Optimal Order Quantities of Style Products

Walid R. Abillama

Department of Industrial and Operations Engineering
University of Michigan
Ann Arbor, Michigan 48109-2117

December 1993

Technical Report 93-37

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Abstract

We consider the retailer's problem of ordering quantities of a variety of style products of the same family from a manufacturer. The demand for each product is random and occurs at the beginning of the season, which is well ahead in the future. The manufacturer has a limited capacity but places no limit on the quantities it is willing to accept since the final delivery date of these quantities lies well ahead in the future. Orders are placed to the manufacturer with the final delivery date set well ahead in the future, yet before the start of the season so that when these orders are delivered, more accurate information about the demand for each product type is available. Once the final delivery of these orders is made, the retailer places a second order to the manufacturer to make up for the various types of products of which it believes it is still short. This time, more accurate information about the demand for the various products is available but the manufacturer's capacity is limited since little time is left before the start of the season. The retailer's problem becomes to determine the various quantities that must be ordered the first time around, in order to allocate the manufacturer's capacity more effectively as the start of the

season draws near and more accurate information about the demand for the various products becomes available.

1 Introduction

Consider a retailer that sells a variety of style products of the same familty and the manufacturer that produces them. Management at the retail store is planning for the purchase of these products well in advance to meet the demand that occurs at the start of the season. The manufacturer has a limited production capacity but will accept any order quantities that the retailer will place, as long as the final delivery date of these quantities lies well ahead in the future. Currently, with the start of the season being well ahead in the future, very little information is available about the demand for each type of product. Therefore, the demand for each type of product is considered uncertain, possibly dependent on the demand of other products of the same family under consideration as well.

However, Marketing at the retail store believes that it would have a very accurate estimate of the demand for each product as the start of the season draws near. Once this information is passed onto Management, this latter is required to allocate the manufacturer production capacity to the various products, so that the ordered quantities will meet the demand for the various types of products in a most efficient manner. However, due to the limited manufacturer's production capacity, many products may not have any of their demand satisfied when demand occurs at the start of the season, resulting in huge profit losses.

Therefore, Management decides not to let the season draw hazardously near without placing any order, waiting for Marketing to provide it with accurate information about the demand for the various types of products. Instead, it would place an order for the production of some quantities in spite of the high risk associated with the demand uncertainty. These orders would be delivered at the time accurate information about the demand of the various types of products would be available.

In doing this, the retailer would overcome the manufacturer's capacity problem and order "off-season" unlimited quantities way before the start of the season, at the price of having inaccurate information about the demand for each product. Whenever these quantities are delivered, Management will allocate the limited production capacity of the manufacturer to the various types of products and order "in-season" quantities, only to make up for quantities of the various products of which it is still short at the light of accurate information about the demand for each type of product. The challenge that Management faces is to determine the off-season order quantities while assuming the demand for each type of product to be uncertain. As a result, capacity allocation decisions are made only when Marketing is able to provide accurate information about the demand for all types of products, at the time when Management has set the delivery of the off-season order quantities to take place.

Ordering very large off-season quantities may result in overage when more accurate information about the demand is known just before the start of the season. In this case, disposal costs are incurred because it is customary that the price of an excess style product is reduced until it is eventually sold. Ordering very small in-season quantities may result in a lack of capacity to make up for quantities of the various types of products of which the retailer is still short. However the crucial issue that this paper considers is the dependence of the off-season order quantities of each product on the cost structure of the various types of products and the limited production capacity of the manufacturer. For instance, suppose that demand for the various types of products is known with certainty and the off-season order quantities have been delivered. Furthermore, suppose that the available limited capacity is not sufficient to make up for all types of products of which the retailer is still short. In this case, the shortage cost of a particular type of product among those latter may be so high compared to the rest that it may be more economical to allocate the entire in-season capacity to the production of the short quantities of that particular type of product. Therefore, since the off-season order quantities must be decided upon before the demand for the various types of products is known with certainty, these order quantities depend on the capacity allocation policy that is going to be applied after the demand for the various types of products is known with certainty. If the available limited capacity is not sufficient to make up for all products still short of, the capacity allocation policy will depend on the cost structure of these latter products, hence the impact of these latter on the off-season order quantities.

Another factor that will heavily impact on the off-season order quantities is the ratio of the in-season to the off-season unit acquisition cost. A higher in-season unit acquisition cost of a certain type of product will encourage ordering large off-season quantities of that type of product, particularly if that type of product is highly profitable. However, the opposite is not necessarily true for lower in-season unit acquisition cost due to the limited in-season capacity and hence is dependent on the demand distribution of that particular type of product and on the in-season capacity allocation policy. It is quite realistic to assume that for each of the various types of products, the unit off-season purchase cost is less than the unit in-season purchase cost. This is true since the manufacturer is benefiting from the fact that the season is far ahead in the future and hence has enough time to plan the production of these orders efficiently. On the other hand, in-season orders must be produced and delivered in a very short time and as a result, the manufacturer might impose a higher price to the retailer for the purchase of in-season products. However, since off-season are produced way before the season starts, the retailer incurs a holding cost due to the long duration of the off-season period. This cost is added to the unit off-season purchase cost to obtain the unit off-season acquisition, which becomes difficult to compare with the unit in-season acquisition cost since this latter is the same as the unit in-season purchase cost due to the short duration of the in-season period.

2 Literature Review

Production and Inventory problems for style products have been analyzed, among others, by Murray and Silver (1966), Hausman and Peterson (1972), Bradford and Sugrue (1990) and Fisher and Raman (1992).

Murray and Silver (1966) consider a multiple production periods, single product bayesian model with limited production capacity in each period. The demand distribution is updated in each period, based on the demand realization in the previous period. Their model assumes that the numbers of customers in each period is known. The conditional probability that exactly j customers out of N customers will buy the product in the next period given the probability of a customer buying this product is binomial with parameters N and p, the probability of a customer buying this product. Furthermore, they assume that p is Beta distributed with parameters: The total number of customers who bought the product in the previous periods and the total number of customers in the previous periods (these parameters are chosen arbitrarily for the first production period). They decide in each period on the order quantity based on: Initial inventory in that period and the total number of customers who bought the product in the previous periods. This latter state variable, along with the total number of customers in the previous periods are the parameters of the revised Beta distribution . Their cost function includes a unit purchase cost incurred in each buying period, a unit shortage and a unit disposal cost incurred when demand is realized at the end of the last period. They recognize the prohibitive computational cost of the model and they illustrate their analysis by solving for some simple examples.

Hausman and Peterson (1972) consider the same problem, but they generalize it for multiple products. In this model, the ratio of demand forecast in a period relative to the previous period is a Lognormal random variable. They assume a perfect forecast in the period following the last period, hence production in the last period does not occur under a perfect forecast. Ratios are independent across periods and products, but there is no word on how to get the distributions parameters i.e. the means and variances of the Lognormal distributions in each production period. They decide in each period on the production quantity based on: Initial inventory in that period and demand forecast for that period. The cost function includes only a shortage and a disposal costs, incurred when the demand is realized at the end of last period. They recognize the prohibitive computational cost of the model, they propose 3 heuristics which they illustrate using few examples.

Bradford and Sugrue (1990) propose the classical multiple periods inventory problem with random demand. The new assumption is that it is a bayesian model. Their model is a 2 production periods model, single product, where delivery occurs in both periods, with unlimited production capacity in each period. The demand is revised in the second period in the following manner. The conditional demand given the mean is heterogenous Poisson with mean Gamma distributed, hence demand in first period is Negative Binomial. From bayes rule, conditional distribution of mean demand in the second period given first period demand is Gamma, hence the demand in second period is Negative Binomial (since conditional demand in the second period is also assumed heterogenous Poisson with mean Gamma distributed). They decide in each period on the production quantity based on the initial inventory in that period and the revised Negative Binomial distribution. Their cost function includes a purchase, a shortage cost and a selling price in the first period, a purchase, a shortage, a disposal cost, a selling price and a salvage value in the second period. They illustrate their analysis using some examples.

Fisher and Raman (1992) consider a 2 production periods model with n products, an unlimited capacity in the first period, a limited capacity in second period and where delivery occurs only at the end of the second period. They consider the case of Sport Obermayer, a skiwear manufacturer. They treat the demand forecast in the first period and the total demand forecast in both periods as dependent random variables distributed according to a bivariate normal distribution. The means, the variances and the covariance are estimated from historical data and expert opinion. Their cost function includes only a unit shortage and a unit disposal cost incurred at the end of the last period, after demand is realized. They recognize that it is the second period capacity that makes the problem intractable and they use an approximation scheme to solve it. The approximation scheme consists of throwing away the capacity constraint in the second period and replacing it by a minimum production requirement in the first period. The minimum production level is considered a decision variable along with the production quantities for the various types of products, with no word on how good is this approximation. They test the model succesfully for the 92/93 season at Sport Obermayer and they report a 50% increase in profit compared with what is actually done and 400% increase compared with producing only during the off-season and not allowing in-season production when more accurate information about the demand is available.

Our work is different from the studies listed above in the sense that it provides an explanation on the impact of the off-season and in-season costs, in-season production capacity and demand variability on the optimal off-season order quantities. The model used in this paper is simpler than the ones used in the above listed papers, but still captures the effect of a long off-season and a short in-season period, profitability among products, demand variability during a long off-season period and the trade-off between imperfect information/unlimited capacity and perfect information/limited capacity, all issues that are proper to style products inventory problems. In all the work cited above, none attempts to penalize early off-season acquisitions while they all recognize that the off-season period is much longer than the in-season period. It is different from Murray and Silver (1966) and Hausman and Peterson (1972) in the sense that these two papers describe a complex model that include forecasts revisions and excess/shortage inventory costs after demand realization, without being able to make any statement about the issues that govern style products inventory problems. Bradford and Sugrue (1990) consider a production system with unlimited capacity in both periods, hence the problem reduces to the classical multiple production periods problem with random demand with the exception that demand in the second period is revised in light of the demand realization in the first period. Finally in Fisher and Raman (1992), the treatment of the demand is narrowly focused on the case of skiwear manufacturing and hence the model does not provide insights to optimal ordering policies for other types of style products. Furthermore, they formulate a model to which they derive an aproximation and solve computationally, without saying how close is this solution to the optimal solution. Their optimal off-season ordering quantities is insensitive to the production capacity available in the second period. It can be inferred from their discussion of the approximation scheme that the larger the production capacity, the further is their solution from the optimal solution. In fact, the approximation leads to the optimal solution for a zero production capacity. On the other hand the optimal solution of the model improves as the production capacity increases. At the limit, assuming an infinite production capacity, the optimal off-season ordering quantities are zeros and the optimal in-season ordering quantity for each product is the corresponding Newsboy solution. Although one does not expect (relative to the forecasted demand) large production capacities in the various style products industries, this approximation encourages implicitly ordering large off-season quantities. However, by assuming a much longer off-season period than the in-season period, there are instances when high holding costs are incurred off-season if acquisition occurs early during the off-season period, hence discouraging ordering large off-season quantities, particularly if the production capacity is sufficiently large to protect against off-season demand variability. This latter issue is not addressed by the model since costs are incurred in the model only at the end of the second period and include only shortage/disposal costs. Our model addresses this issue because off-season and in-season acquisitions costs are incorporated in it, in addition to shortage and diposal costs incurred after demand realization. We are ready now to formulate the model.

3 The Model

Let p_i be the unit shortage cost, $\overline{c}_i$ be the unit in-season production cost, h_i be the unit disposal cost, $\tilde{c}_i$ be the unit off-season production cost and $\tilde{h}_i$ be the average holding rate for each unit of product i ordered off-season, for $i = 1,..,N$. Let $c_i = \tilde{c}_i + \tilde{h}_i$ be the total unit off-season ordering cost. It is reasonable to assume that $\tilde{c}_i \leq \overline{c}_i$. This is true because off-season orders are placed well ahead the start of the season, and so the manufacturer benefits from the long off-season period to plan efficiently for the production of these orders, hence the resulting low unit production cost relatively to the in-season unit production cost where units must be produced and delivered in a very short period. However, it is not clear how c_i and $\overline{c}_i$ compare. We also assume quite realistically $p_i \geq Max\{c_i, \overline{c}_i\}$ and $Min\{c_i, \overline{c}_i\} \geq h_i$. Let D_i be the demand for product i and $f(x_i,..,x_j)$ be the joint mass probability distribution of products i through j, $1 \leq i < j \leq N$. Let Q_i be the off-season order quantity of product i and I_i be the initial on-hand inventory of product i. Let d_i be the realization of D_i, $\overline{I}_i$ be the on-hand inventory of product i after Q_i is delivered and $\overline{Q}_i$ be the in-season order quantity of product i, i.e. after Q_i is delivered and d_i is realized. Clearly $\overline{I}_i = I_i + Q_i$. Let $\overline{J}\left(\overline{I}_1,..,\overline{I}_N\right)$ be the cost of ordering $\left(\overline{Q}_1,..,\overline{Q}_N\right)$ given $\left(\overline{I}_1,..,\overline{I}_N\right)$. Let $J(I_1,..,I_N)$ be the cost of ordering $(Q_1,..,Q_N)$ given $(I_1,..,I_N)$ and assuming that $\left(\overline{Q}_1^*,..,\overline{Q}_N^*\right)$ are produced. Finally let K be the available production capacity of the manufacturer. To determine Q_i^*, $i = 1,..,N$ we solve the following non-linear programming problem denoted by $\mathcal{P}(1)$:

