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The Support Points of the Unit Ball in Bloch Space

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Let $H(\mathbf{D})$ be the topological vector space of all functions F holomorphic in the unit disc $\mathbf{D}$. We consider the compact convex subset $\tilde{\mathscr{B}}_1 = \{F \in H(\mathbf{D}) : F(0) = 0 \wedge |F'(z)|\,(1-|z|^2) \leqslant 1 \text{ for } z \in \mathbf{D}\}$ of $H(\mathbf{D})$ and show that $G \in \tilde{\mathscr{B}}_1$ is a support point of $\tilde{\mathscr{B}}_1$ if and only if $\Lambda(G) = \{z \in \mathbf{D} : |G'(z)|\,(1-|z|^2) = 1\} \neq \varnothing$. This is an application of a more general result which is concerned with the maximization of continuous linear functionals on a set $\mathscr{K}_1$ related to $\tilde{\mathscr{B}}_1$.

1. INTRODUCTION

Let $H(\mathbf{D})$ be the set of functions holomorphic in the unit disc $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. Endowed with the topology of locally uniform convergence $H(\mathbf{D})$ is a complex topological vector space. For $F \in H(\mathbf{D})$ and $z \in \mathbf{D}$ we introduce the notation

$$\mu_F(z) = |F'(z)|\,(1-|z|^2).$$

The Bloch space $\mathscr{B}$ is the set of all functions $F \in H(\mathbf{D})$ for which the Bloch norm

$$\|F\|_{\mathscr{B}} = |F(0)| + \sup_{z \in \mathbf{D}} \mu_F(z)$$

is finite. Here we consider the unit ball $\mathscr{B}_1 = \{F \in \mathscr{B} : \|F\|_{\mathscr{B}} \leqslant 1\}$ of $\mathscr{B}$. This set is a compact convex subset of $H(\mathbf{D})$ and occurred for the first time in connection with lower bounds for Bloch's constant. We recall some basic facts of the theory of convex sets.

Suppose C is a convex compact subset of a complex topological vector space V. A point $x \in C$ is called an extreme point of C, if it does not belong to the interior of a segment lying in C. Equivalently, $x \in C$ is an extreme point of C, if and only if $x \pm y \in C$ with $y \in V$ implies $y = 0$.

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A point $x \in C$ is called a support point of C, if there exists a closed hyperplane H passing through x such that C is contained in exactly one of the half-spaces determined by H. Equivalently, $x \in C$ is a support point of C, if and only if there exists a continuous linear functional $L: V \to \mathbf{C}$ such that the real part $\operatorname{Re} L$ of L is not constant on C and $\operatorname{Re} L(y) \leqslant \operatorname{Re} L(x)$ for all $y \in C$. If C has nonempty interior, then it follows from the Hahn–Banach type separation theorems that the set of support points of C coincides with the set of boundary points of C. These sets will be different in general (cf. [Köt, p. 193 ff.]).

In [C-W] it is shown that the set of extreme points of $\mathscr{B}_1$ is the union of the set of unimodular constants and the set of extreme points of the convex compact subset $\tilde{\mathscr{B}}_1 = \{F \in \mathscr{B}_1 : F(0) = 0\}$ of $\mathscr{B}_1$.

There are results which indicate that for a function $F \in \tilde{\mathscr{B}}_1$ to be an extreme point of $\tilde{\mathscr{B}}_1$ the set

$$\Lambda(F) = \{z \in \mathbf{D} : \mu_F(z) = 1\},$$

where μ_F attains its maximum, has to be "large." For example, if $\Lambda(F)$ has a limit point in $\mathbf{D}$, then F is an extreme point of $\tilde{\mathscr{B}}_1$. Under the additional assumption $\lim_{|z| \to 1} \mu_F(z) = 0$ this condition is also necessary [C-W]. The Ahlfors–Grunsky function [A-G] is an example of an extreme point of $\tilde{\mathscr{B}}_1$, for which $\Lambda(F)$ has no limit point in $\mathbf{D}$ [Bo2]. In this case the set $\Lambda(F)$ is a discrete subset of $\mathbf{D}$ related to a certain non-euclidean triangulation of $\mathbf{D}$.

A simple characterization of the extreme points of $\tilde{\mathscr{B}}_1$ in terms of the set $\Lambda(F)$ is not known. It is still an open problem whether there are extreme points of $\tilde{\mathscr{B}}_1$ for which $\Lambda(F)$ is empty [C-W].

The situation is much clearer for the support points of $\mathscr{B}_1$. A characterization of these points is given in Theorem 3 below. This is a corollary of Theorem 1, which is concerned with the maximization of real linear functionals on a certain convex set $\mathscr{K}_1$ related to $\mathscr{B}_1$. An application to coefficient problems is given in Theorem 2.

2. The Class $\mathscr{K}_1$

For the formulation and proof of the next theorem we fix notation and state some needed facts.

If $a \in \mathbf{D}$ and $r > 0$ we denote by $D(a, r) = \{z \in \mathbf{C} : |z - a| < r\}$ the open disc with center a and radius r, by $\mathbf{C}^*$ the set of complex numbers different from 0, and by $\bar{\mathbf{C}}$ the Riemann sphere.

For $F \in H(\mathbf{D})$ let $Z(F)$ be the zero set of the function F and let $\operatorname{ord}_z(F)$ be the order of a zero $z \in \mathbf{D}$. We put $\operatorname{ord}_z(F) = 0$ if $F(z) \neq 0$ and $\operatorname{ord}_z(F) = -\infty$ if $F \equiv 0$.

In the following it is more convenient to work with the derivatives of Bloch functions and not with the Bloch functions themselves. So for $F \in H(\mathbf{D})$ we define

$$M(F) = \sup_{z \in \mathbf{D}} |F(z)| \, (1 - |z|^2)$$

and introduce the class $\mathscr{K} = \{F \in H(\mathbf{D}) : M(F) < \infty\}$ consisting of the derivatives of Bloch functions. We will be concerned with the function class $\mathscr{K}_1 = \{F \in \mathscr{K} : M(F) \leqslant 1\}$. Note that $\mathscr{K}$ is a subpace and $\mathscr{K}_1$ a compact convex subset of $H(\mathbf{D})$. For $F \in \mathscr{K}_1$ we define

$$\Gamma(F) = \{z \in \mathbf{D} : |F(z)| \, (1 - |z|^2) = 1\}.$$

A set $S \subseteq \mathbf{D}$ will be called a set of uniqueness (for $\mathscr{K}$) if and only if $F \mid S \equiv 0$ for $F \in \mathscr{K}$ implies $F \equiv 0$. For example, if $S \subseteq \mathbf{D}$ has a limit point in $\mathbf{D}$, then S is a set of uniqueness. A necessary condition for a set $S \subseteq \mathbf{D}$ to be a set of uniqueness is that S be infinite. Here we will not give a more detailed analysis of the conditions under which a set $S \subseteq \mathbf{D}$ is a set of uniqueness.

