JOURNAL OF COMBINATORIAL THEORY, Series A 67, 100–115 (1994)

Perfect Matchings and Perfect Squares

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Communicated by the Managing Editors

Received May 22, 1992

In 1961, P. W. Kasteleyn enumerated the domino tilings of a $2n \times 2n$ chessboard. His answer was always a square or double a square (we call such a number "squarish"), but he did not provide a combinatorial explanation for this. In the present thesis, we prove by mostly combinatorial arguments that the number of matchings of a large class of graphs with 4-fold rotational symmetry is squarish; our result includes the squarishness of Kasteleyn's domino tilings as a special case and provides a combinatorial interpretation for the square root. We then extend our result to graphs with other rotational symmetries.

1. INTRODUCTION

Suppose you had an ordinary chessboard and 32 indistinguishable dominoes, and that each domino were exactly big enough to cover two squares of the chessboard. You could then use your dominoes to tile the chessboard (left side of Fig. 1). You might wonder how many different ways you could do this. This is an example of a *domino tiling problem*: given a subset of the lattice squares of the plane, in how many ways can it be tiled by dominoes? For the $2n \times 2n$ chessboard, the problem was first solved by P. W. Kasteleyn in 1961 [K61]; the answer is

$$2^{2n^2} \prod_{k=1}^{n} \prod_{l=1}^{n} \left(\cos^2\left(\frac{\pi k}{2n+1}\right) + \cos^2\left(\frac{\pi l}{2n+1}\right) \right). \tag{1}$$

The formula (1) is always a square or double a square (this latter result is due to E. W. Montrall; for an exposition, see [Lo, Problem 4.29]); we will call such a number *squarish*. There should be a combinatorial reason why the formula is squarish. However, Kasteleyn's proof of (1) used the *Pfaffian method*, which entails showing that the number of tilings is the *Pfaffian*

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0097-3165/94 $6.00

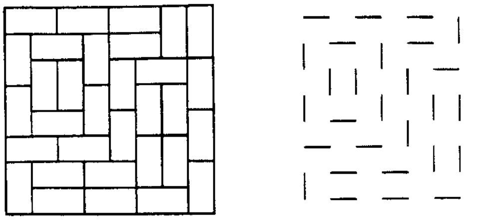


FIG. 1. A domino tiling of the ordinary chessboard, and the corresponding matching of its dual graph.

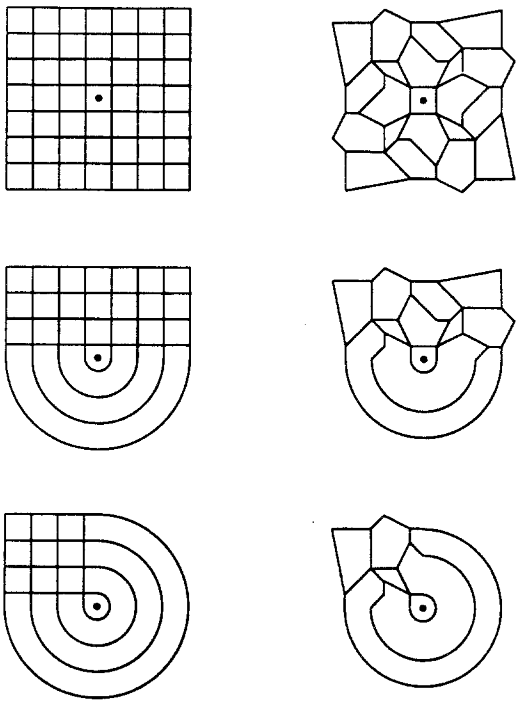


FIG. 2. Sample graphs G (top row), G_2 (middle row), and G_4 (bottom row). The graph G in the left column is the dual of the ordinary chessboard; G in the right column is a "general" 4-odd-symmetric graph. In each case, the dot in the center of the graph denotes the origin. The points where the circular parts of edges join the straight parts are *not vertices*, even when the two pieces meet at an angle. All other points where lines come together at an angle are vertices.

(i.e., the square root of the determinant) of a certain matrix A. Kasteleyn then computes the determinant by finding the eigenvalues of A. The argument is highly noncombinatorial. Other proofs have been found since, using transfer matrices for instance [Li; S1, pp. 273–274], but none is combinatorial, and all depend heavily on linear algebra.