$$
\begin{aligned}
J^*(I_1,..,I_N) = \quad & \text{Min} \quad \sum_{i=1}^{N} c_i Q_i + E\left[\overline{J}^*\left(\overline{I}_1,..,\overline{I}_N\right)\right] \\
& \text{s.t.} \quad Q_i \geq 0, i = 1,..,N
\end{aligned}
$$

where $\overline{I}_i = I_i + Q_i$, $i = 1,..,N$. To determine $\overline{J}^*\left(\overline{I}_1,..,\overline{I}_N\right)$ we solve the following non-linear programming problem:

$$\begin{aligned} \overline{J}^*\left(\overline{I}_1,..,\overline{I}_N\right) = \ & \text{Min} \ \sum_{i=1}^{N}\left[\overline{c}_i\overline{Q}_i + h_i\left(\overline{I}_i+\overline{Q}_i-d_i\right)^+ + p_i\left(d_i-\overline{I}_i-\overline{Q}_i\right)^+\right] \\ & \text{s.t.} \ \sum_{i=1}^{N}\overline{Q}_i \le K \\ & \qquad \overline{Q}_i \ge 0, i=1,..,N \end{aligned}$$

which can formulated as a dynamic programming problem. For $i = N, N-1, .., 1$:

$$\begin{aligned} \overline{J}_i^*\left(\overline{I}_i, K_i\right) = \ & \text{Min}_{0\le\overline{Q}_i\le K_i} \ \overline{c}_i\overline{Q}_i + h_i\left(\overline{I}_i+\overline{Q}_i-d_i\right)^+ + p_i\left(d_i-\overline{I}_i-\overline{Q}_i\right)^+ + \\ & \overline{J}_{i+1}^*\left(\overline{I}_{i+1}, K_i-\overline{Q}_i\right) \end{aligned}$$

with $\overline{J}_{N+1}^*\left(\overline{I}_{N+1}, K_N-\overline{Q}_N\right) = 0$, $K_1 = K$ and $\overline{J}_1^*\left(\overline{I}_1, K\right) \equiv \overline{J}^*\left(\overline{I}_1,..,\overline{I}_N\right)$. Denote the dynamic programming formulation by $\mathcal{P}(2)$.

4 Single Product Case

The solution to $\mathcal{P}(2)$ is clearly

$$\overline{Q}^*\left(\overline{I}\right) = \begin{cases} 0 & \overline{I} \ge d \\ d-\overline{I} & d-K \le \overline{I} \le d \\ K & \overline{I} \le d-K \end{cases} \tag{1}$$

hence

$$\overline{J}^*(I+Q) = \begin{cases} h(I+Q-D) & 0 \le D \le I+Q \\ \overline{c}(D-I-Q) & I+Q \le D \le I+Q+K \\ \overline{c}K + p(D-I-Q-K) & D \ge I+Q+K \end{cases} \tag{2}$$

Substituting in $\mathcal{P}(1)$ we get

$$\begin{aligned}
J(I) &= cQ + h\int_0^{I+Q}(I+Q-x)\,f(x)\,dx + \bar{c}\int_{I+Q}^{I+Q+K}(x-I-Q)\,f(x)\,dx + \\
&\quad \int_{I+Q+K}^{\infty}\left[\bar{c}K + p\,(x-I-Q-k)\right]f(x)\,dx \qquad (3)\\
\frac{dJ(I)}{dQ} &= c + h\int_0^{I+Q} f(x)\,dx - \bar{c}\int_{I+Q}^{I+Q+K} f(x)\,dx - p\int_{I+Q+K}^{\infty} f(x)\,dx \qquad (4)\\
\frac{d^2J(I)}{dQ^2} &= (\bar{c}+h)\,f(I+Q) + (p-\bar{c})\,f(I+Q+K) \qquad (5)
\end{aligned}$$

Since $p \geq \bar{c}$, then $J(I)$ is convex in Q and hence Q^* is obtained by setting (4) to zero and solving for Q^*. Doing this we get that $Q^*(I)$ is described by the following order-up-to policy:

$$Q^*(I) = \begin{cases} X^* - I & I \leq X^* \\ 0 & \text{otherwise} \end{cases} \qquad (6)$$

where after substituting $(I+Q)$ by X in (4), X^* satisfies the following first-order condition:

$$\frac{dJ(I)}{dX} = (p-\bar{c})\,F[X+K] + (\bar{c}+h)\,F[X] - (p-c) = 0 \qquad (7)$$

An intuitive result that falls from the analysis of the single product case is that the optimal off-season order quantity in the event that Management wants only to order off-season, without choosing to use the manufacturer's limited capacity after the demand becomes known, must be at least Q^*. In this latter case the problem reduces to the classical Newsvendor problem, with an underage cost of $(p-c)$ and an overage cost of $(c+h)$. The optimal off-season ordering policy is similar to (6) and the order point $X'^* = F^{-1}[(p-c)/(p+h)]$. This result is intuitive since the extra capacity K represents potential inventory that can be used in the event that demand is greater than Q^*, while ordering off-season only is equivalent to the manufacturer having a zero production capacity. Hence the order quantity for $K = 0$ must be at least equal to Q^* to account for any shortages since there is no opportunity to make up for short quantities after the demand is known. The proof follows by substituting X by X'^* in (7). Doing this we get:

$$\begin{aligned} \frac{dJ(I)}{dX}|_{X=X'^*} &= (p-\bar{c})\left(F[X'^*+K] - \left[\frac{p-c}{p+h}\right]\right) \\ &\geq (p-\bar{c})\left(F[X'^*] - \left[\frac{p-c}{p+h}\right]\right) = 0 \end{aligned}$$

hence $X^* \leq X'^*$ since $J(I)$ is convex in Q. Another intuitive result is that $X^* \geq X'^* - K$. The proof follows by substituting X by $X'^* - K$ in (7). Doing this we get:

$$\begin{aligned} \frac{dJ(I)}{dX}|_{X=X'^*} &= (\bar{c}+h)\left(F[X'^*-K] - \left[\frac{p-c}{p+h}\right]\right) \\ &\leq (\bar{c}+h)\left(F[X'^*] - \left[\frac{p-c}{p+h}\right]\right) = 0 \end{aligned}$$

A practical interpretation of this result is the following. If the entire production capacity K must be allocated *a priori* towards meeting the demand, hence no unused capacity remains as a second chance for production after the demand is known, then the optimal off-season order quantity is nothing but the Newsvendor solution less the entire manufacturer's production capacity, i.e. $X'^* - K$. As a result, the optimal off-season order quantity must be at least that amount since not all the production capacity will necessarily be used after the demand is realized, hence more should have been ordered off-season.

4.1 Effect of Capacity

We show the rather intuitive result that the optimal order point decreases with capacity. This is intuitive because the larger is the manufacturer's production capacity, the higher the chance that demand will be met after it is known, hence the smaller the off-season order quantity. We show this by differentiating (7) with respect to K. Doing this we get

$$(p-\overline{c})\left[\frac{dX}{dK}+1\right]f(X+K)+(\overline{c}+h)\left[\frac{dX}{dK}\right]f(X)=0 \tag{8}$$

and hence

$$\frac{dX}{dK}=\frac{-(p-\overline{c})f(X+K)}{(p-\overline{c})f(X+K)+(\overline{c}+h)F(X)}\leq 0 \tag{9}$$

Denote by η the fraction of off-season to in-season ordered quantities, a random variable. η is then defined as

$$\eta=\frac{Q^*}{Q^*+\overline{Q}^*(Q^*)} \tag{10}$$

$\overline{Q}^*$ is given by (1), which can be rewritten as

$$\overline{Q}^*(Q^*)=\begin{cases}0 & D\leq I+Q^*\\ D-I-Q^* & I+Q^*\leq D\leq I+Q^*+K\\ K & D\geq I+Q^*+K\end{cases} \tag{11}$$

We differentiate between two cases: $X^*\geq I$ and $X^*\leq I$. If $X^*\geq I$, then η becomes

$$\eta=\begin{cases}1 & D\leq X^*\\ \frac{X^*-I}{D-I} & X^*\leq D\leq X^*+K\\ \frac{X^*-I}{X^*-I+K} & D\geq X^*+K\end{cases} \tag{12}$$

If $X^*\leq I$, then $Q^*=0$. Therefore

$$\begin{aligned}E[\eta] &= \int_0^{X^*}f(x)\,dx+\left[\frac{X^*-I}{X^*-I+K}\right]\int_{X^*+K}^{\infty}f(x)\,dx+\\ &\quad \int_{X^*}^{X^*+K}\left[\frac{X^*-I}{x-I}\right]f(x)\,dx\end{aligned} \tag{13}$$

Differentiating (13) with respect to K, we get

$$\frac{dE[\eta]}{dK} = \frac{dX^*}{dK}f(X^*)-\left[\frac{X^*-I}{X^*-I+K}\right]\left(\frac{dX^*}{dK}+1\right)f(X^*+K)+ \tag{14}$$

$$\left[\frac{K\frac{dX^*}{dK}-(X^*-I)}{(X^*-I+K)^2}\right]\int_{X^*+K}^{\infty} f(x)\,dx + \int_{X^*}^{X^*+K}\left[\frac{\frac{dX^*}{dK}}{x-I}\right] f(x)\,dx$$

$$\left[\frac{X^*-I}{X^*-I+K}\right]\left(\frac{dX^*}{dK}+1\right) f(X^*+K) - \frac{dX^*}{dK} f(X^*) +$$

$$= \left[\frac{K\frac{dX^*}{dK}-(X^*-I)}{(X^*-I+K)^2}\right]\int_{X^*+K}^{\infty} f(x)\,dx + \int_{X^*}^{X^*+K}\left[\frac{\frac{dX^*}{dK}}{x-I}\right] f(x)\,dx \leq 0$$

since $X^* \geq I$ and $dX^*/dK \leq 0$.

4.2 Effect of Demand Variance

In this section, we study the effect of the demand variance on X^*. To do this, we will use a simple mean-preserving transformation of a random variable. This transformation was first used by Baron (1970), Rothschild and Stiglitz (1970) and Sandmo (1971) in Economic Theory, and was first used by Gerchak and Mossman (1991) in Iventory Theory to show the effect of the demand variance on the optimal solution to the classical Newsvendor problem. With D as the demand, the transformation is

$$D_\alpha = \alpha(D-\mu)+\mu \tag{15}$$

where μ is the demand mean. It is clear that (15) implies $E[D_\alpha] = E[D]$ and $Var[D_\alpha] = \alpha^2 Var[D]$. Hence we increase or decrease the demand variance by assigning values for α larger or smaller than 1 respectively. Substituting back in (7), the first-order condition becomes

$$\frac{dJ(I)}{dX_\alpha} = (p-\bar{c})\int_0^{\frac{X_\alpha-\mu}{\alpha}+\frac{K}{\alpha}+\mu} f(x)\,dx + (\bar{c}+h)\int_0^{\frac{X_\alpha-\mu}{\alpha}+\mu} f(x)\,dx - (p-c) \tag{16}$$

We want to study the variation of X_α with α. Intuitively, $X_\alpha = \mu$ and $X_\alpha = \mu - K$ at $\alpha = 0$, for $\bar{c} \geq c$ and $\bar{c} \leq c$ respectively. In the rest of this section, we will differentiate between these two cases and study the variation of X_α for these two cases separately.

4.2.1 Case Of $\bar{c} \geq c$

We will first prove that $X_\alpha = \mu$ at $\alpha = 0$. Suppose that $X_\alpha \neq \mu$ at $\alpha = 0$. Then from (16), $X_0 > \mu$ gives

$$\frac{dJ(I)}{dX_\alpha}|_{X_0>\mu} = c + h \geq 0 \tag{17}$$

For $\mu - K < X_0 < \mu$, (16) gives

$$\frac{dJ(I)}{dX_\alpha}|_{\mu-K<X_0<\mu} = c - \bar{c} \leq 0 \tag{18}$$

For $X_0 = \mu - K$, (16) gives

$$\frac{dJ(I)}{dX_\alpha}|_{X_0=\mu-K} = (p - \bar{c}) \int_0^{\frac{dX_\alpha}{d\alpha}|_{\alpha=0}+\mu} f(x)\, dx - (p - c) \leq 0 \tag{19}$$

Finally, for $X_0 < \mu - K$, (16) gives

$$\frac{dJ(I)}{dX_\alpha}|_{X_0<\mu-K} = -(p - c) \leq 0 \tag{20}$$

As a result, it must be that $X_0 = \mu$. Having shown this intuitive fact, note that the first integral in (16) is equal to one whenever its upper limit is larger than $\overline{D}$, the largest demand realization, that is for

$$X_\alpha \geq \alpha\left(\overline{D} - \mu\right) + \mu - K \tag{21}$$

This condition is satisfied at $\alpha = 0$. Moreover, since X_α is continuous in α, then from (16) we have

$$\frac{dJ(I)}{dX_\alpha} = (\bar{c} + h) \int_0^{\frac{X_\alpha-\mu}{\alpha}+\mu} f(x)\, dx - (\bar{c} - c) = 0$$

and hence

$$X_\alpha = \alpha \left(F^{-1} \left[\frac{\overline{c} - c}{\overline{c} + h} \right] - \mu \right) + \mu \tag{22}$$

to the right of $\alpha = 0$. Since $F^{-1} \left[(\overline{c} - c) / (\overline{c} + h) \right] \leq \overline{D}$, then (22) is true for $0 \leq \alpha \leq \alpha_{cr}$, where α_{cr} is the intersection point of (22) and (21) and is given by

$$\alpha_{cr} = \frac{K}{\overline{D} - F^{-1} \left[\frac{\overline{c} - c}{\overline{c} + h} \right]} \tag{23}$$

