THREE SOLUBLE PROBLEMS IN LINEAR TRANSPORT THEORY*

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I humbly dedicate this work to my wife, Donna, without whose love, understanding and silent sacrifice this work would never have been completed. My only hope is that some day these past years will prove worthwhile to her.

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ABSTRACT

In this study three exactly soluble problems in linear transport theory are considered.

In the first problem, the one velocity, time independent, isotropic scattering, neutron transport equation with r^{-1} cross sections is solved exactly by two-sided Laplace transforms. The homogeneous solutions and the Green's functions are exhibited for both the infinite medium and the finite sphere. The diffusion approximation is compared to the exact result.

In the second problem, a typical translationally invariant linear transport problem for a half-space is solved by the eigenfunction method. Using some derived identities for the solution of the associated homogeneous Hilbert problem it is shown that the usual cumbersome integrals occurring in half-space problems can be reduced to a simple form.

For the third problem the particle density of a simple model of the exosphere is obtained by solving exactly the collisionless Boltzmann equation. The main point of the solution is that it is a discontinuous, multivalued function of the constants of motion. Results, of course, agree with those of other methods based on Newtonian mechanics.

CHAPTER I

INTRODUCTION

The complexity of the integro-differential equations occurring in linear transport phenomena permit few problems to be exactly soluble. However, as has been suggested by Wigner (1), use of invariance properties permit not only simplification of the equation, but may in fact allow complete solutions to be found. The first two problems in this thesis are problems which do have the "complete" symmetry necessary for solubility.

The first problem that is considered is the one velocity, time independent, isotropic scattering, neutron transport equations with cross sections which vary as r^{-1}. The invariance of this equation under the transformation $r \rightarrow \alpha r$, $\alpha > 0$ suggests (1) simplicity of the equation in spherical coordinates and the use of Mellin transforms, but for convenience the usual transformation is made so that Laplace transforms are applicable. Solutions to the infinite and finite sphere problems are then shown to be expressible in quadrature and are compared to the usual diffusion approximation.

The second problem considered is a typical one dimensional translationally invariant transport equation. Instead of exploiting the full symmetry of the problem we solve the two independent variables equation by the usual eigenfunction technique. The completeness of these eigenfunctions is then used to solve some typical half-space problems. Several identities derived for the solution of the associated homogeneous Hilbert problem are used to simplify the resulting cumbersome integrals.

In the third problem, a simple model of an exosphere, the linear collisionless Boltzmann equation for particles in an external gravitational field is solved exactly by using the method of characteristics. The solution is shown to be a discontinuous, multivalued function of the constants of motion.

CHAPTER II

THE NEUTRON TRANSPORT EQUATION WITH r^{-1} CROSS SECTIONS

A. PRELIMINARY REMARKS

In this chapter the one velocity, time independent, isotropic scattering neutron transport equation with cross sections which vary as r^{-1} is solved by Laplace transform techniques. While the r^{-1} dependence of the cross sections makes this model mainly an exercise in mathematical physics; with little modification the results can be applied to electron transport in velocity space with cross sections which vary as v^{-2}.

The time independent integral equation for the stationary neutron density, $\rho(\vec{r})$, in the one velocity approximation can be written:

$$\rho(\vec{r}) = \int_{-\infty}^{\infty} d\vec{r}\,' \frac{e^{-\int_0^{|\vec{r}-\vec{r}\,'|} \sigma\left(\vec{r}-\frac{(\vec{r}-\vec{r}\,')s}{|\vec{r}-\vec{r}\,'|}\right) ds}}{4\pi|\vec{r}-\vec{r}\,'|^2} \left\{C(\vec{r}\,')\sigma(\vec{r}\,')\rho(\vec{r}\,') + \frac{4\pi S(\vec{r}\,')}{v}\right\}. \tag{II-1}$$

Where $S(\vec{r})$ is the isotropic source density, $C(\vec{r})$ is the average number of secondaries emitted after a collsion occurs at $\vec{r}$, and $\sigma(\vec{r})$ is the probability that a neutron suffer a collision at $\vec{r}$ for small $\vec{r}$. For the model being considered, $C(\vec{r}) = C$, $\sigma(\vec{r}) = \sigma r^{-1}$, C and σ being dimensionless constants. Using the spherical symmetry of the proglem the homogeneous equation for an infinite medium can be reduced to: (see Appendix A)

$$\eta(\theta) = C\sigma \int_{-\infty}^{\infty} d\theta'\ \eta(\theta')\ K(|\theta - \theta'|) \tag{II-2a}$$

$$\eta(\Theta) = r\rho(r) \Big|_{r \to ae^{-2\Theta}} \tag{II-2b}$$

and

$$K(\Theta) = \int_{\tanh\Theta}^{1} \frac{dx}{x} \left[\frac{1 - x}{1 + x}\right]^{\sigma} \tag{II-2c}$$

(a is an arbitrary constant with length dimensions).

As two-sided Laplace transforms will be used exclusively in this chapter, the notation and analyticity properties will be standardized and exhibited now.

The transform of the kernel,

$$\tilde{K}(z) = \int_{-\infty}^{\infty} e^{-z\Theta}\, K(|\Theta|)\, d\Theta \tag{II-3}$$

has the following representation: (see Appendix B)

$$K(z) = \frac{1}{z}\left[\psi\left(\frac{\alpha + z}{4}\right) - \psi\left(\frac{\alpha - z}{4}\right)\right] \tag{II-4}$$

for $-\alpha < \text{Re}z < \alpha$, $\alpha = 2(\sigma + 1)$, and $\psi(z) = \frac{d\ell n}{dz}\Gamma(z)$. $\tilde{K}(z)$ is analytic in in the strip $-\alpha < \text{Re } z < \alpha$, and the analytic continuation of $\tilde{K}(z)$ defined by (II-4) is analytic everywhere except for simple poles at $z = \pm(4m + \alpha)$, $m = 0, 1, 2, \ldots$. Clearly $\tilde{K}(z)$ is a real, even function of z.

The zeros of $1 - C\sigma\,\tilde{K}(z)$ for $-\alpha < \text{Re } z < \alpha$, and its analytic continuation for $|\text{Re } z| > \alpha$ are needed for inverting the transforms. These are discussed in Appendix B and can be summarized as follows:

(a) if $(\sigma C)^{-1} > \tilde{K}(0)$, $1 - C\sigma\,\tilde{K}(z)$ has two real zeros, $\pm z_0$, $0 < z_0 < \alpha$, for $-\alpha < \text{Re } z < \alpha$;

(b) if $(\sigma C)^{-1} < \tilde{K}(0)$, $1 - C\sigma\,\tilde{K}(z) = 0$ has two imaginary roots, $\pm z_0$, for $-\alpha < \text{Re } z < \alpha$;

(c) if $(\sigma C)^{-1} = \tilde{K}(0)$, $1 - C\sigma\,\tilde{K}(z) = 0$ has a double root at $z = \pm z_0 = 0$;

(d) and for any σ, $C \neq 0$, there are an infinite number of real roots, $\pm z_n$, $n = 1,2,3,\ldots$ to $1 - C\sigma\,\tilde{K}(z) = 0$, $z_n > 0$, where z_n lies in the interval $4(n-1) + \alpha < z_n < 4n + \alpha$.

As a convenient reference point, we note that the representation of $\tilde{K}(z)$ given by (B-2) simply gives $\tilde{K}(2) = 1/\sigma$. Therefore, if $C > 1$, $z_0 < 2$, and if $C < 1$, $z_0 > 2$.

As $\tilde{K}(z)$ is analytic in the strip $-\alpha < \text{Re } z < \alpha$, the use of Laplace transforms requires that we look for solutions such that

$$\lim_{|\theta|\to\infty} |\eta(\theta)| \leq A\, e^{\beta|\theta|} \tag{II-5}$$

where β is real and less than α. Then

$$f_+(z) \equiv \int_0^\infty e^{-z\theta}\,\eta(\theta)\,d\theta \tag{II-6a}$$

and

$$f_-(z) \equiv \int_{-\infty}^0 e^{-z\theta}\,\eta(\theta)\,d\theta \tag{II-6b}$$

will be analytic for Re $z > \beta$, and Re $z < -\beta$, respectively (2). Moreover the inversion theorem gives us:

$$\text{for } \theta > 0,\ \eta(\theta) = \lim_{\lambda\to\infty} \frac{1}{2\pi i}\int_{\alpha-\epsilon-i\lambda}^{\alpha-\epsilon+i\lambda} f_+(z)\, e^{z\theta} dz \tag{II-7a}$$

and for $\theta < 0$,

$$\eta(\theta) = \lim_{\lambda\to\infty} \frac{1}{2\pi i} \int_{-\alpha+\epsilon-i\lambda}^{-\alpha+\epsilon+i\lambda} f_-(z)\, e^{z\theta}, \quad \epsilon > 0 \ . \qquad \text{(II-7b)}$$

B. SOLUTIONS FOR THE INFINITE MEDIUM

In the usual way (2) the homogeneous equation for the infinite medium, Eq. (II-2), defines an analytic continuation of f+(z) and f-(z) in transform space such that

$$f_+(z)[1 - C\sigma\tilde{K}(z)] = - f_-(z)\ [1 - C\sigma\tilde{K}(z)] \qquad \text{(II-8)}$$

for $-\alpha < \mathrm{Re}\ z < \alpha$, and both sides are analytic in this strip. Hence, in this strip $f_+(z) = -f_-(z)$ except at the zeros, $\pm z_0$, of $1 - C\sigma\tilde{K}(z)$, and $f_+(z)$ is analytic except for possible poles at $z = \pm z_0$. Using the inversion theorem, (II-7), and converting back to r and $\rho(r)$, we find the general solution of the homogeneous equation is given by:

$$\rho(r) = A_1\left(\frac{r}{a}\right)^{z_0/2-1} + B_1\left(\frac{r}{a}\right)^{-z_0/2-1}, \text{ if } (C\sigma)^{-1} \neq \tilde{K}(0) \qquad \text{(II-9a)}$$

and

$$\rho(r) = A_2\left(\frac{r}{a}\right)^{-1} + B_2\left(\frac{r}{a}\right)^{-1} \ell n\, \frac{r}{a}\ , \text{ if } (C\sigma)^{-1} = \tilde{K}(0) \ . \qquad \text{(II-9b)}$$

For the infinite medium Green's function or shell source solution we introduce into Eq. (II-1) the source:

$$S(r;r_0) = \frac{\delta(r - r_0)}{(4\pi)^2\ r_0^{\ 2}} \ . \qquad \text{(II-10)}$$

Using the same substitutions as in the homogeneous equation, the integral equation for the shell source solution can be written as:

$$\eta(\theta;\theta) = C\sigma \int_{-\infty}^{\infty} d\theta' \; \eta(\theta',\theta_o) \; K(|\theta - \theta'|) + \frac{1}{\lambda} K(|\theta - \theta_o|) \qquad \text{(II-11)}$$

$r_o = ae^{-2\theta_o}$, $\lambda = 8\pi v r_o$. In transform space, this equation again defines an analytic continuation of $f_+(z)$, $f_-(z)$, such that

$$f_+(z)[1 - \sigma\tilde{K}(z)] = -f_-(z)[1 - C\sigma\tilde{K}(z)] + e^{-z\theta_o} \frac{\tilde{K}(z)}{\lambda} \qquad \text{(II-12)}$$

for $-\alpha < \mathrm{Re}\, z < \alpha$, and both sides are analytic in this region. Inverting the transform, we obtain the general solution to (II-11):

$$\eta(\theta;\theta_o) = A_1 \, e^{-z_o\theta} + B_1 \, e^{z_o\theta} + \frac{1}{2\pi i\lambda} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{z(\theta-\theta_o)} \, \tilde{K}(z)\, dz}{1 - C\sigma\tilde{K}(z)} \; . \qquad \text{(II-13)}$$

For $z_o < \gamma < \alpha$, $(C\sigma)^{-1} \neq K(0)$. To determine A_1 and B_1 boundary conditions must be applied. If we require $r\rho(r;r_o)$ to finite as $r \to 0$ and $r \to \infty$, Eq. (II-13) becomes (for $(\sigma C)^{-1} > \tilde{K}(0)$):

$$\eta(\theta;\theta_o) = \frac{1}{2\pi i\lambda} \int_{-i\infty}^{+i\infty} \frac{e^{z(\theta-\theta_o)} \, \tilde{K}(z)\, dz}{1 - C\sigma\tilde{K}(z)} \; . \qquad \text{(II-14)}$$

