Note

R-sequenceability and $R^*$-sequenceability of abelian 2-groups

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Abstract

A group of order $n$ is said to be $R$-sequenceable if the nonidentity elements of the group can be listed in a sequence $a_1, a_2, \ldots, a_{n-1}$ such that the quotients $a_1^{-1}a_2, a_2^{-1}a_3, \ldots, a_{n-2}^{-1}a_{n-1}, a_{n-1}^{-1}a_1$ are distinct. An abelian group is $R^*$-sequenceable if it has an $R$-sequencing $\sigma, \tau, \ldots, \rho, \rho$ such that $a_{i-1}^{-1}a_i = a_j$ for some $i$ (subscripts are read modulo $n-1$). Friedlander, Gordon and Miller (1978) showed that an $R^*$-sequenceable Sylow 2-subgroup is a sufficient condition for a group to be $R$-sequenceable. In this paper we also show that all noncyclic abelian 2-groups are $R^*$-sequenceable except for $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

A group of order $n$ is said to be $R$-sequenceable if the nonidentity elements of the group can be listed in a sequence $a_1, a_2, \ldots, a_n$ such that the quotients $a_1^{-1}a_2, a_2^{-1}a_3, \ldots, a_{n-1}^{-1}a_n, a_n^{-1}a_1$ are distinct. The concept of $R$-sequenceability has been around for more than 40 years in one form or another. In 1951 Paige observed that it is a sufficient condition for a group to have a complete mapping. In 1955 Hall and Paige [3] showed that a solvable group has a complete mapping if and only if its Sylow 2-subgroup is either trivial or noncyclic. In 1974 Ringel [5] was led to the concept of $R$-sequenceability in his solution of the map coloring problem for all compact two-dimensional manifolds except the sphere. In their book [1] Dénes and Keedwell used an alternative definition of $R$-sequenceable and discussed the topic in great depth. They also showed that an abelian group is a super $P$-group if and only if it is either $R$-sequenceable or sequenceable. Friedlander et al. [2] showed that the following types of abelian groups are $R$-sequenceable: cyclic groups of odd order greater than 1; groups of odd order whose Sylow 3-subgroup is cyclic; groups whose orders are relatively prime to 6; elementary abelian $p$-groups, except the group of order 2; groups of type $\mathbb{Z}_2 \times \mathbb{Z}_k$, $k \geq 1$; groups whose Sylow $p$-subgroup has the form $\mathbb{Z}_m^m$, $m > 1$ but $m \neq 3$; groups $G$ whose Sylow $p$-subgroup has the form $S = \mathbb{Z}_2 \times \mathbb{Z}_n$. 
where \( n = 2^k \) and either \( k \) is odd or \( k \geq 2 \) is even and \( G/S \) has a direct cyclic factor of order congruent to 2 modulo 3. Ringel \([1]\) has claimed that abelian groups of the form \( \mathcal{Z}_2 \times \mathcal{Z}_{6^k+2} \) are \( R \)-sequenceable.

Friedlander et al. \([2]\) conjectured that an abelian group is \( R \)-sequenceable if and only if its Sylow 2-subgroup is either trivial or noncyclic. This paper proves the conjecture for abelian 2-groups.

The following types of nonabelian groups are known to be \( R \)-sequenceable: groups of order \( pq \) where \( p \) and \( q \) are odd primes, \( p < q \), and \( p \) has 2 as a primitive root \([4]\); dihedral groups of order \( 2n \) where \( n \) is even \([4]\); dicyclic groups of order \( 4n \) where \( n \) is divisible by 4 \([6]\).

An abelian group is \( R^* \)-sequenceable if it has an \( R \)-sequencing \( a_1, a_2, \ldots, a_{n-1} \) such that \( a_{i-1}a_{i+1} = a_i \) for some \( i \) (subscripts are read modulo \( n-1 \)). The term was introduced by Friedlander et al. \([2]\), who showed that the existence of an \( R^* \)-sequenceable Sylow 2-subgroup is a sufficient condition for a group to be \( R \)-sequenceable. In this paper we also show that all noncyclic abelian 2-groups are \( R^* \)-sequenceable except for \( \mathcal{Z}_2 \times \mathcal{Z}_4 \) and \( \mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{Z}_2 \).