Finally, X_α is indeed given by (16) set to zero for $\alpha \geq \alpha_{cr}$, and is a nonlinear function of α in that range. Using the same arguments as when we showed that $X'^* - K \leq X^* \leq X'^*$ in equation (7) and where $X'^* = F^{-1} \left[(p - c) / (p + h) \right]$, we can show that

$$\alpha \left(F^{-1} \left[\frac{p - c}{p + h} \right] - \mu \right) + \mu - K \leq X_\alpha \leq \alpha \left(F^{-1} \left[\frac{p - c}{p + h} \right] - \mu \right) + \mu \tag{24}$$

The lower and upper bounds represent the solution to the Newsvendor when the mean preserving transformation defined in (15) is applied to the Demand. The intuitive interpretation of these bounds is similar to the interpretation provided for $X'^* - K \leq X^* \leq X'^*$, however here it is true $\forall \alpha \geq 0$ and not just for $\alpha = 1$. (16) gives $lim_{\alpha \to \infty} dX_\alpha / d\alpha = X'^* - \mu$. Differentating (16) with respect to α we get

$$\begin{aligned} \frac{(p - \overline{c})}{\alpha^2} \left[\alpha \frac{dX_\alpha}{d\alpha} - (X_\alpha - \mu + K) \right] f \left(\frac{X_\alpha - \mu}{\alpha} + \frac{K}{\alpha} + \mu \right) + \\ \frac{(\overline{c} + h)}{\alpha^2} \left[\alpha \frac{dX_\alpha}{d\alpha} - (X_\alpha - \mu) \right] f \left(\frac{X_\alpha - \mu}{\alpha} + \mu \right) = 0 \end{aligned} \tag{25}$$

For equation (25) to be true, we must have

$$\frac{X_\alpha - \mu}{\alpha} \leq \frac{dX_\alpha}{d\alpha} \leq \frac{X_\alpha - \mu + K}{\alpha} \tag{26}$$

Therefore, if there exists α such that $dX_\alpha/d\alpha = 0$, then $\mu \leq X_\alpha \leq \mu - K$. As a result, once X_α hits $X_\alpha = \mu$ at $\alpha^* > 0$, there does not exist $\alpha > \alpha^*$ for which $dX_\alpha/d\alpha = 0$, and hence X_α is strictly increasing in that range. Similarly, once X_α hits $X_\alpha = \mu - K$ at $\tilde{\alpha}^* > 0$, there does not exist $\alpha > \tilde{\alpha}^*$ for which $dX_\alpha/d\alpha = 0$, and hence X_α is strictly decreasing in that range. Differentiating (25) with respect to α we get

$$(p-\overline{c})\left[\frac{\alpha\frac{dX_\alpha}{d\alpha}-(X_\alpha-\mu+K)}{\alpha}\right]^2\frac{df(.)}{d(.)}+(\overline{c}+h)\left[\frac{\alpha\frac{dX_\alpha}{d\alpha}-(X_\alpha-\mu)}{\alpha}\right]^2\frac{df(.)}{d(.)}+ \tag{27}$$
$$\frac{d^2X_\alpha}{d\alpha^2}\left[(p-\overline{c})\,f\left(\frac{X_\alpha-\mu}{\alpha}+\frac{K}{\alpha}+\mu\right)+(\overline{c}+h)\,f\left(\frac{X_\alpha-\mu}{\alpha}+\mu\right)\right]=0$$

If the demand density function is non-increasing (non-decreasing), then X_α is convex (concave) in α. The shape of X_α is far less obvious when the demand density function has modes. If it is unimodal, then if there exists α such that $d^2X_\alpha/d\alpha^2 = 0$, then the inflection point must occur in the region bounded by $X_\alpha \geq \alpha(\overline{x}-\mu)+\mu-K$ and $X_\alpha \leq \alpha(\overline{x}-\mu)+\mu$, $\overline{x}$ being the mode of the demand density function. Figure 1 shows X_α versus α when $F^{-1}[(p-c)/(p+h)] \geq \mu$ and $F^{-1}[(\overline{c}-c)/(\overline{c}+h)] \leq \mu$, figure 2 shows X_α versus α when $F^{-1}[(p-c)/(p+h)] \geq \mu$ and $F^{-1}[(\overline{c}-c)/(\overline{c}+h)] \geq \mu$ and figure 3 shows X_α versus α when $F^{-1}[(p-c)/(p+h)] \leq \mu$ and $F^{-1}[(\overline{c}-c)/(\overline{c}+h)] \leq \mu$. In all three figures, X_α is bounded from above by $X'_\alpha = \alpha(X'^*-\mu)+\mu$, the α-dependent Newsvendor solution, and by $X'_\alpha - K$ from below. To show that $X_\alpha \leq \alpha\left(\overline{D}-\mu\right)+\mu-K$ for $\alpha \geq \alpha_{cr}$, we substitute X_α by $\alpha\left(\overline{D}-\mu\right)+\mu-K$ in (16) and get

$$\frac{dJ(I)}{dX_\alpha}|_{X_\alpha=\alpha\left(\overline{D}-\mu\right)+\mu-K}=(\overline{c}+h)\int_0^{\overline{D}-\frac{K}{\alpha}}f(x)\,dx+(c-\overline{c})\geq 0 \tag{28}$$

To show that $X_\alpha \geq \alpha(F^{-1}[(\overline{c}-c)/(\overline{c}+h)]-\mu)+\mu$ for $\alpha \geq \alpha_{cr}$, we substitute X_α by $\alpha(F^{-1}[(\overline{c}-c)/(\overline{c}+h)]-\mu)+\mu$ in (16) and get

$$\frac{dJ(I)}{dX_\alpha}|_{X_\alpha=\alpha\left(F^{-1}\left[\frac{\overline{c}-c}{\overline{c}+h}\right]-\mu\right)+\mu-K}=(p-\overline{c})\left[\int_0^{F^{-1}\left[\frac{\overline{c}-c}{\overline{c}+h}\right]+\frac{K}{\alpha}}f(x)\,dx-1\right]\leq 0 \tag{29}$$

For nonmonotonous density functions, extremums in figure 1 occur only in the region bounded by

$$
\begin{aligned}
X_\alpha &= \mu \\
X_\alpha &= \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right]-\mu\right)+\mu-K \\
X_\alpha &= \alpha\left(F^{-1}\left[\frac{\overline{c}-c}{c+h}\right]-\mu\right)+\mu \\
X_\alpha &= \alpha\left(\overline{D}-\mu\right)+\mu
\end{aligned}
$$

No extremums occur in figure 2 because of (26), and extremums in figure 3 in the region bounded by

$$
\begin{aligned}
X_\alpha &= \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right]-\mu\right)+\mu \\
X_\alpha &= \mu-K \\
X_\alpha &= \alpha\left(F^{-1}\left[\frac{\overline{c}-c}{c+h}\right]-\mu\right)+\mu \\
X_\alpha &= \alpha\left(\overline{D}-\mu\right)+\mu
\end{aligned}
$$

Computational experience suggests that X_α is convex for $\alpha \geq \alpha_{cr}$ when the demand distribution is assumed unimodal.

4.2.2 Case Of $\overline{c} \leq c$

We will first prove that $X_\alpha = \mu - K$ at $\alpha = 0$. Suppose that $X_\alpha \neq \mu - K$ at $\alpha = 0$. Then from (16), $X_0 < \mu - K$ gives

$$\frac{dJ(I)}{dX_\alpha}|_{X_0<\mu-K} = -(p-c) \leq 0 \tag{30}$$

For $\mu - K < X_0 < \mu$, (16) gives

$$\frac{dJ(I)}{dX_\alpha}|_{\mu-K<X_0<\mu} = c - \overline{c} \geq 0 \tag{31}$$

For $X_0 = \mu$, (16) gives

$$\frac{dJ(I)}{dX_\alpha}|_{X_0=\mu} = (\overline{c} + h)\int_0^{\frac{dX_\alpha}{d\alpha}|_{\alpha=0}+\mu} f(x)\,dx + (c - \overline{c}) \geq 0 \tag{32}$$

Finally, for $X_0 > \mu$, (16) gives

$$\frac{dJ(I)}{dX_\alpha}|_{X_0>\mu} = c + h \geq 0 \tag{33}$$

As a result, it must be that $X_0 = \mu - K$. Having shown this intuitive fact, note that the first integral in (16) is equal to zero whenever its upper limit is smaller than 0, the smallest demand realization, that is for

$$X_\alpha \leq -\alpha\mu + \mu \tag{34}$$

This condition is satisfied at $\alpha = 0$. Moreover, since X_α is continuous in α, then from (16) we have

$$\frac{dJ(I)}{dX_\alpha} = (p - \overline{c})\int_0^{\frac{X_\alpha-\mu+K}{\alpha}+\mu} f(x)\,dx - (p - c) = 0$$

and hence

$$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p-\overline{c}}\right] - \mu\right) + \mu - K \tag{35}$$

to the right of $\alpha = 0$. Since $F^{-1}\left[(p-c)/(p-\overline{c})\right] \geq 0$, then (35) is true for $0 \leq \alpha \leq \tilde{\alpha}_{cr}$, where $\tilde{\alpha}_{cr}$ is the intersection point of (35) and (34) and is given by

$$\tilde{\alpha}_{cr} = \frac{K}{F^{-1}\left[\frac{p-c}{p-\bar{c}}\right]} \tag{36}$$

Finally, X_α is indeed given by (16) set to zero for $\alpha \geq \tilde{\alpha}_{cr}$, and is a nonlinear function of α in that range. Equation (24) still holds, as well as (25), (26) and (27). Figure 4 shows X_α versus α when $F^{-1}\left[(p-c)\,/\,(p+h)\right] \leq \mu$ and $F^{-1}\left[(p-c)\,/\,(p-\bar{c})\right] \geq \mu$, figure 5 shows X_α versus α when $F^{-1}\left[(p-c)\,/\,(p+h)\right] \leq \mu$ and $F^{-1}\left[(p-c)\,/\,(p-\bar{c})\right] \leq \mu$ and figure 6 shows X_α versus α when $F^{-1}\left[(p-c)\,/\,(p+h)\right] \geq \mu$ and $F^{-1}\left[(p-c)\,/\,(p-\bar{c})\right] \geq \mu$. In all three figures, X_α is bounded from above by $X_\alpha^{'} = \alpha\left(X'^{*} - \mu\right) + \mu$, the α-dependent Newsvendor solution, and by $X_\alpha^{'} - K$ from below. To show that $X_\alpha \geq -\alpha\mu + \mu$ for $\alpha \geq \alpha_{cr}$, we substitute X_α by $-\alpha\mu + \mu$ in (16) and get

$$\frac{dJ\left(I\right)}{dX_\alpha}|_{X_\alpha = -\alpha\mu+\mu} = \left(p - \bar{c}\right)\int_0^{\frac{K}{\alpha}} f\left(x\right)dx - \left(p - c\right) \leq 0 \tag{37}$$

since $\alpha \geq \alpha_{cr}$. To show that $X_\alpha \leq \alpha\left(F^{-1}\left[(p-c)\,/\,(p-\bar{c})\right] - \mu\right) + \mu$ for $\alpha \geq \alpha_{cr}$, we substitute X_α by $\alpha\left(F^{-1}\left[(p-c)\,/\,(p-\bar{c})\right] - \mu\right) + \mu$ in (16) and get

$$\frac{dJ\left(I\right)}{dX_\alpha}|_{X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p-\bar{c}}\right]-\mu\right)+\mu-K} = \left(\bar{c} + h\right)\int_0^{F^{-1}\left[\frac{p-c}{p-\bar{c}}\right]-\frac{K}{\alpha}} f\left(x\right)dx \geq 0 \tag{38}$$

For nonmonotonous density functions, extremums in figure 4 occur only in the region bounded by

$$\begin{aligned}
X_\alpha &= \mu - K \\
X_\alpha &= \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right] - \mu\right) + \mu \\
X_\alpha &= \alpha\left(F^{-1}\left[\frac{p-c}{p-\bar{c}}\right] - \mu\right) + \mu - K \\
X_\alpha &= -\alpha\mu + \mu
\end{aligned}$$

No extremums occur in figure 5 because of (26), and extremums in figure 6 occur in the region bounded by

$$
\begin{aligned}
X_\alpha &= \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right]-\mu\right)+\mu-K \\
X_\alpha &= \mu \\
X_\alpha &= \alpha\left(F^{-1}\left[\frac{p-c}{p-\overline{c}}\right]-\mu\right)+\mu-K \\
X_\alpha &= -\alpha\mu+\mu
\end{aligned}
$$

Computational experience suggests that X_α is convex for $\alpha \geq \tilde{\alpha}_{cr}$ when the demand distribution is assumed unimodal.

5 Single product Case: Practical Interpretation

We provide in this section a practical interpretation to the behavior of the optimal subcontracted amount as a function of demand variance.

