Optimal Assignment of Due Dates and Starting Times to Identical Jobs on a Single Machine with Random Processing Time

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Abstract

We consider the problem of assigning optimal due dates and optimal starting times to a set of identical jobs on a single machine when processing time on the machine is random. There are N identical jobs ready to be scheduled on the machine and processing time on the machine is random with known distribution. We assume that the same raw material is required for all jobs to start and that it is available at no additional cost. There is an earliness/tardiness cost for finishing a job one unit of time past/prior to its due date . There is also a cost for quoting an uncompetitive due date for each job in the set, this cost being zero if the quoted due date does not exceed a certain "acceptable" value A. The objective is to minimize the expected total cost of quoting the due dates and scheduling the jobs in the set. The optimal due dates and the optimal starting times are determined analytically. They are

the unique solutions to a set of first order conditions. We show that there exists an optimal solution where the due date of the last job to be processed is at least equal to A. We also show that the optimal starting time for a particular job in the set is described by a simple *wait-until* policy. This optimal policy is completely determined by a single critical number, which represents the optimal planned lead time for that job. We also show that the optimal planned lead times are non-increasing with the position of the job in the sequence, with the exception of the planned lead time of the first job to be processed being the smallest. Finally we show that adding a job to a preexisting set of jobs results in quoting earlier (or the same) due dates to the jobs in the former set.

1 The Problem

1.1 Introduction

A set of N identical jobs are ready to be scheduled for processing on a machine. The optimal due dates for these identical jobs need to be quoted before any processing occurs on the machine. We assume that the same raw material is required for all jobs to start and that it is available at no additional cost. The machine cannot process more than one job at a time. The jobs consist of projects that must be completed once started in order to be delivered to different customers, hence preemption is not allowed. Since all jobs are identical with the same cost structure, they will be scheduled in an Earliest Due Date (EDD) sequence. Denote by job N the job with the earliest due date, hence the job to be started first. The processing time τ at the machine is random with known distribution F. Once the due dates d_i^*, $i = 1,..,N$ of the jobs are quoted, it is required to determine the optimal starting policy $y_i^*(l_i, d_{i-1}, ..., d_1)$, i.e. the optimal waiting time before starting job i, $i = 1, ..., N$, given that d_i is l_i units of time away and given the quoted due dates $d_{i-1}, ..., d_1$. Obviously, $y_N^*(l_N, d_{N-1}, ..., d_1) = 0$. This observation stems from the fact that having assumed job N is ready to be processed, we would like to quote its due date as early as possible, hence $l_N \equiv d_N$. The objective is to minimize the cost of quoting the due dates and scheduling jobs N through 1. A holding cost h per unit time is incurred if a job is completed before its quoted due date and a shortage cost p per unit time is incurred otherwise. The cost of quoting an uncompetitive due date is $C(.)$, assumed to be a strictly increasing function of the due date, convex, continuous, twice differentiable, and zero for a due date no greater than the acceptable limit A (Jones [10]). A is a value determined by the market and by the customer conception of how long is he or she willing to wait before her order is delivered.

1.2 Background

Considerable research has been done on assigning optimal due-dates for the single machine scheduling problem with earliness/tardiness penalties. In their surveys, Baker [1] and Cheng [3] report of no analytical work done with the machine having random processing time. Further work with deterministic processing time have been done by De, assuming a given common due date in [7] and assigning distinct due dates in [8], and by Cheng [4] assigning the same time window (flow allowance) to all jobs. Random machine processing time has been considered in conjunction with random due dates as in De [6] and Emmons [9] with the objective of minimizing the weighted number of tardy jobs. We are not aware of any past research that considers random processing time and assigns distinct optimal due dates with earliness and tardiness penalties. Cheng [5] considers a model with random due dates and tardiness/earliness penalties. However, the model does not assign optimal due dates and neither does it determine optimal starting times on the machine for the different jobs in the set. The objective of our paper is to develop a methodology for determining optimal starting times and optimal due dates for a set of jobs ready to be processed on a single machine, in order to minimize the total production and due dates quoting costs in a random processing time environment. This paper is organized as follows. In section 2, we analyze the case when $N = 2$. We determine the optimal due dates d_2^* and d_1^* and the optimal starting policy $y_1^*(l_1)$ for job 1. We show in this section that for $N = 2$, there exists an optimal solution where the due date of the second job to be processed is at least equal to A and that it is bounded by the sum of the highest realization of the processing time on the machine and the optimal planned lead time of the second job to be processed. In section 3, we analyze the case when $N = 3$ to illustrate the derivation of the optimal starting policy whenever there is more than one remaining job to be processed, a situation that does not occur when $N = 2$. Hence for $N = 3$, we determine $y_2^*(l_2, d_1)$, $y_1^*(l_1)$, d_3^*, d_2^* and d_1^*. We show in this section that the optimal planned lead time of the second job to be processed, which completely determines the optimal starting policy of that job, is at least equal to the optimal planned lead time of the third job to be processed. We also discuss the effect of adding a third job on the optimal due date of the second job to be processed and show that it decreases (or stays the same). In section 4, we discuss the economic interpretation of the first order conditions that give rise to the optimal due dates and to the optimal starting policy and we discuss in section 5 the managerial insights provided by the practical results obtained in sections 2 and 3. We generalize in section 6 for $N > 3$. Section 6 may be skipped if the reader is not interested in the mathematics. We conclude in section 7 by suggesting some further directions in research.

2 Two-Jobs Model

Suppose that $N = 2$. We use backward stochastic dynamic programming to determine d_2^*, d_1^* and $y_1^*(l_1)$. The first stage is triggered when job 2 is done processing. Figure 1 depicts the time advances in a two-jobs model. The first stage problem is defined as following:

$$J_1^*(l_1) = \text{Min}_{y_1 \geq 0} h \int_0^{l_1 - y_1} [(l_1 - y_1) - t] f_1(t)\, dt + p \int_{l_1 - y_1}^{\infty} [t - (l_1 - y_1)] f_1(t)\, dt \qquad (1)$$

where the first term is the expected holding cost and the second term is the expected shortage cost. It can be easily checked that $J_1(l_1)$ is convex in y_1 by differentiating it twice. Therefore, the optimal solution $y_1^*(l_1)$ to the first stage problem is obtained by differentiating equation (1) with respect to y_1 and setting to zero. Doing this we get the following *wait-until* policy, where we wait $l_1 - X_1^*$ units of time before processing the job if $l_1 \geq X_1^*$, and process immediately otherwise.

$$y_1^*(l_1) = \begin{cases} l_1 - X_1^* & \text{if } l_1 \geq X_1^* \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

where $X_1^* = F_1^{-1}[p/(p+h)]$ is called the optimal planned lead time for job 1. Figure 1 shows that $l_1 = d_1 - \tau_2$. Hence the second stage problem is defined as following:

$$\begin{aligned} \text{Min } J_2(d_2, d_1) &= C(d_2) + C(d_1) + h\int_0^{d_2}(d_2 - u) f_2(u)\,du + \\ & \quad p\int_{d_2}^{\infty}(u - d_2) f_2(u)\,du + E[J_1^*(d_1 - \tau_2)] \\ & \quad \text{s.t. } d_1 \geq d_2 \geq 0 \end{aligned} \quad (3)$$

Our goal is to show that the Hessian of $J_2(d_2, d_1)$ is non-negative. The Hessian of the first four terms is non-negative by assumption and from the first stage analysis. Suppose that $J_1^*(l_1)$ is convex in l_1, then we are done. Our goal is to show that $J_1^*(l_1)$ is convex in l_1. Substituting (2) in (1), we get

$$J_1^*(l_1) = \begin{cases} h\int_0^{X_1^*}(X_1^* - t) f_1(t)\,dt + p\int_{X_1^*}^{\infty}(t - X_1^*) f_1(t)\,dt & l_1 \geq X_1^* \\ h\int_0^{l_1}(l_1 - t) f_1(t)\,dt + p\int_{l_1}^{\infty}(t - l_1) f_1(t)\,dt & X_1^* \geq l_1 \end{cases} \quad (4)$$

It is easy to see that (4) is continuous and differentiable at $l_1 = X_1^*$. Finally, differentiating $J_1^*(l_1)$ twice shows that it is convex in l_1 and hence the Hessian of $J_2(d_2, d_1)$ is non-negative. To determine d_2^* and d_1^*, we substitute l_1 by $(d_1 - \tau_2)$ in (4), apply the expectation operator, differentiate (3) with respect to d_2 and d_1 and set to zero. Doing this we get

$$
\begin{aligned}
E\left[J_1^*\left(d_1-\tau_2\right)\right] = {} & h \int_{d_1-X_1^*}^{d_1} \int_0^{d_1-u} \left(d_1-u-t\right) f_1(t) f_2(u)\, dt du + \\
& p \int_{d_1-X_1^*}^{d_1} \int_{d_1-u}^{\infty} \left(t+u-d_1\right) f_1(t) f_2(u)\, dt du + \\
& p \int_{d_1}^{\infty} \left(\mu_1+u-d_1\right) f_2(u)\, du + \\
& \left[h \int_0^{X_1^*} \left(X_1^*-t\right) f_1(t)\, dt + p \int_{X_1^*}^{\infty} \left(t-X_1^*\right) f_1(t)\, dt\right] \int_0^{d_1-X_1^*} f_2(u)\, du
\end{aligned} \tag{5}
$$

and hence

$$
\begin{aligned}
\frac{\delta J_2\left(d_2,d_1\right)}{\delta d_1} = {} & C'\left(d_1\right) + E'\left[J_1^*\left(d_1-\tau_2\right)\right] \\
= {} & C'\left(d_1\right) + \left[h \int_0^{X_1^*} \left(X_1^*-t\right) f_1(t)\, dt + p \int_{X_1^*}^{\infty} \left(t-X_1^*\right) f_1(t)\, dt\right] f_2\left(d_1-X_1^*\right) + \\
& h \int_{d_1-X_1^*}^{d_1} \int_0^{d_1-u} f_1(t) f_2(u)\, dt du - h f_2\left(d_1-X_1^*\right) \int_0^{X_1^*} \left(X_1^*-t\right) f_1(t)\, dt - \\
& p \int_{d_1-X_1^*}^{d_1} \int_{d_1-u}^{\infty} f_1(t) f_2(u)\, dt du + p f_2\left(d_1\right) \int_0^{\infty} t f_1(t)\, dt - \\
& p f_2\left(d_1-X_1^*\right) \int_{X_1^*}^{\infty} \left(t-X_1^*\right) f_1(t)\, dt - p \int_{d_1}^{\infty} f_2(u)\, du - p\mu_1 f_2\left(d_1\right) = 0
\end{aligned} \tag{6}
$$

$$
\frac{\delta J_2\left(d_2,d_1\right)}{\delta d_2} = C'\left(d_2\right) + h \int_0^{d_2} f_2(u)\, du - p \int_{d_2}^{\infty} f_2(u)\, du = 0 \tag{7}
$$

which reduce to

$$
\begin{aligned}
\frac{\delta J_2\left(d_2,d_1\right)}{\delta d_1} = {} & C'\left(d_1\right) + E'\left[J_1^*\left(d_1-\tau_2\right)\right] \\
= {} & C'\left(d_1\right) + h \int_{d_1-X_1^*}^{d_1} \int_0^{d_1-u} f_1(t) f_2(u)\, dt du - \\
& p \int_{d_1-X_1^*}^{d_1} \int_{d_1-u}^{\infty} f_1(t) f_2(u)\, dt du - p \int_{d_1}^{\infty} f_2(u)\, du = 0
\end{aligned} \tag{8}
$$

$$
\frac{\delta J_2\left(d_2,d_1\right)}{\delta d_2} = C'\left(d_2\right) + h \int_0^{d_2} f_2(u)\, du - p \int_{d_2}^{\infty} f_2(u)\, du = 0 \tag{9}
$$

$d_2^* \geq 0$ can be determined easily from (9). d_2^* satisfies

$$
d_2^* = F^{-1}\left[\frac{p-C'\left(d_2^*\right)}{p+h}\right] \leq X_1^* \tag{10}
$$

Proposition 1 $d_1^* \geq d_2^*$

Proof: To show that $d_1^* \geq d_2^*$, we substitute d_1 by d_2^* in (8). Doing this we get

$$\begin{aligned}
\frac{\delta J_2(d_2,d_1)}{\delta d_1}|_{d_1=d_2^*} &= C'(d_2^*) + h\int_0^{d_2^*}\int_0^{d_2^*-u} f_1(t) f_2(u)\, dtdu - \\
& p\int_0^{d_2^*}\int_{d_2^*-u}^{\infty} f_1(t) f_2(u)\, dtdu - p\int_{d_2^*}^{\infty} f_2(u)\, du \\
&= C'(d_2^*) + (h+p)\int_0^{d_2^*}\int_0^{d_2^*-u} f_1(t) f_2(u)\, dtdu - p \\
&\leq C'(d_2^*) + (h+p)\int_0^{d_2^*} f_2(u)\, du - p = \frac{\delta J_2(d_2,d_1)}{\delta d_2}|_{d_2=d_2^*} = 0
\end{aligned}$$

Proposition 2 *There exists an optimal solution with* $d_2^* \geq A$.