We begin with two results of Friedlander et al. concerning abelian 2-groups.

**Lemma 1** (Friedlander et al. \([2]\)). \((\mathcal{Z}_2)^m \) is \( R^* \)-sequenceable for \( m > 1, m \neq 3, \mathcal{Z}_2 \times \mathcal{Z}_2 \), is \( R^* \)-sequenceable for \( k \) odd, and \( R \)-sequenceable for all \( k \).

**Lemma 2** (Friedlander et al. \([2]\)). \( \mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{Z}_2 \) and \( \mathcal{Z}_2 \times \mathcal{Z}_4 \) are \( R \)-sequenceable but not \( R^* \)-sequenceable.

**Lemma 3.** If an abelian group \( G \) is an extension of \( \mathcal{Z}_2 \times \mathcal{Z}_2 \) by an \( R^* \)-sequenceable group \( H \), then \( G \) is \( R^* \)-sequenceable.

**Proof of Lemma 3.** Let \( n = |H| \). Since \( H \) is \( R^* \)-sequenceable, the cosets of \( \mathcal{Z}_2 \times \mathcal{Z}_2 \), excluding \( \mathcal{Z}_2 \times \mathcal{Z}_2 \) itself, have an ordering \( K_1, \ldots, K_{n-1} \) that is an \( R \)-sequence with \( K_{n-1}K_2 = K_1 \). Choose \( k_i, 1 \leq i \leq n-1 \), such that \( k_i \in K_i \) and \( K_{i-1}K_2 = K_1 \). Then any element in \( G \) can be uniquely expressed as a product of an element in \( \mathcal{Z}_2 \times \mathcal{Z}_2 \) and an element in \( \{K_1, \ldots, K_{n-1}, e\} \). Let \( \{y_1^{4n-1}_i\} \) be the sequence \( k_1, k_2, \ldots, k_{n-1}, e, k_1, k_1, k_1, k_3, \ldots, k_{n-1}, e, k_2, k_3, \ldots, k_{n-1}, e, k_2, k_3, \ldots, k_{n-1}, e \). Let \( a \) and \( b \) be generators of the \( \mathcal{Z}_2 \times \mathcal{Z}_2 \) subgroup of \( G \). Define \( \{x_i^{4n-1}\} \) as follows.

**Case 1:** \(|H| \mod 3 = 0 \). Let \( 3k = |H|, \{x_i\} \) is given by the successive rows of the \( 4 \times n \) matrix

\[
\begin{pmatrix}
  e & e & \cdots & b & a \\
  ab & k-2 \text{ copies of } \{a, b, ab\} & a & ab & ab & b & a \\
  ab & b & k-2 \text{ copies of } \{ab, a, b\} & ab & b & a & ab \\
  b & a & k-2 \text{ copies of } \{b, ab, a\} & b & a & e
\end{pmatrix}
\]
If \( k = 1 \), then \( H = \mathcal{A}_3 \), so \( G = \mathcal{A}_2 \times \mathcal{A}_6 \), which is \( R^\ast \)-sequenceable since its Sylow 2-subgroup is \( R^\ast \)-sequenceable.

**Case 2:** \(|H| \mod 3 = 1\). Let \( 3k + 1 = |H| \). \( \{x_i\} \) is read from the successive rows of the \( 4 \times n \) matrix.

\[
\begin{pmatrix}
e & e & \cdots & b & a \\
ak-1 \text{ copies of } \{b,a,ab\} & ab & b & a \\
b & b & k-1 \text{ copies of } \{a,ab,b\} & a & ab \\
b & a & k-1 \text{ copies of } \{ab,b,a\} & e
\end{pmatrix}
\]

**Case 3:** \(|H| \mod 3 = 2\). Let \( 3k + 2 = |H| \). \( \{x_i\} \) is read from the successive rows of the \( 4 \times n \) matrix.

\[
\begin{pmatrix}
e & e & \cdots & b & a \\
k-1 \text{ copies of } \{b,a,ab\} & ab & b & a \\k-1 \text{ copies of } \{a,ab,b\} & a & ab \\
a & a & k-1 \text{ copies of } \{ab,b,a\} & e
\end{pmatrix}
\]