5.1 Case Of $\overline{c} \geq c$

The variation of X_α versus α whenever $\overline{c} \geq c$ can be interpreted as follows. For simplicity, assume the initial stock is zero. We order μ off-season if there is no uncertainty about the demand to take advantage of two things: the unlimited manufacturer's production capacity and the fact that the unit off-season ordering cost c is less than the unit in-season ordering cost $\overline{c}$. As the demand becomes uncertain, the marginal underage cost is the difference between the unit in-season and the unit off-season ordering cost and the marginal overage cost is the unit off-season ordering cost, as long as the demand variance does not exceed $\alpha_{cr}^2 Var(D)$. For $\alpha \leq \alpha_{cr}$, it is guaranteed that the demand be smaller than the optimal off-season order quantity plus the in-season production capacity. However, there is still the chance that demand might be on either side of the optimal off-season order quantity. As a result, if the off-season order quantity is short in one unit of demand, it can always be made up from the manufacturer's in-season production capacity, hence the underlined marginal underage cost $c - \bar{c}$. On the other hand, if the off-season order quantity is over in one unit of demand, then an unnecessary order has been placed off-season and must be disposed, hence the underlined marginal overage cost $c + h$. As in the Newsvendor problem, the off-season order quantity varies linearly with α, with the understanding that it may either be decreasing or increasing, depending on how the critical ratio compares with μ. In other words, the less attractive is the unit off-season ordering cost compared to the unit in-season ordering cost as in figures 1 and 3, the more likely it is that we order less off-season as demand becomes increasingly uncertain and rely on the in-season production capacity to make up for shortages, assuming of course that shortages will not exceed the in-season production capacity. If on the other hand, it is much cheaper to order off-season than to order in-season as in figure 2, then it is profitable to order more off-season as demand becomes increasingly uncertain. α_{cr} represents that level of demand variance for which there is a chance that the demand be larger than the optimal off-season order quantity plus the in-season production capacity. As expected, the larger the in-season production capacity, the wider is the range of α over which X_α is linear. Beyond α_{cr}, there is a chance that the demand be larger than the optimal off-season order quantity plus the in-season production capacity. In figure 1, depending on the demand density function, the in-season production capacity can still continue to protect against growing demand uncertainty up to certain extent, as in density functions with $f\left(\overline{D}\right) = 0$, or it can increase abruptly to the right of α_{cr}, as in density functions with $f\left(\overline{D}\right) > 0$. We showed that the former case applies to decreasing densities where X_α is convex, and the latter case applies to increasing densities where X_α is concave. As more uncertainty in the demand is introduced, the need for ordering large quantities off-season grows stronger because the fixed in-season capacity is not large enough to protect against growing uncertainty, i.e. the increasing risk of lost demand beyond the in-season production capacity forces us to order large quantities off-season. The larger is the in-season production capacity, the more uncertainty is required for the off-season order quantity to start increasing with more uncertainty. In figure 2, even with a nonexistant chance of shortages, would they occur, exceeding the in-season production capacity, the off-season order quantity is increased just as demand becomes uncertain. It follows that with a nonzero chance of that occurring, there is more reason to keep increasing the off-season order quantity as demand becomes increasingly uncertain. Finally, figure 3 assumes that if no in-season capacity is available for production after the demand is known, the product is so little profitable that the off-season order quantity decreases with increasing demand variance. Hence some capacity will only encourage ordering a smaller off-season quantity. However, there are instances when the chance of shortages, would they occur, exceeding the in-season production capacity is so high, as in density functions with $f\left(\overline{D}\right) > 0$, that X_α increases at the right of α_{cr}, only to level off and to decrease eventually as the demand becomes more variable due to the low profitability of the product.

5.2 Case Of $\overline{c} \leq c$

The variation of X_α versus α whenever $\overline{c} \leq c$ can be interpreted as follows. For simplicity, assume the initial stock is zero. We order $\mu - K$ if there is no uncertainty about the demand to take advantage of two things: the unlimited off-season manufacturer's production capacity and the fact that the unit in-season ordering cost $\overline{c}$ is less than the unit off-season ordering cost c. As the demand becomes uncertain, the marginal underage cost is $p - c$, the product profitability, and the marginal overage cost is the difference between the unit off-season ordering cost and the unit in-season ordering cost, as long as the demand variance does not exceed $\tilde{\alpha}_{cr}^2 Var(D)$. For $\alpha \leq \tilde{\alpha}_{cr}$, it is guaranteed that the demand be larger than the optimal off-season order quantity. However, there is still the chance that demand might be on either side of the optimal off-season order quantity plus the production capacity. As a result, if the off-season order quantity plus the in-season production capacity are short in one unit of demand, it must be that the in-season capacity is depleted and that one additional unit should have been ordered off-season, hence the underlined marginal underage cost $p - c$, the product profitability. On the other hand, if the off-season order quantity plus the in-season production capacity are over in one unit of demand, then an additional unit should not have been ordered off-season but should have been ordered in-season instead at a lower cost, hence the underlined marginal overage cost $c - \bar{c}$. As in the Newsvendor problem, the off-season order quantity varies linearly with α, with the understanding that it may either be decreasing or increasing, depending on how the critical ratio compares with μ. In other words, the more profitable is the product as in figures 4 and 6, the more likely it is that we order more off-season as demand becomes increasingly uncertain while allocating the entire in-season production capacity to satisfy K units of demand, given of course that demand variance is small enough such that demand will always exceed the off-season order quantity. If on the other hand, product profitability is low as in figure 5, then it is more profitable to order less off-season as demand becomes increasingly uncertain. $\tilde{\alpha}_{cr}$ represents that level of demand variance for which there is a chance that the demand be smaller than the optimal off-season order quantity. As expected, the larger the in-season production capacity, the wider is the range of α over which X_α is linear since a larger fraction of the demand would be absorbed by the in-season production capacity and hence keeping a zero chance that demand less K be smaller than the off-season order quantity. Beyond α_{cr}, there is a chance that the demand be smaller than the optimal off-season order quantity. In figure 4, depending on the demand density function, the in-season production capacity can still continue to protect against growing demand uncertainty and absorb K units, as in density functions with $f(0) = 0$, up to certain extent. After that, demand stands a high chance of being smaller than the off-season order quantity and it becomes more profitable to order a smaller off-season quantity. The second alternative is that it can decrease abruptly to the right of

α_{cr}, as in density functions with $f(0) > 0$. We showed that the former case applies to increasing densities where X_α is convex, and the latter case applies to decreasing densities where X_α is concave. As more uncertainty in the demand is introduced, the need for ordering large off-season quantities grows weaker because the fixed in-season production capacity is not large enough to protect against growing uncertainty, i.e. the increasing risk of having demand below the off-season order quantity forces us to order a smaller off-season quantity. The larger is the fixed in-season capacity, the more uncertainty is required for the off-season order quantity to start decreasing with more uncertainty. In figure 5, even with a zero chance of demand being smaller than the off-season order quantity, the product is so little profitable that the off-season order quantity is increased just as demand becomes uncertain. It follows that with a nonzero chance of that occurring, there is more reason to keep decreasing the off-season order quantity as demand becomes increasingly uncertain. Finally, figure 6 assumes that if no in-season capacity is available for production after the demand is known and shifted by K units, the product is so highly profitable that the off-season order quantity increases with increasing demand variance. Hence some in-season capacity at a cheaper price will decrease the marginal overage cost and hence encourage ordering a larger off-season quantity. However, there are instances when the demand has high chances of being smaller than the off-season order quantity, as in density functions with $f(0) > 0$, that X_α decreases at the right of α_{cr}, only to level off and to increase eventually as the demand becomes more variable due to the high profitability of the product.

6 Two-Product Case

It is easy to see that the solution to $\mathcal{P}(2)$ is $\overline{Q_1^*} = \overline{Q_2^*} = 0$ if $\overline{I}_1 \geq d_1$ and $\overline{I}_2 \geq d_2$. Also obvious is the case when $\overline{I}_1 \leq d_1$, $\overline{I}_2 \leq d_2$ and $K \geq \left(d_1 - \overline{I}_1\right) + \left(d_2 - \overline{I}_2\right)$, which results in $\overline{Q_1^*} = d_1 - \overline{I}_1$ and $\overline{Q_2^*} = d_2 - \overline{I}_2$. If $\overline{I}_1 \leq d_1$ and $\overline{I}_2 \geq d_2$, then the problem reduces to the single product case with $\overline{Q}_2^* = 0$ and $\overline{Q}_1^* = \text{Min}\left\{d_1 - \overline{I}_1, K\right\}$. Similarly, if $\overline{I}_2 \leq d_2$ and $\overline{I}_1 \geq d_1$, then $\overline{Q}_1^* = 0$ and $\overline{Q}_2^* = \text{Min}\left\{d_2 - \overline{I}_2, K\right\}$. The most interesting case occurs whenever $\overline{I}_1 \leq d_1, \overline{I}_2 \leq d_2$ but $K \leq \left(d_1 - \overline{I}_1\right) + \left(d_2 - \overline{I}_2\right)$. It can be shown that $\overline{Q}_i^* \leq d_i - \overline{I}_i$, $i = 1, 2$. To show that, suppose that $\overline{Q}_1^* > d_1 - \overline{I}_1$. Then there is some extra capacity that is held as finished inventory of product 1. We also have that $\overline{Q}_2^* \leq d_2 - \overline{I}_2$, otherwise $K > \overline{Q}_1^* + \overline{Q}_2^* \left(d_1 - \overline{I}_1\right) + \left(d_2 - \overline{I}_2\right)$: contradiction. Therefore, some extra capacity is needed to make up for the shortage in product 2. As a result, the total cost strictly decreases if the extra capacity invested in holding product 1 were to be used to makeup for shortages in product 2 and hence $\overline{Q}_1^* \leq d_1 - \overline{I}_1$. Similar arguments can be used to show that $\overline{Q}_2^* \leq d_2 - \overline{I}_2$. Hence the problem is reduced to a simple linear programming problem whose optimal solution depend on the products cost structure: If $p_1 - \overline{c}_1 \geq p_2 - \overline{c}_2$, then $\overline{Q}_1^* = \text{Min}\left\{d_1 - \overline{I}_1, K\right\}$ and $\overline{Q}_2^* = K - \overline{Q}_1^*$. Otherwise, $\overline{Q}_2^* = \text{Min}\left\{d_2 - \overline{I}_2, K\right\}$ and $\overline{Q}_1^* = K - \overline{Q}_2^*$.

Substituting in $\overline{J}^* (I_1 + Q_1, I_2 + Q_2)$ we get for $p_1 - \overline{c}_1 \geq p_2 - \overline{c}_2$:

$$\frac{d\overline{J}^* (I_1 + Q_1, I_2 + Q_2)}{dQ_1} = \begin{cases} h_1 & 0 \leq D_1 \leq I_1 + Q_1 \\ & \\ -\overline{c}_1 & I_1 + Q_1 \leq D_1 \leq I_1 + Q_1 + K \\ & D_2 \geq 0, D_1 + D_2 \leq K + I_1 + Q_1 + I_2 + Q_2 \\ & \\ -\overline{c}_1 - (p_2 - \overline{c}_2) & I_1 + Q_1 \leq D_1 \leq I_1 + Q_1 + K \\ & D_1 + D_2 \geq K + I_1 + Q_1 + I_2 + Q_2 \\ & \\ -p_1 & D_1 \geq I_1 + Q_1 + K \end{cases} \tag{39}$$

and

$$
\frac{d\overline{J}^{*}\left(I_1+Q_1, I_2+Q_2\right)}{dQ_2} = \begin{cases} h_2 & 0 \le D_2 \le I_2+Q_2 \\ -\overline{c}_2 & I_2+Q_2 \le D_2 \le I_2+Q_2+K \\ & D_1 \ge 0, D_1+D_2 \le K+I_1+Q_1+I_2+Q_2 \\ -p_2 & \text{otherwise} \end{cases} \tag{40}
$$

After making the substitutions $X_1 = I_1 + Q_1$ and $X_2 = I_2 + Q_2$, we differentiate $J(I_1, I_2)$ with respect to X_1 and X_2 using (39) and (40) and get

$$
\begin{aligned}
\frac{\delta J\left(I_1, I_2\right)}{\delta X_1} &= c_1 + h_1 F_1\left[X_1\right] - \overline{c}_1 F_1\left[X_1+K\right] + \overline{c}_1 F_1\left[X_1\right] \\
&\quad - \left(p_2 - \overline{c}_2\right)\int_{X_1}^{X_1+K}\int_{K+X_1+X_2-x_1}^{\infty} f\left(x_1, x_2\right) dx_2 dx_1 - p_1 \overline{F}_1\left[X_1+K\right]
\end{aligned} \tag{41}
$$

and

$$
\begin{aligned}
\frac{\delta J\left(I_1, I_2\right)}{\delta X_2} &= \left(p_2 - \overline{c}_2\right)\int_{X_2}^{X_2+K}\int_{0}^{K+X_1+X_2-x_2} f\left(x_1, x_2\right) dx_1 dx_2 + \\
&\quad \left(p_2 + h_2\right) F_2\left[X_2\right] - \left(p_2 - c_2\right) \\
&= \left(p_2 - \overline{c}_2\right)\int_{X_1}^{X_1+K}\int_{X_2}^{K+X_1+X_2-x_1} f\left(x_1, x_2\right) dx_2 dx_1 + \\
&\quad \left(p_2 - \overline{c}_2\right)\int_{0}^{X_1}\int_{X_2}^{X_2+K} f\left(x_1, x_2\right) dx_2 dx_1 + \\
&\quad \left(p_2 + h_2\right) F_2\left[X_2\right] - \left(p_2 - c_2\right)
\end{aligned} \tag{42}
$$