A (perfect) *matching* of a graph G is a subset of the edges of G which includes exactly one of the edges at each vertex of G and does not include any loops. The above domino tiling problem is equivalent to a matching problem: If we let G be the dual graph of the $n \times n$ chessboard, a domino tiling determines a matching in a natural way—include an edge in the matching if its endpoints lie on the same domino in the tiling (Fig. 1). This is a bijection between the domino tilings of the chessboard and the matchings of G, so counting the domino tilings of the chessboard is equivalent to counting the matchings of G. Problems involving the matchings of some graph arise in many contexts, and matching theory is a thriving subject; for a good general reference, see [LP].

The dual graph of the chessboard is an example of a *4-odd-symmetric* graph (defined in the next section; for an example, see Fig. 2.) Several people have counted the matchings of specific 4-odd-symmetric graphs [EKLP, Ku, Y]; in each case, they found that the number of matchings is squarish. Our main result is that the number of matchings of any 4-odd-symmetric graph is squarish. Although our proof is not strictly bijective, we feel it does provide combinatorial insight. Furthermore, it provides a combinatorial interpretation of the square root. We then prove some related results for graphs with other rotational symmetries. We conclude by mentioning some related unsolved problems and possible avenues for further investigation.

2. Definitions and Notation

The *punctured plane* is the plane with the origin removed. A *plane graph* is a graph embedded in the plane or the punctured plane whose edges do not intersect, except at their endpoints. Let G be a plane graph; loops and multiple edges are allowed. Then $V(G)$ and $E(G)$ denote the vertex and edge sets of G. We say that G is *bipartite* if the $V(G)$ can be partitioned into two sets (the *parts*) such that every edge of G has one endpoint in each part. A *k-factor* of G is a multiset of edges of G which includes exactly k edges indicent on each vertex v of G (a loop at v counts as two edges incident on v). A 1-factor is the same as a matching. $M(G)$ denotes the number of matchings of G.

If G is embedded in the punctured plane, we can classify the cycles of G as *contractible* and *noncontractible* cycles. Note that a 2-factor is a union of disjoint cycles; we say that a 2-factor is *contractible* if all its cycles are contractible. We say that a cycle is *trivial* if it consists of two copies of the same edge, together with its endpoints. An *alternating set of edges* in a cycle is a subset of the edges of the cycle which includes exactly one edge incident on each vertex of the cycle. A cycle of odd length has no alternating sets of edges. A cycle of even length has one alternating set of edges if it is trivial and two otherwise. Given a 2-factor in a bipartite graph G, we can find a matching in G by arbitrarily selecting an alternating set of edges from each cycle.

We say that a graph G is *k-symmetric* if G is a bipartite graph, embedded in the punctured plane with noncrossing edges, and a $360/k$ degree rotation R_k of the punctured planed about the origin maps G to itself. If G is connected, we say that the symmetry is *even* or *odd* according to whether the number of edges in a path between a vertex v of G and $R_k(v)$ is even or odd. Given a k-symmetric graph G, we let G_k denote the graph whose vertices and edges are, respectively, the R_k-orbits of vertices and edges of G. For instance, the vertices of G_2 are sets $\{v, R_2(v)\}$, where v is a vertex of G. Note that in the 4-odd-symmetric case, G_2 is bipartite but G_4 is not. Sample graphs G, G_2, and G_4 are shown in Fig. 2. We will always assume that the G_k are embedded in the punctured plane as shown (copy a $1/k$ pie piece of G, with no vertices on the boundary of the copied region, then put in circular arcs centered at the origin to connect up edges which were cut by the copying). There are natural graph maps $\pi_k: G \to G_k$ defined by $\pi_k(v) = w$ iff $v \in w$, etc. If G is 4-symmetric, we also define $\pi'_2: G_2 \to G_4$ by $\pi'_2(v) = w$ iff $v \subset w$. If m is a matching (1-factor) in G, then $\pi_2(m)$ is a 2-factor in G_2. π'_2 behaves similarly. When thinking about the G_k, it is helpful to think of the edges into which circular arcs were inserted as being in no way fundamentally different from the others; indeed, the choice of where to insert the arcs has no effect on the structure of the graph.

We define the *winding number* w of an oriented edge e of G in the usual manner: $w(e) = \int_e d\theta/2\pi$. However, if e is an edge of G_k, then we let $w(e) = w(f)$, where f is any edge of G with $\pi_k(f) = e$. We say that a cycle is *oriented* if its edges are assigned compatible orientations, the head of one serving as the tail of the next. The *winding number* of an oriented cycle is the sum of the winding numbers of its edges. Contractible cycles always have winding number 0; since the graphs are planar, noncontractible cycles always wind once around the origin, which means that noncontractible cycles in G and G_k have winding numbers ± 1 and $\pm 1/k$, respectively. Lastly, we say that a 2-factor is *oriented* if each of its cycles is oriented independently, and the *winding number* of an oriented 2-factor is the sum of the winding numbers of its component cycles.