Proof: $E'[J_1^*(d_1-\tau_2)]$ is non-decreasing in d_1 and $E'[J_1^*(d_1-\tau_2)] = 0$ for $d_1 \geq \overline{\tau} + X_1^*$ where $\overline{\tau}$ is the largest realization of the machine processing time. This can be shown by differentiating $E[J_1^*(d_1-\tau_2)]$ twice with respect to d_1. Doing this we get

$$\begin{aligned}
\frac{\delta^2 E[J_1^*(d_1-\tau_2)]}{\delta d_1^2} &= h\int_{d_1-X_1^*}^{d_1} f_1(d_1-u) f_2(u)\, du - h\int_0^{X_1^*} f_2(d_1-X_1^*) f_1(t)\, dt \\
&+p\int_{d_1-X_1^*}^{d_1} f_1(d_1-u) f_2(u)\, du - p\int_0^{\infty} f_2(d_1-X_1^*) f_1(t)\, dt \\
&+p\int_{X_1^*}^{\infty} f_2(d_1-X_1^*) f_1(t)\, dt + pf_2(d_1-X_1^*) \\
&= (h+p)\int_{d_1-X_1^*}^{d_1} f_1(d_1-u) f_2(u)\, du \geq 0
\end{aligned}$$

Hence $E'[J_1^*(d_1-\tau_2)] \leq 0$. Note also that $C'(d_1) = 0$ for $d_1 \leq A$. Therefore

$$d_1^* = \begin{cases} \{x, x \in [\overline{\tau} + X_1^*, A]\} & \text{if } \overline{\tau} + X_1^* \leq A \\ \geq A & \text{otherwise} \end{cases} \tag{11}$$

3 Three-Jobs Model

Before extending the problem to N jobs, it is necessary to analyze the case when there are three jobs in order to illustrate the derivation of $y_2^*(l_2, d_1)$. In a three jobs problem, the starting time of job 2 depends on the optimal planned lead time of job 1. Figure 2 depicts the time advances in a three jobs problem. In a three jobs problem, the decision variables are d_3, d_2 and d_1 at stage 3, y_2 at stage 2 and y_1 at stage 1. y_1^* is given by the optimal starting policy defined in (2). To determine y_2^*, we solve

$$\begin{aligned} J_2^*(l_2, d_1) &= \text{Min}_{y_2 \geq 0} \; h \int_0^{l_2 - y_2} [(l_2 - y_2) - t] \, f_2(t) \, dt + \\ & \quad p \int_{l_2 - y_2}^{\infty} [t - (l_2 - y_2)] \, f_2(t) \, dt + E[J_1^*(l_1)] \end{aligned} \quad (12)$$

To solve (12), we substitute l_1 by $(l_2 - y_2 + d_1 - d_2 - \tau_2)$ in (4), substitute $(l_2 - y_2)$ by X_2 in (12), differentiate (12) with respect to X_2 and set it to zero. Note that convexity in X_2 is conserved since $J_1^*(l_1)$ was shown to be convex in l_1 in a two jobs problem and the first two terms are convex. Doing this, we obtain a first order condition containing all the integral terms in (9) and (8), but with X_2 and $(X_2 + d_1 - d_2)$ instead of d_2 and d_1 respectively. That is

$$\begin{aligned} \frac{dJ_2(l_2, d_1)}{dX_2} &= h \int_0^{X_2} f_2(u) \, du - p \int_{X_2}^{\infty} f_2(u) \, du + E'[J_1^*(X_2 + d_1 - d_2 - \tau_2)] \quad (13) \\ &= h \int_0^{X_2} f_2(u) \, du - p \int_{X_2}^{\infty} f_2(u) \, du + \\ & \quad h \int_{X_2 + d_1 - d_2 - X_1^*}^{X_2 + d_1 - d_2} \int_0^{X_2 + d_1 - d_2 - u} f_1(t) \, f_2(u) \, dt du - \\ & \quad p \int_{X_2 + d_1 - d_2 - X_1^*}^{X_2 + d_1 - d_2} \int_{X_2 + d_1 - d_2 - u}^{\infty} f_1(t) \, f_2(u) \, dt du - \\ & \quad p \int_{X_2 + d_1 - d_2}^{\infty} f_2(u) \, du = 0 \end{aligned} \quad (14)$$

and hence y_2^* is given by the following *wait-until* starting policy:

$$y_2^*(l_2, d_1) = \begin{cases} l_2 - X_2^* & \text{if } l_2 \geq X_2^* \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

where X_2^*, the optimal planned lead time of job 2, satisfies (14).

Proposition 3 $X_1^* \leq X_2^* \leq \overline{\tau} + X_1^*$

Proof: This proposition is true since substituting X_2 by X_1^* in (13) gives

$$\begin{aligned}\frac{dJ_2(l_2,d_1)}{dX_2}|_{X_2=X_1^*} &= h\int_0^{X_1^*} f_2(u)\,du - p\int_{X_1^*}^{\infty} f_2(u)\,du + E'\left[J_1^*(X_1^*+d_1-d_2-\tau_2)\right] \\ &= E'\left[J_1^*(X_1^*+d_1-d_2-\tau_2)\right] \leq 0\end{aligned}$$

Hence $X_2^* \geq X_1^*$. Also, $X_2^* \leq \overline{\tau}+X_1^*$ since substituting X_2 by $(\overline{\tau}+X_1^*)$ in (14)gives $h \geq 0$. To determine d_3^*, d_2^* and d_1^*, we solve

$$\begin{aligned}\text{Min } J_3(d_3,d_2,d_1) &= C(d_3)+C(d_2)+C(d_1)+ \\ & h\int_0^{d_3}(d_3-v)f_3(v)\,dv + p\int_{d_3}^{\infty}(v-d_3)f_3(v)\,dv + \\ & E\left[J_2^*(d_2-\tau_3,d_1)\right] \qquad (16) \\ & \text{s.t. } d_1 \geq d_2 \geq d_3 \geq 0\end{aligned}$$

Our goal is to show that the Hessian of $J_3(d_3,d_2,d_1)$ is non-negative. To show that, we substitute d_2 by (d_3+r_2) and d_1 by $(d_3+r_2+r_1)$ and show that the Hessian of $J_3(d_3,r_2,r_1)$ is non-negative. This implies that the Hessian of $J_3(d_3,d_2,d_1)$ is non-negative by Theorem 3.4 in Rockafellar [12]. After making the abovementionned substitutions the problem becomes

$$\begin{aligned}\text{Min } J_3(d_3,r_2,r_1) &= C(d_3)+C(d_3+r_2)+C(d_3+r_2+r_1)+ \\ & h\int_0^{d_3}(d_3-v)f_3(v)\,dv + p\int_{d_3}^{\infty}(v-d_3)f_3(v)\,dv + \\ & E\left[J_2^*(d_3+r_2-\tau_3,r_1)\right] \qquad (17) \\ & \text{s.t. } d_3,r_2,r_1 \geq 0\end{aligned}$$

Our goal now becomes to show that the Hessian of $J_3(d_3,r_2,r_1)$ is non-negative. Clearly the Hessian of the first five terms is non-negative. Suppose that the Hessian of $J_2^*(l_2, r_1)$ is non-negative, then we are done. To show that the Hessian of $J_2^*(l_2, r_1)$ is non-negative we substitute (15) in (12) and get

$$J_2^*(l_2, r_1) = \begin{cases} h\int_0^{X_2^*}(X_2^* - u) f_2(u)\, du + p\int_{X_2^*}^{\infty}(u - X_2^*) f_2(u)\, du + & \\ E[J_1^*(X_2^* + r_1 - \tau_2)] & l_2 \geq X_2^* \\ h\int_0^{l_2}(l_2 - u) f_2(u)\, du + p\int_{l_2}^{\infty}(u - l_2) f_2(u)\, du + & \\ E[J_1^*(l_2 + r_1 - \tau_2)] & X_2^* \geq l_2 \end{cases} \tag{18}$$

Using (14), it is easy to see that (18) is continuous and differentiable at $l_2 = X_2^*$. Since $J_1^*(l_1)$ is convex, then the Hessian of $J_2^*(l_2, r_1)$ is non-negative and therefore the Hessian of $J_3(d_3, r_2, r_1)$ is non-negative. As a result, the Hessian of $J_3(d_3, d_2, d_1)$ is non-negative using Theorem 3.4 in Rockafellar [12]. To determine d_3^*, d_2^* and d_1^*, we substitute l_2 by $(d_2 - \tau_3)$, apply the expectation operator, differentiate (16) with respect to d_3, d_2 and d_1 and set to zero. Doing this we get

$$\begin{aligned} E[J_2^*(d_2 - \tau_3, d_1)] &= h\int_0^{d_2 - X_2}\int_0^{X_2}(X_2 - u) f_2(u) f_3(v)\, du dv + \\ & p\int_0^{d_2 - X_2}\int_{X_2}^{\infty}(u - X_2) f_2(u) f_3(v)\, du f dv + \\ & h\int_{d_2 - X_2}^{d_2}\int_0^{d_2 - v}(d_2 - v - u) f_2(u) f_3(v)\, du dv + \\ & p\int_{d_2 - X_2}^{d_2}\int_{d_2 - v}^{\infty}(u + v - d_2) f_2(u) f_3(v)\, du dv + \\ & p\int_{d_2}^{\infty}(\mu_2 + v - d_2) f_3(v)\, dv + \\ & E[J_1^*(X_2 + d_1 - d_2 - \tau_2)]\int_0^{d_2 - X_2} f_3(v)\, dv + \\ & \int_{d_2 - X_2}^{\infty} E[J_1^*(d_1 - v - \tau_2)] f_3(v)\, dv \end{aligned}$$

and hence

$$\begin{aligned} \frac{\delta J_3(d_3, d_2, d_1)}{\delta d_1} &= C'(d_1) + E'[J_1^*(X_2 + d_1 - d_2 - \tau_2)]\int_0^{d_2 - X_2} f_3(v)\, dv + \\ & \int_{d_2 - X_2}^{\infty} E'[J_1^*(d_1 - \tau_2 - v)] f_3(v)\, dv \end{aligned} \tag{19}$$