To show convexity, we take the second derivatives and get

$$
\begin{aligned}
\frac{\delta^2 J\left(I_1, I_2\right)}{\delta X_1^2} &= \left(\overline{c}_1 + h_1\right) f_1\left(X_1\right) + \left(p_2 - \overline{c}_2\right)\int_{K+X_2}^{\infty} f\left(X_1, x_2\right) dx_2 + \\
&\quad \left(p_2 - \overline{c}_2\right)\int_{X_1}^{X_1+K} f\left(x_1, K+X_1+X_2-x_1\right) dx_1 + \\
&\quad \left(p_1 - \overline{c}_1\right) f_1\left(X_1+K\right) - \left(p_2 - \overline{c}_2\right)\int_{X_2}^{\infty} f\left(X_1+K, x_2\right) dx_2
\end{aligned} \tag{43}
$$

$$
\frac{\delta^2 J\left(I_1, I_2\right)}{\delta X_1 \delta X_2} = \left(p_2 - \overline{c}_2\right)\int_{X_1}^{X_1+K} f\left(x_1, K+X_1+X_2-x_1\right) dx_1 \tag{44}
$$

and

$$\frac{\delta^2 J(I_1,I_2)}{\delta X_2^2} = (p_2-\overline{c}_2)\int_{X_1}^{X_1+K} f(x_1, K+X_1+X_2-x_1)\,dx_1 + \tag{45}$$
$$(p_2+h_2) f_2(X_2) - (p_2-\overline{c}_2)\int_0^{X_1+K} f(x_1,X_2)\,dx_1 +$$
$$(p_2-\overline{c}_2)\int_0^{X_1} f(x_1, X_2+K)\,dx_1$$

The only term in (44) is present in (43) and (45). Furthermore, in (43) we have

$$\begin{aligned}
& (p_1-\overline{c}_1) f_1(X_1+K) \\
\geq\ & (p_1-\overline{c}_1)\int_{X_2}^{\infty} f(X_1+K,x_2)\,dx_2 \\
\geq\ & (p_2-\overline{c}_2)\int_{X_2}^{\infty} f(X_1+K,x_2)\,dx_2
\end{aligned}$$

and in (45) we have

$$(p_2+h_2) f_2(X_2) \geq (p_2+h_2)\int_0^{X_1+K} f(x_1,X_2)\,dx_1 \geq (p_2-\overline{c}_2)\int_0^{X_1+K} f(x_1,X_2)\,dx_1$$

Therefore, the Hessian matrix of $J(I_1,I_2)$ is positive definite and hence the optimal off-season order quantities are described by the following order-up-to policy for $i=1,2$.

$$Q_i^*(I_i) = \begin{cases} X_i^* - I_i & I_i \leq X_i^* \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

where X_i^*, $i=1,2$ are the unique solutions to the set of first-order conditions expressed in (41) and (42), and rewritten as

$$\frac{\delta J(I_1,I_2)}{\delta X_1} = (p_1-\overline{c}_1) F_1[X_1+K] + (\overline{c}_1+h_1) F_1[X_1] - (p_1-c_1) - \tag{47}$$
$$(p_2-\overline{c}_2)\int_{X_1}^{X_1+K}\int_{K+X_1+X_2-x_1}^{\infty} f(x_1,x_2)\,dx_2 dx_1 = 0$$

$$\frac{\delta J(I_1,I_2)}{\delta X_2} = (p_2-\overline{c}_2) F_2[X_2+K] + (\overline{c}_2+h_2) F_2[X_2] - (p_2-c_2) - \tag{48}$$
$$(p_2-\overline{c}_2)\int_{X_2}^{X_2+K}\int_{K+X_1+X_2-x_2}^{\infty} f(x_1,x_2)\,dx_1 dx_2 = 0$$

If the in-season capacity was zero, the optimal off-season order quantities would be the solution to the classical Newsvendor problem, that is $X_i'^* = F_i^{-1}\left[(p_i - c_i)/(p_i + h_i)\right]$, $i = 1,2$. To show that $X_i^* \leq X_i'^*$, $i = 1,2$, we substitute X_i by $X_i'^*$, $i = 1,2$, in (47) and (48). Doing this we get

$$\begin{aligned}
\frac{\delta J(I_1,I_2)}{\delta X_1}|_{X_1=X_1'^*} &= (p_1-\overline{c}_1)\,F_1\left[X_1'^*+K\right]+(\overline{c}_1+h_1)\,F_1\left[X_1'^*\right]-(p_1-c_1)- \\
&\quad (p_2-\overline{c}_2)\int_{X_1'^*}^{X_1'^*+K}\int_{K+X_1'^*+X_2-x_2}^{\infty} f(x_1,x_2)\,dx_2dx_1 \\
&= \frac{(p_1-\overline{c}_1)(p_1-c_1)}{(p_1+h_1)}+(p_1-\overline{c}_1)\int_{X_1'^*}^{X_1'^*+K} f_1(x_1)\,dx_1+ \\
&\quad \frac{(\overline{c}_1+h_1)(p_1-c_1)}{(p_1+h_1)}-(p_1-c_1)- \\
&\quad (p_2-\overline{c}_2)\int_{X_1'^*}^{X_1'^*+K}\int_{K+X_1'^*+X_2-x_2}^{\infty} f(x_1,x_2)\,dx_2dx_1 \\
&= (p_1-\overline{c}_1)\int_{X_1'^*}^{X_1'^*+K} f_1(x_1)\,dx_1- \\
&\quad (p_2-\overline{c}_2)\int_{X_1'^*}^{X_1'^*+K}\int_{K+X_1'^*+X_2-x_2}^{\infty} f(x_1,x_2)\,dx_2dx_1 \\
&\geq (p_1-\overline{c}_1)\Bigg[\int_{X_1'^*}^{X_1'^*+K} f_1(x_1)\,dx_1- \\
&\quad \int_{X_1'^*}^{X_1'^*+K}\int_{K+X_1'^*+X_2-x_2}^{\infty} f(x_1,x_2)\,dx_2dx_1\Bigg]\geq 0
\end{aligned}$$

and

$$\frac{\delta J(I_1,I_2)}{\delta X_2}|_{X_2=X_2'^*} = (p_2-\overline{c}_2)\int_{X_2'^*}^{X_2'^*+K}\int_{0}^{K+X_1+X_2'^*-x_2} f(x_1,x_2)\,dx_1dx_2 \geq 0$$

Therefore $X_i^* \leq X_i'^*$, $i = 1,2$. Denote by $\overline{X_1^*}$ the optimal off-season order quantity of product 1 in the absence of product 2, i.e. the solution to (7). Define $\overline{X_2^*}$ similarly. We want to show that $X_i^* \geq \overline{X_i^*}$, $i = 1,2$. Rewrite (7), (47) and (48) respectively as

$$g(X) = p-c \tag{49}$$

$$g_1(X_1) = p_1-c_1 \tag{50}$$

$$g_2(X_2) = p_2-c_2 \tag{51}$$

Note that $dg(X)/dX \geq 0$, $dg_1(X_1)/dX_1 \geq 0$ and $dg_2(X_2)/dX_2 \geq 0$ as a result of convexity. Therefore if the single product in (49) was product 1, then clearly $g_1(X_1) \leq g(X_1)$, which implies that $g_1(X_1)$ intersects $p_1 - c_1$ in (50) at a value at least equal to the value at which $g(X_1)$ intersects it in (49), hence $X_1^* \geq \overline{X_1^*}$. Suppose the single product in (49) is product 2 and rewrite (49) as

$$(p_2 + h_2) F_2[X_2] + (p_2 - \overline{c}_2) \int_{X_2}^{X_2+K} f_2(x_2)\, dx_2 = g(X_2) = p_2 - c_2 \tag{52}$$

Thus $g_2(X_2) \leq g(X_2)$, which implies that $g_2(X_2)$ intersects $p_2 - c_2$ in (50) at a value at least equal to the value at which $g(X_2)$ intersects it in (52), hence $X_2^* \geq \overline{X_2^*}$.

6.1 Effect of Demand Variance

In this section, we shall study the effect of product 1 demand variance and product 2 demand variance on the optimal off-season order points X_1^* and X_2^*. Beginning by product 1 demand variance, we use the same transformation defined in (15), that is

$$D_{1\alpha_1} = \alpha_1 (D_1 - \mu_1) + \mu_1 \tag{53}$$

Equations (47) and (48) can be written as

$$\begin{aligned} \frac{\delta J(I_1, I_2)}{\delta X_1} &= (p_1 - \overline{c}_1) Pr[D_1 \leq X_1 + K] + \qquad (54) \\ &\quad (\overline{c}_1 + h_1) Pr[D_1 \leq X_1] - (p_1 - c_1) - \\ &\quad (p_2 - \overline{c}_2) Pr[X_1 \leq D_1 \leq X_1 + K, D_1 + D_2 \geq X_1 + X_2 + K] = 0 \\ \frac{\delta J(I_1, I_2)}{\delta X_2} &= (p_2 - \overline{c}_2) Pr[D_2 \leq X_2 + K] + \qquad (55) \\ &\quad (\overline{c}_2 + h_2) Pr[D_2 \leq X_2] - (p_2 - c_2) - \\ &\quad (p_2 - \overline{c}_2) Pr[X_2 \leq D_2 \leq X_2 + K, D_1 + D_2 \geq X_1 + X_2 + K] = 0 \end{aligned}$$

Using (53), we substitute $D_{1\alpha_1}$ in (54) and (55) and get

$$\begin{aligned}\frac{\delta J(I_1,I_2)}{\delta X_{1\alpha_1}} &= (p_1-\bar{c}_1)F_1\left[\frac{X_{1\alpha_1}-\mu_1+K}{\alpha_1}+\mu_1\right]+ \\ &(\bar{c}_1+h_1)F_1\left[\frac{X_{1\alpha_1}-\mu_1}{\alpha_1}+\mu_1\right]-(p_1-c_1)- \\ &(p_2-\bar{c}_2)\int_{\frac{X_{1\alpha_1}-\mu_1}{\alpha_1}+\mu_1}^{\frac{X_{1\alpha_1}-\mu_1+K}{\alpha_1}+\mu_1}\int_{K+X_{1\alpha_1}+X_{2\alpha_1}-\alpha_1(x_1-\mu_1)-\mu_1}^{\infty} f(x_1,x_2)\,dx_2dx_1=0\end{aligned} \tag{56}$$

$$\begin{aligned}\frac{\delta J(I_1,I_2)}{\delta X_{2\alpha_1}} &= (p_2-\bar{c}_2)F_2\left[X_{2\alpha_1}+K\right]+ \\ &(\bar{c}_2+h_2)F_2\left[X_{2\alpha_1}\right]-(p_2-c_2)- \\ &(p_2-\bar{c}_2)\int_{X_{2\alpha_1}}^{X_{2\alpha_1}+K}\int_{\frac{K+X_{1\alpha_1}+X_{2\alpha_1}-x_2-\mu_1}{\alpha_1}+\mu_1}^{\infty} f(x_1,x_2)\,dx_1dx_2=0\end{aligned} \tag{57}$$

We have already shown that $X_i'^* \geq X_i^* \geq \overline{X}_i^*, i=1,2$ for all distributions $F_i, i=1,2$. Therefore, $X_{1\alpha_1}$ is bounded from above by $X_{1\alpha_1}' = \alpha_1(X_1'^*-\mu_1)+\mu_1$, the α_1-dependent Newsvendor solution, and from below by $\overline{X}_{1\alpha_1}$, the α_1-dependent single product solution. It can be shown that $X_{1\alpha_1}=\mu_1$ at $\alpha_1=0$ for $\bar{c}_1 \geq c_1$. $X_{1\alpha_1}=\mu_1-K$ at $\alpha_1=0$ for $\bar{c}_1 \leq c_1$. As a result, $dX_{1\alpha_1}/d\alpha_1 \geq F^{-1}\left[(\bar{c}_1-c_1)/(\bar{c}_1+h_1)\right]-\mu_1$ at $\alpha_1=0$. Similarly, it can be shown that $X_{1\alpha_1}=\mu_1-K$ at $\alpha_1=0$ for $\bar{c}_1 \leq c_1$ and hence $dX_{1\alpha_1}/d\alpha_1 \geq F^{-1}\left[(p_1-c_1)/(p_1-\bar{c}_1)\right]-\mu_1$ at $\alpha_1=0$. In both cases, $\lim_{\alpha_1\to\infty} dX_{1\alpha_1}/d\alpha_1$ is nothing but the slope of the α_1-dependent Newsvendor solution. The shape of $X_{2\alpha_1}$ is more interesting. We have $X_2'^* \geq X_{2\alpha_1} \geq \overline{X}_2^*, \forall\alpha_1 \geq 0$ and $X_{2\alpha_1}=\overline{X}_2^*$ at $\alpha_1=0$. Denote the limit of $X_{2\alpha_1}$ as α_1 approaches ∞ by $\tilde{X}_2$. Rewriting (57), we get

$$\begin{aligned}\frac{\delta J(I_1,I_2)}{\delta X_{2\alpha_1}} &= (p_2-\bar{c}_2)\int_{X_{2\alpha_1}}^{X_{2\alpha_1}+K}\int_0^{\frac{K+X_{1\alpha_1}+X_{2\alpha_1}-x_2-\mu_1}{\alpha_1}+\mu_1} f(x_1,x_2)\,dx_1dx_2+ \\ &(p_2+h_2)\int_0^{X_{2\alpha_1}} f_2(x_2)\,dx_2-(p_2-c_2)=0\end{aligned} \tag{58}$$

Taking the limit in (58) as α_1 approaches ∞, $\tilde{X}_2$ satisfies

$$(p_2-\bar{c}_2)\int_{\tilde{X}_2}^{\tilde{X}_2+K}\int_0^{X_1'^*} f(x_1,x_2)\,dx_1dx_2+(p_2+h_2)\int_0^{\tilde{X}_2} f_2(x_2)\,dx_2=(p_2-c_2) \tag{59}$$

We have $X_2'^* \geq \tilde{X}_2 \geq \overline{X}_2^*$. Computational experience with unimodal demand density functions suggests that $X_{2\alpha_1}$ increases as α_1 increases, overshoots $\tilde{X}_2$, reaches a maximum and approaches $\tilde{X}_2$ as α_1 approaches infinity.