For $\theta > \theta_o$ ($\theta < \theta_o$) the contour integral can be completed in the left (right) half plane. In terms of the density, Eq. (II-14) is:

$$\rho(r;r_o) = \frac{1}{\lambda r (C\sigma)^2} \sum_{n=0}^{\infty} \frac{1}{K'(z_n)} \left(\frac{r}{r_o}\right)^{z_n/2} , \; r < r_o \qquad \text{(II-15a)}$$

and

$$\rho(r;r_o) = \frac{1}{\lambda r(C\sigma)^2} \sum_{n=0}^{\infty} \frac{1}{\tilde{K}'(z_n)} \left(\frac{r_o}{r}\right)^{z_n/2} \text{ for } r > r_o . \quad \text{(II-15b)}$$

In (II-15) we have required $(\sigma C)^{-1} > \tilde{K}(0)$, and used the property $\tilde{K}'(z) = -\tilde{K}'(z)$, $\tilde{K}'(z) = d/dz\, \tilde{K}(z)$. The boundary condition that we have used is that the total number of collisions per unit radial distance, $4\pi r^2\sigma(r)\rho(r;r_o)$, be finite at the origin and infinity. If instead of this condition we had required that $\rho(r;r_o)$ be everywhere non-negative and vanish as $r \to \infty$, for all C, then we would have found the same solutions as Eq. (II-15). For $(C\sigma)^{-1} < \tilde{K}(0)$ the boundary conditions that $r\rho(r;r_o)$ be finite at the origin and infinity are not enough to uniquely determine the solution. In order to specify a unique solution one would need some kind of a "radiation condition." However, it is evident that there are no solutions for $(\sigma C)^{-1} < K(0)$ for which the density is everywhere non-negative. Hence, for $C > C_{min} \equiv 1/\sigma\tilde{K}(0)$ there are no physically acceptable stationary solutions to the transport equation. This is normally referred to as the "criticality" condition. That is, if $C > C_{min}$ the sphere is "super-critical" and if $C < C_{min}$ the sphere is "sub-critical." In Table I we have calculated C_{min} for various values of σ.

It is interesting to note that unlike the constant cross section case, which has a spectrum consisting of two discrete eigenvalues and a continuum on $(1,\infty)$ and $(-1,\infty)$, (3), the r^{-1} transport equation has a purely discrete unbounded spectrum.

TABLE I

C_{min}

σ	C_{min}	σ	C_{min}
.25	2.355	2.5	1.050
.50	1.571	3.0	1.032
.75	1.328	4.0	1.024
1	1.218	5.0	1.018
1.2	1.171	$\sigma \gg 1$	$1+(3\sigma^2)^{-1}$
1.5	1.118		
1.7	1.092		
2.0	1.071		

C. DIFFUSION SOLUTION FOR THE SHELL SOURCE

For most realizable systems (e.g., a reactor) the transport equation is too complex to be solved by analytical techniques; one normally has to resort to a diffusion calculation. It is interesting therefore to see just how the diffusion solution approximates the transport solution for the model that we are considering.

The diffusion equation for the r^{-1} cross sections with a shell source can be derived by the usual methods. The equation is:

$$\nabla \cdot \vec{j} + \frac{\sigma_a}{r} \rho_D(r;r_o) = \frac{\delta(r - r_o)}{4\pi v r^2} , \qquad \text{(II-16a)}$$

$$\vec{j} \simeq - \frac{rC}{3\sigma} \nabla \rho_D(r;r_o) \qquad \text{(II-16b)}$$

and $\sigma_a = \sigma(1-C)$. In spherical coordinates, and using the substitution $x = \ell n\, r_o/r$, Eq. (II-16a) becomes:

$$\left(\frac{d^2}{dx^2} + \frac{2d}{dx} - \alpha\right) \rho_D(x) = \frac{-3\sigma\, \delta(x)}{4\pi C v r_o{}^2} \qquad \text{(II-17)}$$

where $\alpha = 3\sigma\,\sigma_a/C$. Solving this equation by Fourier transforms, and requiring that $r\rho_D(r;r_o)$ be finite at the origin and infinity, we find:

$$\rho_D(r;r_o) = \frac{3\sigma}{C\lambda r\sqrt{1+\alpha}}\left(\frac{r}{r_o}\right)^{\sqrt{1+\alpha}}, \quad r < r_o \qquad \text{(II-18a)}$$

$$\rho_D(r;r_o) = \frac{3\sigma}{C\lambda r\sqrt{1+\alpha}}\left(\frac{r_o}{r}\right)^{\sqrt{1+\alpha}}, \quad r > r_o \ . \qquad \text{(II-18b)}$$

It is interesting to note that the solutions given by Eq. (II-18) are everwhere non-negative if $\alpha > -1$. This condition is just

$$C < \frac{1}{1 - \frac{1}{3\sigma^2}} \simeq 1 + \frac{1}{3\sigma^2}, \qquad \text{(II-19)}$$

which is the "sub-criticality" condition for the transport equation when $\sigma \gtrsim 5$.

For $C < C_{min}$, the asymptotic transport solutions given by Eq. (II-15) are:

$$\rho_{as}(r;r_o) \simeq \frac{1}{\lambda r (C\sigma)^2\,\tilde{K}'(z_o)}\left(\frac{r_o}{r}\right)^{-z_o/2} \quad r \ll r_o \qquad \text{(II-20a)}$$

$$\rho_{as}(r:r_o) \simeq \frac{1}{\lambda r (C\sigma)^2\,\tilde{K}'(z_o)}\left(\frac{r_o}{r}\right)^{z_o/2} \quad r \gg r_o . \qquad \text{(II-20b)}$$

In the limit of no absorption, $C = 1$, $z_o = 2$, $\alpha = 0$, the asymptotic transport solutions have the same dependence on r as the diffusion solutions. For $\sigma \gg 1$ Eq. (B-2) gives that $\tilde{K}'(2) \simeq 1/3\sigma^3$, and in this limit ($C = 1$) the asymptotic solutions become exactly the diffusion solutions. Similarly, if $\sigma \ll 1$, we find $\tilde{K}'(2) \simeq \frac{1}{2\sigma^2}$, and in this limit we see that the diffusion solution is smaller than the asymptotic solutions by a factor of $3\sigma/2$. For C = 1 we have several ratios of ρ_D/ρ_{as} listed in Table II.

TABLE II

$\left.\rho_D/\rho_{as}\right)$ C = 1

σ	$3\sigma^3\,\tilde{K}'(2) = \left.\frac{\rho_D}{\rho_{as}}\right) C = 1$
<<1	$3\sigma/2$
1	.70
2	.86
3	.93
4	.96
>>1	1.0

From Table II it is evident that the diffusion approximation is reliable for C = 1, and $\sigma \gtrsim 3$. If $C \neq 1$, the root z_0 becomes approximately:

$$\frac{z_0}{2} \simeq 1 + \frac{3\sigma_a\sigma}{2C}\left(\frac{1}{3\sigma^3\,\tilde{K}'(2)}\right), \qquad \text{(II-21)}$$

if $\sigma_a/2\sigma^2\,C\tilde{K}'(2) \ll 1$. Likewise, for $\alpha \ll 1$, the exponent of r in the diffusion solution becomes approximately $1 + 3\sigma_a\sigma/2C$. Therefore, if $3\sigma^3\tilde{K}'(2) \simeq 1$, (i.e., $\sigma \gtrsim 4$), and $|1 - C| \ll \frac{1}{3\sigma^2}$, the diffusion solution gives an excellent asymptotic representation to the neutron density.

D. FINITE SPHERE PROBLEM

The equation for the neutron density in a finite sphere of radius "a" surrounded by a vacuum can be derived from (II-1). It is:

$$r\rho(r) = \frac{C\sigma}{2}\int_0^a dr'\rho(r')\int_{\frac{|r-r'|}{r+r'}}^{1} \frac{dx}{x}\left[\frac{1-x}{1+x}\right]^{\sigma} , \qquad \text{(II-22)}$$

for $r < a$. Using the same substitutions as used to derive Eq. (II-2), Eq. (II-22) becomes:

$$\eta(\theta) = C\sigma \int_0^{\infty} d\theta'\ \eta(\theta')\ K(|\theta - \theta'|), \text{ for } \theta > 0 . \qquad \text{(II-23)}$$

This equation can be solved by the Wiener-Hopf method (2). Defining an $\eta(\theta)$ for $\theta < 0$ by the right hand side of Eq. (II-23) and using Laplace transforms as before, we obtain:

$$f_+(z)\ [1 - C\sigma\ \tilde{K}(z)] = - f_-(z) \qquad \text{(II-24)}$$

for $-\alpha < \text{Re } z < \alpha$ as $1 - C\sigma\,\tilde{K}(z)$ has two zeros at $z = \pm z_o$ in this strip, we look for a Wiener-Hopf decomposition of the form:

$$1 - C\sigma\ \tilde{K}(z) = \frac{X_-(z)}{X_+(z)}\ (z^2 - z_o^2) \qquad \text{(II-25)}$$

such that $X_-(z)$ is analytic and non-zero for $\text{Re } z < \beta$, z finite, and $X_+(z)$ is analytic and non-zero for $\text{Re } z > -\beta$, z finite. For this purpose, consider:

$$\tau(z) \equiv \frac{[1 - C\sigma\ \tilde{K}(z)]\,[z^2 - \alpha^2]}{z^2 - z_o^2} . \qquad \text{(II-26)}$$

We note:

(a) $\tau(z)$ is analytic and non-zero for $-\alpha < \text{Re } z < \alpha$.

(b) $\tau(z)$ is a real even function of z.

(c) $\lim_{z\to\infty} \tau(z) = 1$ in the strip.

Decomposing
$$\tau(z) = \frac{\sigma_-(z)}{\sigma_+(z)} \qquad \text{(II-27)}$$

where $\sigma_-(z)$, $\sigma_+(z)$ have the same properties as $X_-(z)$, $X_+(z)$ respectively, the unique solution to Eq. (II-27) is given by Cauchy's theorem as:

$$\sigma_-(z) = e^{-\Gamma_-(z)}, \qquad \text{(II-28a)}$$

$$\sigma_+(z) = e^{-\Gamma_+(z)}, \qquad \text{(II-28b)}$$

where for $-\lambda < \text{Re } z < \lambda$

$$\Gamma_-(z) = -\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\ln \tau(z')dz'}{z' - z} \qquad \text{(II-29a)}$$

and

$$\Gamma_+(z) = \frac{1}{2\pi i}\int_{-\lambda-i\infty}^{-\lambda+i\infty} \frac{\ln \tau(z')dz'}{z' - z} \qquad \text{(II-29b)}$$

and $\beta < \lambda < \alpha$. By construction $\ln \tau(z)$ is single valued in the strip $-\alpha < \text{Re } z < \alpha$, so the branch of $\ln \tau(z)$ can be chosen to be the one which vanishes as $z \to \pm i\infty$. Therefore, $\Gamma_-(z)$ and $\Gamma_+(z)$ exist and go to zero for Re z going to minus and plus infinity respectively. The unique solutions for $X_+(z)$ and $X_-(z)$ are now given by:

$$X_-(z) = \frac{\sigma_-(z)}{z - \alpha}, \qquad \text{(II-30a)}$$

and

$$X_+(z) = [z + \alpha]\, \sigma_+(z) \; . \qquad \text{(II-30b)}$$

It is clear that $X_{\pm}$ are real functions of z. Also for z real, we have that $X_-(z) < 0$ for $z < \lambda$, and $X_+(z) > 0$ for $z > -\lambda$.