Substituting we get

$$\begin{aligned}
\frac{\delta J_3(d_3,d_2,d_1)}{\delta d_1} &= C'(d_1) + \left[h\int_{X_2+d_1-d_2-X_1^*}^{X_2+d_1-d_2}\int_0^{X_2+d_1-d_2-u} f_1(t) f_2(u)\,dtdu - \right.\\
&p\int_{X_2+d_1-d_2-X_1^*}^{X_2+d_1-d_2}\int_{X_2+d_1-d_2-u}^{\infty} f_1(t) f_2(u)\,dtdu - \\
&\left. p\int_{X_2+d_1-d_2}^{\infty} f_2(u)\,du\right]\int_0^{d_2-X_2} f_3(v)\,dv + \\
&h\int_{d_2-X_2}^{d_1-X_1^*}\int_{d_1-X_1^*-v}^{d_1-v}\int_0^{d_1-u-v} f_1(t) f_2(u) f_3(v)\,dtdudv + \\
&h\int_{d_1-X_1^*}^{d_1}\int_0^{d_1-v}\int_0^{d_1-u-v} f_1(t) f_2(u) f_3(v)\,dtdudv - \\
&p\int_{d_2-X_2}^{d_1-X_1^*}\int_{d_1-X_1^*-v}^{d_1-v}\int_{d_1-u-v}^{\infty} f_1(t) f_2(u) f_3(v)\,dtdudv - \\
&p\int_{d_2-X_2}^{d_1-X_1^*}\int_{d_1-v}^{\infty} f_2(u) f_3(v)\,dudv - \\
&p\int_{d_1-X_1^*}^{d_1}\int_0^{d_1-v}\int_{d_1-u-v}^{\infty} f_1(t) f_2(u) f_3(v)\,dtdudv - \\
&p\int_{d_1-X_1^*}^{d_1}\int_{d_1-v}^{\infty} f_2(u) f_3(v)\,dudv - p\int_{d_1}^{\infty} f_3(v)\,dv = 0 \qquad (20)
\end{aligned}$$

We also have

$$\begin{aligned}
\frac{\delta J_3(d_3,d_2,d_1)}{\delta d_2} &= C'(d_2) + h\int_{d_2-X_2}^{d_2}\int_0^{d_2-v} f_2(u) f_3(v)\,dudv - \\
&p\int_{d_2-X_2}^{d_2}\int_{d_2-v}^{\infty} f_2(u) f_3(v)\,dudv - p\int_{d_2}^{\infty} f_3(v)\,dv - \\
&E'\left[J_1^*(X_2+d_1-d_2-\tau_2)\right]\int_0^{d_2-X_2} f_3(v)\,dv \\
&= C'(d_2) + h\int_{d_2-X_2}^{d_2}\int_0^{d_2-v} f_2(u) f_3(v)\,dudv - \\
&p\int_{d_2-X_2}^{d_2}\int_{d_2-v}^{\infty} f_2(u) f_3(v)\,dudv - p\int_{d_2}^{\infty} f_3(v)\,dv - \\
&\left[h\int_{X_2+d_1-d_2-X_1^*}^{X_2+d_1-d_2}\int_0^{X_2+d_1-d_2-u} f_1(t) f_2(u)\,dtdu - \right.\\
&p\int_{X_2+d_1-d_2-X_1^*}^{X_2+d_1-d_2}\int_{X_2+d_1-d_2-u}^{\infty} f_1(t) f_2(u)\,dtdu - \\
&\left. p\int_{X_2+d_1-d_2}^{\infty} f_2(u)\,du\right]\int_0^{d_2-X_2} f_3(v)\,dv = 0 \qquad (21)
\end{aligned}$$

and finally

$$\frac{\delta J_3(d_3,d_2,d_1)}{\delta d_3} = C'(d_3) + h\int_0^{d_3} f_3(v)\,dv - p\int_{d_3}^{\infty} f_3(v)\,dv = 0 \tag{22}$$

d_3^* is obtained from (22). Clearly d_3^* is equal to the due date of the first job to be processed in a two-jobs problem, hence $d_3^* \leq X_1^* \leq X_2^*$. d_2^*, d_1^* and X_2^* are determined by solving simultaneously (14), (20) and (21).

Proposition 4 $d_3^* \leq d_2^* \leq \overline{\tau} + X_2^*$

Proof: To show that $d_3^* \leq d_2^*$, we substitute d_2 by d_3^* in (21) and use the fact that $d_3^* \leq X_2^*$. Doing this we get

$$\begin{aligned}
\frac{\delta J_3(d_3,d_2,d_1)}{\delta d_2}|_{d_2=d_3^*} &= C'(d_3^*) + h\int_0^{d_3^*}\int_0^{d_3^*-v} f_2(u)\,f_3(v)\,dudv - \\
&\quad p\int_0^{d_3^*}\int_{d_3^*-v}^{\infty} f_2(u)\,f_3(v)\,dudv - p\int_{d_3^*}^{\infty} f_3(v)\,dv \\
&= C'(d_3^*) + (h+p)\int_0^{d_3^*}\int_0^{d_3^*-v} f_2(u)\,f_3(v)\,dudv - p \\
&\leq C'(d_3^*) + (h+p)\int_0^{d_3^*} f_3(v)\,dv - p = \frac{\delta J_3(d_3,r_2,r_1)}{\delta d_3}|_{d_3=d_3^*} = 0
\end{aligned}$$

To show that $d_2^* \leq \overline{\tau} + X_2^*$, we substitute d_2 by $(\overline{\tau} + X_2^*)$ in (21). Doing this we get

$$\frac{\delta J_3(d_3,d_2,d_1)}{\delta d_2}|_{d_2=\overline{\tau}+X_2} = C'(\overline{\tau}+X_2) - E'\left[J_1^*(d_1-\overline{\tau}-\tau_2)\right] \geq 0$$

Proposition 5 $d_2^* \leq d_1^* \leq 2\overline{\tau} + X_1^*$

Proof: To show that $d_1^* \geq d_2^*$, we substitute d_1 by d_2^* in (20), d_2 by d_2^* in (21) and substitute $C'(d_2^*)$ in (20) from (21). Doing this we get

$$\frac{\delta J_3(d_3,d_2,d_1)}{\delta d_1}|_{d_1=d_2^*} = C'(d_2^*) + E'\left[J_1^*(X_2-\tau_2)\right]\int_0^{d_2^*-X_2} f_3(v)\,dv +$$

$$
\begin{aligned}
& h\int_{d_2^*-X_2}^{d_2^*-X_1^*}\int_{d_2^*-X_1^*-v}^{d_2^*-v}\int_0^{d_2^*-u-v} f_1(t)\,f_2(u)\,f_3(v)\,dtdudv + \\
& h\int_{d_2^*-X_1^*}^{d_2^*}\int_0^{d_2^*-v}\int_0^{d_2^*-u-v} f_1(t)\,f_2(u)\,f_3(v)\,dtdudv - \\
& p\int_{d_2^*-X_2}^{d_2^*-X_1^*}\int_{d_2^*-X_1^*-v}^{d_2^*-v}\int_{d_2^*-u-v}^{\infty} f_1(t)\,f_2(u)\,f_3(v)\,dtdudv - \\
& p\int_{d_2^*-X_2}^{d_2^*-X_1^*}\int_{d_2^*-v}^{\infty} f_2(u)\,f_3(v)\,dudv - \\
& p\int_{d_2^*-X_1^*}^{d_2^*}\int_0^{d_2^*-v}\int_{d_2^*-u-v}^{\infty} f_1(t)\,f_2(u)\,f_3(v)\,dtdudv - \\
& p\int_{d_2^*-X_1^*}^{d_2^*}\int_{d_2^*-v}^{\infty} f_2(u)\,f_3(v)\,dudv - p\int_{d_2^*}^{\infty} f_3(v)\,dv = 0 \\
= \;& 2E'\left[J_1^*(X_2-\tau_2)\right]\int_0^{d_2^*-X_2} f_3(v)\,dv + \\
& h\int_{d_2^*-X_2}^{d_2^*-X_1^*}\int_{d_2^*-X_1^*-v}^{d_2^*-v}\int_0^{d_2^*-u-v} f_1(t)\,f_2(u)\,f_3(v)\,dtdudv + \\
& h\int_{d_2^*-X_1^*}^{d_2^*}\int_0^{d_2^*-v}\int_0^{d_2^*-u-v} f_1(t)\,f_2(u)\,f_3(v)\,dtdudv - \\
& p\int_{d_2^*-X_2}^{d_2^*-X_1^*}\int_{d_2^*-X_1^*-v}^{d_2^*-v}\int_{d_2^*-u-v}^{\infty} f_1(t)\,f_2(u)\,f_3(v)\,dtdudv - \\
& p\int_{d_2^*-X_2}^{d_2^*-X_1^*}\int_{d_2^*-v}^{\infty} f_2(u)\,f_3(v)\,dudv - \\
& p\int_{d_2^*-X_1^*}^{d_2^*}\int_0^{d_2^*-v}\int_{d_2^*-u-v}^{\infty} f_1(t)\,f_2(u)\,f_3(v)\,dtdudv - \\
& p\int_{d_2^*-X_1^*}^{d_2^*}\int_{d_2^*-v}^{\infty} f_2(u)\,f_3(v)\,dudv - p\int_{d_2^*}^{\infty} f_3(v)\,dv - \\
& h\int_{d_2^*-X_2}^{d_2^*}\int_0^{d_2^*-v} f_2(u)\,f_3(v)\,dudv + \\
& p\int_{d_2^*-X_2}^{d_2^*}\int_{d_2^*-v}^{\infty} f_2(u)\,f_3(v)\,dudv + p\int_{d_2^*}^{\infty} f_3(v)\,dv \qquad (23)
\end{aligned}
$$

We have shown that $E'\left[J_1^*(X_2-\tau_2)\right] \leq 0$ in proposition 2. Furthermore, comparing the holding cost coefficients in (23) we get

$$
\begin{aligned}
& \int_{d_2^*-X_2}^{d_2^*-X_1^*}\int_{d_2^*-X_1^*-v}^{d_2^*-v}\int_0^{d_2^*-u-v} f_1(t)\,f_2(u)\,f_3(v)\,dtdudv + \\
& \quad \int_{d_2^*-X_1^*}^{d_2^*}\int_0^{d_2^*-v}\int_0^{d_2^*-u-v} f_1(t)\,f_2(u)\,f_3(v)\,dtdudv \leq
\end{aligned}
$$

$$\int_{d_2^*-X_2}^{d_2^*-X_1^*} \int_0^{d_2^*-v} \int_0^{d_2^*-u-v} f_1(t) f_2(u) f_3(v)\, dtdudv +$$
$$\int_{d_2^*-X_1^*}^{d_2^*} \int_0^{d_2^*-v} \int_0^{d_2^*-u-v} f_1(t) f_2(u) f_3(v)\, dtdudv =$$
$$\int_{d_2^*-X_2^*}^{d_2^*} \int_0^{d_2^*-v} \int_0^{d_2^*-u-v} f_1(t) f_2(u) f_3(v)\, dtdudv \leq$$
$$\int_{d_2^*-X_2}^{d_2^*} \int_0^{d_2^*-v} f_2(u) f_3(v)\, dudv$$

Comparing the shortage cost coefficients in (23), the terms with single and double integrals cancel and only the non-positive triple integrals terms remain, resulting in $d_1^* \geq d_2^*$. To show $d_1^* \leq 2\overline{\tau} + X_1^*$, we substitute d_1 by $(2\overline{\tau} + X_1^*)$ in (19). Doing this we get

$$\begin{aligned}\frac{\delta J_3(d_3, d_2, d_1)}{\delta d_1}\Big|_{d_1=2\overline{\tau}+X_1^*} &= C'(2\overline{\tau} + X_1^*) + \\ &\quad E'[J_1^*(X_2 + 2\overline{\tau} + X_1^* - d_2 - \tau_2)] \int_0^{d_2-X_2} f_3(v)\, dv + \\ &\quad \int_{d_2-X_2}^{\infty} E'[J_1^*(2\overline{\tau} + X_1^* - \tau_2 - v)] f_3(v)\, dv \\ &= C'(2\overline{\tau} + X_1^*) \geq 0\end{aligned}$$

since $E'[J_1^*(d_1 - \tau_2)] = 0$ for $d_1 \geq \overline{\tau} + X_1^*$ from proposition 2 and $d_2^* \leq \overline{\tau} + X_2$ from proposition 4.