We use two methods to construct new functions in $\mathscr{K}$ from given ones.

If $F_1 \in \mathscr{K}$ and $P \in H(\mathbf{D})$ is bounded on $\mathbf{D}$, then $F_2 = PF_1 \in \mathscr{K}$. In particular, this applies to a polynomial P.

If $F_1 \in H(\mathbf{D})$ and $P \not\equiv 0$ is a polynomial with

$$\operatorname{ord}_z(P) \leqslant \operatorname{ord}_z(F_1) \qquad \text{for} \quad z \in \mathbf{D},$$

then there exists a unique function $F_2 \in H(\mathbf{D})$ with $F_1 = PF_2$. If furthermore $F_1 \in \mathscr{K}$ and $P(z) \neq 0$ for $z \in \partial \mathbf{D}$, then $F_2 \in \mathscr{K}$. This is seen as follows.

There exists a constant $M_1 \geqslant 0$ such that

$$|F_1(z)| \leqslant \frac{M_1}{1 - |z|^2} \qquad \text{for} \quad z \in \mathbf{D}$$

and a number $r \in (0, 1)$ such that $P(z) \neq 0$ for $r \leqslant |z| \leqslant 1$. Then there is a constant $M_2 > 0$ such that

$$1/|P(z)| \leqslant M_2 \qquad \text{for} \quad r \leqslant |z| \leqslant 1.$$

The function F_2 is bounded on the compact set $\overline{D(0, r)}$. So it is possible to choose a number $M_3 \geqslant 0$ such that $|F_2(z)| \leqslant M_3$ for $z \in \overline{D(0, r)}$. Now define $M_4 = \max\{M_1 M_2, M_3\}$. Then

$$|F_2(z)| \leqslant \frac{M_4}{1 - |z|^2} \qquad \text{for} \quad z \in \mathbf{D}$$

and so $F_2 \in \mathscr{K}$.

A theorem of Toeplitz (cf. [Sch, p. 36]) states that there is a one-to-one correspondence between continuous linear functionals $L\colon H(\mathbf{D})\to\mathbf{C}$ and sequences $(a_\nu)_{\nu\in\mathbf{N}_0}$ of complex numbers with

$$\limsup_{\nu\to\infty} |a_\nu|^{1/\nu}<1. \tag{1}$$

If $(a_\nu)_{\nu\in\mathbf{N}_0}$ is such a sequence, then the corresponding functional is given by

$$L(F)=\sum_{\nu=0}^{\infty} a_\nu b_\nu$$

for every function $F\in H(\mathbf{D})$ with Taylor expansion $F(z)=\sum_{\nu=0}^{\infty} b_\nu z^\nu$ at 0. Examples of continuous linear functionals on $H(\mathbf{D})$ are evaluation functionals $F\mapsto F^{(n)}(z_0)$ with fixed $n\in\mathbf{N}_0$ and $z_0\in\mathbf{D}$ and linear combinations of evaluation functionals, which are called functionals of rational type. The representation of a functional of rational type as a linear combination of evaluation functionals is unique. This is equivalent to the following statement. If $n\in\mathbf{N}$, $z_1, ..., z_n\in\mathbf{D}$ are pairwise distinct, $k_1, ..., k_n\in\mathbf{N}_0$, $\lambda_{1,0}, ..., \lambda_{1,k_1}, ..., \lambda_{n,0}, ..., \lambda_{n,k_n}\in\mathbf{C}$, and the continuous linear functional $L\colon H(\mathbf{D})\to\mathbf{C}$ is defined as

$$L(F)=\sum_{\nu=1}^{n}\sum_{\mu=0}^{k_\nu}\lambda_{\nu,\mu}F^{(\mu)}(z_\nu) \qquad \text{for} \quad F\in H(\mathbf{D}), \tag{2}$$

then $L\equiv 0$ implies $\lambda_{1,0}=\cdots=\lambda_{1,k_1}=\cdots=\lambda_{n,0}=\cdots=\lambda_{n,k_n}=0$.

To see this note that there exists a holomorphic function $G\in H(\mathbf{D})$ with

$$G^{(\mu)}(z_\nu)=\overline{\lambda_{\nu,\mu}} \qquad \text{for} \quad \nu\in\{1, ..., n\}, \quad \mu\in\{0, ..., k_\nu\}.$$

If at least one of the coefficients $\lambda_{\nu,\mu}$ is different from 0, then $L(G)>0$ and so $L\not\equiv 0$.

A continuous linear functional L on $H(\mathbf{D})$ can also be represented as an integral. We formulate this as a lemma.

LEMMA 1. *Suppose $L\colon H(\mathbf{D})\to\mathbf{C}$ is a continuous linear functional. Then there exist a number $r\in(0,1)$ and a function F_1 holomorphic in a region containing $\mathbf{C}\backslash D(0,r)$ such that if we define $\gamma(t)=re^{it}$ for $t\in[0,2\pi]$, then*

$$L(F)=\frac{1}{2\pi i}\int_\gamma F(z)\,F_1(z)\,dz \qquad \textit{for} \quad F\in H(\mathbf{D}). \tag{3}$$

Proof. Represent the functional L by a sequence $(a_\nu)_{\nu \in \mathbf{N}_0}$ satisfying (1). There exists a number $r_1 > 1$ such that the sequence $(a_\nu r_1^\nu)_{\nu \in \mathbf{N}_0}$ is bounded. Then the function F_1 defined by

$$F_1(z) = \sum_{\nu=0}^{\infty} \frac{a_\nu}{z^{\nu+1}} \qquad \text{for} \quad z \in \bar{\mathbf{C}}, |z| > 1/r_1$$

is holomorphic in $\{z \in \bar{\mathbf{C}} : 1/r_1 < |z|\}$. If we now choose r with $1/r_1 < r < 1$, then (3) is true. ∎

We need the following results about functionals of rational type.

PROPOSITION. *A continuous linear functional L: $H(\mathbf{D}) \to \mathbf{C}$ is of rational type, if and only if there exists a function $H \in H(\mathbf{D})$, $H \not\equiv 0$, such that $L(PH) = 0$ for all polynomials P.*

LEMMA 2. *Suppose the continuous linear functional L: $H(\mathbf{D}) \to \mathbf{C}$ is of rational type, has an integral representation as in Lemma* 1, *and the function $H \in H(\mathbf{D})$ of the proposition can be chosen to have simple zeros in $D(0, r)$. Then L is a linear combination of point evaluation functionals, where the evaluation points are zeros of H in $D(0, r)$; i.e., there exist points $z_1, ..., z_n \in Z(H) \cap D(0, r)$ and complex numbers $\lambda_1, ..., \lambda_n \in \mathbf{C}$ such that*

$$L(F) = \sum_{\nu=1}^{n} \lambda_\nu F(z_\nu) \qquad \textit{for} \quad F \in H(\mathbf{D}).$$

Proof of the Proposition and of Lemma 2. Suppose L is of rational type. Then L has a representation as in (2). Choose a polynomial $H \not\equiv 0$ with $\operatorname{ord}_{z_\nu}(H) \geqslant k_\nu + 1$ for $\nu \in \{1, ..., n\}$. If P is an arbitrary polynomial, then $\operatorname{ord}_{z_\nu}(PH) \geqslant k_\nu + 1$ for $\nu \in \{1, ..., n\}$ and so $L(PH) = 0$.