Inserting (II-30) into (II-24) we obtain:

$$\frac{f_+(z)\ [z^2 - z_0{}^2]}{X_+(z)} = -\frac{f_-(z)}{X_-(z)} \qquad \text{(II-31)}$$

for $-\alpha < \text{Re } z < \alpha$. The left hand side of (II-31) is analytic for Re $z > -\beta$; the right hand side is analytic for Re $z < \beta$; they are equal in the strip, and hence define an entire function everywhere. By the extended Liouville theorem, we have:

$$\frac{f_+(z)\ (z^2 - z_0{}^2)}{X_+(z)} = P_m(z), \qquad \text{(II-32)}$$

where $P_m(z)$ is a polynomial of order m. As $\eta(\theta)$ is finite for $\theta \to 0$ ($r\rho(r)$ is finite as $r \to a$), then for Re $z \to +\infty$ $f\ (z) \simeq \eta(0)/z + O(z^{1+\epsilon})$ $\epsilon > 0$. Also $X\ (z) \underset{\text{Re } z\to\infty}{\simeq} z$, hence we find that $P_m(z) = \text{constant} = \eta(0)$. The for $f_+(z)$ is therefore given by:

$$f_+(z) = \frac{\eta(0)\ X_+(z)}{z^2 - z_0{}^2} \qquad \text{(II-33a)}$$

or

$$f_+(z) = \frac{\eta(0)\ X_-(z)}{1 - C\sigma\widetilde{K}(z)}\ . \qquad \text{(II-33b)}$$

Using the inversion theorem, for $\theta > 0$, the contour can be deformed into the left half plane where the only singularities of the integrand are the zeros of $1 - C\sigma\widetilde{K}(z)$. We find the density is given by

$$\rho(r) = \frac{a\rho(a)}{r}\left\{\sum_{n=0}^{\infty}\frac{X_-(-z_n)}{C\sigma\,\tilde{K}'(z_n)}\left(\frac{r}{a}\right)^{z_n/2} - \frac{X_-(z_o)}{C\sigma\,\tilde{K}'(z_o)}\left(\frac{a}{r}\right)^{z_n/2}\right\} \qquad \text{(II-34)}$$

for $r \leq a$, and $(C\sigma)^{-1} \neq \tilde{K}(0)$. The obvious identity:

$$\sum_{n=0}^{\infty}\frac{X_-(-z_n)}{C\sigma\,\tilde{K}'(z_n)} - \frac{X_-(z_o)}{C\sigma\,\tilde{K}'(z_o)} \equiv 1 \qquad \text{(II-35)}$$

plus the fact that $X_-(z) < 0$, z real and less than β insure that the neutron density is non-negative for $r \leq a$ when $C \leq C_{min}$. Similarly, for $C > C_{min}$ the density becomes negative somewhere in the sphere. This is the same "criticality" condition as for the infinite medium, as of course it must be because of the invariance of the equation under the transformation $r \to \alpha r$, $\alpha > 0$.

As the neutron density is non-negative for $C \leq C_{min}$, for $r \geq a$, we can ask for the distance $r^o > a$, where the asymptotic solution vanishes. Asymptotically, for $r \to 0$ the solution (II-34) becomes:

$$\rho(r) \simeq \frac{2a\rho(a)}{C\sigma\,\tilde{K}'(z_o)r}\,[X_-(z_o)\;X_-(-z_o)]^{1/2}\,\sinh\;\frac{z_o}{2}\,\ell n\,\frac{r}{r^o}\;, \qquad \text{(II-36)}$$

where

$$r^o = a\left[\frac{X_-(z_o)}{X_-(-z_o)}\right]^{1/z_o} \qquad \text{(II-37)}$$

and $(C\sigma)^{-1} > \tilde{K}(0)$. In the "just critical" limit ($C = C_{min}$), the extrapolation radius, r^o, becomes simply:

$$r^o = a\,e^{\frac{2X_-'(0)}{X_-(0)}}\;. \qquad \text{(II-38)}$$

For $C = C_{min} = 1.218$, $\sigma = 1$, the extrapolated radius is computed approximately in Appendix C. We find that $r^o = (1.64)a$

We can also find the shell-source solution for the finite sphere. As this problem is so similar to the previous, we will just sketch the method and give the result. Using the same notation and substitutions as before, we find the integral equation:

$$\eta(\theta;\theta_0) = C\sigma \int_0^\infty \eta(\theta';\theta_0)d\theta' \, K(|\theta - \theta'|) + \frac{1}{\lambda} K(|\theta - \theta_0|) \,, \qquad \text{(II-39)}$$

for $\theta > 0$, $\theta_0 > 0$ (i.e., shell source inside of the sphere). In transform space we then have:

$$f_+(z) \, [1 - C\sigma \, \widetilde{K}(z)] = - f_-(z) + \frac{\widetilde{K}(z) \, e^{-z\theta_0}}{\lambda} \qquad \text{(II-40)}$$

for $-\alpha < \text{Re } z < \alpha$. Using the decompositon of $1\text{-}C\sigma \, \widetilde{K}(z)$ made before, Eq. (II-40) becomes:

$$\frac{f_+(z)}{X_+(z)} (z^2 - z_0^2) = - \frac{f_-(z)}{X_-(z)} + \frac{K(z) \, e^{-z\theta_0}}{\lambda X_-(z)} \,. \qquad \text{(II-41)}$$

Making another decomposition such that

$$\frac{\widetilde{K}(z) \, e^{-z\theta_0}}{\lambda X_-(z)} = \Omega_+(z) + \Omega_-(z) \qquad \text{(II-42)}$$

where $\Omega_+(z)$ is analytic for Re $z \geq -\beta$ and goes to zero as Re $z \to \infty$, and $\Omega_-(z)$ is analytic for Re $z < \beta$, and goes to zero Re $z \to -\infty$, we find that

$$f_+(z) = \frac{\eta(0) \; X_+(z)}{z^2 - z_0^2} + \frac{\widetilde{K}(z) \, e^{-z\theta_0}}{\lambda[1 - C\sigma \, \widetilde{K}(z)]} - \frac{\Omega_-(z) \; X_-(z)}{1 - C\sigma \, \widetilde{K}(z)} \,. \qquad \text{(II-43)}$$

The unique solutions for $\Omega_+(z)$ and $\Omega_-(z)$, for $-\beta < \text{Re } z < \beta$ are given by:

$$\Omega_+(z) = -\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty} \frac{\tilde{K}(z')\, e^{-z'\theta_0}\, dz'}{\lambda X_-(z')\,[z'-z]} , \qquad \text{(II-44a)}$$

and

$$\Omega_-(z) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{\tilde{K}(z')\, dz'\, e^{-z'\theta_0}}{\lambda X_-(z')\,[z'-z]} . \qquad \text{(II-44b)}$$

Now, using the inversion theorem and requiring that $r\rho(r;r_o)$ be finite at the origin we obtain for $C < C_m$:

$$\rho(r;r_o) = \sum_{n=0}^{\infty} \frac{X_-(z_n)}{rC\sigma\,\tilde{K}'(z_n)}\left(\frac{r}{a}\right)^{z_n/2}[a\rho(a) - \Omega_-(-z_n)]$$

$$+ \frac{1}{\lambda r(C\sigma)^2}\sum_{n=0}^{\infty}\frac{1}{\tilde{K}'(z_n)}\left(\frac{r}{r_o}\right)^{z_n/2}, \text{ for } r < r_o < a \qquad \text{(II-45a)}$$

$$\rho(r;r_o) = \sum_{n=0}^{\infty} \frac{X_-(-z_n)}{rC\sigma\,\tilde{K}'(z\ )}\left(\frac{r}{a}\right)^{z_n/2}[a\rho(a) - \Omega_-(-z_n)]$$

$$+ \frac{1}{\lambda r(C\sigma)^2}\sum_{n=0}^{\infty}\frac{1}{\tilde{K}'(z_n)}\left(\frac{r_o}{r}\right)^{z_n/2}, \ r_o < r \leq a \ , \qquad \text{(II-45b)}$$

where

$$a\rho(a) = \Omega\ (z_o) - \frac{1}{C\sigma\lambda X_-(z\ )}\left(\frac{r_o}{a}\right)^{z_o/2} . \qquad \text{(II-46)}$$

E. CONCLUSION

In conclusion it should be pointed out that the totally black sphere problem can now be easily done. In this problem, there is a black ($\sigma = \infty$, $C = 0$) sphere of radius "a" inbedded in a medium with r^{-1} cross sections. For the homogeneous problem, the solution is simply given by

Eq. (II-34) with r/a replaced by a/r ($\Theta \to -\Theta$). The shell source solution can also be obtained from the previous solution by requiring now that $r\rho(r;r_0)$ be finite at infinity.

We also note that for applications to electron transport in velocity space with total cross section which vary as v^{-2} a class of slowing-down problems can be solved with formally little modification. In particular, problems can be solved in which $\sigma(\vec{v}' \to \vec{v})d^3\vec{v}$, the probability per sec that an electron with velocity $\vec{v}'$ suffers a collision and has a final velocity in $d\vec{v}$ about $\vec{v}$, has the invariance property that $d\vec{v}''v''' \sigma(\vec{v}''' \to \vec{v}'')$ $= \alpha^{-1}v' d\vec{v}\; \sigma(\vec{v}' \to \vec{v})$, where $\vec{v}''' = \alpha\vec{v}'$, $\vec{v}'' = \alpha\vec{v}$, $\alpha > 0$, and this "slowing-down" function is isotropic. This invariance requirement on $\sigma(\vec{v}' \to \vec{v})$ simply insures that the transport equation is invariant under the transformation $\vec{v} \to \alpha\vec{v}$. Using substitutions similar to those used in the neutron problem, one can derive an integral equation with translational symmetry and solve it by means of the Laplace transform. Evidently, the only significant differences of this class of electron slowing-down problems from the r^{-1} neutron problem are the properties of the transform of the kernel, $\tilde{K}(z)$, and of course as a result, a change in the spectrum.

CHAPTER III

USEFUL IDENTITIES FOR HALF-SPACE PROBLEMS IN LINEAR TRANSPORT THEORY

A. PRELIMINARY REMARKS

A large class of problems in linear transport phenomena can be formulated as boundary value problems and solved by the eigenfunction technique (7) through (12). Typically, the governing equation possesses translational invariance and can be reduced to the form:

$$\left(\frac{\mu\partial}{\partial x} + 1\right) \psi(x,\mu) = f_1(\mu) \int_{-\beta}^{\beta} f_2(\mu') \psi(x,\mu')d\mu', \qquad \text{(III-1)}$$

where $-\beta \leq \mu \leq \beta$, and $f_1(\mu)\cdot f_2(\mu)$ is a real even function of μ. In this chapter we first show that with suitable restrictions on $f_1(\mu)\cdot f_2(\mu)$ Eq. (III-1) always generates a complete set of solutions. Using these solutions, some typical half-space problems are considered and it is shown that some identities established for the solution of the associated homogeneous Hilbert problem allow the usual cumbersome integrals occurring in half-space problems to be considerably simplified.

B. SOLUTIONS OF EQUATION (III-1)

The translational symmetry of Eq. (III-1) suggests looking for solutions of the form:

$$\psi_\nu(x,\mu) = e^{-x/\nu} \phi_\nu(\mu) . \qquad \text{(III-2)}$$

For ν finite (III-1) becomes:

$$(\nu - \mu)\, \phi_\nu(\mu) = \nu f_1(\mu) \int_{-\beta}^{\beta} f_2(\mu')\, \phi_\nu(\mu')\, d\mu' \qquad \text{(III-3)}$$

choosing the convenient normalization of $\phi_\nu(\mu)$;

$$\int_{-\beta}^{\beta} f_2(\mu)\, \phi_\nu(\mu)\, d\mu = 1 \qquad \text{(III-4)}$$

(Eq. III-3) can be solved by the standard procedure (8). The usual discrete eigenvalues are then given by the zeros of $\Lambda(\nu)$, where

$$\Lambda(\nu) \equiv 1 - \int_{-\beta}^{\beta} f_2(\mu)\, \phi_\nu(\mu)\, d\mu = 1 + \nu \int_{-\beta}^{\beta} \frac{f_1(\mu)\, f_2(\mu)\, d\mu}{\mu - \nu} . \qquad \text{(III-5)}$$

We will always assume that $f_1(\mu)\cdot f_2(\mu)$ belongs to the class H* (13) on $(-\beta,\beta)$, for β finite; and for β infinite that $f_1(\mu)\cdot f_2(\mu)$ belongs to the class H and satisfies:

$$\lim_{|\mu|\to\infty} f_1(\mu)\, f_2(\mu) \le C|\mu|^{-(2+\alpha)},\ \alpha > 0 . \qquad \text{(III-6)}$$

Then $\Lambda(\nu)$ is a real even function of ν, sectionally holomorphic with boundary $(-\beta,\beta)$, and asymptotically,

$$\Lambda(\nu) \underset{\nu\to\infty}{\simeq} 1 - \int_{-\beta}^{\beta} f_1(\mu)\, f_2(\mu)\, d\mu + O\left(\frac{1}{\nu^2}\right) . \qquad \text{(III-7)}$$

Clearly, if ν_i is a root of $\Lambda(\nu) = 0$, so is $-\nu_i$ and $\pm\nu_i^*$. Defining $\underline{\nu_i} = -\nu_i$, it will be assumed (only for convenience) that the ν_i can be labeled so that Re $\nu_i > 0$ $i > 0$, and if $\nu^*_{i+1} \neq \nu_i$, then $|\text{Re } \nu_i| < |\text{Re } \nu_{i+1}|$.