Before leaving the three jobs problem we want to compare the optimal due date of the second job to be processed in a two jobs problem to the optimal due date of the second job to be processed in a three jobs problem to study the effect of adding a third job on the optimal due date of the second job to be processed.

Proposition 6 *Adding a third job results in quoting earlier (or the same) due dates to the two preexisting jobs in the set.*

Proof: In a two-jobs problems, d_1^* is given by (8) rewritten as

$$C'(d_1) = -E'[J_1^*(d_1 - \tau_2)] \tag{24}$$

where

$$\frac{\delta^2 E\left[J_1^*\left(d_1-\tau_2\right)\right]}{\delta d_1^2}=(h+p)\int_{d_1-X_1^*}^{d_1} f_1\left(d_1-u\right) f_2(u)\, du \geq 0 \tag{25}$$

as was shown in proposition 2. In a three jobs problem, d_2^* satisfies equation (21) rewritten as

$$\begin{aligned} C'\left(d_2\right) &= -h\int_{d_2-X_2}^{d_2}\int_0^{d_2-v} f_2(u) f_3(v)\, du dv+ \\ & p\int_{d_2-X_2}^{d_2}\int_{d_2-v}^{\infty} f_2(u) f_3(v)\, du dv+p\int_{d_2}^{\infty} f_3(v)\, dv+ \\ & E'\left[J_1^*\left(X_2+d_1-d_2-\tau_2\right)\right]\int_0^{d_2-X_2} f_3(v)\, dv \end{aligned} \tag{26}$$

Differentiating the right-hand side of (26) with respect to d_2 we get

$$\begin{aligned} & -h\int_{d_2-X_2}^{d_2} f_2\left(d_2-v\right) f_3(v)\, dv+h\int_0^{X_2} f_3\left(d_2-X_2\right) f_2(u)\, du- \\ & p\int_{d_2-X_2}^{d_2} f_2\left(d_2-v\right) f_3(v)\, dv+p\int_0^{\infty} f_3\left(d_2-X_2\right) f_2(u)\, du- \\ & p\int_{X_2}^{\infty} f_3\left(d_2-X_2\right) f_2(u)\, du-p f_3\left(d_2-X_2\right)+ \\ & f_3\left(d_2-X_2\right) E'\left[J_1^*\left(X_2+d_1-d_2-\tau_2\right)\right] \\ = & -(h+p)\int_{d_2-X_2}^{d_2} f_2\left(d_2-v\right) f_3(v)\, dv+ \\ & f_3\left(d_2-X_2\right)\left[h\int_0^{X_2} f_2(u)\, du-p\int_{X_2}^{\infty} f_2(u)\, du-E'\left[J_1^*\left(X_2+d_1-d_2-\tau_2\right)\right]\right] \\ = & -(h+p)\int_{d_2-X_2}^{d_2} f_2\left(d_2-v\right) f_3(v)\, dv \leq 0 \end{aligned} \tag{27}$$

The right-hand sides of (24) and (26) are equal at $d_1=0$ and $d_2=0$ respectively. Furthermore, the derivative of the right-hand side of equation (26) given by (27), is steeper than the derivative of the right-hand side of equation (24) given by the negative of (25), that is

$$\int_{d_2-X_2}^{d_2} f_2\left(d_2-v\right) f_3(v)\, dv \geq \int_{d_1-X_1^*}^{d_1} f_1\left(d_1-u\right) f_2(u)\, du$$

since $X_1^* \leq X_2^*$. As a result, the right-hand side of (26) intersects $C'(d_2)$ at a smaller value than the one at which the right-hand side of (24) intersects $C'(d_1)$, i.e. $d_2^* \leq d_1^*$ and hence the optimal due date of the second job to be processed in a two jobs problem is at least equal to the optimal due date of the second job to be processed in a three jobs problem.

4 Economic Interpretation

In this problem, the due dates must be quoted before any processing occurs on the machine. However, due to the randomness in the processing times, once the due dates have been quoted and processing has started, then the starting time of the next job in the sequence must be determined given the quoted optimal due dates and the optimal planned lead times of the jobs remaining to be processed. In this section we shall provide an economic interpretation to the first-order conditions that give rise to the optimal due dates (equations (21), (20) and (8)) and the optimal planned lead times (equation (14)), in the two-jobs and three-jobs problems analyzed in the previous sections.

4.1 Optimal Starting Times

Consider the three jobs problem. Suppose that there remains one job that has not been processed yet and whose due date have been already set. Then its starting time is determined by (2), determined completely by the solution of the classical Newsvendor problem which balances the tardiness cost p and the earliness cost h to find the optimal starting time X_1^*. The problem is more complicated when there remains two unprocessed jobs whose due dates have already been set. As in the previous case, the starting time for the next job is determined by (15), determined completely by the solution to (14).

Equation (14) has a very appealing economic interpretation. It can be rewritten as

$$h\left[Pr\left\{\tau_2 \leq X_2\right\} + Pr\left\{\tau_2 \geq X_2 - X_1^* + r_1, \tau_1 + \tau_2 \leq X_2 + r_1\right\}\right]$$
$$- p\left[Pr\left\{\tau_2 \geq X_2\right\} + Pr\left\{\tau_2 \geq X_2 - X_1^* + r_1, \tau_1 + \tau_2 \geq X_2 + r_1\right\}\right] = 0 \qquad (28)$$

where r_1 is defined as in figure 2. Equation (28) illustrates the combined impact of the marginal holding and shortage costs associated with each of the two jobs, on the optimal planned lead time decision X_2^*, i.e. the time window comprising the next job (job 2 in our case). The effect of job 2 is the one of the Newsvendor problem, indicated by the first probability term inside the marginal holding and shortages cost brackets in the left-hand side of (28). The effect of the second job (job 1) on the current decision is less myopic in nature. Marginal savings in holding cost due to waiting an extra unit of time before starting job 2 are achieved only if the processing time of job 2 continues past the predetermined starting time of the next job *and* job 1 processing time does not end past its due date. While the second condition is a reminder of the savings achieved in the Newsvendor problem, the first condition stresses the non-myopicity of the decision process, in the sense that no marginal savings in holding cost of job 1 due to waiting an extra unit of time before starting job 2 are achieved if some slack time is realized between the completion of job 2 and the start of job 1. Similarly, the marginal increases in shortage cost of job 1 due to waiting an extra unit of time before starting job 2 occur only if the processing time of job 2 continues past the predetermined starting time of the next job *and* job 1 processing time does end past its due date. Equivalently, no marginal increases in shortage cost of job 1 due to waiting an extra unit of time before starting job 2 are incurred if some slack time is realized between the completion of job 2 and the start of job 1. This information agrees with the intuition that job 1 has no impact on the starting time of job 2 if it is certain that some slack time will be realized after the completion of job 2. In other words, if $X_2^* \leq X_1^* - r_1$, then it is predetermined *a priori* that no slack is allowed between the two jobs and job 1 is rushed immediately after the completion of job

2. In that case

$$Pr\{y_1^* \leq 0\} = Pr\{\tau_2 \geq X_2^* - (X_1^* - r_1^*)\} = 1$$

For a three jobs problem, we have shown that $X_2^* \geq X_1^* \geq X_1^* - r_1^*$ hence we never decide *a priori* to rush the next job and X_2^* is indeed determined by (28). This property can be generalized for larger number of jobs. We prove it for any number of jobs in the next section. There exist scenarios in production where the manager must plan for the processing of two jobs in series and must decide *a priori* to rush immediately the second job independently of the realization of the first job.

4.1.1 A Serial Production System Revisited

Consider for example a serial production line consisitng of two stages, where the processing time at stage 2 and 1 (stage 2 is the upstream stage), τ_2 and τ_1 are random with distribution F_2 and F_1. Suppose that an order has been placed and a due date has been set at some point in the future. Assume that the order consists of a project and that once production is started at a stage, it must be completed. Assume also that raw material is available at no additional cost, there is a cost h_2 and h_1 for holding an extra unit of time the semi-finished product at stage 2 and the finished product at stage 1 respectively, there is a penalty p for finishing production one unit of time beyond the quoted due date d and naturally $h_2 \leq h_1 \leq p$. Yano [11] considers this problem and solves the two-stage problem. However, the mathematical approach that Yano adopted renders the analysis of three-stage problem and larger extremely difficult. By formulating the problem using backwards dynamic programming we are able to provide additional insight into the optimal solution of larger problems. What is required to be determined in this problem is $y_2^*(l_2)$ and $y_1^*(l_1)$, the optimal starting time at stage 2 and 1 with the due date being l_2 and l_1 units of time away respectively. To determine $y_1^*(l_1)$ we solve

$$J_1^*(l_1) = \text{Min}_{y_1 \geq 0} \quad h_2(l_1 - y_1) + h_1 \int_0^{l_1 - y_1} [(l_1 - y_1) - t] f_1(t)\, dt +$$

$$p\int_{l_1-y_1}^{\infty}\left[t-(l_1-y_1)\right]f_1(t)\,dt \tag{29}$$

The objective function is clearly convex in y_1 and therefore $y_1^*(l_1)$ is given by (2) where $X_1^* = F^{-1}\left[(h_2+p)/(h_1+p)\right]$ is the optimal planned lead time for job 1. To determine $y_2^*(l_2)$ we solve

$$J_2^*(l_2) = \text{Min}_{y_2\geq 0}E\left[J_1^*(l_2-y_2-\tau_2)\right] \tag{30}$$

$J_1^*(l_1)$ is convex in l_1 using the same arguments as in (4), hence $J_2(l_2)$ is convex in y_2. We substitute (l_2-y_2) by X_{21}, differentiate with respect to X_{21} and set to zero (note that convexity in X_{21} is conserved). The result is that $y_2^*(l_2)$ is given by (15), where X_{21}^*, the optimal cumulative planned lead time for jobs 2 and 1, satisfies

$$\begin{aligned}\frac{dJ_2(X_2)}{dX_2} &= h_1\int_{X_{21}-X_1^*}^{X_{21}}\int_0^{X_{21}-u}f_1(t)\,f_2(u)\,dtdu + h_2\int_0^{X_{21}-X_1^*}f_2(u)\,du \\ &-p\left(\int_{X_{21}-X_1^*}^{X_{21}}\int_{X_{21}-u}^{\infty}f_1(t)\,f_2(u)\,dtdu + \int_{X_{21}}^{\infty}f_2(u)\,du\right)=0\end{aligned} \tag{31}$$

which may be interpreted as

$$\begin{aligned}\frac{dJ_2(X_{21})}{dX_{21}} &= h_1Pr\{\tau_2\geq X_{21}-X_1^*,\tau_2+\tau_1\leq X_{21}\} + h_3Pr\{\tau_2\geq X_{21}-X_1^*\} \\ &-pPr\{\tau_2\geq X_{21}-X_1^*,\tau_2+\tau_1\geq X_{21}\}=0\end{aligned} \tag{32}$$