Given a cycle c in G_k, it will be helpful to understand the structure of $\pi_k^{-1}(c)$. To this end, we say that a path q in G is a *lift* of a path p in G_k

if $\pi_k(q)=p$. We can cut an oriented cycle c in G_k at any vertex to obtain a path p; we say that a path q in G is a lift of c if q is the lift of such a path p. If the endpoints of the path q are the same, we say that the cycle d thus obtained is a lift of c. Note that a lift of a cycle c contains exactly one of the k inverse images (under π_k) of each edge of c, and that lift preserves winding number. Hence, the structure of $\pi_k^{-1}(c)$ is determined by the structure of c. That is, if c has winding number 0 (i.e., if c is contractible), then a lift of c also has winding number 0 and is therefore a contractible cycle; hence $\pi_k^{-1}(c)$ consists of k contractible cycles. However, if c has winding $\pm 1/k$, a lift of c also has winding number $\pm 1/k$ and cannot be a cycle. Therefore, $\pi_k^{-1}(c)$ must be a single cycle with winding number ± 1, i.e., a noncontractible cycle.

We define lifts from G_4 to G_2 similarly. A similar argument then shows that the same conclusions hold for π_2': if c is a contractible cycle in G_4, then $\pi_2'^{-1}(c)$ consists of two contractible cycles in G_2, and if c is a noncontractible cycle in G_4, then $\pi_2'^{-1}(c)$ is a single noncontractible cycle in G_2.

3. Perfect Matchings and Perfect Squares

We are now ready to state and prove our principal result.

THEOREM 1. *Let G be a 4-odd-symmetric graph. Then*

$$M(G)=\left(\sum_{\text{2-factors of } G_4} 2^{\text{number of nontrivial, contractible cycles} + 1/2 \text{ number of noncontractible cycles}}\right)^2. \tag{2}$$

This is a square if the number of vertices of G_4 is even and double a square otherwise.

Remark. The author has examples which show that all of the conditions in the definition of a 4-odd-symmetric graph are needed for the conclusion of Theorem 1. It is conceivable, however, that a weaker result could be established for a larger class of graphs.

We prove (2) by the following sequence of identities, whose proofs are given below:

$$M(G)=\sum_{\text{contractible 2-factors of } G_2} 2^{\text{number of nontrivial cycles in } f} \tag{3}$$

$$=\sum_{\text{oriented 2-factors } h \text{ of } G_2} e^{\pi i w(h)} \tag{4}$$

$$=\sum_{\text{ordered pairs } (h_1,\, h_2) \text{ of oriented 2-factors of } G_4} e^{\pi i (w(h_1)+w(h_2))} \tag{5}$$

$$M(G)=\Bigg(\sum_{\text{oriented 2-factors } h \text{ of } G_4} e^{\pi i w(h)}\Bigg)^2 \tag{6}$$

$$=\Bigg(\sum_{\text{2-factors } f \text{ of } G_4}\ \prod_{\text{cycles } c \text{ of } f}\ \sum_{\text{orientations of } c} e^{\pi i w(c)}\Bigg)^2 \tag{7}$$

$$=\Bigg(\sum_{\text{2-factors of } G_4} 2^{\text{number of nontrivial, contractible cycles} + 1/2 \text{ number of noncontractible cycles}}\Bigg)^2. \tag{8}$$

Proof of (3). If m is a matching (i.e., a 1-factor) of G, then $\pi_2(m)$ is a 2-factor of G_2. An example is in Fig. 3. Given a 2-factor f of G_2, we count the matchings m of G with $\pi_2(m)=f$. Suppose f contains a nontractible cycle c. The inverse image $\pi_2^{-1}(c)$ is a single noncontractible cycle d. Any matching m of G with $\pi_2(m)=f$ would have to include half of the edges of d, i.e., an alternating set of edges of d, but the image of such a set is two copies of the same alternating set of edges of c, where one copy of each alternating set is needed. Hence there is no such matching m.

Now suppose f is contractible. For each trivial cycle in f, both edges in the inverse image must be in m; there is one way to do this. Now consider a nontrivial, contractible cycle c of f. The inverse image of c consists of two disjoint cycles in G. We need to pick an alternating set of edges in each such that the image of the union of the alternating sets is c. There are two ways to do this. This choice can be made independently for each nontrivial, contractible cycle of f. Equation (3) follows.