Conversely, suppose that there exists a function $H \in H(\mathbf{D})$, $H \not\equiv 0$, such that $L(PH) = 0$ for all polynomials P. The functional L has an integral representation as in Lemma 1 (this includes the definition of a number $r \in (0, 1)$ as described). Then we have

$$\int_\gamma z^n H(z)\, F_1(z)\, dz = 0 \qquad \text{for} \quad n \in \mathbf{N}_0. \tag{4}$$

There is a number $r' \in (0, r)$ such that F_1 is holomorphic in $\{z \in \bar{\mathbf{C}}: r' < |z|\}$. Then the function HF_1 is holomorpic in the annulus $\{z \in \mathbf{C} : r' < |z| < 1\}$. So it has a Laurent expansion $H(z)\, F_1(z) = \sum_{\nu=-\infty}^{\infty} d_\nu z^\nu$ converging for $r' < |z| < 1$. From (4) it follows that $d_{-\mu} = 0$ for $\mu \in \mathbf{N}$. This shows that HF_1 has a holomorphic continuation to the unit disc $\mathbf{D}$. Denote this extension of HF_1 to $\mathbf{D}$ by $F_2 \in H(\mathbf{D})$. Then we have $F_1(z) = F_2(z)/H(z)$ for $r' < |z| < 1$

and we see that F_1 has a meromorphic extension to $\mathbf{D}$. Since F_1 is holomorphic in $\{z \in \bar{\mathbf{C}} : r' < |z|\}$, the function F_1 has a meromorphic extension to $\bar{\mathbf{C}}$. This extension will also be denoted by F_1 (by abuse of language).

The function F_1 is rational and poles can only occur in $\overline{D(0, r')} \subseteq D(0, r)$. For $z \in D(0, r)$ we have $F_1(z) = F_2(z)/H(z)$. This shows that we can have a pole of F_1 in $D(0, r)$ only where H vanishes and the order of the pole of F_1 cannot exceed the order of the zero of H. From this and the integral representation of L we conclude by an application of the Residue Theorem that L is of rational type. If H can be chosen to have simple zeros in $D(0, r)$, then F_1 can only have simple poles at these zeros. So L is a point evaluation functional of the described type. ▌

We need two more lemmas.

LEMMA 3. *Suppose $L: H(\mathbf{D}) \to \mathbf{C}$ is a continuous linear functional and let $G \in H(\mathbf{D})$. For arbitrary $\varepsilon \in (0, 1]$ define $G_\varepsilon \in H(\mathbf{D})$ by $G_\varepsilon(z) = G((1-\varepsilon)z)$ for $z \in \mathbf{D}$. Then there exists a constant $K > 0$ such that*

$$|L(G_\varepsilon - G)| \leqslant \varepsilon K \qquad \textit{for} \quad \varepsilon \in (0, 1]. \tag{5}$$

Proof. Assume that the Taylor expansion of G at 0 is given by $G(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$. The functional L can be represented by a sequence $(a_\nu)_{\nu \in \mathbf{N}_0}$ of complex numbers satisfying (1). There are numbers $r_1 > 1$ and $K_1 > 0$ such that

$$|a_\nu|\, r_1^{2\nu} \leqslant K_1 \qquad \text{for} \quad \nu \in \mathbf{N}_0.$$

Since $1/r_1 \in (0, 1)$, the Taylor expansion of G at 0 converges for $z = 1/r_1$. Hence the sequence $(c_\nu / r_1^\nu)_{\nu \in \mathbf{N}_0}$ is bounded and so there is a constant $K_2 > 0$ such that

$$|c_\nu| / r_1^\nu \leqslant K_2 \qquad \text{for} \quad \nu \in \mathbf{N}_0.$$

Now define $K = K_1 K_2 \sum_{\nu=1}^{\infty} \nu / r_1^\nu \in (0, \infty)$. Then for $\varepsilon \in (0, 1]$ we get

$$\begin{aligned} |L(G_\varepsilon - G)| &\leqslant \sum_{\nu=1}^{\infty} |a_\nu c_\nu|\, (1 - (1-\varepsilon)^\nu) \leqslant \varepsilon \sum_{\nu=1}^{\infty} \nu\, |a_\nu c_\nu| \\ &= \varepsilon \sum_{\nu=1}^{\infty} |a_\nu|\, r_1^{2\nu} \frac{|c_\nu|}{r_1^\nu} \frac{\nu}{r_1^\nu} \leqslant \varepsilon K_1 K_2 \sum_{\nu=1}^{\infty} \frac{\nu}{r_1^\nu} = \varepsilon K. \end{aligned}$$ ▌

LEMMA 4. *Suppose $M \geqslant 0$. Then there exist numbers ε_1, $R \in (0, 1)$ such that*

$$\frac{1}{1 - (1-\varepsilon)^2 |z|^2} + \frac{\varepsilon M}{1 - |z|^2} \leqslant \frac{1}{1 - |z|^2} \qquad \textit{for} \quad 0 < \varepsilon \leqslant \varepsilon_1 \textit{ and } R \leqslant |z| < 1. \tag{6}$$

Proof. Choose $R\in(0,1)$ with $R^2>M/(M+2)$. Then there exists a number $\varepsilon_1\in(0,1)$ such that $M\leqslant R^2((1-\varepsilon)^2M+2-\varepsilon)$ for $\varepsilon\in(0,\varepsilon_1]$. Inequality (6) now follows by direct computation. ▌

For a continuous linear functional $L\colon H(\mathbf{D})\to\mathbf{C}$ we define

$$\mathcal{M}_L=\{G\in\mathcal{K}_1:\sup_{F\in\mathcal{K}_1}\operatorname{Re}L(F)=\operatorname{Re}L(G)\}.$$

Since the set $\mathcal{K}_1$ is compact, we have $\mathcal{M}_L\neq\varnothing$.

Let L be a functional for which there exist complex numbers $z_1,\ldots,z_n\in\mathbf{D}$ and $\lambda_1,\ldots,\lambda_n\in\mathbf{C}^*$ such that

$$L(F)=\sum_{\nu=1}^{n}\lambda_\nu F(z_\nu)\qquad\text{for}\quad F\in H(\mathbf{D}).$$

Then we get the estimate

$$\sup_{F\in\mathcal{K}_1}\operatorname{Re}L(F)\leqslant\sum_{\nu=1}^{n}\frac{|\lambda_\nu|}{1-|z_\nu|^2}.$$

The case where we here have equality will be important for us. We say that a continuous linear functional $L\colon H(\mathbf{D})\to\mathbf{C}$ is of "special type," if there exist a natural number $n\in\mathbf{N}$, pairwise distinct points $z_1,\ldots,z_n\in\mathbf{D}$, and complex numbers $\lambda_1,\ldots,\lambda_n\in\mathbf{C}^*$ such that

$$L(F)=\sum_{\nu=1}^{n}\lambda_\nu F(z_\nu)\qquad\text{for}\quad F\in H(\mathbf{D})\qquad\text{and}$$
$$\sup_{F\in\mathcal{K}_1}\operatorname{Re}L(F)=\sum_{\nu=1}^{n}\frac{|\lambda_\nu|}{1-|z_\nu|^2}.\tag{7}$$

Now we can state our main result.