The obective of this paper is not to analyze the single period serial production system, but we have carried this analysis of the serial production system to illustrate a situation when it is decided *a priori* to rush the processing at stage 1 immedialtely after processing is done at stage 2, i.e. when no slack is allowed between two successive jobs independently of the processing time realization of the first job to have been processed. In this serial production system, the slack time between job 2 and job 1 is y_1 and no slack is achieved with certainty by having $Pr\{y_1^* \leq 0\} = 1$. Equivalently

$$Pr\{y_1^* \leq 0\} = Pr\{\tau_2 \geq X_{21}^* - X_1^*\} = 1$$

which does occur, but only whenever $X_{21}^* \leq X_1^*$. This result is counter intuitive since it means that the presence of job 2 has resulted in the allocation of a smaller safety time for both jobs 2 and 1 than the safety time that was originally allocated to job 1 alone. Mathematically, evaluating the first-order derivative at $X_{21} = X_1^*$ gives

$$\frac{dJ_2(X_{21})}{dX_{21}}|_{X_{21}=X_1^*} = (h_1 + p)\, Pr\{\tau_2 + \tau_1 \leq X_1^*\} - p \tag{33}$$

which may be either positive or negative. Hence there may be instances when h_2, the holding cost at stage 2, is so high (with respect to the holding cost of raw materials assumed zero here) that it becomes more economical to wait long enough before releasing job 2 so as to ensure with certainty that when the processing at stage 2 is completed, the semi-finished product is not held at stage 2 but rather rushed immediately because the time remaining till the due date is less than its allocated planned lead time. It turns out that to ensure no waiting at stage 2, we must not release job 2 before we are X_1^* units of time away from the due date, hence the cumulative planned lead time of job 2 and 1 being smaller than the planned lead time that was originally allocated to job 1 alone. In this case, $X_{21}^* \leq X_1^*$ is determined by solving

$$\frac{dJ_2(X_{21})}{dX_{21}} = (h_1 + p)\, Pr\{\tau_2 + \tau_1 \leq X_{21}\} - p = 0$$

since (33) is non-negative and $J_2(X_2)$ is convex. Here X_{21}^* is simply $F_{21}^{-1}[p/(h_1 + p)]$ and the two stages are pooled into a single stage whose processing time distribution is the convolution of the individual processing time distributions at stage 2 and 1, as Yano indicated in [11]. The problem becomes significantly more complicated for serial production systems with more than two stages. Consider for instance a three stage problem with

$h_3 \leq h_2$. Using the same arguments as in a two-stage problem, the first order condition in a 3-stage problem can take one of the two forms, depending on whether or not stages 2 and 1 were pooled. If they were not pooled and X_{21}^* is indeed determined by (32), i.e. $X_{21}^* \geq X_1^*$, then the first order condition is given by

$$\begin{aligned}\frac{dJ_3(X_{31})}{dX_{31}} = \; & h_1 Pr\{\tau_3 \geq X_{31} - X_{21}^*, \tau_3 + \tau_2 \geq X_{31} - X_1^*, \tau_3 + \tau_2 + \tau_1 \leq X_{31}\} + \\ & h_2 Pr\{\tau_3 \geq X_{31} - X_{21}^*, \tau_3 + \tau_2 \leq X_{31} - X_1^*\} + \\ & h_3 Pr\{\tau_3 \leq X_{31} - X_{21}^*\} - \\ & p Pr\{\tau_3 \geq X_{31} - X_{21}^*, \tau_3 + \tau_2 \geq X_{31} - X_1^*, \tau_3 + \tau_2 + \tau_1 \geq X_{31}\} \qquad (34)\end{aligned}$$

If they were pooled, then the 3-stage problem is treated as a 2-stage problem where lead time τ_{21} at the new stage 21 has a distribution F_{21}, the convolution of F_1 and F_2, and lead time τ_3 at stage 3 has a distribution F_3. We also have if stages 2 and 1 are pooled that $X_{21}^* = F_{21}^{-1}[(p+h_3)/(p+h_1)] \leq F_1^{-1}[(p+h_2)/(p+h_1)] = X_1^*$ and in (34), $\tau_3 \geq X_{31} - X_{21}^* \Rightarrow \tau_3 + \tau_2 \geq X_{31} - X_1^*$. Thus the first-order condition becomes

$$\begin{aligned}\frac{dJ_3(X_{31})}{dX_{31}} = \; & h_1 Pr\{\tau_3 \geq X_{31} - X_{21}^*, \tau_3 + \tau_{21} \leq X_{31}\} + \\ & h_3 Pr\{\tau_3 \leq X_{31} - X_{21}^*\} - \\ & p Pr\{\tau_3 \geq X_{31} - X_{21}^*, \tau_3 + \tau_{21} \geq X_{31}\} \qquad (35)\end{aligned}$$

In this latter case, it should be clear by now that due to convexity, stage 3 is pooled with the new stage 21 if and only if (35) is nonnegative at $X_{31} = X_{21}^*$. If this is the case, we pool stages 1,2,3 and set $X_{31}^* = F_{31}^{-1}[p/(p+h_1)] \leq X_{21}^* \leq X_1^*$. Otherwise, $X_{31}^* \geq X_{21}^*$ and X_{31}^* is the unique solution to (35) set to zero. The case when stage 1 and 2 are not pooled is more complicated. Here, one of the three things can happen: a) Pool 3, 2 and 1, b) Pool 3 and 2, c) do not pool. We present the algorithm that determines the optimal configuration and hence X_{31}^*. The algorithm is based on the convexity of $J_3(X_{31})$ in X_{31}, and the fact that stage 1 and 2 are not pooled i.e. $X_{21}^* \geq X_1^*$ where X_{21}^* solves (32) and

$X_1^* = F_1^{-1}\left[(p+h_2)\,/\,(p+h_1)\right]$.

Algorithm

1) If (34) is nonnegative at $X_{31} = X_1^*$, then a) is optimal and $X_{31}^* = F_{31}^{-1}\left[p/\left(p+h_1\right)\right] \leq X_1^*$.

2) Otherwise, if (34) is negative at $X_{31} = X_{21}^*$, then b) is optimal, $X_1^* \leq X_{31}^* \leq X_{21}^*$ and we set X_{31}^* at the unique root of

$$\begin{aligned}\frac{dJ_3\left(X_{31}\right)}{dX_{31}} &= h_1 Pr\left\{\tau_3+\tau_2 \geq X_{31}-X_1^*, \tau_3+\tau_2+\tau_1 \leq X_{31}\right\}+ \\ &\quad h_2 Pr\left\{\tau_3+\tau_2 \leq X_{31}-X_1^*\right\}+ \\ &\quad pPr\left\{\tau_3+\tau_2 \geq X_{31}-X_1^*, \tau_3+\tau_2+\tau_1 \geq X_{31}\right\}=0\end{aligned}$$

3) Otherwise, c) is optimal, $X_1^* \leq X_{21}^* \leq X_{31}^*$ and X_{31}^* is the unique solution to (34) set to zero.

In contrast, the solution to the problem considered in this paper is never such that we decide *a priori* to rush the second job immediately after the completion of the first job and independently of the realization of the processing time. However, $X_2^* \geq X_1^* - r_1^*$ and X_{21}^* satisfies (28) indeed. The reason why it is never economical to pool the two jobs into a single job whose processing time distribution is the convolution of the individual processing time distributions is the following. Assume that the same due date has been quoted for both jobs, i.e. $r_1^* = 0$. Then (28) becomes

$$\begin{aligned}&h\left[Pr\left\{\tau_2 \leq X_2\right\}+Pr\left\{\tau_2 \geq X_2-X_1^*, \tau_1+\tau_2 \leq X_2\right\}\right] \\ &-p\left[Pr\left\{\tau_2 \geq X_2\right\}+Pr\left\{\tau_2 \geq X_2-X_1^*, \tau_1+\tau_2 \geq X_2\right\}\right]=0 \qquad (36)\end{aligned}$$

and X_2 can be seen now as the cumulative planned lead time for both jobs. Comparing (36) with (32), one can see from the joint events terms that the impact of job 1 on the starting time of job 2 is the same. However, while in (36) h_2 may be so high that it becomes more economical to start the processing of job 2 together with having less than X_1^* remaining till the due date, it is never the case in (32) since no matter how high is h, the holding cost of job 2, the shortage cost guarantees that safety time must be increased when a second job is added. It is the shortage cost that forces us to have safety time in the Newsvendor problem, and it is the shortage cost that guarantees that safety time must be increased when a second job with the same cost structure is added at the same due date. As a result, $X_2^* \geq X_1^* - r_1^*$, X_{21}^* satisfies (28) indeed and hence computations of optimal planned lead times when the due dates have been set in advance for problems with more than two jobs are significantly simplified, knowing that $X_j^* \geq X_i^*$ for $j \geq i$.

4.2 Optimal Due Dates

Equations (21), (20) and (8) also have an appealing economic interpretation. They can be rewritten respectively as

$$C'(d_2^*) = -hPr\{\tau_3 \geq d_2^* - X_2, \tau_{32} \leq d_2^*\} + pPr\{\tau_3 \geq d_2^* - X_2, \tau_{32} \geq d_2^*\} \quad (37)$$

$$C'(d_1^*) = -hPr\{\tau_3 \geq d_2^* - X_2, \tau_{32} \leq d_1^* - X_1^*, \tau_{31} \leq d_1^*\} + pPr\{\tau_3 \geq d_2^* - X_2, \tau_{32} \geq d_1^* - X_1^*, \tau_{31} \geq d_1^*\} \quad (38)$$

$$C'(d_1^*) = -hPr\{\tau_2 \geq d_1^* - X_1^*, \tau_{21} \leq d_1^*\} + pPr\{\tau_2 \geq d_1^* - X_1^*, \tau_{21} \geq d_1^*\} \quad (39)$$

The marginal costs associated with the first job to be processed are obvious as illustrated in (9) and (22) for a two and three jobs problem respectively. Determining the optimal due date for the next job in the sequence is slightly more complicated. Consider (37). For a three jobs problem, the marginal increases in holding cost associated with job 2 due to quoting a due date one unit of time longer are incurred only if job 3 is completed past the predetermined starting time of job 2 *and* job 3 is completed before its quoted due date. In other words, no marginal costs in holding cost are incurred due to delaying delivery one unit of time if some slack is realized after the completion of job 3. On the other hand, marginal savings in shortage cost associated with job 2 due to quoting a due date one unit of time longer are acheived only if job 3 is completed past the predetermined starting time of job 2 *and* job 3 is completed after its due date. For each job, the combined marginal effects of increases in holding cost and savings in shortage cost is negatively decreasing with increasing values for the quoted due date of that job, as the positively decreasing right-hand side of equations (37) and (38) indicate. In other words, the tardiness argument is stronger than the earliness argument for job 2 and 1. Consider job 2 and equation (37). This is due to the fact that marginal savings and marginal increases occur jointly, only when there is no slack after the completion of job 3. Moreover, savings occur only if the processing time of job 2 exceeds its due date, while increases occur only if it does not. As a result, the rate of the marginal savings is positive and the rate of marginal increases is negative because the higher the due date of job 2, the more likely the processing time of job 2 will exceeds it if no slack is going to be realized after the completion of job 3. Equation (37) and (38) illustrate the intuitive fact that if there was no cost for quoting an uncompetitive due date, then one would quote due date values at least equal to $\overline{\tau} + X_2^*$ for job 2 and $2\overline{\tau} + X_1^*$ for job 1. However if that cost exists and $A \geq 2\overline{\tau} + X_1^*$, then $\overline{\tau} + X_2^* \leq d_2^* \leq A$ and $\overline{\tau} + X_1^* - X_2^* + d_2^* \leq d_1^* \leq A$ as shown by the shaded area in figure 3a. In figure 3a, the shaded area represents the set of multiple optima (d_1^*, d_2^*) when the first order conditions of the unrealistic problem corresponding to the case of $A \to \infty$ are set to zero. This area of multiple optimal due date values represents the set of slacks $s_2 = d_2^* - (\overline{\tau} + X_2^*) \leq A - (\overline{\tau} + X_2^*)$ and $s_1 = d_1^* - (\overline{\tau} + X_1^* - X_2^* + d_2^*) \leq A - (\overline{\tau} + X_1^* - X_2^* + d_2^*)$ that the manager will have with probability 1 after the completion of job 3 and after the completion of job 2 respectively under the optimal starting policy. Figures 3b, c and d show how the feasible region varies with A. Note that we cannot have $d_1^* \geq d_2^* + \overline{\tau} + X_1^* - X_2^*$ and $d_2^* \leq \overline{\tau} + X_2^*$ in figure 3b, i.e. when $\overline{\tau} + X_2^* \leq A \leq 2\overline{\tau} + X_1^*$. If $d_1^* \geq d_2^* + \overline{\tau} + X_1^* - X_2^*$, then $E'\left[J_1^*\left(X_2 + d_1 - d_2 - \tau_2\right)\right] = 0$ in (21) and hence $d_2^* = \{x, x \in [\overline{\tau} + X_2^*, A]\}$. As a result, we have that if $A \leq 2\overline{\tau} + X_1^*$, then $\text{Max}\{A, d_2^*\} \leq d_1^* \leq 2\overline{\tau} + X_1^*$ and $d_3^* \leq d_2^* \leq \overline{\tau} + X_2^*$. Otherwise, the darkened area in figure 3a represents the set of multiple optima (d_1^*, d_2^*) where there exists an optimal solution $d_2^* \leq d_1^* = A$. Therefore there exists an optimal solution where $d_1^* \geq A$. For all due dates to be less than A, we must have that $d_1^* \leq A$, hence $2\overline{\tau} + X_1^* \leq A$. For all due dates to be more than A we must have $d_3^* \geq A$, hence $X_1^* \geq A$. $X_1^* \leq A \leq 2\overline{\tau} + X_1^*$ implies merely that $d_3^* \leq A$ and $d_1^* \geq A$ and does not provide additional information about d_2^*. $\overline{\tau} + X_2^* \leq A \leq 2\overline{\tau} + X_1^*$ implies naturally that $d_2^* \leq A$ since we have shown that $d_2^* \leq \overline{\tau} + X_2^*$, but $A \leq \overline{\tau} + X_2^*$ can result in $d_2^* \leq A$ as well as $d_2^* \geq A$ as can be seen in figure 3c and d. As a result, if the cost of quoting an uncompetitive the due date is linear to the right of A with slope c, it is more likely to quote A for the last job to be processed when c is high. Several additional observations can be made from equations (37) and (38). The higher is p and the smaller is h, the slower is the rate of negatively decreasing combined marginal effects of savings in holding cost and increases in shortage cost, hence the more likely that it is higher in abslolute value than $C'(A)$ and the further is the due date. Furthermore, the larger is the processing time variance, the higher is the term containing p and the less likely is that the due date is A. Therefore, the tradeoffs are that high p/low h and high variance increase the quoted due date, force us to produce early and keep a high chance of introducing slack time between the processing of consecutive jobs, while the cost of quoting an uncompetitive due date have the opposite effect and ensures that jobs are rushed without any slack in between. Finally, the analysis presented shows that $r_i^* \geq 0$, $i = 1, ..., N-1$ and hence quoting a common due date for all the jobs is suboptimal in single machine problems with random processing time and earliness/tardiness costs.

