begin{pmatrix} e & e & \cdots & & b & a \\ ab & k-1 \text{ copies of } \{b,a,ab\} & b & ab & b & a \\ ab & b & k-1 \text{ copies of } \{a,ab,b\} & a & a & ab \\ b & a & k-1 \text{ copies of } \{ab,b,a\} & ab & e & \end{pmatrix}.$$

Then $\{x_iy_i\}$ is an R^*-sequence. Clearly $(x_{4n-1}y_{4n-1})(x_2y_2)=x_1y_1$. Verifying that $\{x_iy_i\}$ is an R-sequence is straightforward with the following observations:

(i) $k_{n-1}^{-1}e=k_1^{-1}k_2$ and $e^{-1}k_2=k_{n-1}^{-1}k_1$, so $\{y_i^{-1}y_{i+1}\}_{i=1}^{4n-1}$ (with $y_{4n}=y_1$) is the sequence $k_1^{-1}k_2, k_2^{-1}k_3, \ldots, k_{n-2}^{-1}k_{n-1}, k_1^{-1}k_2, k_{n-1}^{-1}k_1, k_2^{-1}k_3, k_3^{-1}k_4, \ldots, k_{n-2}^{-1}k_{n-1}$, $k_{n-1}^{-1}k_1, e, e, k_1^{-1}k_2, k_2^{-1}k_3, \ldots, k_{n-2}^{-1}k_{n-1}, k_1^{-1}k_2, e, k_{n-1}^{-1}k_1, k_2^{-1}k_3, k_3^{-1}k_4, \ldots, k_{n-2}^{-1}k_{n-1}, k_{n-1}^{-1}k_1$.

(ii) If x_m is the first element of the first copy of one of the repeated 3-element sequences in $\{x_i\}$, then $y_m=k_3$, and the sequence $\{a,b,ab\}$ is itself an R-sequence. □

Lemma 4. *$\mathscr{Z}_2\times\mathscr{Z}_{2^n}$ is R^*-sequenceable for $n\geqslant 1$, $n\neq 2$.*

Proof of Lemma 4. Any sequence of the nonidentity elements of $\mathscr{Z}_2\times\mathscr{Z}_2$ is an R^*-sequence. $\mathscr{Z}_2\times\mathscr{Z}_8\cong\langle a,b\,|\,a^8=b^2=e, ab=ba\rangle$ has the R^*-sequence ba^7, b, a^5, a^3, ba^6, ba, a^2, a^6, ba^5, ba^2, a^4, ba^4, ba^3, a^7, a. The relevant triple is ba^4, ba^3 and a^7.

For $n\geqslant 4$, $\mathscr{Z}_2\times\mathscr{Z}_{2^n}\cong\langle a,b\,|\,a^{2^n}=b^2=e, ab=ba\rangle$, an R^*-sequence can be read from the successive rows of this $2m\times 8$ matrix, where $m=2^{n-3}$:

ba^{8m-1}	b	a^{3m}	a^{5m}	ba^{8m-2}	ba	a^{m-2}	a^{7m+2}
ba^{8m-3}	ba^{2}	a^{3m-2}	a^{5m+2}	ba^{8m-4}	ba^{3}	a^{m-4}	a^{7m+4}
			$\vdots$				
ba^{7m+3}	ba^{m-4}	a^{2m+4}	a^{6m-4}	ba^{7m+2}	ba^{m-3}	a^{2}	a^{8m-2}
ba^{7m+1}	ba^{m-2}	a^{2m+2}	a^{6m-2}	ba^{7m}	ba^{m-1}	a^{8m-1}	a
ba^{7m-1}	ba^{m}	a^{6m-1}	a^{2m+1}	ba^{7m-2}	ba^{m+1}	a^{8m-3}	a^{3}
ba^{7m-3}	ba^{m+2}	a^{6m-3}	a^{2m+3}	ba^{7m-4}	ba^{m+3}	a^{8m-5}	a^{5}
			$\vdots$				
ba^{5m+3}	ba^{3m-4}	a^{4m+3}	a^{4m-3}	ba^{5m+2}	ba^{3m-3}	a^{6m+1}	a^{2m-1}
ba^{5m+1}	ba^{3m-2}	a^{4m+1}	a^{4m-1}	ba^{5m}	ba^{3m-1}	a^{2m}	a^{6m}
ba^{5m-1}	ba^{3m}	a^{4m}	—	ba^{5m-2}	ba^{3m+1}	a^{2m-2}	a^{6m+2}
ba^{5m-3}	ba^{3m+2}	a^{4m-2}	a^{4m+2}	ba^{5m-4}	ba^{3m+3}	a^{2m-4}	a^{6m+4}
			$\vdots$				
ba^{4m+1}	ba^{4m-2}	a^{3m+2}	a^{5m-2}	ba^{4m}	ba^{4m-1}	a^{m}	a^{7m}