For ν not on the interval $(-\beta,\beta)$ $\Lambda(\nu)$ will in general have 2N' zeros, where N' includes the order of the zero. That is:

$$\Lambda^{(j-1)}(\nu_i) = \left.\frac{d^{(j-1)}\Lambda(\nu)}{d\nu^{(j-1)}}\right|_{\nu=\nu_i} = 0, \quad \begin{array}{l} j = 1,2,\cdots m_i, \\ \pm i = 1,2,\cdots n', \end{array} \tag{III-8}$$

and $N' = \sum_{i=1}^{n'} m_i$. Also, $\Lambda_\pm(\nu)$ will have 2L zeros for ν an $[-\beta,\beta]$, where $\Lambda_+(\nu)$ and $\Lambda_-(\nu)$ denote the boundary values of $\Lambda(\nu)$ from above and below respectively. Explicitly, we assume:

$$\Lambda_\pm^{(j-1)}(\pm\mu_i) = 0 \quad j = 1,2,\cdots m_i, \; i = 1,2,\cdots \ell, \tag{III-9}$$

and $L = \sum_{i=1}^{\ell} m_i$. If we consider the function

$$\tilde{\Lambda}(\nu) \equiv \frac{\Lambda(\nu)}{\prod_{i=1}^{\ell} (\mu_i^2 - \nu^2)^{m_i}}, \tag{III-10}$$

we see that $\tilde{\Lambda}(\nu)$ has no zeros on $[-\beta,\beta]$, is analytic, and has 2N' + 2L zeros outside of $[-\beta,\beta]$. Choosing arg $\tilde{\Lambda}_+(0) = 0$, and using the usual change of argument theorem, we find:

$$N' + L = \frac{1}{\pi}\Delta_{(0,\beta)} \arg \tilde{\Lambda}_+(\nu) = \frac{1}{\pi} \arg \tilde{\Lambda}_+(\beta) . \tag{III-11}$$

($\Delta_{(L)} \equiv$ change along L). If we label the zeros of $f_1(\mu)$ and $f_2(\mu)$ such that:

$$f_1^{(j-1)}(v_i) = 0, \; j = 1,2,\cdots m_i, \; i = 1,2,\cdots n_1$$

$$f_2^{(j-1)}(v_i) = 0, \; j = 1,2,\cdots m_i, \; i = n_1+1,\cdots n_2, \qquad \text{(III-12)}$$

and $2M = \sum_{i=1}^{n_1} m_i + \sum_{i=n_1+1}^{n_2} m_i$, then clearly $L \leq M$, as $[f_1(\mu)\; f_2(\mu)]^{\ell}_{\mu=\mu_i} = 0$ is a necessary condition for $\Lambda_+^{(\ell)}(\mu_i) = 0$. Also, as the $\mathrm{Im}\,\tilde{\Lambda}_+(\nu) = 0$, $0 < \nu < \beta$, M-L times, we have by Eq. (III-11) that $N'+L \leq M-L+1$. Throughout this chapter we will assume that N' and L are finite. A sufficient condition for this to be true is that M is finite.

With these properties of $\Lambda(\nu)$ we can now construct the solutions of Eq. (III-1). We will separate them into three classes.

Class I—Discrete Eigenvalues outside of $[-\beta,\beta]$

Corresponding to the 2n' eigenvalues outside of the interval $[-\beta,\beta]$, one finds the 2n' solutions:

$$\psi_{\nu_i}^{(0)}(x,\mu) = \frac{e^{-x/\nu_i}\,\nu_i f_1(\mu)}{\nu_i - \mu}, \quad \pm i = 1,2,\cdots n' \qquad \text{(III-13)}$$

where $\Lambda(\nu_i) = 0$, for $|\nu_i|$ finite. As the i^{th} zero is m_i degenerate, the 2n' eigenfunctions given by (III-13) are supplemented by:

$$\psi_{\nu_i}^{(j-1)}(x,\mu) = \frac{d^{(j-1)}}{d\nu^{(j-1)}}\,\psi_{\nu}^{(0)}(x,\mu)\Bigg|_{\nu=\nu_i}, \quad j = 2,3,\cdots m_i, \;\; \pm i = 1,2,\cdots n'. \qquad \text{(III-14)}$$

The 2N' solutions given by Eqs. (III-13) and (III-14) are linearly independent and by Eq. (III-5) clearly satisfy Eq. (III-1). For x = 0, these eigenfunctions will be denoted by $\phi_{\nu_i}^{(j-1)}(\mu)$.

If $\int_{-\beta}^{\beta} f_1(\mu)\, f_2(\mu) d\mu = 1$, then $\Lambda(\nu)$ has a double zero at infinity (assuming $\int_{-\beta}^{\beta} \mu^2 f_1(\mu)\, f_2(\mu) d\mu \neq 0$). In this case Eq. (III-1) has the $2(N'-1)$ solutions given by (III-13) and (III-14), and the two linearly independent solutions:

$$\psi_{\infty}^{(1)}(x,\mu) = f_1(\mu),\ \psi_{\infty}^{(2)}(x,\mu) = f_1(\mu)\,[x - \mu]\ . \tag{III-15}$$

Class II—Discrete Eigenvalues on $[-\beta,\beta]$

In our previous discussion we observed that in general we have 2ℓ eigenvalues on the interval $[-\beta,\beta]$ given by $\Lambda_+(\mu_i) = 0$. Corresponding to the zeros of $\Lambda_+(\mu)$, for $\mu = \mu_i = \nu_i$, $i = 1,2,\cdots n_1'$, $n_1' \leq n_1$ we have the set of $\sum_{i=1}^{n_1'} m_i$ linearly independent eigenfunctions:

$$\psi_{\mu_i}^{(j-1)}(x,\mu) = \frac{d^{(j-1)}}{d\nu^{(j-1)}} \left[e^{-x/\nu}\, \phi_{\nu}^{(o)}(\mu)\right]_{\nu=\mu_i},\quad \begin{array}{l} j = 1,2,\cdots m_i \\ i = 1,2,\cdots n_1' \end{array} \tag{III-16a}$$

where

$$\phi_{\mu_i}^{(o)}(\mu) = \frac{\mu_i f_1(\mu)}{\mu_i - \mu}\ . \tag{III-16b}$$

Similarly, corresponding to the zeros of $\Lambda_+(\mu)$, for $\mu = \mu_i = \nu_i$, $i = n_1+1,\cdots n_2'$, $n_1+1 \leq n_2' \leq n_2$, we have the set of $2\sum_{i=n_1+1}^{n_2'} m_i$ linearly independent eigenfunctions:

$$\psi_{\mu_i}^{\pm(j-1)}(x,\mu) = \frac{d^{(j-1)}}{d\nu^{(j-1)}} e^{-x/\nu} \phi_{\nu}^{(o)\pm}(\mu) \Big|_{\nu=\mu_i} , \quad \begin{matrix} j = 1,2,\cdots m_i \\ i = n_1+1,\cdots n_2' \end{matrix} \tag{III-17a}$$

where

$$\phi_{\mu_i}^{(o)\pm}(\mu) = \frac{\mu_i f_1(\mu)}{\mu_i \pm i\ \epsilon - \mu} . \tag{III-17b}$$

We will show in Section C that all of the solutions in Class II are linearly dependent on the solutions of Class III and hence are never explicitly needed.

Class III—Continuum Solutions

In addition to the solutions of Classes I and II which are associated with the zeros of $\Lambda(\nu)$, there are the continuum of solutions:

$$\psi_{\nu}(x,\mu) = e^{-x/\nu} \phi_{\nu}(\mu), \text{all } \nu \text{-}\beta \le \nu \le \beta \tag{III-18a}$$

where

$$\phi_{\nu}(\mu) = \nu f_1(\mu)\ P \frac{1}{\nu-\mu} + \lambda(\nu)\delta(\nu - \mu) \tag{III-18b}$$

corresponding to the branch cut of $\Lambda(\nu)$. By Eq. (III-4) we find:

$$\lambda(\nu)\ f_2(\nu) = \frac{\Lambda_+(\nu) + \Lambda_-(\nu)}{2} . \tag{III-19}$$

It should be noted that Eq. (III-19) defines $\lambda(\nu)$ only up to delta functions in $\nu-\mu_i$, $i = n_1+1,\cdots n_2$. Therefore, inplicit in the solutions (III-18) are the solutions:

$$\psi_{\mu_i}(x,\mu) = e^{-x/\mu_i}\ \delta(\mu - \mu_i) \quad i = n_1 + 1,\cdots n_2 . \tag{III-20}$$

These will always be explicitly factored out of the continuum solutions.

C. COMPLETENESS THEOREM

Theorem: The set of solutions of Class III for $\gamma_1 \leq \nu \leq \gamma_2$, $\gamma_1 < \gamma_2$, $-\beta \leq \gamma_1$, $\gamma_2 \leq \beta$, plus the solutions in Class I form a complete set of functions on the class of linear continuous functionals, $\psi(\mu)$, with compact support in $[\gamma_1, \gamma_2]$ and such that:

$$\lim_{\mu \to \gamma_{1,2}} \psi(\mu) \leq C|\mu - \gamma_{1,2}|^{-\epsilon}, \quad \epsilon < 1 \tag{III-21}$$

and for either (or both) γ_1, γ_2 infinite

$$\lim_{\substack{\mu \to \gamma_1 \\ \text{and/or} \\ \gamma_2 \to \infty}} \mu^k \, \psi(\mu) \, f_2(\mu) = 0 \tag{III-22}$$

for some k to be specified later.

In the following proof we will always assume that none of the zeros of $\Lambda_+(\mu)$ coincide with γ_1 or γ_2. This condition and the restructions implied by Eqs. (III-21) and (III-22) can be relaxed within the framework of distribution theory (14), however, for convenience and simplicity this will not be done here.

To prove the theorem we have to show that the singular integral equation

$$\psi(\mu) = \sum_{i,j} a_{ij} \, \phi_{\nu_i}^{(j-i)}(\mu) + \int_{\gamma_1}^{\gamma_2} A(\nu) \, \phi_\nu(\mu) d\nu, \tag{III-23}$$

has a solution when $\psi(\mu)$ is subject to the restrictions of the theorem. For the present it is required that γ_1 and γ_2 be finite.

Letting $\psi'(\mu) = \psi(\mu) - \sum a_{ij}\, \phi_{\nu_i}^{(j-1)}(\mu)$, Eq. (III-23) becomes:

$$\psi'(\mu) = f_1(\mu)\, P \int_{\gamma_2}^{\gamma_1} \frac{\nu A(\nu)\, d\nu}{\nu - \mu} + \lambda(\mu)\, A(\mu) + \sum_i b_i\, \delta(\mu - \mu_i). \qquad \text{(III-24)}$$

clearly the b_i are given by:

$$b_i = \lim_{\epsilon \to 0} \int_{\mu_i - \epsilon}^{\mu_i + \epsilon} d\mu \psi(\mu), \text{ all } \gamma_1 \leq \mu_i \leq \gamma_2 \quad . \qquad \text{(III-25)}$$

Assuming that $A(\nu)$ has the same properties as $\psi(\mu)$ the function

$$N(z) \quad \frac{1}{2\pi i} \int_{\gamma_1}^{\gamma_2} \frac{A(\nu)\, d\nu}{\nu - z} \qquad \text{(III-26)}$$

has the properties (15):

(a) $N(z)$ is analytic except on the interval $[\gamma_1, \gamma_2]$, and $N(z)$ has at most, polar behavior in this interval.

(b) $\lim_{z \to \infty} N(z) = 0$

(c) $\lim_{z \to \gamma_{1,2}} N(z) \leq C[z - \gamma_{1,2}]^{-\epsilon}$.