6.2 Approximation for the Case of Highly Variable Hot Product

If the demand variance of the hot product is much higher than the demand variance of the other product, then an approximation for X_2^* is given by $\tilde{X}_2$, which is in turn given by (59), and an approximation for X_1^* is given by substituting $\tilde{X}_2$ in (47) and solving for X_1. Computational experience shows that this approximation is very close to the optimal solution, even for small diference in the variances of the two products.

6.3 Case of $p_1 - \overline{c}_1 = p_2 - \overline{c}_2$

Suppose that both products have equal priority but their demand distribution is different. Equations (47) and (48) become respectively

$$\begin{aligned}(p_1 - \overline{c}_1)\, Pr\,[X_1 \leq D_1 \leq X_1 + K] + (p_1 + h_1)\, Pr\,[D_1 \leq X_1] - \\ (p_2 - \overline{c}_2)\, Pr\,[X_1 \leq D_1 \leq X_1 + K, D_1 + D_2 \geq X_1 + X_2 + K] &= (p_1 - c_1) \qquad (60)\end{aligned}$$

$$\begin{aligned}(p_2 - \overline{c}_2)\, Pr\,[X_2 \leq D_2 \leq X_2 + K] + (p_2 + h_2)\, Pr\,[D_2 \leq X_2] - \\ (p_2 - \overline{c}_2)\, Pr\,[X_2 \leq D_2 \leq X_2 + K, D_1 + D_2 \geq X_1 + X_2 + K] &= (p_2 - c_2) \qquad (61)\end{aligned}$$

and hence assuming $p_1 - \overline{c}_1 = p_2 - \overline{c}_2$ we get

$$\begin{aligned}(p + h)\, Pr\,[D_1 \leq X_1] - (p - \overline{c})\, Pr\,[D_1 \leq X_1, D_1 + D_2 \leq X_1 + X_2 + K] + \\ (p - \overline{c})\, Pr\,[D_1 \leq X_1 + K, D_1 + D_2 \leq X_1 + X_2 + K] &= (p - c) \qquad (62)\end{aligned}$$

$$(p+h)\, Pr\,[D_2 \leq X_2] - (p-\bar{c})\, Pr\,[D_2 \leq X_2, D_1 + D_2 \leq X_1 + X_2 + K] + \qquad (63)$$
$$(p-\bar{c})\, Pr\,[D_2 \leq X_2 + K, D_1 + D_2 \leq X_1 + X_2 + K] \;=\; (p-c)$$

The left-hand side of (62) is non-decreasing in X_1 since (43) is non-negative, and the left-hand side of (63) is non-decreasing in X_2 since (45) is non-negative. Moreover, the sum of the first two terms in the left-hand sides of both (62) and (63) is non-negative. As a result, if product 1 demand is stochastically larger than product 2 demand, then the left-hand side of (62) is at most equal to the left hand of (62), $\forall\,(X_1, X_2)$. Hence the left-hand side of (62) intersects the line $(p-c)$ at a point X_1^* at least equal to X_2^*, the point of intersection of the left-hand side of (63) with the line $(p-c)$. Therefore if D_1 is stochastically larger than D_2, then $X_1^* \geq X_2^*$, which is a fairly intuitive result.

6.4 Two-Product Case: Practical Interpretation

We provide in this section a practical interpretation to the results obtained in the previous section on the behavior of the optimal off-season order quantities of both products as a function of the more profitable demand variance.

For both products, the optimal off-season order quantities are larger than in the single product case. In other words, the effect of adding another product increases the optimal off-season order quantities. This is an intuitive result since the two products will be competing for in-season capacity if a second product is added and hence the increase in off-season order quantities. This fact is evidently true for the less profitable product since it must yield for the hot product shortages first before using the in-season production capacity. Product 1 off-season order quantities increase because of product 2 shortage cost. It is this latter cost that forces the hot product off-season order quantities to increase in spite of the fact that this latter has priority to the in-season production capacity. Another intuitive result is that for both products, the optimal off-season order quantities are smaller than in the Newsvendor solution, i.e. assuming a zero in-season production capacity.

Therefore, as the hot product demand variance increases, the off-season production quantities fall between the newsvendor solution and the single product case solution as described in figures 1 through 6. If the demand for product 1 is known with certainty, then the off-season production quantity for product 2 is the same as in the single product case if $\bar{c}_1 \geq c_1$, and the same as in the Newsvendor problem otherwise. As uncertainty is introduced in the product 1 demand, the off-season production quantity for product 2 increases if $\bar{c}_1 \geq c_1$ and decreases otherwise. As the demand variance of product 1 is increased further, the off-season production quantity for product 2 approaches a limit and the dependence of this latter on the demand variance of product 1 becomes very weak for larger product 1 demand variance. In essence, there is so much uncertainty in the hot product demand that the optimal off-season order quantity for product 2 is determined independently of product 1, using only a fraction of the in-season production capacity to make up for shortages of product 2 when demand is known. This fraction is determined by the limit that the optimal off-season order quantity approaches as product 1 demand variance approaches infinity, and lies somewhere between the Newsvendor solution, which assumes zero in-season production capacity, and the single product solution, which uses the entire production capacity to make up for shortages of product 2 when demand is known.

7 Extension to $N > 2$ Products

The first-order conditions for the case of three products can be written as:

$$\begin{aligned}\frac{\delta J(I_1, I_2, I_3)}{\delta X_1} &= (p_1 - \bar{c}_1)\, Pr\,[X_1 \leq D_1 \leq X_1 + K] - \\ &\quad (p_2 - \bar{c}_2)\, \{Pr\,[D_1 \geq X_1, D_2 \geq X_2, D_3 \geq X_3,\end{aligned}$$

$$(D_1 - X_1) \leq K \leq (D_1 - X_1) + (D_2 - X_2)] +$$
$$Pr\,[D_1 \geq X_1, D_2 \geq X_2, D_3 \leq X_3,$$
$$(D_1 - X_1) \leq K \leq (D_1 - X_1) + (D_2 - X_2)]\} -$$
$$(p_3 - \bar{c}_3)\,\{Pr\,[D_1 \geq X_1, D_2 \geq X_2, D_3 \geq X_3,$$
$$(D_1 - X_1) + (D_2 - X_2) \leq K \leq (D_1 - X_1) + (D_2 - X_2) +$$
$$(D_3 - X_3)] + Pr\,[D_1 \geq X_1, D_2 \leq X_2, D_3 \geq X_3,$$
$$(D_1 - X_1) \leq K \leq (D_1 - X_1) + (D_3 - X_3)]\} +$$
$$(p_1 + h_1)\,Pr\,[D_1 \leq X_1] - (p_1 - c_1) \quad (64)$$

$$\frac{\delta J\,(I_1, I_2, I_3)}{\delta X_2} = (p_2 - \bar{c}_2)\,\{Pr\,[D_1 \geq X_1, D_2 \geq X_2, K \geq (D_1 - X_1) + (D_2 - X_2)] +$$
$$Pr\,[D_1 \leq X_1, D_2 \leq X_2, K \geq (D_2 - X_2)]\} -$$
$$(p_3 - \bar{c}_3)\,\{Pr\,[D_1 \leq X_1, D_2 \geq X_2, D_3 \geq X_3,$$
$$(D_2 - X_2) \leq K \leq (D_2 - X_2) + (D_3 - X_3)] +$$
$$Pr\,[D_1 \geq X_1, D_2 \geq X_2, D_3 \geq X_3, (D_1 - X_1) +$$
$$(D_2 - X_2) \leq K \leq (D_1 - X_1) + (D_2 - X_2) + (D_3 - X_3)]\} +$$
$$(p_2 + h_2)\,Pr\,[D_2 \leq X_2] - (p_2 - c_2) \quad (65)$$

$$\frac{\delta J\,(I_1, I_2, I_3)}{\delta X_3} = (p_3 - \bar{c}_3)\,\{Pr\,[D_1 \leq X_1, D_2 \leq X_2, D_3 \geq X_3, K \geq (D_3 - X_3)] +$$
$$Pr\,[D_1 \geq X_1, D_2 \leq X_2, D_3 \geq X_3, K \geq (D_1 - X_1) + (D_3 - X_3)] +$$
$$Pr\,[D_1 \leq X_1, D_2 \geq X_2, D_3 \geq X_3, K \geq (D_2 - X_2) + (D_3 - X_3)] +$$
$$Pr\,[D_1 \geq X_1, D_2 \geq X_2, D_3 \geq X_3, K \geq (D_1 - X_1) +$$
$$(D_2 - X_2) + (D_3 - X_3)]\} + (p_3 + h_3)\,Pr\,[D_3 \leq X_3] - (p_3 - c_3) \quad (66)$$

To interpret this set of first-order conditions, we compare it to the first-order condition of the single product case, i.e. equation (7) rewritten as

$$\frac{dJ(I)}{dX} = (p - \bar{c}) \, Pr\left[X \leq D \leq X + K\right] + (p + h) \, Pr\left[D \leq X\right] - (p - c) \qquad (67)$$

If the in-season capacity is zero, then (67) would reduce to the Newsvendor problem first-order condition, where there is only one production opportunity to meet the demand. With a non-zero in-season capacity, the model assumes that another production opportunity occurs after demand is known. Therefore if demand exceeds the off-season order quantity, but is less than the off-season order quantity plus the in-season capacity, then a profit $(p - \bar{c})$ per unit is made by ordering in-season, hence the first probability term in (67).

For the case of two products, the first-order condition corresponding to the hot product, given by equation (60), is similar to the single product case except for the last probability term of the left-hand side. This term represents the dependence of the hot product optimal off-season quantity on the second product. This term can be interpreted as the lost profit for missing one unit of product 2 due to assigning a higher priority to product 1. This lost profit occurs whenever the demands for both products exceed their respective off-season order quantity, in-season production capacity is higher than the shortage amount of the hot product, but less than the sum of both products shortages. As a result, a lost profit per unit of product 2 $(p_2 - \overline{c_2})$ is incurred by satisfying the demand for product 1 first, leaving some demand units of product 2 unsatisfied because of its lower priority.

The first-order condition corresponding to the second product, given by equation (61) can be written as

$$\begin{aligned} \frac{\delta J(I_1, I_2)}{\delta X_2} = {} & (p_2 - \bar{c}_2) \{ Pr\left[D_1 \geq X_1, D_2 \geq X_2, K \geq (D_1 - X_1) + (D_2 - X_2)\right] + \\ & Pr\left[D_1 \leq X_1, D_2 \leq X_2, K \geq (D_2 - X_2)\right] \} + \\ & (p_2 + h_2) \, Pr\left[D_2 \leq X_2\right] - (p_2 - c_2) \end{aligned} \qquad (68)$$

The first probability term in (68) represents the added profit from having a second production opportunity after demand is known. This term is interpreted as the unit profit $(p_2 - \overline{c_2})$ made whenever in-season capacity is enough to cover for short units of product 2 as in the single product case. The difference here is that since product 2 does not have first priority in the capacity allocation policy, then this term must account for both cases: Demand for product 1 being more than its off-season order quantity where product 2 would benefit only from the remaining in-season capacity that has not been allocated to product 1, and vice-versa where product 2 may benefit from the entire in-season capacity.

For the three products case, the first-order condition corresponding to the hottest product, given by equation (64), is similar to the single product case except for the second and third cost terms. As in the two products case, the second cost term represents the lost profit for missing one unit of product 2 due to assigning a higher priority to product 1. This lost profit occurs whenever the demands for both products exceed their respective off-season order quantity, in-season production capacity is higher than the shortage amount of the hot product, but less than the sum of both products shortages. As a result, a lost profit per unit of product 2 $(p_2 - \overline{c_2})$ is incurred by satisfying the demand for product 1 first, leaving some demand units of product 2 unsatisfied because of its lower priority. Similarly, the third cost term represents the lost profit for missing one unit of product 3 due to assigning a higher priority to product 1. However, since product 2 has a lower priority than product 1 but a higher priority than product 3, then both cases must be considered: Having shortages of product 2 and not having shortages of product 2. If there are no shortages in the demand of product 2, this lost profit occurs whenever the demands for products 1 and 3 exceed their respective off-season order quantity, in-season production capacity is higher than the shortage amount of the hot product, but less than the sum of both products shortages. As a result, a lost profit per unit of product 3 $(p_3 - \overline{c_3})$ is incurred by satisfying the demand for product 1 first, leaving some demand units of product 3 unsatisfied because of its lower priority. If there are shortages in the demand of product 2, this lost profit occurs whenever the demands for the three products exceed their respective off-season order quantity, in-season production capacity is higher than the shortage amount of products 1 and 2, but less than the sum of the three products shortages. As a result, a lost profit per unit of product 3 $(p_3 - \overline{c_3})$ is incurred by satisfying the demand for product 1 first, the demand for product 2 second and leaving some demand units of product 3 unsatisfied because of its lowest priority.

