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Curvature and flatness in a Brans–Dicke universe

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Abstract

The evolution of a universe with Brans–Dicke gravity and nonzero curvature is investigated here. We find solutions to the equations of motion during the radiation dominated era. In a Friedman–Robertson–Walker cosmology we show explicitly that the three possible values of curvature $\kappa = +1, 0, -1$ divide the evolution of the Brans–Dicke universe into dynamically distinct classes just as for the standard model. Subsequently we discuss the flatness problem which exists in Brans–Dicke gravity as it does in the standard model. We also demonstrate a flatness problem in MAD Brans–Dicke gravity. In general, in any model that addresses the horizon problem, including inflation, there are two components to the flatness issue: (i) at the Planck epoch curvature gains importance, and (ii) during accelerated expansion curvature becomes less important and the universe flattens. In many cases the universe must be very flat at the Planck scale in order for the accelerated epoch to be reached; thus there can be a residual flatness problem.

1. Introduction

In the Brans–Dicke theory of gravity, the constant Planck mass of the Einstein theory is replaced with a massless scalar field [1]. As a result, the gravitational constant is not a fundamental constant of the theory but instead, the strength of gravity evolves dynamically. Interest in alterations to Einstein gravity has arisen in a variety of contexts. The cosmological importance of such theories has been investigated in inflationary models such as (hyper)-extended inflation [2] and Starobinsky's cosmology [3]. Other cosmological implications of modifying gravity have been indicated in attempted alternatives to inflation such as the MAD

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prescription [4,5]. In addition, some innovative theories which try to reconcile particle physics with gravity lead to low-energy theories which behave like the Brans–Dicke model. For instance, higher-dimensional theories or Kaluza–Klein theories [6] can lead to a dynamical Planck mass.

In the first part of this paper, we describe the evolution of a universe with Brans–Dicke gravity and nonzero curvature. We present the equations of motion and their solutions during the radiation dominated era (see also refs. [7,8]). We find the evolution of the scale factor, the temperature, and the Hubble constant as a function of the changing Planck mass rather than explicitly as a function of time – see ref. [4] for some discussion of explicit time dependence for a flat Brans–Dicke cosmology. In ref. [4] we presented solutions for a flat universe and, in an appendix, briefly outlined the solutions for the case of nonzero curvature. Solutions for nonzero curvature have been previously stated in the appendix of ref. [7] in a different form; in addition, the results of ref. [8] were obtained concurrently and are in agreement with our work in ref. [4]. In this paper we provide a complete detailed discussion of the evolution of curved Brans–Dicke cosmologies.

We begin by solving the equations of motion for general curvature. As expected, for a Friedmann–Robertson–Walker (FRW) metric, we will see that Brans–Dicke models can be split into three cases as in the standard model: the three possible curvatures $\kappa = +1, 0, -1$ break the universe up into dynamically distinct classes. In the $\kappa = +1$ universe, the energy density in matter exceeds the kinetic energy of the expansion. Eventually the expansion will cease and the universe will collapse under the pull of its own weight. If $\kappa = -1$, the cosmology is open. The energy density in matter is not sufficient to close the universe and it expands forever. The critical case, separating these two is the flat cosmology, $\kappa = 0$, for which there is just enough kinetic energy to escape collapse.

Once we have built a picture of the large-scale behavior of curved Brans–Dicke cosmologies we can ask if these cosmologies have a flatness problem. We devote the latter half of the paper to a study of flatness. The Brans–Dicke cosmology by construction evolves adiabatically and so has a flatness problem, as has been noted in ref. [7] and will be investigated below. On the other hand, if the Planck mass were to couple directly to matter, then the assumption of adiabaticity would be unfounded. It would be interesting in the future to investigate this possibility. Finally, at the end of the paper, we discuss flatness in the MAD solution to the horizon and monopole problems; the MAD proposal also relies on a dynamical Planck mass such as occurs in scalar theories of gravity. (For a discussion of the limitations and future of this model see refs. [9,10]). We show that if the cosmic evolution is adiabatic, as it is for the Brans–Dicke model, then MAD Brans–Dicke gravity cannot resolve the flatness problem.

1.1. Introductory comments on the flatness problem

Before proceeding, we introduce the flatness problem. To begin we review this problem in the context of the standard model. It appears that the universe has survived to a temperature of $T_0 = 2.74$ K and a ripe old age of 10–15 billion years.

The survival of our universe, in the context of the standard hot big bang cosmology, requires extraordinary values of some otherwise arbitrary constants. That is, for our universe to survive with these conditions it must be that curvature does not completely dominate the cosmic evolution. Yet, in the standard cosmology, the universe should quickly veer away from a flat appearance unless extraordinary initial conditions are imposed which render the universe extremely close to flat at its inception.