5 Managerial Insights

In this section, we discuss the managerial insights provided by the two main practical results obtained in sections 2 and 3, namely a) the optimal planned lead times are non-increasing with the position of the job in the sequence, with the exception of the planned lead time of the first job to be processed being the smallest and b) adding a job to a preexisting set of jobs results in quoting earlier (or the same) due dates to the jobs in the former set.

5.1 Effect of Additional Job on Optimal Planned Lead Times

The optimal planned lead times are non-increasing with the position of the job in the sequence, with the exception of the planned lead time of the first job to be processed being the smallest. This agrees with the intuition that jobs are processed earlier as their number increase. For instance, the second job is processed earlier if it is not the last one in line. This increase in planned lead time represents the added protection required due to the presence of a third job down the line, and the uncertainty induced with it. The planned lead times of the first and the last job would be the same if we would know the starting time of the last job with certainty. Intuititvely the first job to be processed has the smallest planned lead time because the starting time of the last job to be processed is uncertain whereas the starting time of the first job to be processed is immediate.

5.2 Effect of Additional Job on Optimal Due Dates

Adding a job to a preexisting set of jobs results in quoting earlier (or the same) due dates to the jobs in the former set. For instance, the due date of the second job to be processed was shown to decrease when a third job is added. This agrees with the intuition that due dates need to be quoted the earliest possible to avoid the due date cost. Hence if a job is added at the last minute, the manager ought to revise the quoted due dates for the preexisting jobs and reduce them since more jobs now are sharing time, the single resource in this problem.

6 Extension to N-Jobs

Extending the problem to $N > 3$ jobs, the problem becomes for $1 \leq i \leq N-1$:

$$\begin{aligned} J_i^*(l_i, r_{i-1}, ..., r_1) = \ \mathrm{Min}_{y_i \geq 0} \ & h \int_0^{l_i - y_i} [(l_i - y_i) - t] f_i(t)\, dt + p \int_{l_i - y_i}^{\infty} [t - (l_i - y_i)] f_i(t)\, dt + \\ & E\left[J_{i-1}^*(l_i - y_i + r_{i-1} - \tau_i, r_{i-2}, ..., r_1)\right] \end{aligned} \tag{40}$$

and for $i = N$:

$$\begin{aligned} \mathrm{Min}\ J_N(d_N, r_{N-1}, ..., r_1) \ = \ & C(d_N) + C(d_N + r_{N-1}) + ... + C(d_N + r_{N-1} + ... + r_1) + \\ & h \int_0^{d_N} (d_N - t) f_N(t)\, dt + p \int_{d_N}^{\infty} (t - d_N) f_N(t)\, dt + \\ & E\left[J_{N-1}^*(d_N + r_{N-1} - \tau_N, r_{N-2}, ..., r_1)\right] \\ & \text{s.t. } d_N, r_{N-1}, ..., r_1 \geq 0 \end{aligned} \tag{41}$$

where $r_i = d_{i+1} - d_i$, $i = 1, ..., N-1$.

Proposition 7 $y_i^*(l_i, d_{i-1}, ..., d_1) \equiv y_i^*(l_i, r_{i-1}, ..., r_1)$, *the optimal waiting time before processing of job i is started, given that d_i is l_i units of time away and given the quoted due dates $d_{i-1}, ..., d_1$, is expressed by*

$$y_i^*(l_i, r_{i-1}, ..., r_1) = \begin{cases} l_i - X_i^* & \text{if } l_i \geq X_i^* \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

where X_i^*, the optimal planned lead time of job i, solves $dJ_i(l_i, r_{i-1}, ..., r_1)/dX_i = 0$ (after subtituting $l_i - y_i$ by X_i).

It is true for $i = 1$ and 2. To prove this for $3 \leq i \leq N$, we assume that $J_{i-1}^*(l_{i-1}, r_{i-2}, ..., r_1)$ is convex in l_{i-1}, hence (42) is true for job i, and show that this implies $J_i^*(l_i, r_{i-1}, ..., r_1)$ is convex in l_i, hence (42) is true for job $i+1$. In fact, substituting $y_i^*(l_i, r_{i-1}, ..., r_1)$ in $J_i(l_i, r_{i-1}, ..., r_1)$, we get

$$J_i^*(l_i, r_{i-1}, ..., r_1) = \begin{cases} h\int_0^{X_i^*}(X_i^* - t) f_i(t)\,dt + p\int_{X_i^*}^{\infty}(t - X_i^*) f_i(t)\,dt + \\ E\left[J_{i-1}^*(X_i^* + r_{i-1} - \tau_i, r_{i-2}, ..., r_1)\right] & l_i \geq X_i^* \\ J_i(X_i + y_i, r_{i-1}, ..., r_1)\,|_{\{y_i=0, X_i=l_i\}} = \\ h\int_0^{l_i}(l_i - t) f_i(t)\,dt + p\int_{l_i}^{\infty}(t - l_i) f_i(t)\,dt + \\ E\left[J_{i-1}^*(l_i + r_{i-1} - \tau_i, r_{i-2}, ..., r_1)\right] & l_i \leq X_i^* \end{cases} \tag{43}$$

It is clearly convex in l_i for $l_i \leq X_i^*$ since the first 2 terms are convex in l_i and we assumed that $J_{i-1}^*(l_{i-1}, r_{i-2}, ..., r_1)$ is convex in l_{i-1}, hence convex in l_i. Furthermore $dJ_i^*(l_i, r_{i-1}, ..., r_1)/dl_i = 0$ at $l_i = X_i^*$ since we assumed that X_i^* solves the equation $dJ_i(l_i, r_{i-1}, ..., r_1)/dX_i = 0$ (after subtituting $l_i - y_i$ by X_i), hence solves the equation $dJ_i(X_i + y_i, r_{i-1}, ..., r_1)\,|_{\{y_i=0, X_i=l_i\}}/dl_i = 0$.

Proposition 8 *X_i^*, $1 \leq i \leq N-1$ solves the following equation (after subtituting $l_i - y_i$ by X_i):*

$$\begin{aligned} \frac{dJ_i(l_i, r_{i-1}, ..., r_1)}{dX_i} &= h\sum_{j=1}^{i} Pr\left\{\sum_{k=j}^{i}\tau_k \leq \sum_{k=j}^{i-1} r_k + X_i, \sum_{k=j+1}^{i}\tau_k \geq \sum_{k=j}^{i-1} r_k + X_i - X_j^*,\right. \\ &\left.\sum_{k=j+2}^{i}\tau_k \geq \sum_{k=j+1}^{i-1} r_k + X_i - X_{j+1}^*, ...\right\} - \\ &p\sum_{j=1}^{i} Pr\left\{\sum_{k=j}^{i}\tau_k \geq \sum_{k=j}^{i-1} r_k + X_i, \sum_{k=j+1}^{i}\tau_k \geq \sum_{k=j}^{i-1} r_k + X_i - X_j^*,\right. \\ &\left.\sum_{k=j+2}^{i}\tau_k \geq \sum_{k=j+1}^{i-1} r_k + X_i - X_{j+1}^*, ...\right\} = 0 \end{aligned} \tag{44}$$

It is true for $i = 1$ and 2. To prove this for $3 \leq i \leq N-1$, assume that it is true for i. $J_i^*(l_i, r_{i-1}, ..., r_1)$ is given by (43) and $J_{i+1}(l_{i+1}, r_i, ..., r_1)$ is given by (40). Substituting

l_i by $l_{i+1} - y_{i+1} + r_i - \tau_{i+1}$ in (40), letting $l_{i+1} - y_{i+1} = X_{i+1}$, differentiating (40) with respect to X^*_{i+1}, setting it to zero, and after doing further manipulations we get

$$\frac{dJ_{i+1}(l_{i+1}, r_i, ..., r_1)}{dX_{i+1}} = hPr\{\tau_{i+1} \le X_{i+1}\} - pPr\{\tau_{i+1} \ge X_{i+1}\} + E\left[\frac{dJ^*_i(X_{i+1} + r_i - \tau_{i+1}, r_{i-1}, ..., r_1)}{dX_{i+1}}\right] = 0 \qquad (45)$$

but from (43), we have that

$$\frac{dJ^*_i(X_{i+1} + r_i - \tau_{i+1}, r_{i-1}, ..., r_1)}{dX_{i+1}} = \begin{cases} 0 & \text{if } \tau_{i+1} \le X_{i+1} + r_i - X^*_i \\ \frac{dJ_i(l_i, r_{i-1}, ..., r_1)}{dl_i}\Big|_{l_i = X_{i+1} + r_i - \tau_{i+1}} & \text{otherwise} \end{cases} \qquad (46)$$