In the first-order condition corresponding to product 2, given by equation (65), the second cost term represents the lost profit for missing one unit of product 3 due to assigning a higher priority to product 2. This lost profit occurs whenever the demands for both products exceed their respective off-season order quantity, in-season production capacity is higher than the shortage amount of product 2, but less than the sum of both products shortages. As a result, a lost profit per unit of product 3 $(p_3 - \overline{c_3})$ is incurred by satisfying the demand for product 2 first, leaving some demand units of product 3 unsatisfied because of its lower priority. However, since shortages for product 2 cannot be satisfied from the in-season production capacity before satisfying shortages of product 1, then both cases must be considered: Having shortages of product 1 and not having shortages of product 1. The first cost term in (65) represents the added profit from having a second production opportunity after demand is known. This term is interpreted as the unit profit $(p_2 - \overline{c_2})$ made whenever in-season capacity is enough to cover for short units of product 2 as in the single product case. The difference here is that since product 2 does not have first priority in the capacity allocation policy, then this term must account for both cases: Demand for product 1 being more than its off-season order quantity where product 2 would benefit only from the remaining in-season capacity that has not been allocated to product 1, and vice-versa where product 2 may benefit from the entire in-season capacity.

In the first-order condition corresponding to product 3, given by equation (66), the first cost term represents the added profit from having a second production opportunity after demand is known. This term is interpreted as the unit profit $(p_3 - \overline{c_3})$ made whenever in-season capacity is enough to cover for short units of product 3 as in the single product case. The added difference between this case and the two products case is that since product 3 has the least priority among the three products in the capacity allocation policy, then this term must account for all four cases: Shortages for products 1 and 2, shortages for product 1 only, shortages for product 2 only and finally no shortages for both products.

8 Conclusion

The purpose of this paper was to analyze the problem of determining optimal ordering quantities for style products. A two periods production model was built with off-season and in-season production costs, and disposal and shortage costs for each product type incurred at the end of the season. The demand in the second period was assumed to be known with certainty and the problem was to determine the optimal off-season ordering quantities that minimize the total expected costs of off-season and in-season production, and the disposal and shortage costsincurred at the end of the season.

The single product case was analyzed and the behavior of the optimal off-season order quantity was studied as a function of the product demand variance. The two products case was analyzed and the optimal off-season order quantities of both products determined analytically, along with the optimal in-season production capacity allocation policy that was shown to give, after the demand for each product is known, priority to the product with the highest profitability. It was shown that the addition of a new product increases the optimal off-season order quantity of the previously existing product, and that the Newsvendor solution for each product is an upper bound on the optimal off-season order quantity of that product. An approximation was provided that decouples the set of first-order conditions used to deterimine the optimal off-season order quantities. This approximation is based on the variation of the optimal off-season order quantity of product 2 with product 1 demand variance. Finally, the set of first-order conditions for more than two products was given and we provided a practical interpretation for it in terms of marginal profit for each product for having the opportunity to produce in-season, and the marginal costs of missing one unit of a certain product due to the production of an extra unit of another product, for all products.

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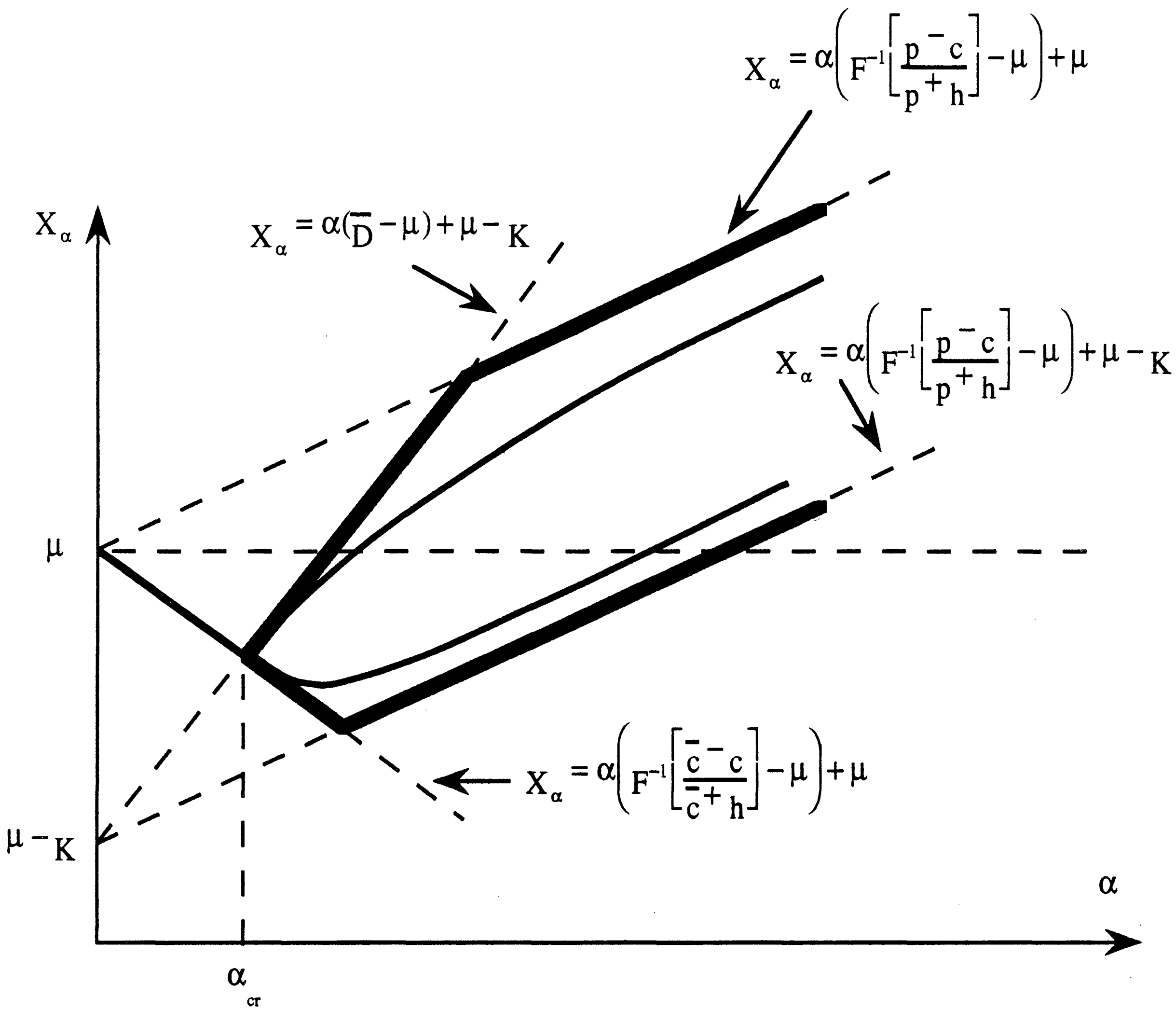
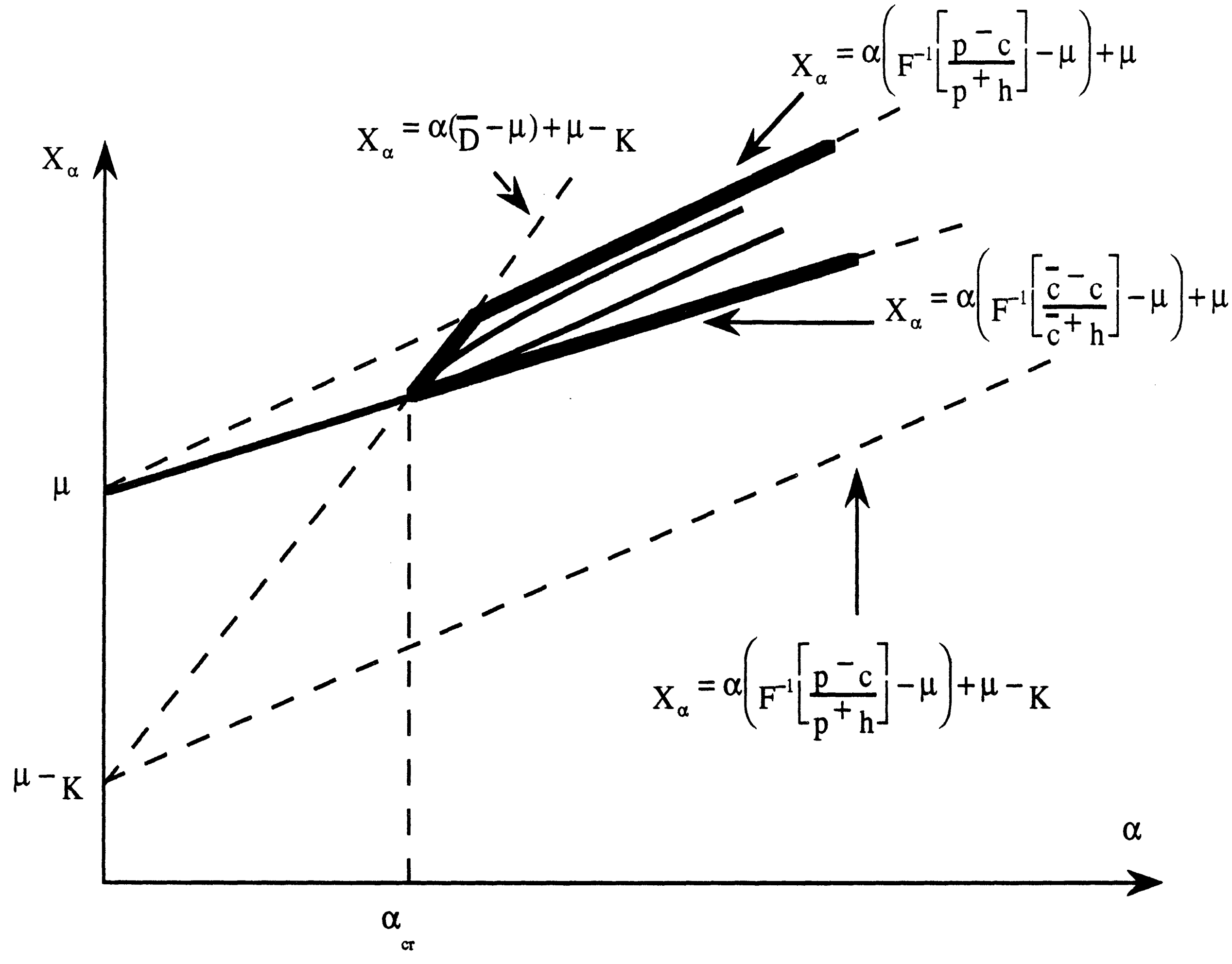


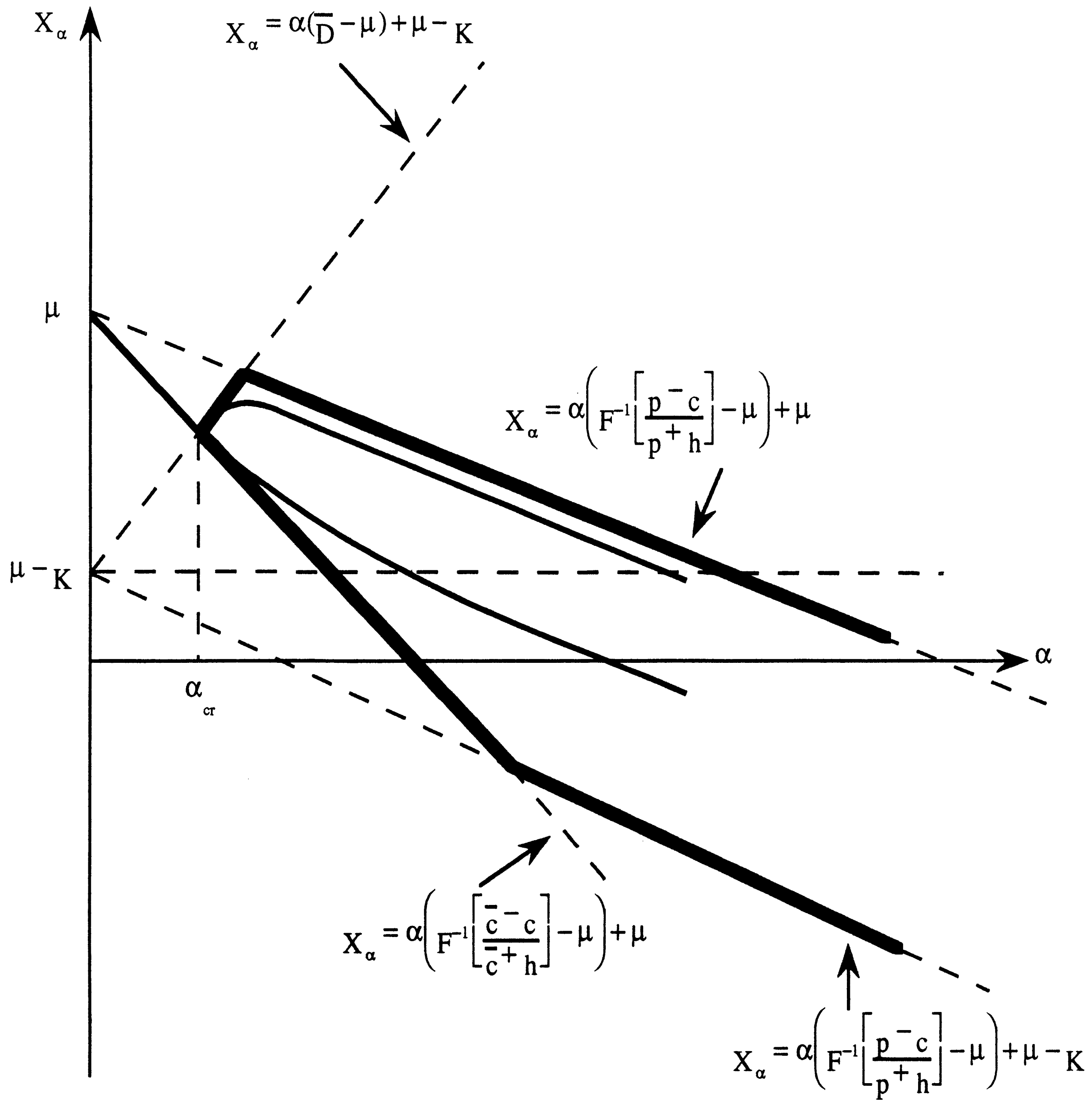
$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right]-\mu\right)+\mu$
X_α
$X_\alpha = \alpha(\bar{D}-\mu)+\mu-K$
$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right]-\mu\right)+\mu-K$
μ
$X_\alpha = \alpha\left(F^{-1}\left[\frac{\bar{c}-c}{\bar{c}+h}\right]-\mu\right)+\mu$
$\mu-K$
α
α_{cr}