Proof of (4). Equation (4) is an immediate consequence of the following claim.

CLAIM. *The sum of the terms on the right-hand side for the orientations h of a 2-factor f is the same as the term for f on the* LHS.

Proof of Claim. Suppose f contains a noncontractible cycle c. Pair up the terms in the RHS in which all cycles except c have the same orientation and c has opposite orientations. The winding numbers of corresponding terms will differ by 1, so they will cancel in the sum; hence the terms on

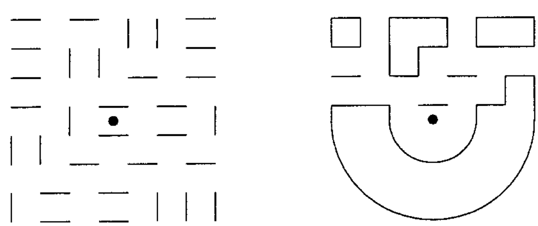


FIG. 3. A sample matching of the dual G of the ordinary chessboard, and the corresponding 2-factor of G_2. Again, the black dot denotes the origin.

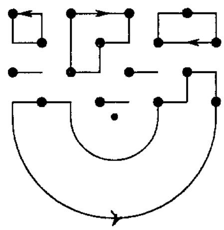
the RHS for a noncontractible 2-factor add to 0. On the other hand, if f is contractible, then every orientation of f has winding number 0, so the sum of the terms for orientations of f on the RHS is just the number of orientations of f, which is $2^{\text{number of nontrivial cycles in } f}$.

Proof of (5). We produce a bijection γ between the set of oriented 2-factors of G_2 and the set of ordered pairs of oriented 2-factors of G_4. To define γ, arbitrarily pick one of the parts of G_2 to be the *black part*; the other part is the *white part*. Since R_4 reverses the parts of G, if e is an oriented edge of G_4, one of the two inverse images (under π_2') of e will point from the white part to the black part and the other will point from the black part to the white part. Given an oriented 2-factor h of G_2, let m_1 be the oriented matching consisting of those edges of h which point from the black part to the white part, and let m_2 be the oriented matching consisting of those edges which point from the white part to the black part. Let $h_1 = \pi_2'(m_1)$ and $h_2 = \pi_2'(m_2)$. Then h_1 and h_2 are oriented 2-factors of G_4. (Why? Consider any vertex v of G_4. Its inverse images are a black vertex and a white vertex of G_2, and there is an edge of h coming into and going out of each. These four edges give rise to an edge coming into and going out of v in each of h_1 and h_2.) Samples h, h_1, and h_2 are shown in Fig. 4. Let $\gamma(h) = (h_1, h_2)$. Inversely, given two oriented 2-factors h_1 and h_2 of G_4, let m_1 consist of those oriented edges in the inverse image of h_1 which point from the black part of G_2 to the white part, let m_2 consist of those oriented edges in the inverse image of h_2 which point from the white part to the black part, and let $\gamma^{-1}(h_1, h_2) = m_1 \cup m_2$. It is easy to check that $\gamma^{-1}(h_1, h_2)$ is an oriented 2-factor, γ is a bijection, and that if $\gamma(f) = (f_1, f_2)$ then $w(f) = w(f_1) + w(f_2)$. Equation (5) follows.

Equations (6) and (7) are clear.

Proof of (8). A trivial cycle has one orientation and this orientation has winding number 0, so trivial cycles have no effect on the product on the LHS. The two orientations of each noncontractible cycle in f have winding numbers $\pm 1/4$ and the two orientations of each contractible cycle have winding number 0. Hence, each noncontractible cycle contributes a factor of $e^{\pi i/4} + e^{-\pi i/4} = 2^{1/2}$ and each contractible cycle contributes a factor of $e^0 + e^0 = 2$. Equation (8) follows.

To complete the proof of Theorem 1, we need to show that the RHS of (8) is a square when G_4 has an even number of vertices and double a square otherwise. To check this, we recall that contractible cycles in G_4 lift with multiplicity 1 to cycles in G_2, which is bipartite, and hence contain an even number of vertices. Noncontractible cycles, by contrast, lift to paths in G_2 with winding number $\pm 1/4$. Since R_4 reverses the parts of G, the endpoints of such paths are in opposite parts of G_2; hence noncontractible