For $\tau_{i+1} \ge X_{i+1} + r_i - X^*_i$, it is given by the middle side of (44) evaluated at $X_i = X_{i+1} + r_i - \tau_{i+1}$. Substituting this latter in (45) gives the following first order condition:

$$h\left(Pr\{\tau_{i+1} \le X_{i+1}\} + \sum_{j=1}^{i}\left[\int_{X_{i+1}+r_i-X^*_i}^{\infty} Pr\left\{\sum_{k=j}^{i+1}\tau_k \le \sum_{k=j}^{i} r_k + X_{i+1}, \sum_{k=j+1}^{i+1}\tau_k \ge \sum_{k=j}^{i} r_k + X_{i+1} - X^*_j, \sum_{k=j+2}^{i+1}\tau_k \ge \sum_{k=j+1}^{i} r_k + X_{i+1} - X^*_{j+1}, ...\right\} f_{i+1}(u)\,du\right]\right) -$$
$$p\left(Pr\{\tau_{i+1} \ge X_{i+1}\} + \sum_{j=1}^{i}\left[\int_{X_{i+1}+r_i-X^*_i}^{\infty} Pr\left\{\sum_{k=j}^{i+1}\tau_k \ge \sum_{k=j}^{i} r_k + X_{i+1}, \sum_{k=j+1}^{i+1}\tau_k \ge \sum_{k=j}^{i} r_k + X_{i+1} - X^*_j, \sum_{k=j+2}^{i+1}\tau_k \ge \sum_{k=j+1}^{i} r_k + X_{i+1} - X^*_{j+1}, ...\right\} f_{i+1}(u)\,du\right]\right) = 0$$

which reduces to

$$\frac{dJ_{i+1}(l_{i+1}, r_i, ..., r_1)}{dX_{i+1}} = hPr\{\tau_{i+1} \le X_{i+1}\} - pPr\{\tau_{i+1} \ge X_{i+1}\} + h\sum_{j=1}^{i} Pr\left\{\sum_{k=j}^{i+1}\tau_k \le \sum_{k=j}^{i} r_k + X_{i+1}, \sum_{k=j+1}^{i+1}\tau_k \ge \sum_{k=j}^{i} r_k + X_{i+1} - X^*_j, \sum_{k=j+2}^{i+1}\tau_k \ge \sum_{k=j+1}^{i} r_k + X_{i+1} - X^*_{j+1}, ...\right\} -$$

$$h\sum_{j=1}^{i} Pr\left\{\sum_{k=j}^{i+1}\tau_k \geq \sum_{k=j}^{i} r_k + X_{i+1}, \sum_{k=j+1}^{i+1}\tau_k \geq \sum_{k=j}^{i} r_k + X_{i+1} - X_j^*, \right.$$
$$\left.\sum_{k=j+2}^{i+1}\tau_k \geq \sum_{k=j+1}^{i} r_k + X_{i+1} - X_{j+1}^*, ...\right\} = 0$$

and finally to

$$\frac{dJ_{i+1}\left(l_{i+1}, r_i, ..., r_1\right)}{dX_{i+1}} = h\sum_{j=1}^{i+1} Pr\left\{\sum_{k=j}^{i+1}\tau_k \leq \sum_{k=j}^{i} r_k + X_{i+1}, \sum_{k=j+1}^{i+1}\tau_k \geq \sum_{k=j}^{i} r_k + X_{i+1} - X_j^*, \right.$$
$$\left.\sum_{k=j+2}^{i+1}\tau_k \geq \sum_{k=j+1}^{i} r_k + X_{i+1} - X_{j+1}^*, ...\right\} -$$
$$h\sum_{j=1}^{i+1} Pr\left\{\sum_{k=j}^{i+1}\tau_k \geq \sum_{k=j}^{i} r_k + X_{i+1}, \sum_{k=j+1}^{i+1}\tau_k \geq \sum_{k=j}^{i} r_k + X_{i+1} - X_j^*, \right.$$
$$\left.\sum_{k=j+2}^{i+1}\tau_k \geq \sum_{k=j+1}^{i} r_k + X_{i+1} - X_{j+1}^*, ...\right\} = 0 \tag{47}$$

and we are done.

Proposition 9 $X_i^* \geq X_{i-1}^*$, $i = 2, ..., N-1$

Proposition 10 r_i^*, $i = 1, .., N-1$ *satisfy the following set of first-order conditions:*

$$C'\left(d_N^* + r_{(N-1)} + ... + r_i\right) = -hPr\left\{\sum_{k=i}^{N}\tau_k \leq \sum_{k=i}^{N-1} r_k + d_N, \sum_{k=i+1}^{n}\tau_k \geq \sum_{k=i}^{N-1} r_k + d_N - X_i, \right.$$
$$\left.\sum_{k=i+2}^{n}\tau_k \geq \sum_{k=i+1}^{N-1} r_k + d_N - X_{i+1}, ...\right\}$$
$$+pPr\left\{\sum_{k=i}^{N}\tau_k \geq \sum_{k=i}^{N-1} r_k + d_N, \sum_{k=i+1}^{N}\tau_k \geq \sum_{k=i}^{N-1} r_k + d_N - X_i, \right.$$
$$\left.\sum_{k=i+2}^{N}\tau_k \geq \sum_{k=i+1}^{N-1} r_k + d_N - X_{i+1}, ...\right\} = 0 \tag{48}$$

The proof is by differentiating (41) with respect to r_i and noting that the expression $\delta J_N\left(d_N, r_{N-1}, ..., r_1\right)/\delta r_i$ is nothing but the due date cost terms plus the terms containing r_i in (44), for $i = 1, ..., N-1$.

Proposition 11 *d_N^* is given by:*

$$C'(d_N) = -hPr\{\tau_N \leq d_N\} + pPr\{\tau_N \geq d_N\} \tag{49}$$

The proof is by differentiating (41) with respect to d_N and noting that the expression $\delta J_N(d_N, r_{N-1}, ..., r_1)/\delta d_N$ is nothing but the due date cost terms plus the terms containing X_N in (44).

Proposition 12 *Let d_i^{N*} be the quoted due date for job i in an N jobs problem.*

$$d_{(i+1)}^{(N+1)*} \leq d_i^{*N}, i = 1, ..., N. \tag{50}$$

7 Conclusion

We have considered the problem of assigning optimal due dates and starting times to a set of identical jobs ready to be processed. We have shown that optimal quoted due dates and optimal planned lead times can be obtained analytically by balancing the marginal effects of holding cost, shortage cost and cost of quoting an uncompetitive due date for each job, due to quoting a due date one unit of time longer. We have also shown that once the optimal due dates have been quoted and the optimal planned lead times have been determined, the optimal starting time for each job is described by a simple wait-until policy. Finally we have shown that the effect of an additional job is to increase the planned lead times and decrease the quoted due dates of the preexisting jobs, except for the first job to be proceesed. The first job to be processed, which is to be started immediately, is not affected by the addition of job to the set, and hence its quoted due date remains the same. The issue of sequencing the jobs was not raised because we assumed the jobs to be identical with same processing time distribution on the machine and same cost structure. However, a future direction on research could be one in which this assumption is relaxed. Hence it would be required to find the optimal sequence in which the jobs must be processed on the machine, their quoted due dates and the optimal starting time policy once processing has started. It would be interesting to determine necessary conditions on the cost structure and/or processing time distributions and parameters that will allow for some specific sequences to be optimal and to question whether these conditions are reasonable. Another direction in research may be the generalization of this model to serial production lines and flow shops.

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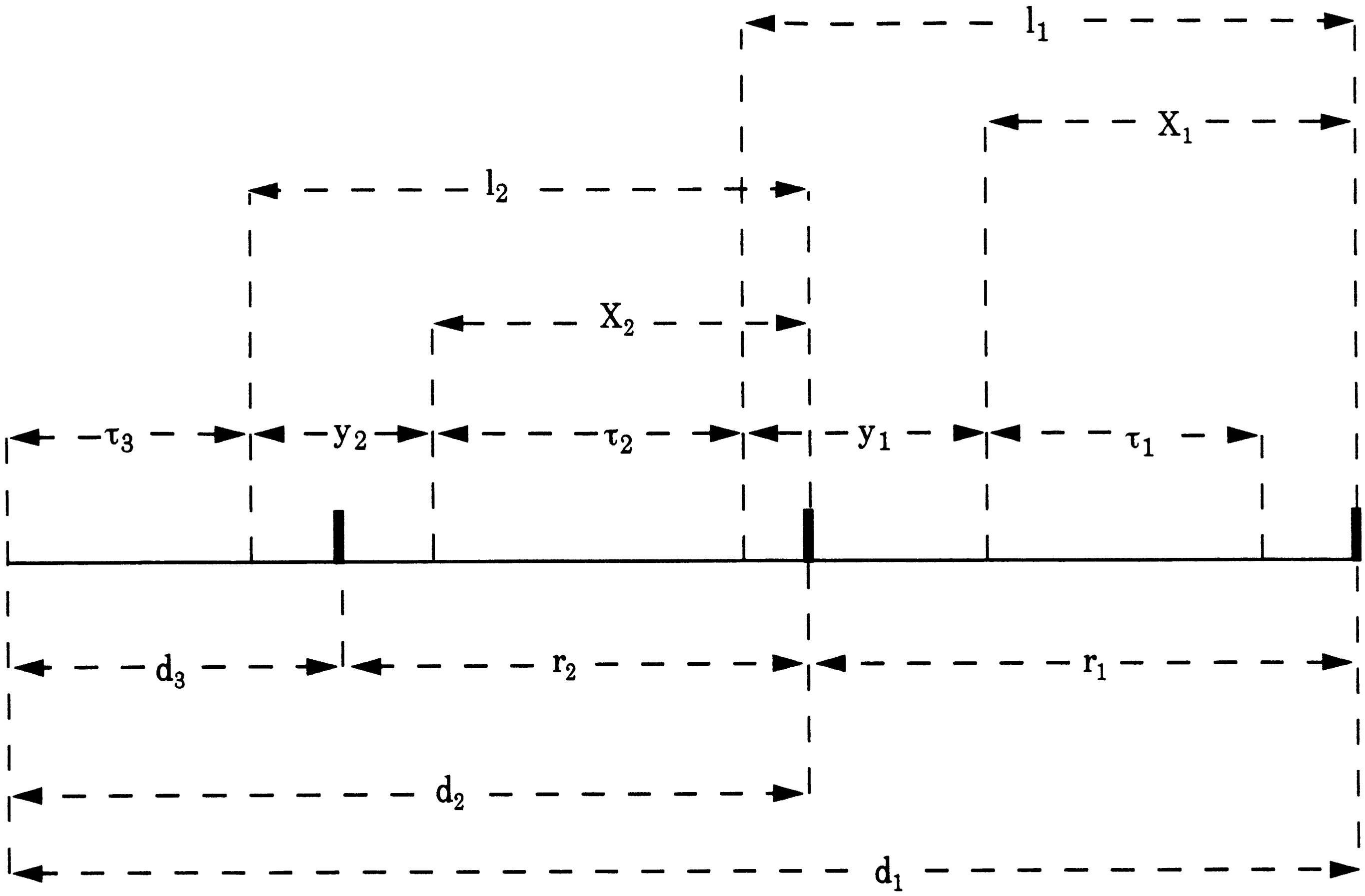