Then \( \{x_i y_i\} \) is an \( R^\ast \)-sequence. Clearly \( (x_{4n-1} y_{4n-1})(x_2 y_2) = x_1 y_1 \). Verifying that \( \{x_i y_i\} \) is an \( R \)-sequence is straightforward with the following observations:

(i) \( k_{i-1}^{-1} e = k_{i-1}^{-1} k_2 \) and \( e^{-1} k_3 = k_{n-1}^{-1} k_1 \), so \( \{y_i^{-1} y_{i+1}\}_{i=1}^{4n-1} \) (with \( y_n = y_1 \)) is the sequence \( k_1^{-1} k_2, k_2^{-1} k_3, \ldots, k_{n-2}^{-1} k_{n-1}, k_1^{-1} k_2, k_2^{-1} k_3, k_3^{-1} k_4, \ldots, k_{n-2}^{-1} k_{n-1}, k_1^{-1} k_2 \), \( e, e, k_3^{-1} k_2, k_2^{-1} k_3, \ldots, k_{n-2}^{-1} k_{n-1}, k_1^{-1} k_2, k_2^{-1} k_3, k_3^{-1} k_4, \ldots, k_{n-2}^{-1} k_{n-1}, k_1^{-1} k_2 \).

(ii) If \( x_m \) is the first element of the first copy of one of the repeated 3-element sequences in \( \{x_i\} \), then \( y_m = k_3 \), and the sequence \( \{a, b, ab\} \) is itself an \( R \)-sequence. □

**Lemma 4.** \( \mathcal{A}_2 \times \mathcal{A}_2 \) is \( R^\ast \)-sequenceable for \( n \geq 1 \), \( n \neq 2 \).

**Proof of Lemma 4.** Any sequence of the nonidentity elements of \( \mathcal{A}_2 \times \mathcal{A}_2 \) is an \( R^\ast \)-sequence. \( \mathcal{A}_2 \times \mathcal{A}_2 \cong \langle a, b \rangle a^3 = b^2 = e, ab = ba \rangle \) has the \( R^\ast \)-sequence \( ba, b, a^2, a^3, ba^2, ba, a^4, ba^4, ba^3, a^2, a \). The relevant triple is \( ba^4, ba^3, a^\gamma \).

For \( n \geq 4 \), \( \mathcal{A}_2 \times \mathcal{A}_2 \cong \langle a, b \rangle a^2n = b^2 = e, ab = ba \rangle \), an \( R^\ast \)-sequence can be read from the successive rows of this \( 2m \times 8 \) matrix, where \( m = 2n-3 \):
To see that the sequence is an $R$-sequence, the successive quotients are listed in the successive rows of this matrix:

\[
\begin{array}{cccccccc}
& ba^{8m-1} & b & a^{3m} & a^{5m} & ba^{8m-2} & ba & a^{m-2} & a^{7m+2} \\
& ba^{8m-3} & ba^2 & a^{3m-2} & a^{5m+2} & ba^{8m-4} & ba^3 & a^{m-4} & a^{7m+4} \\
& & & & & & & & \\
& ba^{7m+3} & ba^{-m-4} & a^{2m+4} & a^{6m-4} & ba^{7m+2} & ba^{m-3} & a^2 & a^{8m-2} \\
& ba^{7m+1} & ba^{m-2} & a^{2m+2} & a^{6m-2} & ba^m & ba^{m-1} & a^{8m-1} & a \\
& ba^{7m-1} & ba^m & a^{6m-1} & a^{2m+1} & ba^{7m-2} & ba^{m+1} & a^{8m-3} & a^3 \\
& ba^{7m-3} & ba^{m+2} & a^{6m-3} & a^{2m+3} & ba^{7m-4} & ba^{m+3} & a^{8m-5} & a^5 \\
& & & & & & & & \\
& ba^{5m+3} & ba^{3m-4} & a^{4m+3} & a^{6m-3} & ba^{5m+2} & ba^{3m-3} & a^{6m+1} & a^{2m-1} \\
& ba^{5m+1} & ba^{3m-2} & a^{4m+1} & a^{6m-1} & ba^{5m} & ba^{m-1} & a^{2m} & a^{6m} \\
& ba^{5m-1} & ba^{3m} & a^{4m} & ba^{5m-2} & ba^{3m+1} & a^{2m-2} & a^{6m+2} \\
& ba^{5m-3} & ba^{3m+2} & a^{4m-2} & a^{6m-2} & ba^{5m-4} & ba^{3m+3} & a^{2m-4} & a^{6m+4} \\
& & & & & & & & \\
& ba^{4m+1} & ba^{4m-2} & a^{3m+2} & a^{5m-2} & ba^{4m} & ba^{4m-1} & a^m & a^{7m} \\
\end{array}
\]