To see that the sequence is an R-sequence, the successive quotients are listed in the successive rows of this matrix:

a	ba^{3m}	a^{2m}	ba^{3m-2}	a^{3}	ba^{m-3}	a^{6m+4}	ba^{m-5}
a^{5}	ba^{3m-4}	a^{2m+4}	ba^{3m-6}	a^{7}	ba^{m-7}	a^{6m+8}	ba^{m-9}
			$\vdots$				
a^{2m-7}	ba^{m+8}	a^{4m-8}	ba^{m+6}	a^{2m-5}	ba^{7m+5}	a^{8m-4}	ba^{7m+3}
a^{2m-3}	ba^{m+4}	a^{4m-4}	ba^{m+2}	a^{2m-1}	ba^{7m}	a^{2}	ba^{7m-2}
a^{2m+1}	ba^{5m-1}	a^{4m+2}	ba^{5m-3}	a^{2m+3}	ba^{7m-4}	a^{6}	ba^{7m-6}
a^{2m+5}	ba^{5m-5}	a^{4m+6}	ba^{5m-7}	a^{2m+7}	ba^{7m-8}	a^{10}	ba^{7m-10}
			$\vdots$				
a^{6m-7}	ba^{m+7}	a^{8m-6}	ba^{m+5}	a^{6m-5}	ba^{3m+4}	a^{4m-2}	ba^{3m+2}
a^{6m-3}	ba^{m+3}	a^{8m-2}	ba^{m+1}	a^{6m-1}	ba^{7m+1}	a^{4m}	ba^{7m-1}
a^{6m+1}	ba^{m}	–	ba^{m-2}	a^{6m+3}	ba^{7m-3}	a^{4m+4}	ba^{7m-5}
a^{6m+5}	ba^{m-4}	a^{4}	ba^{m-6}	a^{6m+7}	ba^{7m-7}	a^{4m+8}	ba^{7m-9}
			$\vdots$				
a^{8m-7}	ba^{7m+8}	a^{2m-8}	ba^{7m+6}	a^{8m-5}	ba^{5m+5}	a^{6m-4}	ba^{5m+3}
a^{8m-3}	ba^{7m+4}	a^{2m-4}	ba^{7m+2}	a^{8m-1}	ba^{5m+1}	a^{6m}	ba^{m-1}

If $m=6k+2$, we have $\ldots, ba^{7m-1-2(4k+1)}, ba^{m+2(4k+1)}, a^{6m-1-2(4k+1)}, \ldots$ and $(ba^{7m-1-2(4k+1)})(a^{6m-1-2(4k+1)})=ba^{14k+4}=ba^{m+2(4k+1)}$. If $m=6k+4$, we have $\ldots, a^{2m+1+2(4k+2)}, ba^{7m-2-2(4k+2)}, ba^{m+1+2(4k+2)}, \ldots$ and $(a^{2m+1+2(4k+2)})(ba^{m+1+2(4k+2)})=ba^{34k+22}=ba^{7m-2-2(4k+2)}$. Thus, the sequence is an R^*-sequence for all $n\geqslant 4$. □

Theorem. If G is a non-cyclic abelian 2-group, then G is R-sequenceable. Moreover, if $|G| \neq 8$, then G is R^*-sequenceable.

Proof. If $|G|=8$, the result follows from Lemma 2. Otherwise, we use induction on n, where $|G|=2^n$. For n even, the base of the induction is $n=2$, so that $G \cong \mathscr{Z}_2 \times \mathscr{Z}_2$, which is R^*-sequenceable by Lemma 1. For n odd, the base of the induction is $n=5$, so that either $G \cong \mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_2$, $G \cong \mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_4$, $G \cong \mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_8$, $G \cong \mathscr{Z}_2 \times \mathscr{Z}_4 \times \mathscr{Z}_4$, $G \cong \mathscr{Z}_2 \times \mathscr{Z}_{16}$ or $G \cong \mathscr{Z}_4 \times \mathscr{Z}_8$, $\mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_2 \times \mathscr{Z}_2$ is R^*-sequenceable by Lemma 1. $\mathscr{Z}_2 \times \mathscr{Z}_{16}$ is R^*-sequenceable by Lemma 4. The other groups are extensions of $\mathscr{Z}_2 \times \mathscr{Z}_4$ by $\mathscr{Z}_2 \times \mathscr{Z}_2$. Let H_1, H_2, H_3 be the cosets, other than $\mathscr{Z}_2 \times \mathscr{Z}_4$ itself, of $\mathscr{Z}_2 \times \mathscr{Z}_4$, and let h_1, h_2, h_3 be elements of H_1, H_2, H_3, respectively, such that $h_1 h_3 = h_2$. This is possible since H_1, H_2, H_3 must be an R^*-sequence of $G/(\mathscr{Z}_2 \times \mathscr{Z}_4)$. Let the subgroup of G isomorphic to $\mathscr{Z}_2 \times \mathscr{Z}_4$ be generated by a and b with $a^4 = b^2 = e$, $ab = ba$. Then the following is an R^*-sequence: h_1, h_2, ba^2h_3, a, bah_2, bah_3, a^2h_1, ah_1, bh_1, ba^2h_2, ba^3h_3, ba^2, b, ba^3h_2, a^2h_3, a^3h_1, bah_1, a^2h_2, ah_3, ba^3, a^3, ah_2, a^3h_3, ba^2h_1, ba^3h_1, a^3h_2, bh_3, ba, a^2, bh_2, h_3. The relevant triple is ba^3h_1, a^3h_2 and bh_3.

To complete the induction, we assume the result is true for n. Let $|G|=2^{n+2}$. If $G \cong \mathscr{Z}_2 \times \mathscr{Z}_{2^{n+1}}$, G is R^*-sequenceable by Lemma 4. Otherwise, G is an extension of $\mathscr{Z}_2 \times \mathscr{Z}_2$ by a noncyclic abelian 2-group H, and $|H|=2^n$. Since H is R^*-sequenceable by assumption, G is R^*-sequenceable by Lemma 3. □

Since Friedlander et al. [2] have shown that an abelian group whose Sylow 2-subgroup is R^*-sequenceable is itself R^*-sequenceable, we have the following corollary.

Corollary. *An abelian group whose Sylow* 2*-subgroup is noncyclic and not of order* 8 *is* R^**-sequenceable.*

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