Consider the standard model Einstein equation in a FRW cosmology,

$$H^2 + \frac{\kappa}{R^2} = \frac{8\pi}{3M_0^2}\rho. \tag{1}$$

The curvature term in the equation of motion (1) scales as $1/R^2$ while the radiation density term scales as $\rho \sim 1/R^4$. Consequently, as we look back in time, when the universe is very small, the energy density dominates over curvature. Initially, curvature is unimportant in determining the dynamics of the scale factor and the universe looks roughly flat. As R grows, the curvature term should quickly come to dominate in the determination of the cosmological evolution. The fact that the matter term is still significant implies that the curvature radius defined from

$$R_{\text{curv}} = R(t)|\kappa|^{-1/2} \tag{2}$$

must be greater than or comparable to the Hubble length H^{-1}

$$R_{\text{curv}} \gtrsim H^{-1}. \tag{3}$$

Multiplying both sides of Eq. (3) by the temperature T and cubing we have the condition that the entropy within a curvature volume,

$$S_{\text{curv}} = R_{\text{curv}}^3 T^3 = R^3(t)T^3(t)|\kappa|^{-3/2} \equiv \bar{S}|\kappa|^{-3/2}, \tag{4}$$

must exceed $H_0^{-3}T_0^3$, which is roughly the entropy within a Hubble volume,

$$\bar{S}\,|\kappa|^{-3/2} \gtrsim H_0^{-3}T_0^3 = \alpha_0^{-3/2}M_0^3T_0^{-3}, \tag{5}$$

with $\alpha_0 = \gamma(t_0)\eta_0 = \frac{8\pi}{3}\cdot\frac{1}{30}\pi^2 g_*(t_0)\eta_0$ where $\eta_0 \sim 10^4$–10^5 is the ratio today of the energy density in matter to that in radiation. Notice that $\bar{S}(\frac{4}{3})\frac{1}{30}\pi^2 g_s \equiv S$, where S is the constant of motion and g_s counts the number of degrees of freedom contributing to the entropy. The constant Planck mass of the Einstein theory is $M_0 = 1.2\times10^{19}$ GeV and the temperature of the cosmic background radiation in units of GeV is $T_0 = 2.3\times10^{-13}$ GeV. Then Eq. (5) demands that $\bar{S}\,|\kappa|^{-3/2} \gtrsim 10^{90}$. As long as the cosmic evolution is adiabatic, then $\bar{S}$ and S_{curv} are constant, up to factors of degrees of freedom. Notice that, if the universe is flat and $\kappa = 0$, Eq. (5) is automatically satisfied. If instead the universe is created with $\kappa = \pm1$, then Eq. (5) tells us that if the universe is to survive until today, with the conditions we observe, then the otherwise arbitrary constant entropy $\bar{S}$ must have a monstrous value in excess of roughly 10^{90}. Thus an extraordinary value of an arbitrary constant of motion is required to preserve our cosmology. The challenge is to explain the enormous value of this otherwise arbitrary constant.

In modifications to the standard model which attempt to address the related horizon and monopole problems, the flatness problem must be reexamined. As we will see, in these dynamical models there are two components to the flatness issue: (1) At some high temperature, the cosmology undergoes an accelerated expansion, as happens for instance when an inflationary epoch begins [11]. During acceleration curvature becomes less important and the universe becomes flatter as we demonstrate below. (2) Above the temperature at which acceleration ensues, there is first an early epoch during which the universe decelerates and curvature gains importance (unless, of course, the accelerated expansion, e.g. inflation, takes place at the Planck scale).

To see these two components to the flatness problem, consider first dynamic solutions to the horizon problem. One can express the causality condition required to solve the horizon problem in a simple way:

$$\frac{1}{H_c R_c} \geqslant \frac{1}{H_0 R_0}. \tag{6}$$

(This equation is not the most general. It holds only if the scale factor of the universe behaves as a simple power law in time before t_c and during matter domination. See also below Eq. (51).) If this equation is satisfied, our observable universe today fits inside a causally connected region at some early time t_c. Note that this equation implies that $\ddot{R} > 0$ for some period between t_c and the present. The most successful model to date that satisfies Eq. (6) is inflation. MAD models attempt to satisfy Eq. (6) by replacing the potential domination in inflationary models with a change in the behavior of gravity. In any case, any model that satisfies this condition will automatically make the universe flatter. We can demonstrate this by comparing the scales R_{curv}^{-1} and H:

$$\frac{R_{\text{curv}}^{-1}}{H} = \frac{|\kappa|^{1/2}}{\dot{R}}. \tag{7}$$

We have argued that any dynamical model which solves the horizon problem will accelerate the cosmic expansion. As the universe accelerates, $\dot{R}$ must in fact grow. The importance of curvature will only diminish as $\dot{R}$ grows, thus rendering the universe flatter. Therefore, any dynamical model that satisfies Eq. (6) inevitably makes the universe flatter.