Multiplying Eq. (III-24) by $\mu f_2(\mu)$ and using Eq. (III-5), the expansion equation becomes the boundary value equation:

$$\gamma_1 \leq \mu \leq \gamma_2$$

$$\mu f_2(\mu)\, \psi'(\mu) = N_+(\mu)\, \Lambda_+(\mu) - N_-(\mu)\, \Lambda_-(\mu) \quad . \qquad \text{(III-27)}$$

Introducing

$$\tilde{N}(z) = \prod_j (\mu_j - z)^{m_j}\, N(z) \qquad \text{(III-28)}$$

and

$$\Lambda_1(z) = \frac{\Lambda(z)}{\prod_j (\mu_j - z)^{m_j}} \quad , \qquad \text{(III-29)}$$

(where μ_j is a zero of $\Lambda_+(\mu)$ of order m_j, and the product in (III-28) and (III-29) are over all j such that $\gamma_1 < \mu_j < \gamma_2$) into Eq. (III-27), the equation can be put in the conventional form:

$$\gamma(\mu)\, \psi'(\mu) = N_+(\mu)\, X_+(\mu) - N_-(\mu)\, X_-(\mu) \quad , \qquad \text{(III-30)}$$

$$\gamma(\mu) = \frac{X_-(\mu)\; \mu f_2(\mu)}{\Lambda_{1-}(\mu)} \quad , \text{ and } X(z)$$

has the following properties:

(a) X(z) is sectionally holomorphic with boundary (γ_1, γ_2).

(b) X(z) is non-zero in the cut plane.

(c) $\dfrac{X_+(\mu)}{X_-(\mu)} = \dfrac{\tilde{\Lambda}_{1+}(\mu)}{\tilde{\Lambda}_{1-}(\mu)} = \dfrac{\tilde{\Lambda}_+(\mu)}{\tilde{\Lambda}_-(\mu)} \equiv G(\mu)$.

(d) X(z) vanishes more slowly than $|z-\gamma_1|$ and $|z-\gamma_2|$ as $z \to \gamma_1, \gamma_2$ respectively.

(e) $\lim_{z\to\infty} X(z)$ = Class (1) const.

Class (2) z^{k_1}, k_1 positive integer.

Class (3) z^{-k_2}, k_2 positive integer.

The problem of finding X(z) subject to these conditions is the "classical" homogeneous Hilbert problem (13). The solution is:

$$X(z) = (z-\gamma_1)^{t_1} (z-\gamma_2)^{t_2} e^{\Gamma(z)} , \qquad \text{(III-31)}$$

$$\Gamma(z) = \frac{1}{2\pi i} \int_{\gamma_1}^{\gamma_2} \frac{\ell n\, G(\mu) d\mu}{\mu - z} , \qquad \text{(III-32)}$$

and t_1, t_2 are integers chosen such that

$$1 > t_2 + \frac{\Theta(\gamma_2)}{\pi} \geq 0$$

$$1 > t_1 - \frac{\Theta(\gamma_1)}{\pi} > 0, \; \Theta(\mu) = \arg \Lambda_+(\mu) \; . \qquad \text{(III-33)}$$

For the entire-space and half-space problems one can easily obtain t_1 and t_2 from Eq. (III-11). They are listed below.

TABLE III

t_1 and t_2

γ_1	γ_2	t_1	t_2	$t_1+t_2 = -k_2$
$-\beta$	β	$-(N'+L)$	$-(N'+L)$	$-2(N'+L)$
0	β	0	$-(N'+L)$	$-(N'+L)$
$-\beta$	0	$-(N'+L)$	0	$-(N'+L)$

If $N'+L > 0$, then X(z) for the half-space and entire-space problems belong to Class (3). Solving Eq. (III-30) for this class, we find:

$$N(z) = \frac{P_{r-k_2-1}(z)}{X(z) \prod_j (\mu_j - z)^{mj}} + \frac{1}{2\pi i \; X(z) \prod_j (\mu_j - z)^{mj}} \int_{\gamma_1}^{\gamma_2} \frac{d\mu \; \gamma(\mu) \psi'(\mu)}{\mu - z} \; . \qquad \text{(III-34)}$$

Here $P_\ell(z)$ is an arbitrary polynomial of degree ℓ, and $r = \sum_j mj$. This is a solution to (III-30) only if the k_2-r additional conditions:

$$\int_{\gamma_1}^{\gamma_2} \mu^\ell \gamma(\mu) \Psi'(\mu) d\mu = 0, \; \ell = 0,1,\cdots k_2 - r - 1 \qquad \text{(III-35)}$$

are satisfied. For the entire-space and half-space problems as r = 2L, $k_2 = 2(N'+L)$; r = L, $k_2 = N'+L$, respectively, we find that $P_{r-1-k_2}(z) \equiv 0$ in both cases.

To satisfy (III-35) we include k_2-r discrete eigenfunctions of Class I in the sum contained in $\psi(\mu)$. As $k_2 - r \leq 2N'$ there are always enough eigenfunctions to do this. Even so, it still must be shown that Eq. (III-35) does not impose any restrictions on $\psi(\mu)$ other than those stated in the theorem. Using the representation of X(z) derived in Appendix D it will be shown that (III-35) is consistent with the theorem. Before proceeding it should be remarked that for X(z) belonging to either Class (1) or (2), an N(z) can be found which satisfies (III-30) without restrictions such as (III-35). In these cases no discrete eigenfunctions are needed and $A(\nu)$ is easily found to satisfy the assumed conditions. Therefore, in the following, only X's belonging to Class (3) will be considered Eq. (III-35) is explicitly:

$$\int_{\gamma_1}^{\gamma_2} \mu^\ell \gamma(\mu) \Psi(\mu) d\mu = \sum_{i,j} a_{i_j} \int_{\gamma_1}^{\gamma_2} \mu^\ell \gamma(\mu) \phi_{\nu_i}^{(j-1)}(\mu) d\mu, \qquad \text{(III-36)}$$

$\ell = 0,1,\cdots k_2$-r-1, and the sum is over k_2-r a_{ij}'s. In order to express

(III-36) in its simpliest form, we first consider the integral

$$I_{\ell,i}^{(j-1)} = \int_{\gamma_1}^{\gamma_2} \mu^{\ell}\gamma(\mu)\, \phi_{\nu_i}^{(j-1)}(\mu)d\mu, \qquad \text{(III-37)}$$

which is:

$$I_{\ell,i}^{(j-1)} = \frac{d^{(j-1)}}{d\nu^{(j-1)}} - \nu \int_{\gamma_1}^{\gamma_2} \frac{\mu^{\ell}\gamma(\mu)\; f_1(\mu)d\mu}{\mu-\nu}\Bigg|_{\nu=\nu_i} . \qquad \text{(III-38)}$$

Using the representation of X(z) given by (D-3), (III-38) becomes:

$$I_{\ell,i}^{(j-1)} = \frac{d^{(j-1)}}{d\nu^{(j-1)}} - \frac{\nu}{2\pi i} \int_{\gamma_1}^{\gamma_2} \frac{\mu^{\ell}d\mu}{\mu-\nu}\prod_j(\mu_j-\mu)^{mj}\,[X_+(\mu) - X_-(\mu)] \qquad \text{(III-39)}$$

or

$$I_{\ell,i}^{(j-1)} = \frac{d^{(j-1)}}{d\nu^{(j-1)}} - \nu^{\ell+1}\prod_j(\mu_j-\nu)^{mj}X(\nu)\Bigg|_{\nu=\nu_i} . \qquad \text{(III-40)}$$

If $\nu=\infty$ is a zero of $\Lambda(\nu)$ we may want or have to include $\phi_{\infty}^{(1)}(\mu)$ and $\phi_{\infty}^{(2)}(\mu)$ in the sum over the discrete solutions. In this case we have to calculate the integrals:

$$I_{\ell,\infty}^{(o)} = \int_{\gamma_1}^{\gamma_2} \mu^{\ell}\gamma(\mu)\; \phi_{\infty}^{(1)}(\mu) = \int_{\gamma_1}^{\gamma_2} \mu^{\ell}\gamma(\mu)\; f_1(\mu)d\mu \qquad \text{(III-41a)}$$

and

$$I_{\ell,\infty}^{(1)} = \int_{\gamma_1}^{\gamma_2} \mu^{\ell}\gamma(\mu)\; \phi_{\infty}^{(2)}(\mu)d\mu \leq - \int_{\gamma_1}^{\gamma_2} \mu^{\ell+1}\gamma(\mu)\; f_1(\mu)d\mu.$$

(III-41b)

By the same method as above we have:

$$\int_{\gamma_1}^{\gamma_2} \mu^t \gamma(\mu)\; f_1(\mu) d\mu \;=\; \frac{1}{2\pi i} \int_C z^t \prod_j (\mu_j - z)^{mj}\; X(z) dz, \qquad \text{(III-42)}$$

where C encircles the (γ_1, γ_2) cut in the negative direction. We have therefore:

$$\begin{aligned} \int_{\gamma_1}^{\gamma_2} \mu^t \gamma(\mu)\; f_1(\mu) d\mu &= 0, \; t < k_2 - r - 1 \\ &= (-)^{r+1} \quad t = k_2 - r - 1 \qquad \text{(III-43)} \\ &= C \equiv -\lim_{z\to\infty} z[z^{k_2 - r} X(z) \prod (\mu_j - z)^{mj} - (-)^r] \\ &\qquad \text{if} \quad t = k_2 - r. \end{aligned}$$

Using the representation of X(z) given by (III-31) we find:

$$C \;=\; (-)^r [\gamma_1 t_1 + \gamma_2 t_2 + \sum_j \mu_j + \frac{1}{2\pi i} \int_{\gamma_1}^{\gamma_2} d\mu \; \ell n \; G(\mu)] \; . \qquad \text{(III-44)}$$

It is now evident that the integral $I_{\ell,i}^{(j-1)}$ is never zero for all i and all j. Therefore the conditions (III-36) can be satisfied without further restrictions on $\psi(\mu)$. It will be shown in Section D that in fact the a_{ij}'s can be uniquely determined.

We also note that the linear dependence of the solutions of Class II on Class III is now evident. Letting $\psi(\mu) = \phi_{\mu_i}^{(j-1)}(\mu)$, it is easy to show that the integral

$$\int_{\gamma_1}^{\gamma_2} \mu^{\ell} \gamma(\mu)\; \phi_{\mu_i}^{(j-1)}(\mu) d\mu \;=\; 0, \; \ell = 0, 1, \cdots k_2 - r - 1. \; \text{(III-45)}$$

Hence, we can choose $a_{ij} = 0$, and the linear dependence follows. Furthermore it is evident from $N(z)$ given by Eq. (III-34) that $\nu A(\nu) = N_+(\nu) - N_-(\nu)$ has a principle value part and a delta function contribution at $\nu = \mu_i$. Putting this $A(\nu)$ into the expansion Eq. (III-24) and factoring the delta functions out of the integral we obtain the usual expansion expression (7) which from the beginning includes all of the discrete eigenvalues, and not the continuum solutions where there is a discrete eigenvalue in the continuum. (The principle value contribution at $\mu = \mu_i$.)

For the entire space problem we can construct $X(z)$ by inspection. The solution is given, up to constant, by:

$$X(z) = \frac{\tilde{\Lambda}(z)}{\prod_{i=1}^{n'} (\nu_i^2 - z^2)^{m_i}} \quad . \qquad \text{(III-46)}$$

It has been specified that γ_1 and γ_2 be finite in the above procedure. It is clear that if $\beta = \infty$, the proof is still true with suitable restrictions on $\Psi(\mu)$. For $\gamma_1 = -\infty$, $\gamma_2 = +\infty$, the solution for $X(z)$ is given by (III-46). If only one end point is finite, say γ_1, the end point condition for $X(z)$ is applied only to the finite end γ_1. $X(z)$ is then given by:

$$X(z) = (z - \gamma_1)^{t_1} e^{\Gamma(z)} \qquad \text{(III-47)}$$

and

$$\Gamma(z) = \frac{1}{2\pi i} \int_{\gamma_1}^{\infty} \frac{\ell n\ G(\mu)\, d\mu}{\mu - z} , \qquad \text{(III-48)}$$

where $\ell n\ G(\mu)$ is that branch which vanishes at infinity. It is clear that N(z) will exist with the correct properties if in (III-2) we choose $k > 2L + 2 + k_1$ (k_1 given by Class (2)) or $k > 2$, if X(z) belongs to Class (1) or (2).