THEOREM 1. *Suppose* $L\colon H(\mathbf{D})\to\mathbf{C}$, $L\not\equiv0$, *is a continuous linear functional. Then*

(a) *L is of special type or*

(b) *the set $\mathcal{M}_L$ consists of a single point $G\in\mathcal{K}_1$. The function G is an extreme point of $\mathcal{K}_1$ and $\Gamma(G)$ is a set of uniqueness.*

Note that if $L\colon H(\mathbf{D})\to\mathbf{C}$ is of special type and has a representation as in (7) and if $G\in\mathcal{M}_L$, then $\{z_1,\ldots,z_n\}\subseteq\Gamma(G)$. In general no further information on $\Gamma(G)$ can be expected in this case.

Proof of Theorem 1. The proof proceeds in several steps.

1. Suppose $L: H(\mathbf{D}) \to \mathbf{C}$, $L \not\equiv 0$, is a continuous linear functional that is not of special type and let $G \in \mathcal{M}_L$ be given. We claim that $\Gamma(G)$ is a set of uniqueness. To obtain a contradiction assume this is not the case. Then there exists a function $H_1 \in \mathcal{K}$, $H_1 \not\equiv 0$, with $H_1 \mid \Gamma(G) \equiv 0$. Note that $\Gamma(G)$ cannot have a limit point in $\mathbf{D}$, for otherwise $H_1 \equiv 0$ by the uniqueness theorem for analytic functions. So $\Gamma(G)$ consists of isolated points or is empty.

The basic idea of the proof is to construct a variation $\tilde{G} \in \mathcal{K}_1$ of G with $\operatorname{Re} L(\tilde{G}) > \operatorname{Re} L(G)$. Since $G \in \mathcal{M}_L$ and so $\operatorname{Re} L(F) \leqslant \operatorname{Re} L(G)$ for all $F \in \mathcal{K}_1$, this will give us a contradiction.

The variation $\tilde{G}$ may be written as

$$\tilde{G}(z) = G((1-\varepsilon)z) + \varepsilon H_4((1-\varepsilon)z) \qquad \text{for} \quad z \in \mathbf{D}$$

with sufficiently small $\varepsilon > 0$. The function H_4 will be obtained from H_1 by dividing out and shifting some of the zeros of H_1.

2. The functional L has an integral representation as in Lemma 1. To be able to apply Lemma 2 we modify the function H_1 as follows.

The number of zeros of H_1 contained in the disc $D(0, r)$ is finite. Hence there exists a polynomial P_1 such that

$$\operatorname{ord}_z(P_1) = \begin{cases} 0 \\ \operatorname{ord}_z(H_1) \\ \operatorname{ord}_z(H_1) - 1 \end{cases} \qquad \text{for} \quad z \in \begin{cases} \mathbf{C} \setminus D(0, r) \\ D(0, r) \setminus \Gamma(G) \\ D(0, r) \cap \Gamma(G). \end{cases}$$

Since $\operatorname{ord}_z(P_1) \leqslant \operatorname{ord}_z(H_1)$ for $z \in \mathbf{D}$, there is a function $H_2 \in H(\mathbf{D})$ with $H_1 = P_1 H_2$. Indeed $H_2 \in \mathcal{K}$, because $H_1 \in \mathcal{K}$ and P_1 has no zeros on the unit circle. Furthermore, $H_2 \not\equiv 0$ and $H_2 \mid \Gamma(G) \equiv 0$. By construction of H_2 a point $z \in D(0, r)$ is a zero of H_2 if and only if $z \in D(0, r) \cap \Gamma(G)$. Each of these zeros is of first order.

3. Let $K > 0$ be a constant chosen according to Lemma 3. Now consider two cases.

(a) There exists a polynomial P_2 such that $\operatorname{Re} L(P_2 H_2) > 0$. In this case define

$$H_3 = \frac{2K}{\operatorname{Re} L(P_2 H_2)} P_2 H_2.$$

Then we have

$$H_3 \in \mathcal{K}, \ H_3 \not\equiv 0, \ \operatorname{Re} L(H_3) = 2K, \ H_3(z) = 0 \qquad \text{for} \quad z \in \Gamma(G). \tag{8}$$

(b) There exists no polynomial P_2 such that $\operatorname{Re} L(P_2 H_2) > 0$. In this case we would also like to have a function H_3 with the properties (8).

Such a function need not exist, but it is possible to single out an element $z_1 \in \Gamma(G)$ and to construct a function H_3 with the following properties

$$\begin{aligned} &\text{(a)}\ H_3 \in \mathscr{K},\ H_3 \not\equiv 0,\ \operatorname{Re} L(H_3) = 2K,\ H_3(z) = 0 \quad \text{for } z \in \Gamma(G) \setminus \{z_1\}, \\ &\text{(b)}\ \operatorname{Re}(H_3(z_1)/G(z_1)) < 0. \end{aligned} \tag{8'}$$

This can be seen as follows. From our assumptions on L we conclude that $\operatorname{Re} L(e^{is} PH_2) \leqslant 0$ for all polynomials P and all numbers $s \in [0, 2\pi]$. This implies $L(PH_2) = 0$ for all polynomials P. Now apply Lemma 2 with $H = H_2$. This shows that there exist a number $m \in \mathbf{N}_0$, pairwise distinct points $z_1, ..., z_m \in Z(H_2) \cap D(0, r) \subseteq \Gamma(G)$, and numbers $\lambda_1, ..., \lambda_m \in \mathbf{C}^*$ such that

$$L(F) = \sum_{\nu=1}^{m} \lambda_\nu F(z_\nu) \qquad \text{for} \quad F \in H(\mathbf{D}). \tag{9}$$

Here $m \neq 0$ since $L \not\equiv 0$.

The functional L is not of special type. Therefore

$$\operatorname{Re}\left(\sum_{\nu=1}^{m} \lambda_\nu G(z_\nu)\right) < \sum_{\nu=1}^{m} \frac{|\lambda_\nu|}{1 - |z_\nu|^2}. \tag{10}$$

We have $\{z_1, ..., z_m\} \subseteq \Gamma(G)$ and so $|G(z_\nu)| = 1/(1 - |z_\nu|^2)$ for $\nu \in \{1, ..., m\}$. Thus inequality (10) is only possible if there exists a number $k \in \{1, ..., m\}$ with

$$\operatorname{Re}(\lambda_k G(z_k)) < \frac{|\lambda_k|}{1 - |z_k|^2}.$$

Without loss of generality we may assume $k = 1$. Now define $a = \overline{\lambda_1}/|\lambda_1|$ and $b = G(z_1)/|G(z_1)|$. Then $|a| = |b| = 1$ and $\operatorname{Re}(\bar{a}b) < 1$. This implies $a \neq b$.