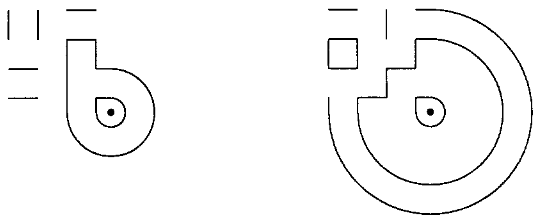


FIG. 4. An orientation h of the 2-factor from Fig. 3, with a selected black part (vertices with heavy dots), and then corresponding ordered pair $\gamma(h)$ of 2-factors of G_4. The "black-to-white" 2-factor h_1 is at the left; h_2 is at the right.

cycles in G_4 pass through an odd number of vertices. It follows that if f is a 2-factor in G_4, the parity of the number of noncontractible cycles in f is the same as the parity of $|V(G_4)|$. Hence the exponent in every term of the sum on the RHS of (2) is an integer if $|V(G_4)|$ is even and half an odd integer otherwise. Theorem 1 follows.

4. Other Symmetries

We now discuss the situation for k-even-symmetric graphs. We assume throughout this section that G is a k-even-symmetric graph. One part of G (and the corresponding part of G_k) has been arbitrarily chosen as the *white part*; the other part is the *black part*. We first show that in the case $k=2$, $M(G)$ is a sum of two squares.

We will need the concept of *weighted matchings* Let A be a commutative ring. A *weight function* u for a graph G is a function u: $E(G) \to A$. If e is an edge of G, we say that $u(e)$ is the *weight* of e. The weight u of a k-factor is the product of the weights of its edges. Finally, the *weight sum* $M_u(G)$ of the matchings of G is $\Sigma_{\text{matchings } m \text{ of } G}\, u(m)$.

Let G be a 2-even-symmetric graph. Let B be a "branch cut" in the plane; that is, let B be a ray with its endpoint at the origin which does not pass through any vertex of G. (More generally, B can be any semi-infinite path which does not intersect any edge of G infinitely often and does not intersect itself or its $360/k$ degree rotation, except at the origin). Let the weight $u(e)$ of an edge e of G be $i^{c-c'}$, where c is the number of times the path traversing e from its white endpoint to its black endpoint crosses B or $R_2(B)$ in the counterclockwise direction, and c' is the number of times the same path crosses B or $R_2(B)$ in the clockwise direction. Let the weight of an edge of G_2 be the weight of one of its inverse images in G. We have the following:

THEOREM 2.

$$M(G)=M_u(G_2)\,\overline{M_u(G_2)},$$

where $\overline{}$ *denotes complex conjugation.*

Proof. We have

$$M(G)=\sum_{\text{oriented 2-factors } h \text{ of } G_2} e^{\pi i w(h)} \tag{9}$$

$$=\sum_{\text{ordered pairs } (h_1,h_2) \text{ of matchings of } G_2} u(h_1)\,\overline{u(h_2)} \tag{10}$$

$$=M_u(G_2)\,\overline{M_u(G_2)}. \tag{11}$$

The proof of (9) is the same as the proof of (4). Equation (10) follows from the bijection between oriented 2-factors h and ordered pairs of matchings given by $h \leftrightarrow (h_1, h_2)$, where h_1 consists of those edges of h which point from the white part to the black part, and h_2 consists of those edges of h which point from the black part to the white part. Equation (11) follows from the definition of weight sum. This completes the proof.

Upon seeing Theorem 1, Richard Stanley suggested that the determinant method, together with the change of basis described below, might allow one to obtain similar theorems for other symmetries. This has proved to be correct. Let G be a k-even-symmetric graph, and pick a branch cut B as above. Let the *crossing number* $\mathrm{cr}(e)$ of an edge e be $c-c'$, where c and c' are the number of times the path traversing e from its white endpoint to its black endpoint crosses one of B, $R_k(B)$, ..., $R_k^{k-1}(B)$ in the counterclockwise and clockwise directions, respectively. Let the weight $u(e)$ of an edge e of G_k be $x^{\mathrm{cr}(e)}$, where x is an indeterminate. By assigning a complex value to x, we obtain a complex weight function u_x on G_k. Let $\omega=e^{\pi i/k}$. We have the following:

THEOREM 3.

$$M(G)=\prod_{x\in S} M_{u_x}(G_k), \tag{12}$$

where

$$S=\begin{cases}\{1, \omega^2, ..., \omega^{2k-2}\} & \textit{if } k \textit{ is odd}\\ \{\omega, \omega^3, ..., \omega^{2k-1}\} & \textit{if } k \textit{ is even.}\end{cases}$$

Note that this specializes to Theorem 2 in the case $k=2$. Before we can prove Theorem 3, we need to sketch the *determinant method* for finding the weight sum of the matchings of a bipartite plane graph. The method is essentially due to Kasteleyn and has been around as "folklore" for some time. The author is unaware of any published proof that the method works, although its correctness follows from that of the very similar but somewhat more general Pfaffian method. The first general description of the Pfaffian method was given by Kasteleyn [K67]; expositions of the Pfaffian method (with proofs) appear in [LP, Sect. 8.3; Ku], among other places. The determinant method has been applied to specific graphs in [Ku] and [Y]. The reader is warned that some some authors use the name "determinant method" for what we and [LP] call the "Pfaffian method."