It is now evident that $A(\nu)$, given by $\nu A(\nu) = N_+(\nu) - N_-(\nu)$ does satisfy the assumed conditions. Therefore, the completeness theorem has been proven.

D. APPLICATIONS TO HALF-SPACE PROBLEMS

In this section the theorem of Section C will be used to obtain solutions to Eq. (III-1) for $0 \leq x \leq \infty$. For simplicity it is assumed that $\nu_i \neq \infty$, $N' = n' > 0$, $L = 0$, and $f_2(\mu) \neq 0$ for $0 < \mu \leq \beta$, except possibly at $\mu = \infty$ if $\beta = \infty$.

We first consider the albedo problem, where we want to find a solution to (III-1) for $0 \leq x \leq \infty$ subject to the boundary condition:

$$\psi(0,\mu) = \delta(\mu - \mu_0), \quad 0 < \mu < \beta, \quad 0 < \mu_0 < \beta \tag{III-49}$$

and

$$\lim_{x \to \infty} \psi(x,\mu) = 0 \quad . \tag{III-50}$$

Using the theorem of Section C we can write:

$$\psi(x,\mu,\mu_0) = \sum_i a_i \, \psi_{\nu_i}^{(0)}(x,\mu) + \int_0^\beta A(\nu) \, \psi_\nu \, (x,\mu) d\nu \; . \tag{III-51}$$

The sum is over N' discrete eigenfunctions and (III-50) implies that the sum is over $i = 1,2,\cdots N'$. Condition (III-49) gives: $\mu > 0$

$$\delta(\mu - \mu_o) = \sum_{i=1}^{N'} a_i \, \phi_{\nu_i}^{(o)}(\mu) + \int_0^\beta A(\nu)\, \phi_\nu(\mu) d\nu \, . \quad \text{(III-52)}$$

The a_i's can be determined by the N' condition of Eq. (III-35). The equations are explicitly:

$$\mu_o^\ell \gamma(\mu_o) = \sum_{i=1}^{N'} a_i \, I_{\ell_i i}^{(o)} \, , \; \ell = 0, \cdots N'-1 \, . \quad \text{(III-53)}$$

It is easy to show that

$$\text{Det } I_{\ell,i}^{(o)} = (-)^{N'} \prod_{i=1}^{N'} \nu_i X(\nu_i) \prod_{\ell > j = 1}^{N'} (\nu_\ell - \nu_j), \quad \text{(III-54)}$$

which is clearly non-zero. Using Cramer's rule the a_i's are found to be:

$$a_i = - \frac{\gamma(\mu_o)}{\nu_i X(\nu_i)} \prod_{j \neq i}^{N'} \frac{\mu_o - \nu_j}{\nu_i - \nu_j} \quad i = 1, \cdots N' \, . \quad \text{(III-55)}$$

It should be mentioned that if v_i is m_i degenerate ($N' \neq n'$) Eq. (III-55) becomes

$$\mu_o^\ell \gamma(\mu_o) = \sum_{i=1}^{N'} b_i u_{i\ell} \, , \quad \text{(III-56)}$$

here

$$bj = a_{1j}, \; j = 1,2, \cdots m_1$$

$$= a_{2j-m_1}, \; j = m_1+1, \cdots m_2+m_1, \text{ etc.},$$

and

$$\det u_{ij} = (-)^{N'} \prod_{i=1}^{n'} [\nu_i X(\nu_i)]^{m_i} \prod_{\ell > j = 1}^{n'} (\nu_\ell - \nu_j)^{m_\ell m_j} \quad \text{(III-57)}$$

which is never zero and therefore the a_{ij}'s can be uniquely determined in this case also.

In applications we normally want to know the discrete eigenfunction coefficients, because they usually give the asymptotic form of $\psi(x,\mu)$, and the emerging "angular" distribution $\psi(0,\mu)$, $\mu < 0$. The emerging "angular" distribution for the albedo problem is:

$$\psi(0,\mu;\mu_0) = \sum_{i=1}^{N'} a_i \, \phi_{\nu_i}^{(o)}(\mu) + \int_0^{\beta} A(\nu) \, \phi_\nu(\mu) d\nu, \quad \mu < 0. \quad \text{(III-53)}$$

For $\mu < 0$, $\phi_\nu(\mu) = \dfrac{\nu f_1(\mu)}{\nu-\mu}$ and Eq. (III-53) becomes

$$\psi(0,\mu;\mu_0) = \sum_{i=1}^{N'} a_i \, \phi_{\nu_i}^{(o)} (\mu) + 2\pi i f_1(\mu) \, N(\mu). \quad \text{(III-54)}$$

Expressing $N(\mu)$ for $\mu < 0$ by using the representation of $X(z)$ given by Eq. (D-3) we obtain:

$$N(\mu) = \frac{1}{2\pi i X(\mu)} \left[\frac{\gamma(\mu_0)}{\mu_0-\mu} + \sum_{i=1}^{N'} \frac{a_i \, \nu_i}{\nu_i-\mu} \, [X(\mu) - X(\nu_i)] \right]. \quad \text{(III-55)}$$

The emerging "angular" distribution now reduces to: $\mu < 0$

$$\psi(0,\mu;\mu_0) = \frac{\gamma(\mu_0) \, f_1(\mu)}{X(\mu)} \left[\frac{1}{\mu_0-\mu} - \sum_{i=1}^{N'} \frac{1}{\nu_i-\mu} \prod_{j\neq i} \frac{\mu_0-\nu_j}{\nu_i-\nu_j} \right]. \quad \text{(III-56)}$$

The Milne problem can be readily solved by using the formulas derived for the albedo problem. In the Milne problem we want to find the solution to Eq. (III-1) subject to the boundary conditions: $(\nu_{N'} \neq \infty)$

$$\lim_{x\to\infty} \psi(x,\mu) = \Psi^{(o)}_{-\nu_i}(x,\mu) \qquad \mathrm{Re}\ \nu_i > 0 \qquad \text{(III-57)}$$

and

$$\psi(0,\mu) = 0, \quad \mu \geq 0 \ . \qquad \text{(III-58)}$$

Using the completeness theorem we can expand $\psi(x,\mu)$ subject to (III-57) as:

$$\psi(x,\mu) = \Psi^{(o)}_{-\nu_i}(x,\mu) + \sum_{i=1}^{N'} a_i \Psi^{(o)}_{\nu_i}(x,\mu) + \int_0^\beta A(\nu)\, \Psi_\nu(x,\mu)d\nu \qquad \text{(III-59)}$$

and then (III-58) becomes: $\mu \geq 0$,

$$-\phi^{(o)}_{-\nu_i}(\mu) = \sum_{j=1}^{N'} a_j \phi^{(o)}_{\nu_j}(\mu) + \int_0^\beta A(\nu)\, \phi_\nu(\mu)d\nu. \qquad \text{(III-60)}$$

In order to calculate the a_i's and the emerging "angular" distribution we only need integrate the albedo solutions times $-\phi^{(o)}_{-\nu}(\mu_o)$ over μ_o. This can again be done using (D-3) and we find:

$$a_k = \frac{\nu_i\, X(-\nu_i)}{\nu_k\, X(\nu_k)} \prod_{j\neq k} \frac{\nu_i + \nu_j}{\nu_j - \nu_k} \quad , \ k = 1,2,\cdots N' \ , \qquad \text{(III-61)}$$

and the emerging "angular" distribution is: $\mu < 0$

$$\psi(0,\mu) = \frac{X(-\nu_i)}{X(\mu)} \left[\phi_{-\nu_i}(\mu) + \nu_i f_i(\mu) \sum_{j=1}^{N'} \frac{1}{\nu_j - \mu} \prod_{k\neq j} \frac{\nu_i + \nu_k}{\nu_k - \nu_j} \right] . \qquad \text{(III-62)}$$

Before computing some of the moments of the emerging distribution for the Milne problem we first derive some useful identities for $X(z)$.

E. IDENTITIES FOR X(z)

In Section D we showed that the representation (D-3) for X(z) allows the usual combersome integrals involved in a half-space problem to be reduced to a form which involved only X(z) and other, in principle, known functions. In this section we will derive two more identities for X(z), both of which may serve as a means to numerically determined X(z), or as will be shown by way of an example in Section F, permit some quantities of interest to be trivally determined.

Defining $C = \int_{-\beta}^{\beta} f_1(\mu)\ f_2(\mu)d\mu$, and for convenience assuming $C \neq 1$, we want to prove:

$$X(z)X(-z) = \frac{\Lambda(z)}{(1-C)\prod_{i=1}^{N'}(\nu_i^2-z^2)}, (N' = n', L = 0) . \qquad \text{(III-63)}$$

To prove this consider the function

$$R(z) = \frac{\Lambda(z)}{[1-C]X(z)X(-z)\prod_{i=1}^{N'}(\nu_i^2-z^2)} . \qquad \text{(III-64)}$$

It is clear that R(z) is analytic in the plane cut from $-\beta$ to β. Across the cut $R_+(\mu)/R_-(\mu) = 1$, and also $\lim_{z\to\infty} R(z) = 1$. Therefore, $R(z) \equiv 1$, and the identity (III-63) is proved. From (III-63) we see immediately:

$$X^2(0) = \frac{1}{[1-C]}\frac{1}{\prod_{i=1}^{N'}\nu_i^2} . \qquad \text{(III-65)}$$

The phase of X(0) is uniquely determined by the fact that $\lim_{z \to -\infty} X(z) = (-)^{N'} |z|^{-N'}$ (z real) and that X(z) is a non-zero, real function of z. Therefore

$$X(0) = (-)^{N'} [(1-C) \prod_{i=1}^{N'} \nu_i^2]^{-1/2} . \qquad \text{(III-66)}$$

A particularly simple result of this is that $[1-C] \prod_{i=1}^{N'} \nu_i^2$ must be real and non-negative. Of course, if C = 1, an identity similar to (III-63) can be derived.

The identity (III-63) may be useful for calculations X(z) for large z. Expanding both sides of Eq. (III-63) in powers of z^{-1}, and equating coefficients, one finds a set of coupled albegraic equations for the coefficients of z^{-n} in X(z). In particular cases a truncated solution may rapidly converge and the coefficients of low inverse powers of z be easily obtainable.

To obtain the second useful identity for X(z), we use the representation of X(z) given by (D-3) and the identity (III-63). We find:

$$X(z) = \frac{1}{[1-C]} \int_{-\beta}^{0} \frac{\mu d\mu \; f_1(\mu) \; f_2(\mu)}{(\mu+z) \; X(\mu) \prod_{i=1}^{N'} (\nu_i^2-\mu^2)} . \qquad \text{(III-67)}$$

From (III-67) we first notice that if we know $X(\mu)$ for $-\beta \leq \mu \leq 0$, we can readily determine X(z) everywhere. Such being the case, by letting $z = \mu$, $-\beta \leq \mu \leq 0$ in (III-67) we obtain a non-linear integral for $X(\mu)$, $-\beta \leq \mu \leq 0$, for which an iterative solution may rapidly converge. More important is the property that (III-67) allows us to express integrals over $X(\mu)^{-1}$ (which always occur in the emerging "angular" distributions) in terms of $X(z)$, and it is this property that we will exploit in the next section.

F. AN APPLICATION

To illustrate the usefulness of the identities derived in the preceding section we will calculate some moments of the emerging "angular" distribution for the Milne problem. While moments for the general emerging "angular" distribution of Eq. (III-62) can be at least simplified by the identities of Section E we will only consider the usual case; N' = 1.