Since $\operatorname{ord}_{z_1} H_2 = 1$, there exists a function $F_3 \in \mathscr{K}$ with $F_3(z_1) \neq 0$ and $H_2(z) = (z - z_1) F_3(z)$ for $z \in \mathbf{D}$. Then $F_3 \not\equiv 0$ and $F_3(z) = 0$ for $z \in \Gamma(G) \setminus \{z_1\}$. It is possible to choose a number $\delta_1 > 0$ such that $z_1' = z_1 + \delta_1(b - a)/F_3(z_1) \in \mathbf{D}$. Now define $\tilde{H}_3(z) = (z - z_1') F_3(z)$ for $z \in \mathbf{D}$. Then $\tilde{H}_3 \in \mathscr{K}$, $\tilde{H}_3 \not\equiv 0$, and $\tilde{H}_3(z) = 0$ for $z \in \Gamma(G) \setminus \{z_1\}$. Using (9) we get

$$\operatorname{Re} L(\tilde{H}_3) = \operatorname{Re}(\lambda_1(z_1 - z_1') F_3(z_1)) = |\lambda_1|\, \delta_1(1 - \operatorname{Re}(\bar{a}b)) > 0.$$

Finally, we have

$$\operatorname{Re}\left(\frac{\tilde{H}_3(z_1)}{G(z_1)}\right) = -\frac{\delta_1}{|G(z_1)|}(1 - \operatorname{Re}(\bar{a}b)) < 0.$$

If we now define

$$H_3 = \frac{2K}{\operatorname{Re} L(\tilde{H}_3)} \tilde{H}_3,$$

then (8′) is true.

4. Put $M = 1 + M(H_3) < \infty$ and apply Lemma 4 to find constants ε_1, $R \in (0, 1)$ such that (6) is valid.

The set $\Gamma(G) \cap \overline{D(0, R)}$ is finite. So there exist a number $q \in \mathbf{N}_0$ and pairwise distinct points $u_1, ..., u_q \in \mathbf{D}$ such that $\{u_1, ..., u_q\} = \Gamma(G) \cap \overline{D(0, R)}$. We want to construct a function $H_4 \in H(\mathbf{D})$ with the following properties

$$\begin{aligned} &\text{(a)} \quad \operatorname{Re}(H_4(z)/G(z)) < 0 \qquad \text{for} \quad z \in \{u_1, ..., u_q\} = \Gamma(G) \cap \overline{D(0, R)}, \\ &\text{(b)} \quad M(H_4) \leqslant M, \\ &\text{(c)} \quad \operatorname{Re} L(H_4) \geqslant 3K/2. \end{aligned} \tag{11}$$

The function H_4 will be obtained from H_3 by shifting some zeros of H_3. We will give the details of this construction for the first case in 3 and will indicate the slight modifications in the second case.

Put $k_\nu = \operatorname{ord}_{u_\nu}(H_3) \in \mathbf{N}$ for $\nu \in \{1, ..., q\}$. Then there exists a function $F_4 \in \mathscr{K}$ such that

$$H_3(z) = F_4(z) \prod_{\nu=1}^{q} (z - u_\nu)^{k_\nu}$$

$$\text{for} \quad z \in \mathbf{D} \qquad \text{and} \qquad F_4(u_\nu) \neq 0 \qquad \text{for} \quad \nu \in \{1, ..., q\}.$$

Choose numbers $t_1, ..., t_q \in [0, 2\pi]$ with

$$\operatorname{Re}\left(e^{ik_\nu t_\nu} \frac{F_4(u_\nu)}{G(u_\nu)} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{q} (u_\nu - u_\mu)^{k_\mu}\right) < 0 \qquad \text{for} \quad \nu \in \{1, ..., q\}$$

and define $u_{\nu, n} = u_\nu - (1/n)\, e^{it_\nu}$ for $\nu \in \{1, ..., q\}$ and $n \in \mathbf{N}$. Then $u_{\nu, n} \to u_\nu$ for $n \to \infty$. This implies that if n is sufficiently large, then

$$\operatorname{Re}\left(e^{ik_\nu t_\nu} \frac{F_4(u_\nu)}{G(u_\nu)} \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^{q} (u_\nu - u_{\mu, n})^{k_\mu}\right) < 0 \qquad \text{for} \quad \nu \in \{1, ..., q\}. \tag{12}$$

For $n \in \mathbf{N}$ and $z \in \mathbf{D}$ define

$$B_n(z) = F_4(z) \prod_{\nu=1}^{q} (z - u_{\nu, n})^{k_\nu}.$$

Then $B_n \in H(\mathbf{D})$ for $n \in \mathbf{N}$ and $\lim_{n\to\infty} B_n(z) = H_3(z)$ for $z \in \mathbf{D}$. Inequality (12) implies that if n is sufficiently large, then

$$\operatorname{Re}(B_n(u_\nu)/G(u_\nu)) < 0 \qquad \text{for} \quad \nu \in \{1, ..., q\}. \tag{13}$$

For sufficiently large n we have

$$C_n = \sup_{z \in \mathbf{D}} \left| \prod_{\nu=1}^{q} (z - u_{\nu,n})^{k_\nu} - \prod_{\nu=1}^{q} (z - u_\nu)^{k_\nu} \right| \leqslant \frac{1}{1 + M(F_4)} \tag{14}$$

and for these n

$$\begin{aligned} M(B_n) &= \sup_{z \in \mathbf{D}} |B_n(z)| \, (1 - |z|^2) \\ &\leqslant C_n \sup_{z \in \mathbf{D}} |F_4(z)| \, (1 - |z|^2) + \sup_{z \in \mathbf{D}} |H_3(z)| \, (1 - |z|^2) \\ &\leqslant \frac{M(F_4)}{1 + M(F_4)} + M(H_3) \leqslant 1 + M(H_3) = M. \end{aligned} \tag{15}$$

It follows that the sequence $(B_n)_{n \in \mathbf{N}}$ is locally uniformly bounded. Since it converges pointwise to H_3, Vitali's theorem shows that the sequence $(B_n)_{n \in \mathbf{N}}$ converges locally uniformly to H_3. Thus by the continuity of L

$$\operatorname{Re} L(B_n) \to \operatorname{Re} L(H_3) = 2K \qquad \text{for} \quad n \to \infty. \tag{16}$$

From (13), (15), and (16) we finally see that is is possible to choose $N_1 \in \mathbf{N}$ large enough such that the function $H_4 = B_{N_1}$ satisfies the conditions (11).

