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Optimal Solution Characterization for Infinite Positive Semi-Definite Programming

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Abstract—We give a set-theoretic description of the set of optimal solutions to a general positive semi-definite quadratic programming problem over an affine set. We also show that the solution space is again an affine set, thus offering the opportunity to find an optimal solution by solving a corresponding operator equation.

Keywords—Solution description, Quadratic programming, Positive semi-definite, Affine set.

1. INTRODUCTION

Consider a general convex programming problem (C) of the following form:

$$\min_{x \in X} C(x), \tag{C}$$

where X is a nonempty, closed, convex subset of a real Hilbert space H and C is a convex, real-valued function on H. It is not difficult to see that the (possibly empty) set X^* of optimal solutions to (C) must also be convex. If X is affine in particular, then it is also not difficult to see that X^* need *not* be affine. Our objective here is to find additional assumptions on the convex objective function C which are sufficient to conclude that X^* is affine whenever X is.

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To this end, we assume that C is quadratic, i.e., C is of the form

$$C(x) = \frac{1}{2}\langle x, Q(x)\rangle + \langle x, c\rangle, \qquad x \in H,$$

where $Q : H \to H$ is a bounded linear operator on H and $c \in H$. Thus, we are considering a general quadratic programming problem (Q) given by

$$\min_{x\in X} \frac{1}{2}\langle x, Q(x)\rangle + \langle x, c\rangle. \tag{Q}$$

Without loss of generality, we may assume Q is self-adjoint. Recall that C is convex in this case if and only if the operator Q is positive semi-definite, i.e., $\langle x, Q(x)\rangle \geq 0$, $x \in H$. Consequently, we assume that Q is positive semi-definite in what follows. In particular, if Q is positive-definite in the sense of [1] and [2], then X^* can be shown [3] to be a singleton, i.e., (Q) admits a unique optimal solution in this case.

Our objective in this note is to show that for the general convex *quadratic* programming problem (Q), if X is affine, then X^* is also affine. We may thereby compute the optimal solutions to (Q) by solving a corresponding operator equation in H. We accomplish our objective by giving a set-theoretic description of X^* in terms of X, Q and c, in which all the operations on these data are seen to preserve the affine property.

2. DESCRIPTION OF OPTIMAL SOLUTIONS

For the remainder of this note, we assume that X is an affine subset of H.

The Fréchet derivative [4] C' of C is a mapping of H into the topological dual space H' of H given by

$$C'(x) = \langle \cdot\, , Q(x)\rangle + \langle \cdot, c\rangle, \qquad x \in H.$$

Since the feasible region $X \neq \emptyset$, we may fix an arbitrary $v \in X$. We then obtain the equivalent translated problem (Q^v) defined as follows:

$$\min_{x\in K} CT(x), \tag{Q^v}$$

where K is a closed subspace of H (hence, a Hilbert space) satisfying $X = K + v$, and $T : H \to H$ is the translation operator by v. If K^* denotes the set of optimal solutions to (Q^v), then $X^* = K^* + v$.

If we denote by P the orthogonal projection onto K, then we may construct the related *unconstrained* optimization problem (Q_K^v) by projection onto K:

$$\min_{x\in H} CTP(x). \tag{Q_K^v}$$

Note that the objective values attained for (Q^v) are the same as those attained for (Q_K^v). However, the set of feasible solutions for (Q^v) is K, while that for (Q_K^v) is $H = K \oplus K^\perp$. If H^* denotes the set of optimal solutions to (Q_K^v), then $H^* = K^* \oplus K^\perp$, so that $P(H^*) = K^*$.

Let $\mathcal{L}(H, K)$ denote the space of bounded linear operators from H to K (similarly for $\mathcal{L}(H, H)$). Then the Fréchet derivative P' of P is a mapping $P' : H \to \mathcal{L}(H, K)$, while the Fréchet derivative T' of T is a mapping $T' : H \to \mathcal{L}(H, H)$. It is easy to verify that P' and T' are in fact constant mappings, where $P'(x) = P$, $x \in H$, and $T'(x) = I_H$, the identity operator on H, $x \in H$. Consequently, if $F = CTP$, then the Fréchet derivative F' of F is the mapping $F' : H \to H'$, given by

$$\begin{aligned} F'(x) &= C'(P(x) + v)P + \langle P(\cdot), c\rangle \\ &= \langle P(\cdot), Q(P(x) + v) + c\rangle, \qquad x \in H. \end{aligned}$$

By a familiar first-order necessary condition for unconstrained optimization [4, p. 178], we have that if $x \in H$ is optimal for (Q_K^v), then $F'(x)$ equals the zero functional on H, i.e.,

$$\langle P(y), Q(P(x)+v)+c\rangle = 0, \qquad y \in H.$$

If we let S denote the set of solutions to this equation, so that

$$S = \{x \in H \ : \langle P(\cdot), Q(P(x)+v)+c\rangle = 0\},$$

then

$$H^* \subseteq S.$$

Note that the preceding argument does not require that Q be positive semi-definite.

Conversely, since C is convex, T is convex linear and P is linear, it follows that F is convex. Therefore, $S \subseteq H^*$ [4, p. 227], i.e., $H^* = S$. Consequently, from the above discussion,

$$X^* = P(H^*) + v = P(S) + v.$$

We turn next to a description of S and consequently, of X^*. If $A \subseteq H$, then $Q^{-1}(A)$ denotes the usual inverse image of A under Q.

PROPOSITION 2.1. *The subset S of H is given by*

$$S = \left[(Q^{-1}(K^\perp - c) - v) \cap K\right] \oplus K^\perp.$$

Therefore,

$$X^* = \left[(Q^{-1}(K^\perp - c) - v) \cap K\right] + v.$$

PROOF. Suppose $x \in H$ belongs to the set on the right side of the first equation. For convenience, let $L = Q^{-1}((K^\perp - c))$, a closed subset of H. Then there exists $k \in (L - v) \cap K$ and $q \in K^\perp$ such that $x = k + q$. Hence, $P(x) = P(k) + P(q) = k$, which is in $L - v$. Thus, $(P(x) + v) \in L$ and $Q(P(x) + v) + c \in K^\perp$, so that

$$\langle P(y), Q(P(x)+v)+c\rangle = 0, \qquad y \in H.$$

Consequently, $x \in S$. Since this argument is valid in reverse, we have the opposite inclusion as well. For the second part, simply evaluate $P(S)$. ∎

THEOREM 2.2. *Suppose Q is positive semi-definite in problem (Q). If X is affine in H, then the set X^* of all optimal solutions to (Q) is also affine in H.*

PROOF. First observe that X^* is affine in H because K and $K^\perp$ are closed subspaces, translates of affine spaces are affine, inverse images of affine spaces under linear operators are affine and the intersection of an affine space with a subspace is affine. Similarly, S is affine because, in addition, the sum of an affine space and a subspace is affine. ∎

Theorem 2.2 offers the opportunity to solve (Q) by solving an operator equation in H whose solution space is X^*. Specifically, every affine space in H is of the form $\{x \in H : Ax = b\}$, for some bounded linear operator A on H and $b \in H$ (per D. Schmidt).

REFERENCES

1. K.Y. Lee, S. Chow, and R.O. Barr, On the control of discrete-time distributed parameter systems, *SIAM J. Control* **10**, 361–376, (1972).
2. J. Zabczyk, Remarks on the control of discrete-time distributed parameter systems, *SIAM J. Control* **12**, 721–773, (1974).
3. J.L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, New York, (1971).
4. D. Luenberger, *Optimization by Vector Space Methods*, J. Wiley, New York, (1969).