The *biadjacency matrix* $B(G)$ of a graph G is a matrix with rows corresponding to vertices in one part of G, columns corresponding to vertices in the other part, and $B(G)_{v,w}$ equal to the sum of the weights of the edges between v and w. Note that each nonzero term in the permanent of $B(G)$ corresponds to an equal term in the weight sum of the matchings of G, so $M(G)=\operatorname{per}(B(G))$. The idea behind the determinant method is to find another matrix whose determinant is term-by-term the same as $\operatorname{per}(B(G))$.

We say that a cycle F of a plane graph G is a *face* if there is no path in G between two vertices of F which lies (except for its endpoints) strictly in the interior of F. Given a cycle C of G, we let $\operatorname{int}(C)$ denote the number of vertices in the interior of C and $|C|$ denote the number of vertices lying on C. We say that C is *nice* if $G-C$ has a perfect matching. Note that if C is nice, then $\operatorname{int}(C)$ is even. Suppose that a matching m of G contains an alternating set of edges A in some nice cycle C. A *nice cycle move* on m consists of removing A from m and replacing it with $C-A$. Since the union of any two matchings is a union of disjoint nice cycles, it is possible to turn any matching of G into any other by nice cycle moves. Let S^1 denote the set of complex numbers with absolute value one. We say that a function $d\colon E(G)\to S^1$ is a *determinantal function* for a bipartite plane graph G provided that for any face F of G and any alternating set Q of edges of F,

$$\prod_{e\in Q} d(e)=(-1)^{|F|/2+\operatorname{int}(F)+1}\prod_{e\in F-Q} d(e). \tag{13}$$

THEOREM 4. 1. *Every bipartite plane graph has a determinantal function.*

2. *If d is a determinantal function for G, and C is a cycle of G and Q is an alternating set of edges for C, then*

$$\prod_{e \in C} d(e) = (-1)^{|C|/2 + \mathrm{int}(C) + 1} \prod_{e \in C - Q} d(e). \tag{14}$$

In particular, if C is nice, we can drop the term $\mathrm{int}(C)$ *in the exponent.*

3. *If d is a determinantal function for G, then for any weight function u on G, $M_u = c \det(A_d)$, where $c \in S^1$ is a constant which does not depend on the choice of u, and A_d is a matrix with rows corresponding to vertices in the black part of G, columns corresponding to vertices in the white part of G, and entries*

$$A_{d_{v,w}} = \sum_{\text{edges } e \text{ from } v \text{ to } w} u(e)\, d(e).$$

Proof (Sketch). 1. First pick values of d arbitrarily on a spanning tree, then pick values for additional edges one at a time, completing a face at each step, and ensuring at each step that (13) holds for the new face.

2. By induction on $|C| + \mathrm{int}(C)$. If C is a face, we are done. Otherwise, find a path between two vertices of C which lies strictly in the interior of C. We now have a figure shaped like the letter "θ" with C and two smaller cycles; use the truth of (13) for the smaller cycles to prove (13) for C.

3. We know that every matching m of G contributes a term $c_m u(m)$ to $\det(A_d)$, where $c_m \in S^1$. We need to show that c_m does not depend on m. It is enough to show that c_m is not affected by a nice cycle move. But (14) ensures exactly this. Recall that a nice cycle always has an even number of interior vertices. When the length of the cycle is a multiple of 4, the change in sign coming from the parity coefficient in the determinant is cancelled by the factor of $(-1)^{|C|/2 + \mathrm{int}(C) + 1} = -1$ multiplied into $d(m)$ when one alternating set of edges is exchanged for the other. When the length of the cycle is not a multiple of 4, both the sign of the term in the determinant and $d(m)$ are unchanged.