In the second case of Step 3 the function H_3 has zeros at each of the points $u_1, ..., u_q$ with the one possible exception of z_1. If we apply the above zero-shifting technique to the other points, we can again construct a sequence of holomorphic functions $(B_n)_{n \in \mathbf{N}}$ converging locally uniformly to H_3 such that for sufficiently large n inequality (15) is true and inequality (13) is true for all points u_ν different from z_1. For the point z_1 this is also true, because by (8′)(b) we have $\operatorname{Re}(H_3(z_1)/G(z_1)) < 0$ and so $\operatorname{Re}(B_n(z_1)/G(z_1)) < 0$ for sufficiently large n. So in this case, too, it is possible to choose $N_1 \in \mathbf{N}$ large enough such that $H_4 = B_{N_1}$ has the properties (11).

5. We now define $Q_\varepsilon(z) = G((1-\varepsilon) z) + \varepsilon H_4((1-\varepsilon) z)$ for $\varepsilon \in (0, 1)$ and $z \in \mathbf{D}$. Then $Q_\varepsilon \in H(\mathbf{D})$. We want to show that $Q_\varepsilon \in \mathscr{K}_1$ for sufficiently small $\varepsilon > 0$. For this we need inequality (6) and the properties (11)(a) and (11)(b) of the function H_4.

Inequality (11)(a) implies that there exists a number $\varepsilon_2 > 0$ such that

$$\left| 1 + \varepsilon_2 \frac{H_4(u_\nu)}{G(u_\nu)} \right| < 1 \qquad \text{for} \quad \nu \in \{1, ..., q\}.$$

The continuity of the function $z \mapsto H_4(z)/G(z)$ at u_ν for $\nu \in \{1, ..., q\}$ shows that there is a number $\delta_2 > 0$ such that $\bigcup_{\nu=1}^q D(u_\nu, \delta_2) \subseteq \mathbf{D}$, $G(z) \neq 0$ for $z \in \bigcup_{\nu=1}^q D(u_\nu, \delta_2)$ and

$$\left|1 + \varepsilon_2 \frac{H_4(z)}{G(z)}\right| \leqslant 1 \qquad \text{for} \quad z \in \bigcup_{\nu=1}^q D(u_\nu, \delta_2).$$

Then

$$\left|1 + \varepsilon \frac{H_4(z)}{G(z)}\right| \leqslant 1 \qquad \text{for} \quad \varepsilon \in (0, \varepsilon_2] \qquad \text{and} \qquad z \in \bigcup_{\nu=1}^q D(u_\nu, \delta_2). \tag{17}$$

There exists a number $\varepsilon_3 > 0$ such that

$$\varepsilon \in (0, \varepsilon_3] \text{ and } z \in \bigcup_{\nu=1}^q D(u_\nu, \delta_2/2) \quad \text{implies} \quad (1-\varepsilon) z \in \bigcup_{\nu=1}^q D(u_\nu, \delta_2) \tag{18}$$

and

$$\varepsilon \in (0, \varepsilon_3] \text{ and } z \in \overline{D(0, R)} \Big\backslash \left(\bigcup_{\nu=1}^q D(u_\nu, \delta_2/2)\right)$$
$$\text{implies} \quad (1-\varepsilon) z \in \overline{D(0, R)} \Big\backslash \left(\bigcup_{\nu=1}^q D(u_\nu, \delta_2/4)\right). \tag{19}$$

Since $\Gamma(G) \cap \overline{D(0, R)} = \{u_1, ..., u_q\}$ we have

$$|G(z)| < \frac{1}{1-|z|^2} \qquad \text{for} \quad z \in \overline{D(0, R)} \Big\backslash \left(\bigcup_{\nu=1}^q D(u_\nu, \delta_2/4)\right).$$

From the usual compactness and continuity arguments it follows that there exists a number $\delta_3 > 0$ such that

$$|G(z)| + \delta_3 \leqslant \frac{1}{1-|z|^2} \qquad \text{for} \quad z \in \overline{D(0, R)} \Big\backslash \left(\bigcup_{\nu=1}^q D(u_\nu, \delta_2/4)\right). \tag{20}$$

Finally, choose a number $\varepsilon_4 > 0$ such that

$$\varepsilon\, |H_4((1-\varepsilon) z)| \leqslant \delta_3 \qquad \text{for} \quad \varepsilon \in (0, \varepsilon_4] \qquad \text{and} \qquad z \in \overline{D(0, R)}. \tag{21}$$

Now define

$$S_1 = \{z \in \mathbf{D} : R \leqslant |z| < 1\},$$
$$S_2 = \bigcup_{\nu=1}^q D(u_\nu, \delta_2/2),$$
$$S_3 = \overline{D(0, R)} \Big\backslash \bigcup_{\nu=1}^q D(u_\nu, \delta_2/2)$$

and $\varepsilon_5 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} > 0$. We have $S_1 \cup S_2 \cup S_3 = \mathbf{D}$.

Suppose $\varepsilon \in (0, \varepsilon_5]$. Then for $z \in S_1$ we get from (11)(b) and (6)

$$\begin{aligned}|Q_\varepsilon(z)| &\leqslant |G((1-\varepsilon) z)| + \varepsilon\, |H_4((1-\varepsilon) z)| \\ &\leqslant \frac{1}{1-(1-\varepsilon)^2 |z|^2} + \frac{\varepsilon M}{1-(1-\varepsilon)^2 |z|^2} \\ &\leqslant \frac{1}{1-(1-\varepsilon)^2 |z|^2} + \frac{\varepsilon M}{1-|z|^2} \leqslant \frac{1}{1-|z|^2}.\end{aligned}$$

For $z \in S_2$ we have $(1-\varepsilon) z \in \bigcup_{\nu=1}^{q} D(u_\nu, \delta_2)$ by (18) and so by (17)

$$|Q_\varepsilon(z)| = |G((1-\varepsilon) z)| \left| 1 + \varepsilon \frac{H_4((1-\varepsilon) z)}{G((1-\varepsilon) z)} \right| \leqslant \frac{1}{1-(1-\varepsilon)^2 |z|^2} \leqslant \frac{1}{1-|z|^2}.$$

Finally, for $z \in S_3$ we have $(1-\varepsilon) z \in \overline{D(0, R)} \backslash (\bigcup_{\nu=1}^{k} D(u_\nu, \delta_2/4))$ by (19) and so by (21) and (20)

$$|Q_\varepsilon(z)| = |G((1-\varepsilon) z)| + \delta_3 \leqslant \frac{1}{1-(1-\varepsilon)^2 |z|^2} \leqslant \frac{1}{1-|z|^2}.$$

It follows that if $\varepsilon \in (0, \varepsilon_5]$, then

$$|Q_\varepsilon(z)| \leqslant \frac{1}{1-|z|^2} \qquad \text{for} \quad z \in \mathbf{D}$$

and so $Q_\varepsilon \in \mathscr{K}_1$.