If $m = 6k + 2$, we have \( ba^{7m-1-2(4k+1)} \cdot ba^{m+2(4k+1)} \cdot a^{6m-1-2(4k+1)} \) \( ba^{14k+4} \) \( ba^{m+2(4k+1)} \) \( ba^{7m+1} \) \( ba^{m+2(4k+1)} \). If $m = 6k + 4$, we have \(a^{2m+1+2(4k+2)} \cdot ba^{7m-2-2(4k+2)} \cdot ba^{m+1+2(4k+2)} \) \( ba^{7m-2-2(4k+2)} \). Thus, the sequence is an $R^*$-sequence for all $n \geq 4$. $\Box$
Theorem. If \( G \) is a non-cyclic abelian 2-group, then \( G \) is \( R \)-sequenceable. Moreover, if \( |G| \neq 8 \), then \( G \) is \( R^* \)-sequenceable.

Proof. If \( |G| = 8 \), the result follows from Lemma 2. Otherwise, we use induction on \( n \), where \( |G| = 2^n \). For \( n \) even, the base of the induction is \( n = 2 \), so that \( G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), which is \( R^* \)-sequenceable by Lemma 1. For \( n \) odd, the base of the induction is \( n = 5 \), so that either \( G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \), \( G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \), \( G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \), or \( G \cong \mathbb{Z}_4 \times \mathbb{Z}_8 \). \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) is \( R^* \)-sequenceable by Lemma 1. \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \) is \( R^* \)-sequenceable by Lemma 4. The other groups are extensions of \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) by \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). Let \( H_1, H_2, H_3 \) be the cosets, other than \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) itself, of \( \mathbb{Z}_2 \times \mathbb{Z}_4 \), and let \( h_1, h_2, h_3 \) be elements of \( H_1, H_2, H_3 \), respectively, such that \( h_1 h_3 = h_2 \). This is possible since \( H_1, H_2, H_3 \) must be an \( R^* \)-sequence of \( G/(\mathbb{Z}_2 \times \mathbb{Z}_4) \). Let the subgroup of \( G \) isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) be generated by \( a \) and \( b \) with \( a^4 = b^2 = e, ab = ba \). Then the following is an \( R^* \)-sequence: \( h_1, h_2, b a^h_3, a, b a h_2, b a h_3, a^2, b, b a^h_2, a^2 h_3, a^3 h_1, b a h_2, a^2 h_3, a^3, a^3, a h_2, a^3 h_3, b a^h_1, a^3 h_1, a^3 h_2, b h_3, b a^2, b h_2, h_3 \). The relevant triple is \( b a^h_1, a^3 h_2 \) and \( b h_3 \).

To complete the induction, we assume the result is true for \( n \). Let \( |G| = 2^{n+2} \). If \( G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \), \( G \) is \( R^* \)-sequenceable by Lemma 4. Otherwise, \( G \) is an extension of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) by a noncyclic abelian 2-group \( H \), and \( |H| = 2^n \). Since \( H \) is \( R^* \)-sequenceable by assumption, \( G \) is \( R^* \)-sequenceable by Lemma 3. \( \Box \)

Since Friedlander et al. [2] have shown that an abelian group whose Sylow 2-subgroup is \( R^* \)-sequenceable is itself \( R^* \)-sequenceable, we have the following corollary.

Corollary. An abelian group whose Sylow 2-subgroup is noncyclic and not of order 8 is \( R^* \)-sequenceable.

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References