We are now ready to prove Theorem 3. We can assume that every edge of G appears in some matching. (If any edge does not, then neither do any of its images under repeated $360/k$ degree rotation, nor does its image in G_k appear in any matching of G_k, so we can remove them all without affecting either side of (12).) Let d_k be a determinantal function for G_k. Define $d: E(G) \to S^1$ by

$$d(e) = \begin{cases} d_k(\pi_k(e)) & \text{odd } k \\ \omega^{\mathrm{cr}(e)} d_k(\pi_k(e)) & \text{even } k. \end{cases}$$

LEMMA. *The function d is a determinantal function for G.*

Proof. Let F be a face of G. We consider two cases. If F is noncontractible, the fact that F is a face implies that $R_k(F)=F$. In particular, the length of F is a multiple of k. Let $F_k=\pi_k(F)$; then F_k is a face of G_k, and $\pi_k^{-1}(F_k)=F$. We can now check directly from the definition of d that (13) holds for F.

On the other hand, if F is contractible, then F does not contain both v and $R(v)$ for any vertex v. To see this, assume the contrary. Then removing $\pi_k(v)$ from G_k would disconnect G_k. At least one component H_k of the disconnected graph will have an odd number of vertices. Therefore one part of H_k has one vertex more than the other; we assume without loss of generality that the white part has more. Then the white part of $\pi_k^{-1}(H_k)$ has k more vertices than the black part of $\pi_k^{-1}(H_k)$, so the pigeonhole principle ensures that in any matching of G, the vertices v, $R(v)$, ..., $R^{k-1}(v)$ are matched into $\pi_k^{-1}(H_k)$. Hence edges between v and components of G_k other than H_k cannot appear; since there is at least one such edge, this contradicts the assumption that every edge of G appears in some matching.

Hence F is the lift of a face of F' of G_k, and we can use the fact that (13) holds for F' to check that it also holds for F.

Proof of Theorem 3. Put the matrix A_d into the basis consisting of the vectors $v+w^{2j}R_k(v)+w^{4j}R_k^2(v)+\cdots+w^{2(k-1)j}R_k^{k-1}(v)$, where v ranges over the vertices between B and $R_k(B)$ and j ranges from 0 to $k-1$. The resulting matrix has k blocks along the diagonal, and each block is a matrix A_{d_k} for $M_{u_x}(G_k)$, where $x=\omega^{2j}$ (if k is odd) or $x=\omega^{2j+1}$ (if k is even). Hence we have

$$\begin{aligned} M(G) &= c\ \det(A_d) \\ &= c \prod_{j=0}^{k-1} \det(A_{d_j}) \\ &= c' \prod_{x\in S} M_{u_x}(G_k). \end{aligned}$$

Here c and c' are constants in S^1. To complete the proof, we note that $c'=1$, since $M(G)$ and $\prod_{x\in S} M_{u_x}(G_k)$ are both nonnegative reals—the former is a combinatorial quantity; the terms of the latter can be paired off into complex conjugates, and the leftover term in the odd case (corresponding to $x=1$) is $M(G_k)$, another combinatorial quantity.

5. Weighted Versions

Theorems 1–3 and their proofs all hold in the more general context of weighted matchings. We use Theorem 1 to illustrate. Let A be an arbitrary commutative ring with unity, and give the edges of G_4 weights in A. The *weight* of an edge of G or G_2 is defined to be the weight of its image in G_4. If A contains an element ω with $\omega^4 = -1$, then let $B = A$. Otherwise, let $B = A[\omega]$, where $\omega^4 = -1$. Henceforth, ω plays the role of $e^{\pi i/4}$, and $(\omega + \omega^{-1})^2 = \omega^2 + \omega^{-2} + 2 = 2$, so $\omega + \omega^{-1}$ plays the role of $2^{1/2}$. Theorem 1 now reads

Theorem 1′. *Let G be a 4-odd-symmetric graph. Then*

$$\sum_{\text{matchings } m \text{ of } G} u(m) = \left(\sum_{\text{2-factors } f \text{ of } G_4} 2^{\substack{\text{number of nontrivial, contractible cycles} \\ +\ 1/2 \text{ number of noncontractible cycles}}} u(f) \right)^2.$$

This is a square in A if the number of vertices of G_4 is even and double a square otherwise.

The only changes in the proof are that all computations are done in B, and that whenever we sum over the k-factors of some graph, each summand is multiplied by the weight of the corresponding k-factor.

Similarly, Theorems 2 and 3 have weighted versions, which we will call Theorems 2′ and 3′. In these cases, our graphs could now have two different kinds of weights—the ones that arose in the proofs, and the ones we assigned to G_k. Whenever both kinds are present in a summand, we multiply them to find the total weight of the summand.