Figure 2

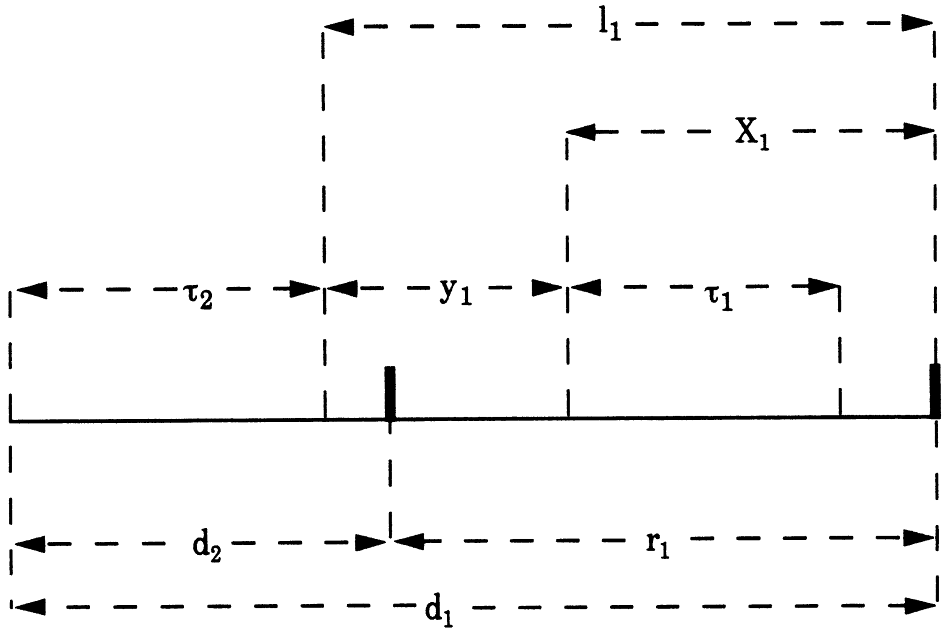

Figure 1

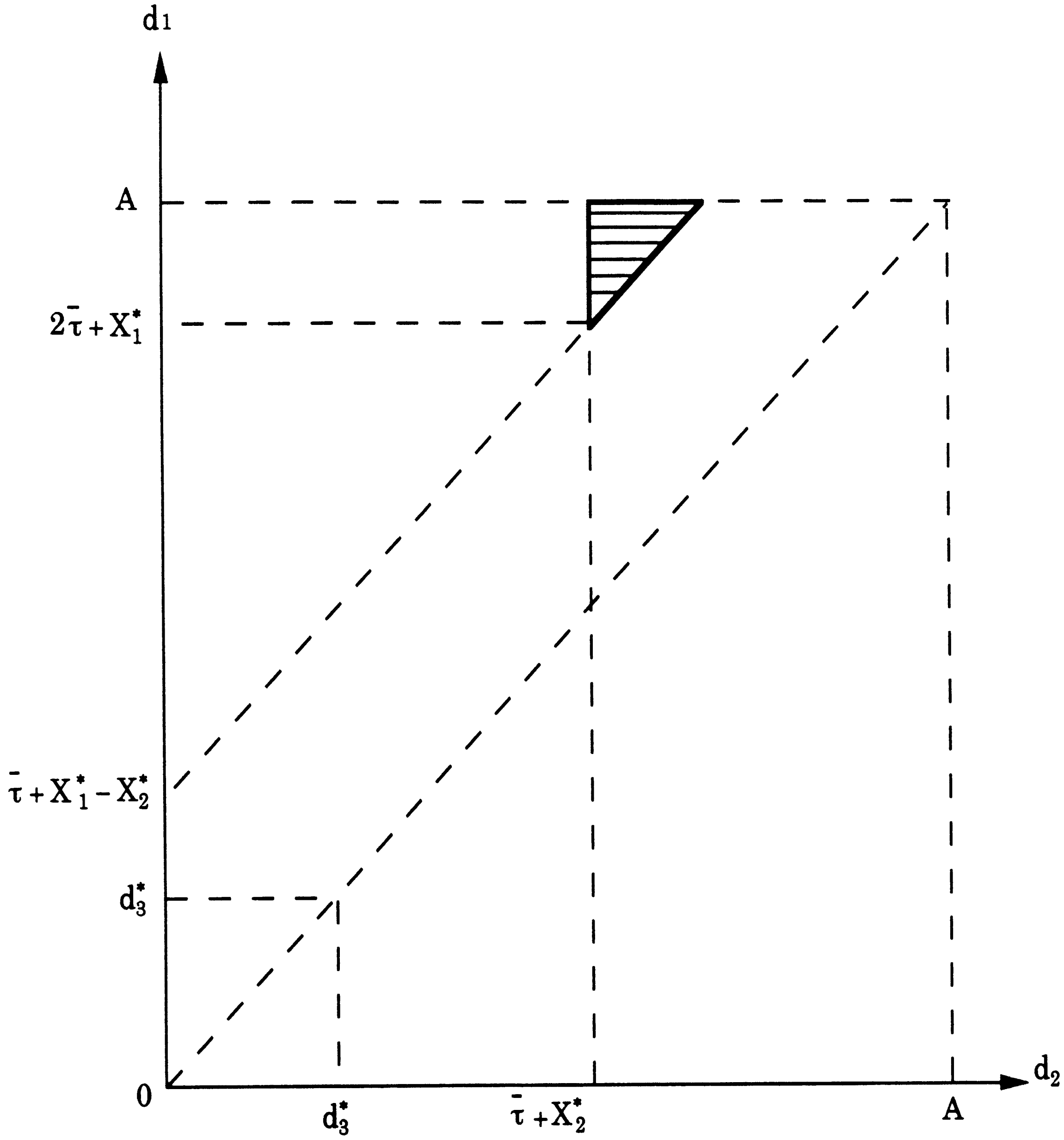

Figure 3a

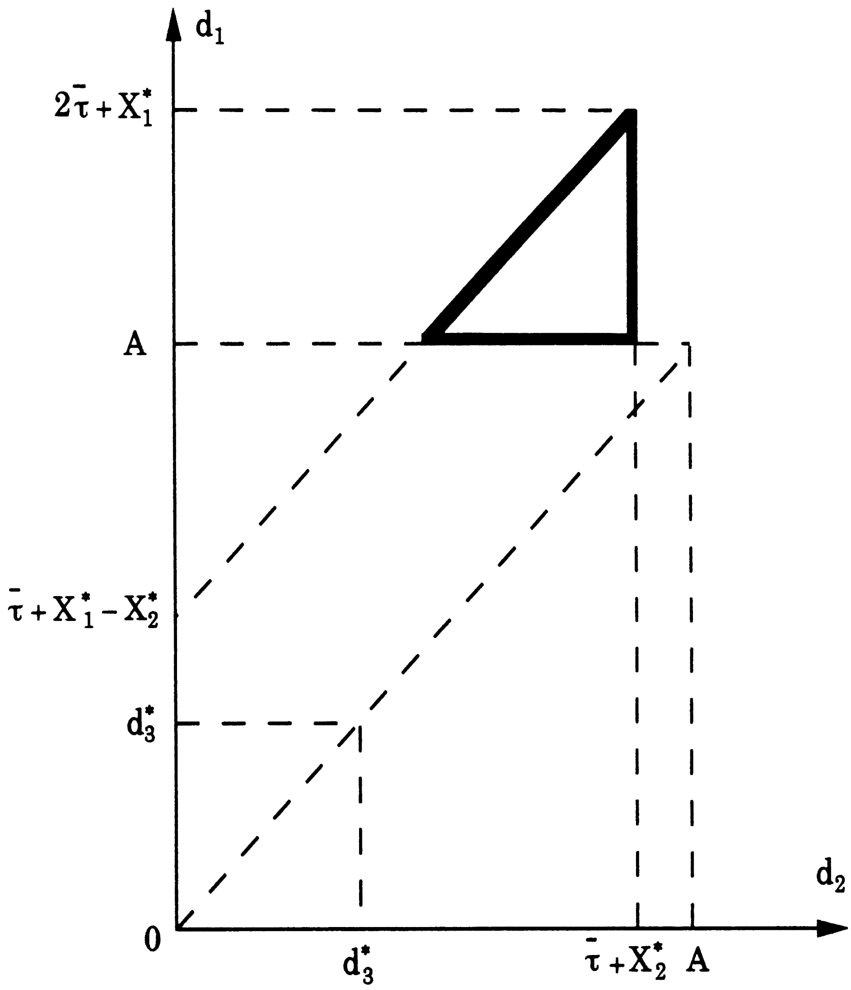

Figure 3b

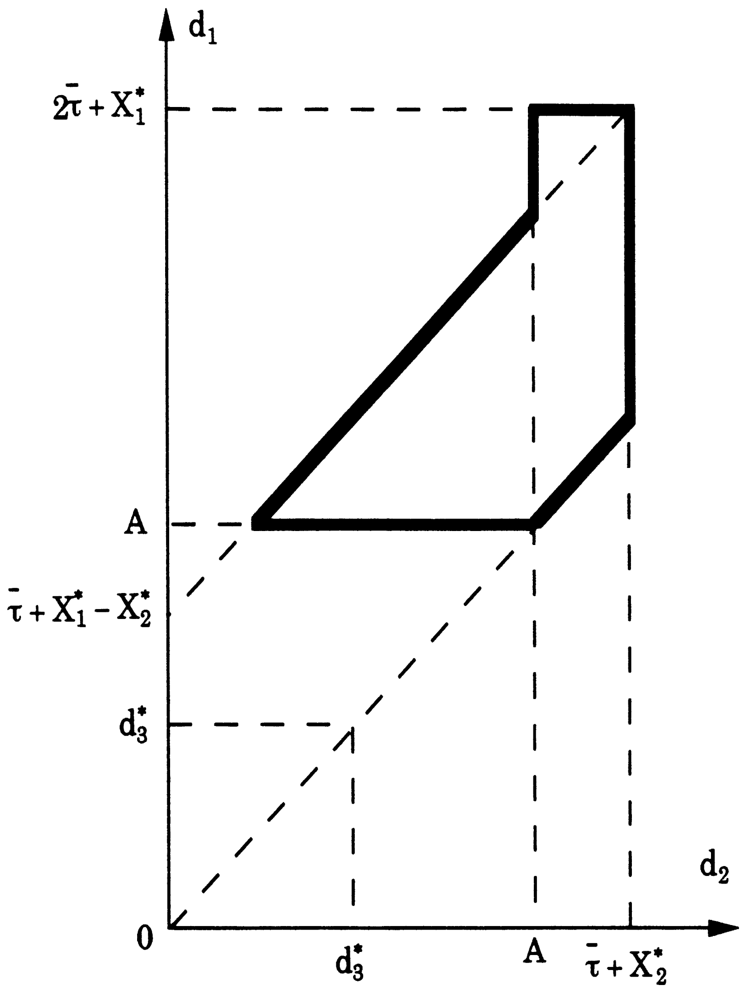

Figure 3c

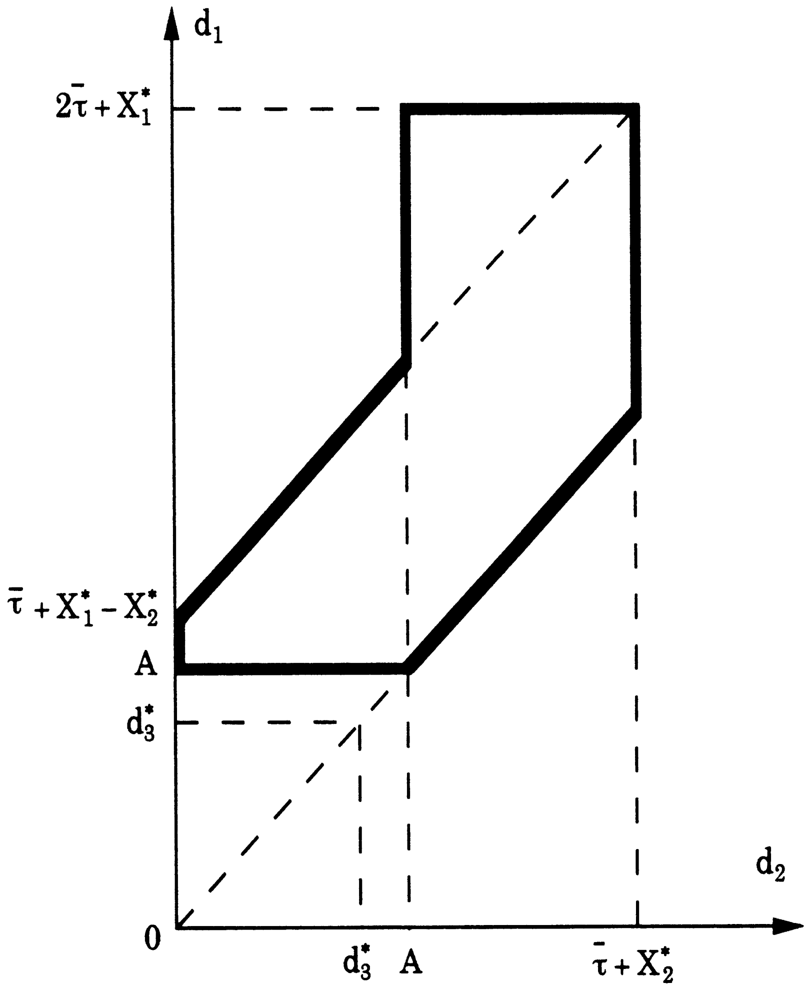

Figure 3d