The emerging "angular" distribution now reduces to:

$$\psi(0,\mu) = \frac{2\nu_1^2 X(-\nu_1)\ f_1(\mu)}{X(\mu)\ (\nu_1^2-\mu^2)} \quad . \qquad \text{(III-68)}$$

The emerging "density" is then given by:

$$\rho_-(0) = \int_{-\beta}^{0} d\mu f_2(\mu)\ \psi(0,\mu) = 2\nu_1^2 X(-\nu_1) \int_{-\beta}^{0} \frac{f_1(\mu)\ f_2(\mu)\ d\mu}{X(\mu)\ (\nu_1^2-\mu^2)} \quad . \qquad \text{(III-69)}$$

However, the integral in (III-69) is simply $X(0)[1 - C]$ (from III-67), and so we find:

$$\rho_-(0) = -2X(-\nu_1)\ [\nu_1^2(1 - C)]^{1/2} \ . \qquad \text{(III-70)}$$

The emerging "current" is:

$$j_-(0) = \int_{-\beta}^{0} d\mu\ \mu f_2(\mu)\ \psi(0,\mu) = 2\nu_1{}^2 X(-\nu_1) \int_{-\beta}^{0} \frac{\mu f_1(\mu)\ f_2(\mu) d\mu}{X(\mu)\ (\nu_1{}^2-\mu^2)} . \qquad \text{(III-71)}$$

From (III-67) we immediately see that the integral in (III-71) is:

$$\lim_{z\to\infty} [1 - C]\ z\ X(z) = (1 - C) . \qquad \text{(III-72)}$$

Therefore we have:

$$j_-(0) = 2\nu_1{}^2 X(-\nu_1)(1 - C) . \qquad \text{(III-73)}$$

We note that the average emerging μ for the half-space problem is given by:

$$\bar{\mu} = \frac{j_-(0)}{\rho_-(0)} = - [\nu_1{}^2(1 - C)]^{1/2} . \qquad \text{(III-74)}$$

Unfortunately, $\overline{u^n}$ for $n > 1$ is not so easily calculable, however the identities of Section E do help to simplify the integrals for these higher moments. For a more illustrative example of the usefulness of the identities we refer to Reference (16).

G. CONCLUSION

While we have considered only the Equation (III-1) it is clear that for any equation or system of equations which have a spectrum similar to Eq. (III-1) and for which a completeness theorem of the form of Section C is valid there are similar identities for the solution of the associated homogeneous Hilbert problem for the half-space. The judicious use of these identities will always simplify the formulas of interest.

CHAPTER IV

DENSITY IN A SIMPLE MODEL OF THE EXOSPHERE

A. DISCUSSION

We consider the following simple model of the planetary exosphere: Exterior to a sphere of radius r_0 we have a gas so rarified that collisions may be neglected. The only force acting on the particles then is the gravitational force, due to the total mass M within r_0. Within the sphere collisions are so frequent that particles emerging from $r = r_0$ have a Maxwell-Boltzmann velocity distribution. The problem is to determine the particle density in the region $r \geq r_0$ subject to the condition that there are no particles present which have not come from within the sphere.

This problem has been treated by Opik and Singer (17) and Brandt and Chamberlain.(18) The first authors find particles densities by straight forward kinetic theory calculation of the numbers of particles which reach a given point in space. The second author start from the collisionless Boltzmann (i.e., Liouville) equation. Since the latter is merely a statement of Newton's laws of motion the two approaches should agree. However, there seems to be some confusion on this point.

In order to clarify the situation we construct a simple explicit solution of the Boltzmann equation subject to the given boundary conditions. From this the particle density is trivially obtained by quadratures.

B. CONSTRUCTION OF THE SOLUTION

The distribution function, $\psi(\vec{r},\vec{v})$, in the exosphere is to satisfy the collisionless Boltzmann equation:

$$\left(\vec{v}\cdot\nabla_{\vec{r}} + \frac{\vec{F}}{m}\cdot\nabla_{\vec{v}}\right)\psi(\vec{r},\vec{v}) = 0 \quad , \qquad \text{(IV-1)}$$

where

$$\vec{F} = -\frac{G\,Mm\,\vec{r}}{r^3} \quad . \qquad \text{(IV-2)}$$

(Here m is the mass of the gas molecules, and $\vec{r}$ is the radius vector from the center of the sphere.) At $r = r_0$ we have the boundary condition that the emerging distribution has the Maxwell form,

i.e. $$\psi(\vec{r}_0,\vec{v}) = N(m\beta/2\pi)^{3/2} \exp-[\beta m\vec{v}^2/2] \quad \text{for } \vec{v}\cdot\vec{r}_0 > 0. \qquad \text{(IV-3)}$$

$$(\beta = 1/kT)$$

Further we have the condition that all particles exterior to the sphere shall have come from within it.

Since the problem has spherical symmetry we know that

$$\psi(\vec{r},\vec{v}) = \psi(r,v,\mu) \quad , \qquad \text{(IV-4)}$$

where

$$r = |\vec{r}|,\ v = |\vec{v}|, \text{ and } \mu = (\vec{r}\cdot\vec{v})/rv \quad . \qquad \text{(IV-5)}$$

In terms of these coordinates Eq. (IV-1) becomes

$$\left\{v\mu\frac{\partial}{\partial r} - \frac{GM\mu}{r^2}\frac{\partial}{\partial v} + \left(\frac{v}{r} - \frac{G\,M}{vr^2}\right)(1-\mu^2)\frac{\partial}{\partial\mu}\right\}\psi(r,v,\mu) = 0 \quad . \qquad \text{(IV-6)}$$

The method of characteristics (<u>19</u>) shows that the only content of Eq. (IV-6) is that ψ is to be an arbitrary function of E and $\vec{L}^2$. Thus $\psi = \psi(E, \vec{L}^2)$ where

$$E = \frac{1}{2} m\vec{v}^2 - \frac{GMm}{r} \quad , \tag{IV-7}$$

and

$$\vec{L}^2 = m^2 v^2 r^2 (1 - \mu^2) \quad . \tag{IV-8}$$

i.e., ψ depends only on the constants of motion—which are the energy and the angular momentum.

We still have to fit the boundary conditions. To do this we note there is no reason for ψ to be a <u>continuous</u> or <u>singlevalued</u> function of its arguments. Consider, therefore, the function

$$\psi(r,v,\mu) = N(m\beta/2\pi)^{3/2} \exp{-\beta[GMm/r_o + E]}$$
$$\cdot\ \theta(E + GMm/r_o - \vec{L}^2/2mr_o^2)\ [1 - \theta\ (E)\ \theta\ (-\mu)] \quad . \tag{IV-9}$$

Here

$$\theta(x) = \begin{array}{ll} 1 & x > 0 \\ 0 & x < 0 \quad . \end{array} \tag{IV-10}$$

The significance of the θ functions is readily seen. Consider first a particle at r_o with energy E. We have

$$\vec{L}^2/2mr_o^2 = (E + GMm/r_o)(1 - \mu^2) \quad . \tag{IV-11}$$

Hence

$$\vec{L}^2/2mr_o^2 \leq (E + GMm/r_o) \quad . \tag{IV-12}$$

Thus, for $\mu > 0$ the function of Eq. (IV-9) does reduce to that of Eq. (IV-3). Further, the factor $\theta(E+GMm/r_o-\vec{L}^2/2mr_o^2)$ guarantees that we have no particles whose orbits do not intersect the sphere of radius r_o. The remaining factor

$$1 - \theta(E)\ \theta(-\mu)$$

arises from the requirement that there be no particles incident from infinity. For $E < 0$ the particle orbits do not reach to infinity and the factor is 1. However, for those orbits which reach to infinity the factor $\theta(-\mu)$ guarantees them to be outgoing.

The function of Eq. (IV-9) thus satisfies all the boundary conditions. It only remains to see whether it satisfies the Boltzmann equation. Except for the dependence on μ this is trivial (since it is a function of the constants of motion). Thus on inserting the function of Eq. (IV-9) into Eq. (IV-6) we need only worry about terms arising from differentiation of $\theta(-\mu)$. Using the result

$$\frac{\partial}{\partial\mu}\theta(-\mu) = -\delta(\mu) \qquad \text{(IV-13)}$$

(where δ denotes the Dirac delta function) we obtain

$$\left(v \cdot\nabla_{\vec{r}} + \frac{\vec{F}}{m}\cdot\nabla_{\vec{v}}\right)\psi$$

$$= N(m/2\pi)^{3/2}\exp-\beta\left[\frac{GMm}{r_o} + E\right]\theta(E)\ \theta(E + GMm/r_o - \vec{L}^2/2mr_o^2) \qquad \text{(IV-14)}$$

$$\cdot\left(\frac{v}{r} - \frac{GM}{vr^2}\right)(1-\mu^2)\ \delta(\mu) \quad .$$

We note the identity

$$\delta(\mu)\ \Theta(E)\ \Theta(E + GMm/r_o - \vec{L}^2/2mr_o^2)$$

$$= \delta(\mu)\ \Theta(E)\ \Theta\left\{\left[E(1 + \frac{r}{r_o}) + \frac{GMm}{r_o}\right]\left[1 - \frac{r}{r_o}\right]\right\} . \qquad \text{(IV-15)}$$

But for $r > r_o$ the argument of the second step function is negative (for E positive). Hence the function of Eq. (IV-15) is identically zero and the Boltzmann equation is satisfied.

We conclude that Eq. (IV-9) does indeed yield a function satisfying all requirements.

C. PARTICLE DENSITY

The calculation of the total position density in the exosphere is now straightforward.

$$\rho(r) = \int d\vec{v}\ \psi(\vec{r},\vec{v}) = 2\pi \int_{-1}^{1} d\mu \int_{0}^{\infty} v^2 dv\ \psi(r,v,\mu) \quad . \text{(IV-16)}$$

The result is:

$$\rho(r) = 2N/(\pi)^{1/2} \exp \alpha(1 - 1/x) \left\{ \int_0^\infty y^2 dy\ e^{-y^2} + \int_0^{\alpha^{1/2}} y^2 dy\ e^{-y^2} \right.$$

$$- (1 - x^2)^{1/2} \left[\int_{\left(\frac{\alpha x}{1+x}\right)^{1/2}}^{\infty} dy\ y^2\ e^{-y^2} \left(1 - \frac{\alpha\ x}{y^2\ (1+x)}\right)^{1/2} \right.$$

$$\left.\left. + \int_{\left(\frac{\alpha x}{1+x}\right)^{1/2}}^{\alpha^{1/2}} dy\ y^2\ e^{-y^2} \left(1 - \frac{\alpha\ x}{y^2\ (1+x)}\right)^{1/2} \right] \right\} \qquad \text{(IV-17)}$$

where

$$\alpha = \beta GMm/r, \qquad x = r_o/r$$

This agrees (up to constants) with the "ballistic density" calculated by Opik and Singer (17) in the 1961 reference (p. 226, formula 31). The expression can be considerably simplified and written in terms of Error functions,

$$\Phi(x) = \left(\frac{4}{\pi}\right)^{1/2} \int_0^x e^{-y^2}\, dy \quad , \tag{IV-18}$$

as

$$\rho(r) = \frac{N}{2} \exp \alpha(1 - 1/x)$$

$$\cdot \left\{ 1 + \Phi(\alpha^{1/2}) - (1 - x^2)^{1/2} \exp(-\alpha x/1 + x) \left[1 + \Phi\left(\left(\frac{\alpha}{1 + x}\right)^{1/2}\right)\right]\right.$$

$$\left. + \left(\frac{4\alpha}{\pi}\right)^{1/2} e^{-\alpha} \left[(1 - x)^{1/2} - 1\right]\right\} \quad . \tag{IV-19}$$

D. CONCLUSION

It has been shown, as might be expected, that there is no difficulty in writing down the solution of the collisionless Boltzmann equation for our simple model of the exosphere. This method, while completely equivalent to any other solution of the problem based on Newtonian mechanics, is probably the quickest and possibly most elegant approach.