6. For $n \in \mathbf{N}$ consider the functions $R_n \in H(\mathbf{D})$ defined by $R_n(z) = H_4((1-1/n) z)$ for $z \in \mathbf{D}$. The sequence $(R_\nu)_{\nu \in \mathbf{N}}$ converges locally uniformly to H_4. Therefore $\operatorname{Re} L(R_n) \to \operatorname{Re} L(H_4)$ for $n \to \infty$. By (11)(c) it is possible to choose $N_2 \in \mathbf{N}$ large enough such that $\operatorname{Re} L(R_{N_2}) > K$ and $1/N_2 \leqslant \varepsilon_5$. Then $\tilde{G} = Q_{1/N_2} \in \mathscr{K}_1$ and so by (5)

$$\begin{aligned}\operatorname{Re} L(\tilde{G}) - \operatorname{Re} L(G) &= \operatorname{Re} L(G_{1/N_2} - G) + \frac{1}{N_2} (\operatorname{Re} L(R_{N_2})) \\ &\geqslant \frac{1}{N_2} (\operatorname{Re} L(R_{N_2})) - |L(G_{1/N_2} - G)| \\ &\geqslant \frac{1}{N_2} (\operatorname{Re} L(R_{N_2}) - K) > 0.\end{aligned}$$

This is a contradiction since $G \in \mathscr{M}_L$ and so

$$\sup_{F \in \mathscr{K}_1} \operatorname{Re} L(F) = \operatorname{Re} L(G).$$

So we have proved that if L is not of special type and if $G \in \mathcal{M}_L$, then $\Gamma(G)$ is a set of uniqueness.

7. If L is not of special type and if $G \in \mathcal{M}_L$, then G is an extreme point of $\mathcal{K}_1$. To see this assume $G \pm F \in \mathcal{K}_1$ with $F \in H(\mathbf{D})$. Then

$$2\,|G(z)|^2 + 2\,|F(z)|^2 = |G(z) + F(z)|^2 + |G(z) - F(z)|^2 \leqslant \frac{2}{(1-|z|^2)^2} \quad \text{for} \quad z \in \mathbf{D}.$$

From this inequality we conclude $F \in \mathcal{K}$ and $F(z) = 0$ for $z \in \Gamma(G)$. But $\Gamma(G)$ is a set of uniqueness and so $F \equiv 0$. Hence G is an extreme point of $\mathcal{K}_1$.

8. If L is not of special type, then $\mathcal{M}_L$ consists of a single point.

Note that $\mathcal{M}_L$ is a convex set. If $G_1, G_2 \in \mathcal{M}_L$ and $G_1 \neq G_2$, then $\frac{1}{2}(G_1 + G_2) \in \mathcal{M}_L$. But $\frac{1}{2}(G_1 + G_2)$ cannot be an extreme point of $\mathcal{M}_L$. This contradicts 7.

The proof is complete. ∎

3. The Coefficient Regions of $\mathcal{K}_1$

For $n \in \mathbf{N}_0$ let $A_n: H(\mathbf{D}) \to \mathbf{C}^{n+1}$ be the continuous linear mapping defined by

$$A_n(F) = (F(0), F'(0), ..., F^{(n)}/n!) \qquad \text{for} \quad F \in H(\mathbf{D}).$$

Then the coefficient regions of $\mathcal{K}_1$ are $K_n = \{A_n(F): F \in \mathcal{K}_1\}$. So far only $K_0 = \bar{\mathbf{D}}$ and K_1 (cf. [Wir]) are explicitly known. It is easy to see that in general the set K_n is a compact convex subset of $\mathbf{C}^{n+1}$ containing $0 \in \mathbf{C}^{n+1}$ in its interior. Obviously, the set K_n is determined by its boundary ∂K_n. As an application of Theorem 1 we can prove the following uniqueness theorem for the boundary points $y \in \partial K_n$.

Theorem 2. *Suppose $n \in \mathbf{N}$ and $y = (c_0, ..., c_n) \in \mathbf{C}^{n+1}$ is a boundary point of K_n with $|c_0| < 1$. Then there exists a unique function $G \in \mathcal{K}_1$ with $A_n(G) = y$. For this function $\Gamma(G)$ is a set of uniqueness.*

Note that if $G \in \mathcal{K}_1$ and $G(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$ is the Taylor expansion of G at 0, then $|c_0| \leqslant 1$.

Without the assumption $|c_0| < 1$ the above uniqueness statement is not true in general. To see this define $G_1(z) = 1$ and $G_2(z) = 1 + z^2$ for $z \in \mathbf{D}$. Then $G_1, G_2 \in \mathcal{K}_1$, $G_1 \neq G_2$, and $A_1(G_1) = A_1(G_2) = (1, 0) \in \partial K_1$.

Proof of Theorem 2. Assume $n \in \mathbf{N}$, $y = (c_0, ..., c_n) \in \partial K_n$, and $|c_0| < 1$. Since K_n has nonempty interior, the set of support points coincides with the set of boundary points of K_n (cf. Introduction). Therefore, y is a support point of K_n and so there exists a continuous linear functional $\tilde{L}: \mathbf{C}^{n+1} \to \mathbf{C}$ such that $\operatorname{Re} \tilde{L}$ is not constant on K_n and

$$\operatorname{Re} \tilde{L}(x) \leqslant \operatorname{Re} \tilde{L}(y) \qquad \text{for} \quad x \in K_n. \tag{22}$$

There are numbers $a_0, ..., a_n \in \mathbf{C}$ such that

$$\tilde{L}((\xi_0, ..., \xi_n)) = \sum_{\nu=0}^{n} a_\nu \xi_\nu \qquad \text{for} \quad (\xi_0, ..., \xi_n) \in \mathbf{C}^{n+1}.$$

Since $\operatorname{Re} \tilde{L}$ is not constant on K_n, at least one of the numbers $a_0, ..., a_n$ is different from 0.

Assume $a_0 \neq 0$ and $a_1 = \cdots = a_n = 0$. Then $x_1 = (\overline{a_0}/|a_0|, 0, ..., 0) \in K_n$ and $\operatorname{Re} \tilde{L}(x_1) \leqslant \operatorname{Re} \tilde{L}(y)$ by (22). On the other hand $\operatorname{Re} \tilde{L}(x_1) = |a_0|$ and $\operatorname{Re} \tilde{L}(y) = \operatorname{Re}(a_0 c_0) \leqslant |a_0 c_0| < |a_0|$. This is a contradiction. Hence at least one of the constants $a_1, ..., a_n$ must be different from 0.

Now define $L = \tilde{L} \circ A_n : H(\mathbf{D}) \to \mathbf{C}$. Then L is a continuous linear functional and we have

$$L(F) = \sum_{\nu=0}^{n} a_\nu \frac{F^{(\nu)}(0)}{\nu!} \qquad \text{for} \quad F \in H(\mathbf{D}).$$

Since one of the numbers $a_1, ..., a_n$ is different from 0 and the representation of a continuous linear functional $L: H(\mathbf{D}) \to \mathbf{C}$ as a sum of evaluation functionals is unique, L is not of special type and $L \not\equiv 0$.