Figure 1



$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right]-\mu\right)+\mu$
$X_\alpha = \alpha(\overline{D}-\mu)+\mu-K$
X_α
$X_\alpha = \alpha\left(F^{-1}\left[\frac{\bar{c}-c}{\bar{c}+h}\right]-\mu\right)+\mu$
μ
$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right]-\mu\right)+\mu-K$
$\mu-K$
α
α_{cr}

Figure 2



X_α
$X_\alpha = \alpha(\overline{D} - \mu) + \mu - K$
μ
$\mu - K$
α
α_{cr}
$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right] - \mu\right) + \mu$
$X_\alpha = \alpha\left(F^{-1}\left[\frac{\bar{c}-c}{\bar{c}+h}\right] - \mu\right) + \mu$
$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right] - \mu\right) + \mu - K$

Figure 3

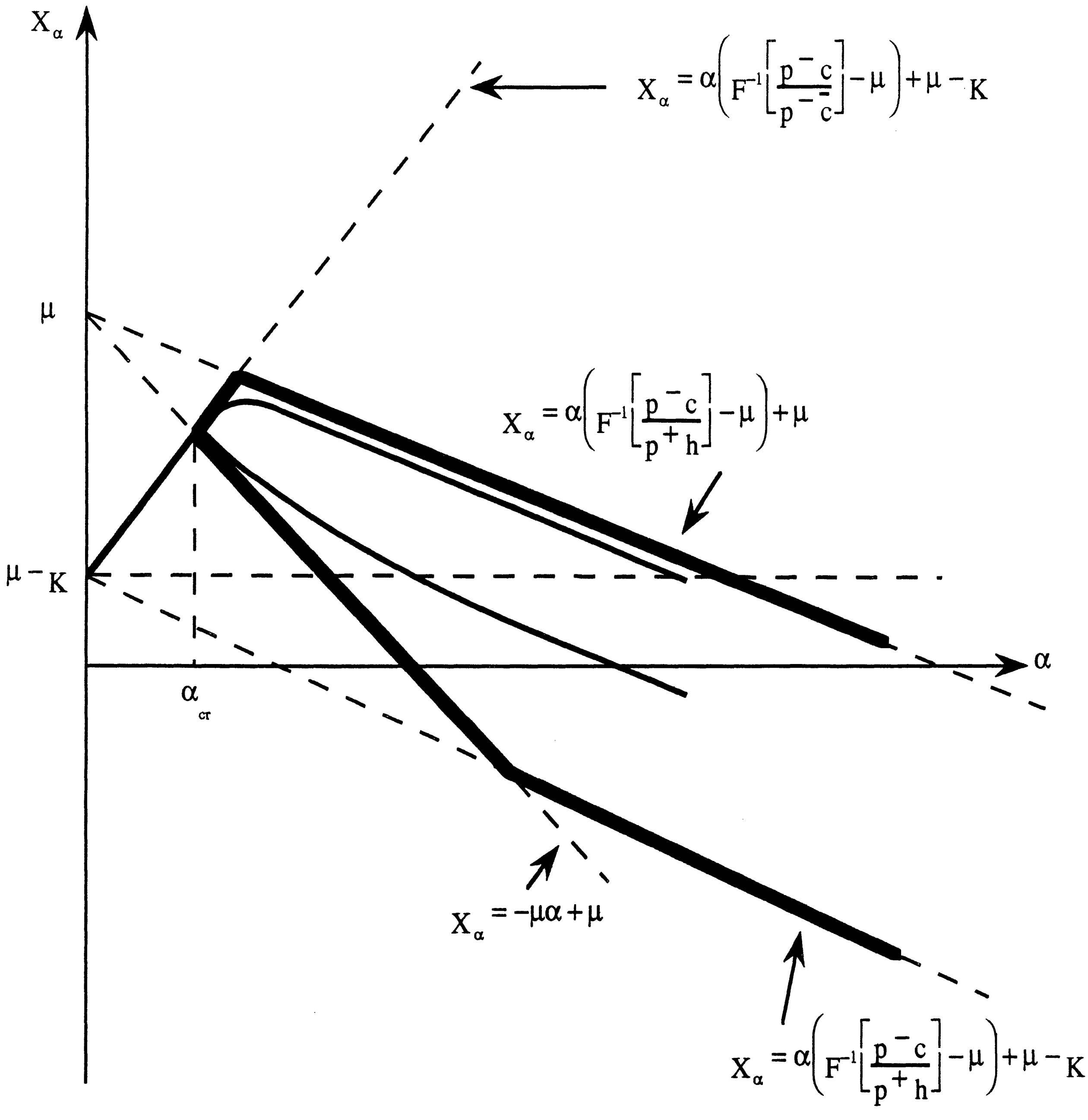
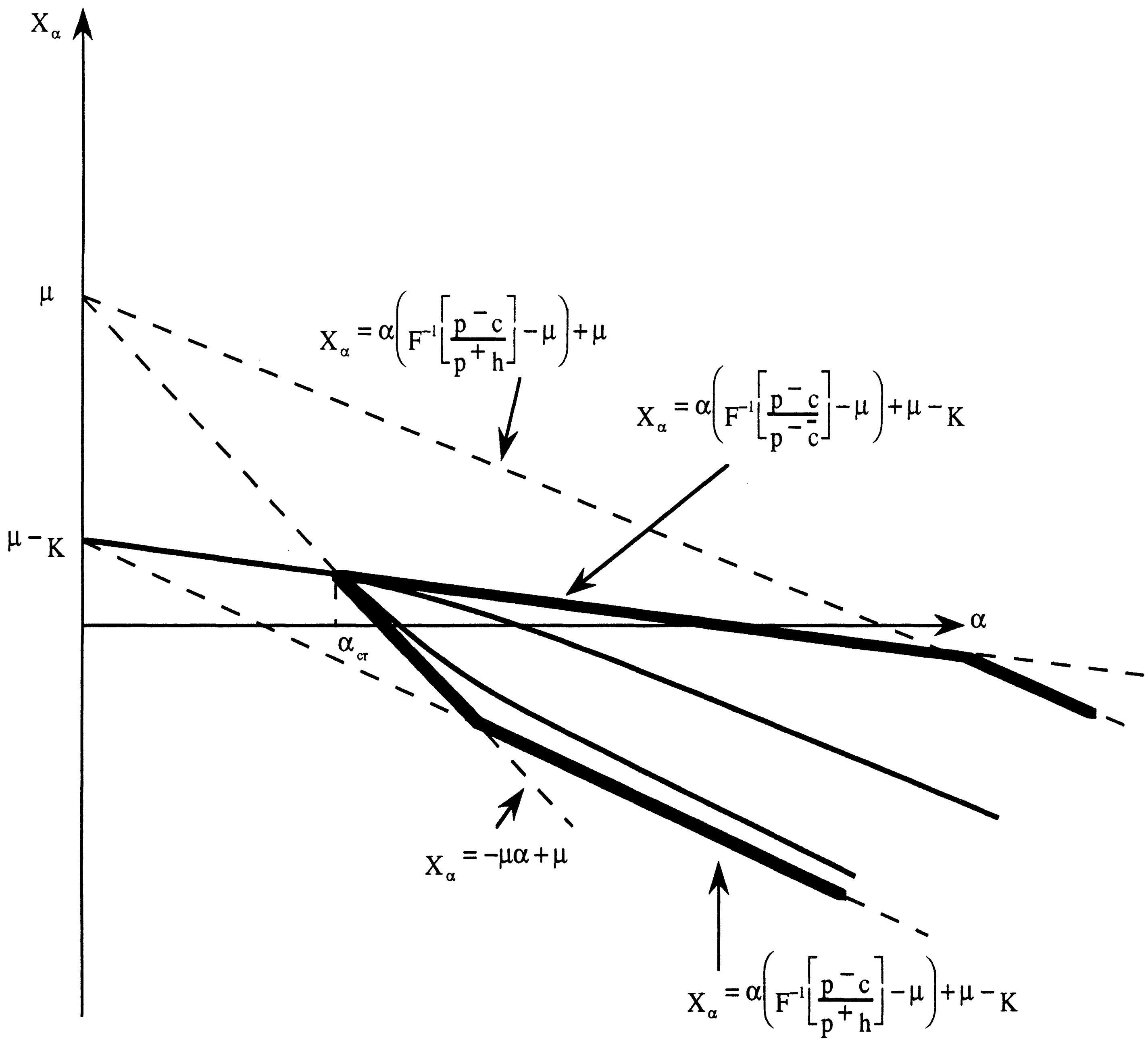


Figure 4



X_α
μ
$\mu - K$
α
α_{cr}
$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right] - \mu\right) + \mu$
$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p-c}\right] - \mu\right) + \mu - K$
$X_\alpha = -\mu\alpha + \mu$
$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right] - \mu\right) + \mu - K$

Figure 5

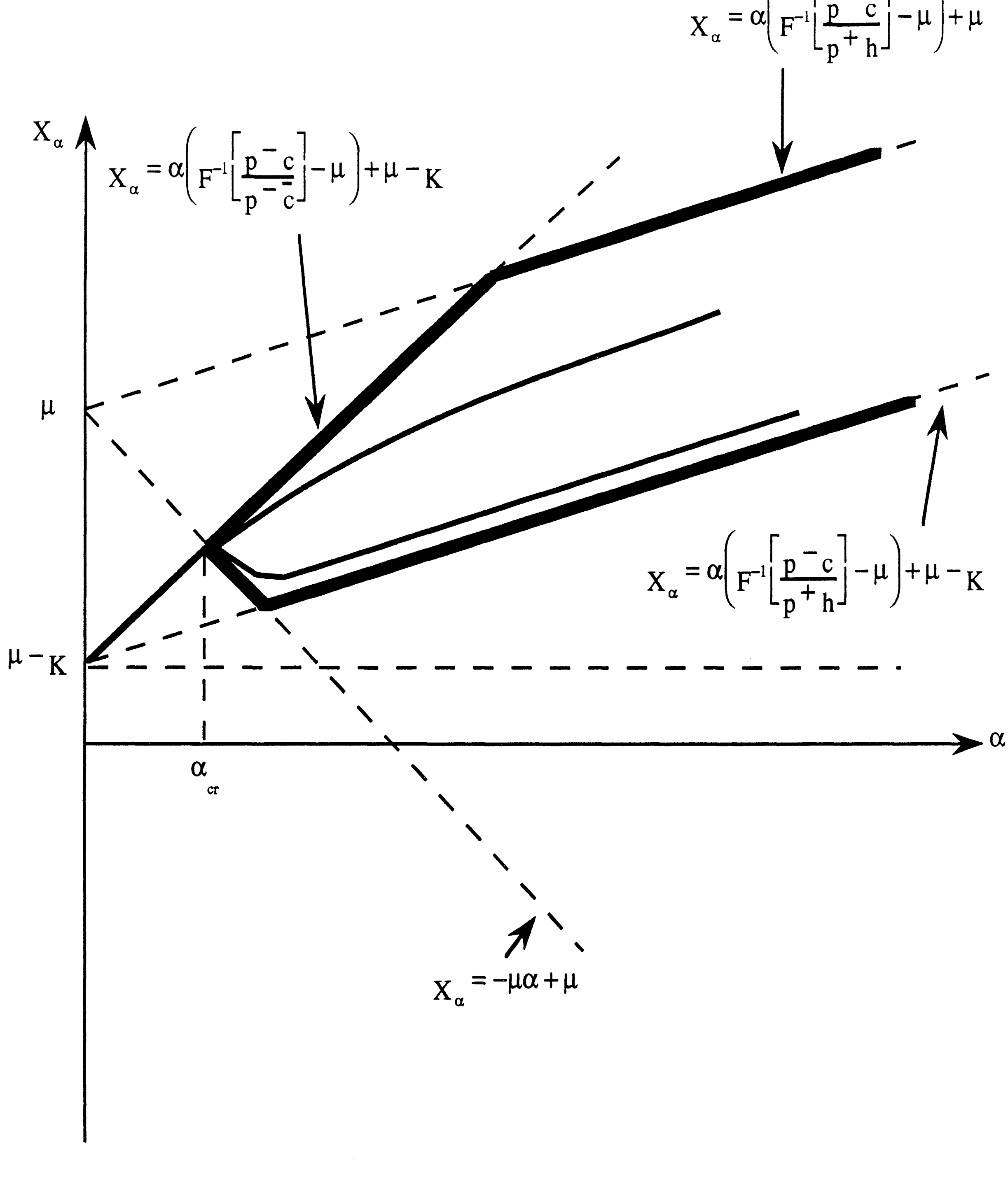

$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right]-\mu\right)+\mu$
X_α
$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p-c}\right]-\mu\right)+\mu-K$
μ
$X_\alpha = \alpha\left(F^{-1}\left[\frac{p-c}{p+h}\right]-\mu\right)+\mu-K$
$\mu-K$
α
α_{cr}
$X_\alpha = -\mu\alpha+\mu$

Figure 6