APPENDIX A

DERIVATION OF THE INTEGRAL EQUATION (II-2)

To derive the integral equation given by (II-2) we start from the Equation (II-1) with the appropriate cross sections for the model. First consider the integral in the exponent,

$$\int_0^{|\vec{r}-\vec{r}\,'|} \sigma\left(\vec{r} - \frac{(\vec{r}-\vec{r}\,')s}{|\vec{r}-\vec{r}\,'|}\right) ds = \sigma \int_0^{|\vec{r}-\vec{r}\,'|} \frac{ds}{\sqrt{r^2 - 2\vec{r}\,\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}s + s^2}} \,. \quad \text{(A-1)}$$

This integral can be done and is:

$$\int_0^{|\vec{r}-\vec{r}\,'|} \sigma\left(\vec{r} - \frac{(\vec{r}-\vec{r}\,')s}{|\vec{r}-\vec{r}\,'|}\right) ds = \sigma\, \ell n \left\{ \frac{r'|\vec{r}-\vec{r}\,'| - \vec{r}\,'\cdot(\vec{r}-\vec{r}\,')}{r|\vec{r}-\vec{r}\,'| - \vec{r}\cdot(\vec{r}-\vec{r}\,')} \right\} . \quad \text{(A-2)}$$

Substituting this back into (II-1), for the shell source, we have:

$$\rho(r) = \frac{1}{2} \int_0^\infty r'^2 dr' \int_{-1}^{1} d\mu \left[\frac{r'|\vec{r}-\vec{r}\,'| - \vec{r}\,'\cdot(\vec{r}-\vec{r}\,')}{r|\vec{r}-\vec{r}\,'| - \vec{r}\cdot(\vec{r}-\vec{r}\,')} \right]^{-\sigma} \frac{1}{|\vec{r}-\vec{r}\,'|^2}$$

$$* \left\{ \frac{C\sigma}{r'} \rho(r') + \frac{\delta(r'-r_o)}{4\pi v\, r_o^2} \right\} . \quad \text{(A-3)}$$

Letting $X = |\vec{r}-\vec{r}\,'|$, $X^2 = r^2 + r'^2 - 2r\, r'\mu$, and then $X = (r+r')y$, Eq. (A-3) becomes:

$$\rho(r) = \frac{1}{2} \int_0^\infty \frac{r' dr'}{r} \left\{ \frac{C\sigma}{r'} \rho(r') + \frac{\delta(r'-r_o)}{4\pi v\, r_o^2} \right\} \int_{\frac{|r-r'|}{|r+r'|}}^{1} \frac{dy}{y} \left\{ \frac{1-y}{1+y} \right\}^{\sigma} . \quad \text{(A-4)}$$

Now introducing $r = a\ e^{-2\theta}$, etc., we find:

$$\eta(\theta) = C\sigma \int_{-\infty}^{\infty} \eta(\theta') \int_{\tanh|\theta-\theta'|}^{1} \frac{dy}{y} \left\{\frac{1-y}{1+y}\right\}^{\sigma} + \frac{1}{8\pi r_o v} \int_{\tanh|\theta-\theta_o|}^{1} \frac{dy}{y} \left\{\frac{1-y}{1+y}\right\}^{\sigma} , \tag{A-5}$$

from which Eq. (II-2) immediately follows.

APPENDIX B

PROPERTIES OF THE TRANSFORMED KERNEL, $\tilde{K}(z)$

The transform of the kernel, $\tilde{K}(z)$, is explicitly:

$$\tilde{K}(z) = \int_{-\infty}^{\infty} e^{-z\theta'} \, d\theta' \int_{\tanh|\theta'|}^{1} \frac{dy}{y} \left\{\frac{1-y}{1+y}\right\}^{\sigma} . \tag{B-1}$$

Letting $e^{-t} = \frac{1-y}{1+y}$, and $\theta' = \theta/2$, one integral in (B-1) can be done, and we find:

$$\tilde{K}(z) = \frac{2}{z} \int_{0}^{\infty} \frac{dt \, e^{-\sigma t} \sinh zt/2}{\sinh t} . \tag{B-2}$$

This function can be written in terms of known functions (4), for $-\alpha < \text{Re } z < \alpha$:

$$Z(z) = \frac{1}{z} \left[\psi\left(\frac{\alpha+z}{4}\right) - \psi\left(\frac{\alpha-z}{4}\right)\right] , \tag{B-3}$$

$\alpha = 2(\sigma+1)$, and $\psi(z) = d/dz \, \ell n \, \Gamma(z)$. As $\psi(z)$ is analytic except for simple poles (4) at $z = 0,-1,-2,\cdots$, $\tilde{K}(z)$ is analytic in the strip $-\alpha <$ Re $z < \alpha$. Defining the analytic continuation of $\tilde{K}(z)$ for $|\text{Re } z| > \alpha$ by the right hand side of (B-3), we can say that $\tilde{K}(z)$ is analytic except for simple poles at $z = \pm(4m+\alpha)$, $m = 0,1,2,\cdots$. Using the formula (4):

$$\psi(z) - \psi(-z) = -\pi \text{Ctn } \pi z - \frac{1}{z} , \tag{B-4}$$

we obtain the asymptotic form of $\tilde{K}(z)$:

$$\underset{|z|\gg\alpha}{K(z)} \simeq -\frac{\pi \text{ ctn}}{z}\frac{\pi z}{4} + 0\left(\frac{1}{z^2}\right). \qquad \text{(B-5)}$$

The zeros of $1 - C\sigma\,\tilde{K}(z)$ are also of interest. The simpliest representation of $\tilde{K}(z)$ for determining these zeros is given by (4) :

$$\tilde{K}(z) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{\alpha}{4})^2 - (\frac{z}{4})^2} \quad . \qquad \text{(B-6)}$$

Using (B-6) it is easy to show that $1 - C\sigma\,\tilde{K}(z) = 0$ has roots only on the real and imaginary axis. For z real and in the strip $-\alpha \leq z \leq \alpha$, we know that $\tilde{K}(z)$ has poles at $z = \pm\alpha$. Also $\partial\tilde{K}(z)/\partial z > 0$ for $z > 0$, and $\tilde{K}(z)$ is and even function of z. Hence $\tilde{K}(z)$ has a relative mininum at $z = 0$. Therefore, if $(C\sigma)^{-1} > \tilde{K}(0)$, there are two real roots, $\pm z_o$, in $-\alpha <$ Re z $< \alpha$. If $(C\sigma)^{-1} = K(0)$, there is a double root at $z = \pm z_o = 0$. For $z = iy$, y real, we similarly find: $\partial\tilde{K}(iy)/\partial y < 0$ for $y > 0$. Also $\tilde{K}(iy)$ is continuous and finite for all real y. It is therefore evident that there will be two imaginary roots, $\pm z_o$, to $1 - C\sigma\,\tilde{K}(z) = 0$, if and only if $(C\sigma)^{-} <$ $\tilde{K}(0)$.

For $|\text{Re } z| > \alpha$, the analytic continuation of $1 - C\sigma\,K(z)$, defined by (B-3) and B-6) has an infinite number of real zeros. As $K(z)$ has poles at $z = \pm(4m+\alpha)$, $m = 0,1,2,\cdots$, and $\partial K(z)/\partial z > 0$, for $z > 0$, z real, there must be one and only one zero to $1 - C\sigma\,K(z)$ in each interval $4(m-1) + \alpha < z < 4m+\alpha$, and similarly for z negative and real. The root in this interval is labeled z_m, and $z_{m+1} > z_m > z_{m-1}$. It is also clear from Eq. (B-5) that

$$\lim_{n\to\infty} (z_n - z_{n-1}) = 4 \quad . \qquad \text{(B-7)}$$

APPENDIX C

CALCULATION OF THE EXTRAPOLATED RADIUS

An exact calculation of the extrapolated radius r^o (Eq. II-37), is clearly a computer task. However, in this appendix we exhibit two approximation methods which converge very rapidly for the r^{-1} cross-section and make this calculation amenable to a slide rule. For simplicity we have taken the case $\sigma = 1$, $C = C_m = 1.218$ ($z_o = 0$). The extrapolated radius is then given by Eq. (II-38), i.e., $r^o/a = e^{2X_-'(0)/X_-(0)}$. By examining Eqs. (II-29) and (II-30), it is easy to prove that

$$X_-(-z) = - X_+^{-1}(z) \quad . \tag{C-1}$$

Equation (II-25) then becomes:

$$X_-(z)\ X_-(-z) = \frac{1 - C_m\tilde{K}(z)}{z^2} \quad . \tag{C-2}$$

Expanding $1 - C_m\tilde{K}(z)/z^2$ in a Taylor series about the origin, we find:

$$\frac{1 - C_m K(z)}{z^2} = \alpha_2 + \alpha_4 z^2 + \alpha_6 z^4 + \alpha_8 z^6 + \alpha_{10} z^8 + \ldots, \tag{C-3}$$

where the α's can be computed by using the representation (B-6) of $\tilde{K}(z)$ and the tables of Reference (5). They are:

$$\alpha_2 = .4112 \times 10^{-1}$$

$$\alpha_4 = .2416 \times 10^{-2}$$

$$\alpha_6 = .1490 \times 10^{-3}$$

$\alpha_8 = .9286 \times 10^{-5}$

$\alpha_{10} = .5799 \times 10^{-6}$.

Now, $X_-(z)$ is expanded in a Taylor series around the origin and coefficients of like powers of z in Eq. (C-2) are equated. The exact value of $X_-(0) = \sqrt{\alpha_2}$ is found. If $X_-^{(n)}(0)/n!$ is neglected for $n \geq 3$, we find: $X_-'(0) = .253$. If $X_-^{(n)}(0)/n!$ for $n \geq 5$ are neglected, a quartic equation must be solved and in this approximation we find: $X_-'(0) = 2.54 \times 10^{-1}$. Therefore, this approximation method converges rapidly and we find that the extrapolated radius is:

$$\frac{r_o}{a} = 1.64 \quad . \qquad \text{(C-4)}$$

Another method of approximating $X_-(z)$ is the use of continued fractions. This method is in general more useful than the previous one as the Taylor series expansion may be slowly convergent for $z = z_o \neq 0$. The function $1 - C\sigma \tilde{K}(z)$ has a continued fraction representation which converges in the entire plane (6) (except at the poles) and the N^{th} approximation is given by:

$$1 - C_m\tilde{K}(z) \simeq \frac{\sum_{n=1}^{N} a_n z^{2n}}{\sum_{n=0}^{N} b_n z^{2n}} \quad . \qquad \text{(C-5)}$$

As $K(z) \underset{z\to\infty}{\sim} O(1/z)$, then we must have $a_N = b_N$. Using this condition and calculating a_n and b_n by an expansion around the origin, or equivalently by the method of Ref. (6), we may approximate $1 - C_m\tilde{K}(z)$ to any order that we desire. Having made this approximation the Wiener-Hopf decomposition given by (II-25) can be trivally done and $X_-(z)$ uniquely determined. Explicitly, we have in first approximation:

$$1 - C_m\tilde{K}(z) \simeq \frac{-z^2}{A-z^2} \quad . \qquad \text{(C-6)}$$

Using this approximation, we find $r_0/a = 1.50$. Going to the next approximation, where only a quadratic equation has to be solved, i.e.,

$$1 - C_m\, K(z) \simeq \frac{\delta z^2+Az^4}{Bz^2+Az^4-1} \quad , \qquad \text{(C-7)}$$

gives $r_0/a = 1.64$. So, this method has rapidly converged to the previous result and is, computational wise, easier.

It should be pointed out that in the constant cross-section case, the convergence of the later approximation scheme is not as rapid as in the r^{-1} case. The extrapolated end in this case is exactly $X_0 = .7104$, (3). In the same approximation as (C-6), we find $X_0 \simeq .577$. In the same approximation as (C-7); we find $X_0 = .694$; and in the next approximation, where a cubic equation must be solved, we obtain $X_0 = .706$.

APPENDIX D

A USEFUL REPRESENTATION FOR X(z)

If $X(z)$ belongs to Class (3), i.e., $\lim_{z\to\infty} X(z) = z^{-k_2}$, $k_2 > 0$, and integer, we have by Cauchy's theorem:

$$X(z) = \frac{1}{2\pi i} \int_C \frac{X(z')dz'}{z' - z} , \qquad \text{(D-1)}$$

where c encircles the (γ_1, γ_2) cut in the negative direction. Equation (D-1) becomes:

$$X(z) = \frac{1}{2\pi i} \int_{\gamma_1}^{\gamma_2} \frac{d\mu}{\mu - z} [X_+(\mu) - X_-(\mu)] = \frac{1}{2\pi i} \int_{\gamma_1}^{\gamma_2} \frac{d\mu \ X_-(\mu) [\Lambda_{1+}^{(\mu)} - \Lambda_{1-}^{(\mu)}]}{(\mu - z)\Lambda_{1-}(\mu)} \qquad \text{(D-2)}$$

or

$$X(z) = \int_{\gamma_1}^{\gamma_2} \frac{d\mu}{\mu - z} \frac{\gamma(\mu) \ f_1(\mu)}{\prod_j (\mu_j - \mu)^{m_j}} . \qquad \text{(D-3)}$$

If $X(z)$ is not in the Class (3), a similar representation of $X(z)$ can be derived by making suitable subtractions.

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FOOTNOTE

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