The author thanks the referee for the following corollary to Theorem 3′. We assume that the reader is acquainted with the definitions of plane partitions and their symmetry classes, which are contained in [S2]. For a class C of plane partitions, C_q denotes $\sum_{PP \in C} q^{|PP|}$.

Corollary.

$$PP(a, a, a)_{q^3} = CSPP(a)_q \, CSPP(a)_{\omega q} \, CSPP(a)_{\omega^2 q}.$$

Proof. By a bijection in [Ku], the left-hand side counts the matchings of the graph G in Fig. 5, with the weights shown; the weights have been chosen so that adding a cube to the plane partition will increase the weight of the corresponding matching by a factor of q^3. The right-hand side is a product of the weight sums of matchings of G_3, with three differents sets of weights coming from Theorem 3′.

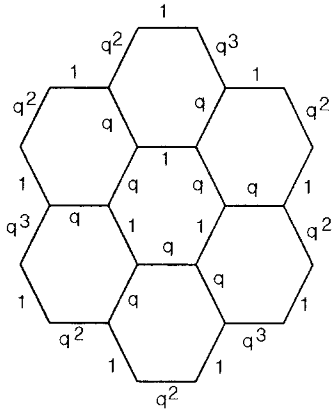


FIG. 5. A graph whose matchings are in bijection [Ku] with the plane partitions in the $2 \times 2 \times 2$ box. The weights have been chosen so that the weight of any matching is $q^{3|PP|}$, where $|PP|$ is the number of cubes in the corresponding plane partition. The bijection may be briefly described as follows. Think of the plane partition as a collection of cubes in the first octant. Look at it along the line $x = y = z$. You will see a tiling of the hexagon. Each tile is the union of two equilateral triangles. The matching is the dual of this tiling.

Remarks. 1. It is known [Ku] known that counting symmetry classes of size-constrained plane partitions is the same as counting symmetry classes of matchings of a certain hexagonal graph. Greg Kuperberg has recently counted the cyclically symmetric, self-complementary plane partitions [Ku], and George Andrews recently counted the totally symmetric, self-complementary plane partitions [A]; they found that the number of the former is the square of the number of the latter. Currently, no combinatorial explanation of this is known.

2. Here are some possible extensions of the present work:

- Can we find analogous of Theorems 1–3 for k-factors of a graph, where $k > 1$?

- Is there an analogue of Theorem 1 for $2k$-odd-symmetric graphs, $k > 2$? What if the graph also has reflective symmetry? (The case $k = 1$ seems unpromising as here the author has examples of such graphs with any arbitrary number of matchings. These graphs also have lines of reflective symmetry.)

- What about graphs on manifolds other than the punctured plane, e.g., the torus? The combinatorial methods of Theorems 1 and 2 do not seem to work because it is possible to have cycles with winding number other than 0 or ± 1. Kasteleyn gives (without proof) a way to write the number of matchings of a torus graph as the sum of four Pfaffians [K61, K67]. Does this help?

• A combinatorial proof of Theorem 3 in the spirit of the proofs of Theorems 1 and 2 would be interesting. The referee points out that in the case where k is a power of 2, Theorem 3 can be proved by induction, where each step is essentially the same as the proof of Theorem 2; the only difference is that we introduce all 2^k copies of the branch cut right from the start. The author's attempts to find such a proof in other cases have been stymied by the fact that in contrast to 2-factors, which decompose into a union of disjoint cycles, k-factors with $k>2$ do not seem to have a nice structure. Even if no such combinatorial proof exists, the author feels that there should be a simpler proof than the one given here. In particular, we should not need to assume that each edge of G appears in at least one matching.

• We still seek a bijective explanation of the mystery cited at the beginning of the Introduction. That is, is there a natural bijection between the domino tilings of the $4n \times 4n$ chessboard and $S(n) \times S(n)$, where $S(n)$ is some set? What about a bijection between the domino tilings of the $(4n+2) \times (4n+2)$ chessboard and $\{0, 1\} \times T(n) \times T(n)$, where $T(n)$ is some set? Is it possible to do this for other 4-odd-symmetric graphs? The problem seems to be difficult; the factor of 2 is especially troublesome.

6. Acknowledgments

This work was originally my Ph.D. thesis at MIT. It began shortly after James Propp, who later became my advisor, suggested that I write a general-purpose MATHEMATICA program to count (via the Pfaffian method) the matchings of plane graphs. Since then, I have had many long, helpful conversations with Propp and my first advisor, Richard Stanley. Propp also carefully read two earlier versions of this paper.

Note added in proof. Greg Kuperberg has found an analogue of Theorem 1 for $2k$-odd-symmetric graphs. [private communication]

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