If $G \in \mathscr{K}_1$ and $A_n(G) = y$ then $G \in \mathscr{M}_L$. To see this note that we have by (22)

$$\operatorname{Re} L(F) = \operatorname{Re} \tilde{L}(A_n(F)) \leqslant \operatorname{Re} \tilde{L}(y) = \operatorname{Re} L(G) \qquad \text{for} \quad F \in \mathscr{K}_1.$$

Theorem 1 shows that G is uniquely determined and that $\Gamma(G)$ is a set of uniqueness. ∎

4. The Support Points of $\mathscr{B}_1$

The results obtained in Sections 2 and 3 for the class $\mathscr{K}_1$ may of course be reformulated for the class $\mathscr{B}_1$. Here we will content ourselves with the following theorem about the support points of $\mathscr{B}_1$.

Theorem 3. (a) *If $F \in \mathscr{B}_1$ is a support point of $\mathscr{B}_1$, then F is a convex combination of a unimodular constant u (identified with the corresponding* *constant function on* $\mathbf{D}$) *and a support point* $G \in \tilde{\mathscr{B}}_1$ *of* $\tilde{\mathscr{B}}_1$; *i.e., there are constants* $\lambda_1, \lambda_2 \in [0, 1]$ *with* $\lambda_1 + \lambda_2 = 1$ *such that* $F = \lambda_1 u + \lambda_2 G$.

Conversely, every convex combination of a unimodular constant and a support point of $\tilde{\mathscr{B}}_1$ *is a support point of* $\mathscr{B}_1$.

(b) *A function* $G \in \tilde{\mathscr{B}}_1$ *is a support point of* $\tilde{\mathscr{B}}_1$ *if and only if* $\Lambda(G) \neq \varnothing$.

Proof. (a) The proof follows from ideas similar to those of Corollary 2 in [C-W]. It offers no serious difficulties, so we omit it.

(b) Assume $G \in \tilde{\mathscr{B}}_1$ and $\Lambda(G) \neq \varnothing$. Then there exists a point $z_0 \in \Lambda(G)$. Hence

$$|G'(z_0)| = 1/(1 - |z_0|^2) = \sup_{F \in \tilde{\mathscr{B}}_1} |F'(z_0)|. \tag{23}$$

If we define $L(F) = \overline{G'(z_0)}\, F'(z_0)$ for $F \in H(\mathbf{D})$, then $L\colon H(\mathbf{D}) \to \mathbf{C}$ is a continuous linear functional. It is clear that $\operatorname{Re} L$ is not constant on $\tilde{\mathscr{B}}_1$ and by (23) we have

$$\operatorname{Re} L(F) \leqslant \operatorname{Re} L(G) \qquad \text{for} \quad F \in \tilde{\mathscr{B}}_1.$$

It follows that G is a support point of $\tilde{\mathscr{B}}_1$.

Conversely, assume that $G \in \tilde{\mathscr{B}}_1$ is a support point of $\tilde{\mathscr{B}}_1$. Then there exists a continuous linear functional $\tilde{L}\colon H(\mathbf{D}) \to \mathbf{C}$ such that $\operatorname{Re} \tilde{L}$ is not constant on $\tilde{\mathscr{B}}_1$ and

$$\operatorname{Re} \tilde{L}(F) \leqslant \operatorname{Re} \tilde{L}(G) \qquad \text{for} \quad F \in \tilde{\mathscr{B}}_1. \tag{24}$$

The functional $\tilde{L}$ can be represented by a sequence $(a_\nu)_{\nu \in \mathbf{N}_0}$ of complex numbers satisfying (1). Define $c_\nu = (1/(\nu + 1))\, a_{\nu+1}$ for $\nu \in \mathbf{N}_0$. Then from (1) it follows that

$$\limsup_{\nu \to \infty} |c_\nu|^{1/\nu} < 1.$$

Consider the continuous linear functional $L\colon H(\mathbf{D}) \to \mathbf{C}$ corresponding to the sequence $(c_\nu)_{\nu \in \mathbf{N}_0}$. Then we have

$$\tilde{L}(F) = L(F') \qquad \text{for all} \quad F \in H(\mathbf{D}) \quad \text{with} \quad F(0) = 0. \tag{25}$$

Since $\operatorname{Re} \tilde{L}$ is not constant on $\tilde{\mathscr{B}}_1$, the functional L is not identically 0. Inequality (24) and equality (25) show that $G' \in \mathscr{M}_L$. If L is not of special type, then $\Lambda(G) = \Gamma(G') \neq \varnothing$ by Theorem 1. If L is of special type, then this is also true by the remark following Theorem 1. ∎

5. Concluding Remarks

(a) Whether a functional given by

$$L(F)=\sum_{\nu=1}^{n}\lambda_\nu F(z_\nu) \qquad \text{for} \quad F\in H(\mathbf{D})$$

is of special type or not, depends on the coefficients $\lambda_1, ..., \lambda_n$ and the points $z_1, ..., z_n$. For $n=1$ the functional is always of special type. For $n=2$ the answer is in principle known and can be obtained from the complete description of the variability regions, which are in our notation defined by $V(z_1; z_2, w_2)=\{F(z_1): F\in\mathcal{H}_1 \wedge F(z_2)=w_2\}$ [Bo1]. Here we will just give two examples.

Fix $r\in(0,1)$ and $c\in\mathbf{C}^*$. Define the functional $L: H(\mathbf{D})\to\mathbf{C}$ by

$$L(F)=F(0)+cF(r) \qquad \text{for} \quad F\in H(\mathbf{D}).$$

If in addition $r\in(0, \sqrt{3}/2)$ and $c<0$, then L is not of special type. To see this assume L is of special type. Then there exists a function $G\in\mathcal{H}_1$ with

$$\operatorname{Re} L(G)=\operatorname{Re}(G(0)+cG(r))=1+|c|/(1-r^2).$$

This is only possible if $G(0)=1$ and $G(r)=-1/(1-r^2)$. Now [Bo1, p. 46, Satz 4.2.1] shows that $G\in\mathcal{H}_1$ and $G(0)=1$ imply

$$\operatorname{Re} G(r)\geqslant\frac{1-\sqrt{3}\,r}{(1-\sqrt{1/3}\,r)^3}>-\frac{1}{1-r^2}.$$

This is a contradiction.

If $r\in[\sqrt{3}/2, 1)$, then L is of special type. To see this apply [Bo1, p. 18, Satz 2.2.1, Case 3]. This shows the existence of a function $G\in\mathcal{H}_1$ with $G(0)=1$ and $G(r)=\bar{c}/(|c|\,(1-r^2))$. It follows that

$$\sup_{F\in\mathcal{H}_1}\operatorname{Re} L(F)=\operatorname{Re} L(G)=1+|c|/(1-r^2)$$

and so L is of special type.

(b) Statements similar to the theorems given above are true for other classes of holomorphic functions satisfying a growth condition. For example, one of these classes is the set of all functions F holomorphic in $\mathbf{C}$ with

$$|F(z)|\leqslant e^{|z|^2} \qquad \text{for} \quad z\in\mathbf{C}.$$

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