However, there is a second component to the flatness problem. Starting at the Planck time, before the onset of the accelerating phase, the universe decelerates and curvature gains importance. Again, we can see this from Eq. (7). As $\dot{R}$ slows, the curvature term grows in importance in determining the cosmic evolution. One has to be cautious that the earliest era during which curvature gains importance does not generate a serious flatness problem. For an adiabatic model, it is this early aspect of the flatness problem which is not escaped [5]. An adiabatic MAD universe therefore has a flatness problem as we will show in detail toward the end of the paper.

Inflation generates a large value for $\bar{S}$ today by dynamically producing entropy. If inflation begins at a temperature $T_c = M_0$, then the flatness problem is solved.

(To reiterate, $M_0 = 1.2 \times 10^{19}$ GeV is the standard Planck mass.) However, if inflation begins significantly below the Planck scale, i.e. $T_c \ll M_0$, and if the universe is closed, then there is a residual, though less severe, flatness problem. In order for the temperature in a closed universe to reach T_c, the temperature at which inflation begins, a large entropy is required, $\bar{S} \gtrsim (M_0/T_c)^3$, as is shown in Subsect. 7.1. Unless inflation begins near the Planck scale, there will be a large constraint on the entropy. For instance, if inflation begins near a temperature of $T_c \sim 10^{14}$ GeV, then the entropy must exceed $\bar{S} \gtrsim 10^{15}$. If the entropy is not at least this large, then the universe collapses before inflation begins. In an open cosmology, the universe will tend away from flatness by the time inflation begins (again, if $T_c < M_0$). To correct for this, inflation requires either (i) extra e-foldings of inflation if initially $\bar{S} \sim 1$ or (ii) an initial value of $\bar{S} \gtrsim (M_0/T_c)^3$. These claims about flatness are explained in detail in the paper.

1.2. Outline

Sect. 2 of this paper shows the equations of motion for Brans–Dicke gravity and their solutions parametrized by the Brans–Dicke field Φ. Sect. 3 presents an interpretation of these solutions for the three cases: (i) flat cosmology, $\kappa = 0$; (ii) open cosmology, $\kappa = -1$; and (iii) closed cosmology, $\kappa = 1$. Sect. 4 reviews the flatness problem in the standard model with Einstein gravity, while the subsequent sections generalize the discussion of flatness. In sect. 5 we define a modified ratio $\bar{\Omega}$ of the energy density of the universe to the critical density. Sect. 6 discusses the flatness problem in Brans–Dicke cosmology for the case where the Planck mass moves slowly and never deviates drastically from today's value. An example of the opposite limit, a large deviation from Einstein gravity, takes place in models of MAD gravity described in sect. 7. The flatness problem in these MAD models, if adiabaticity is assumed, is described. Conclusions are presented in sect. 8.

2. Equations of motion and their solutions

We start with a detailed study of the equations of motion and their solutions in curved Brans–Dicke cosmologies. In a scalar theory of gravity, such as that proposed by Brans and Dicke, the Einstein action, $A_{\mathrm{einst}} = \int \mathrm{d}^4x\sqrt{-g}(-M_0^2/16\pi)\mathscr{R}$ where $\mathscr{R}$ is the Ricci scalar and $M_0 = 10^{19}$ GeV, is modified by introducing a coupling between the Ricci scalar and some function of a scalar field ψ. We will call the function of the scalar field Φ and note that $\langle\Phi\rangle \equiv m_{\mathrm{pl}}^2$. Thus the Planck mass, which dictates the strength of gravity, is determined dynamically by the expectation value of Φ. The $\langle\ \rangle$ will be implicit in the rest of the paper. The action describing the theory is given by

$$A = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{\Phi(\psi)}{16\pi}\mathscr{R} - \frac{\omega}{\Phi}\frac{\partial_\mu\Phi\partial^\mu\Phi}{16\pi} + \mathscr{L}_{\mathrm{m}}\right), \tag{8}$$

where we used the metric convention $(-, +, +, +)$, and $\mathscr{L}_{\mathrm{m}}$ is the lagrangian density for all the matter fields excluding the field ψ. We have assumed zero cosmological constant and also that $\mathscr{L}_{\mathrm{m}}$ describes ideal relativistic or nonrelativistic fluids. The parameter ω is defined by $\omega = 8\pi\Phi/(\partial\Phi/\partial\psi)^2$. In this paper we consider the original proposal of Brans and Dicke,

$$\Phi = \frac{2\pi}{\omega}\psi^2, \tag{9}$$

where ω is a constant parameter of the theory. Notice that there is no direct coupling of the Planck mass to $\mathscr{L}_{\mathrm{m}}$. As a consequence of this, the universe evolves adiabatically so that $R \propto T^{-1}$ as we describe below.

Varying this action with respect to the metric gives the Einstein-like equation

$$G_{\mu\nu} = \frac{8\pi}{\Phi}\left(T^{\mathrm{m}}_{\mu\nu} + T^{\Phi}_{\mu\nu}\right), \tag{10}$$

where $T^{\mathrm{m}}_{\mu\nu}$ is the energy–momentum tensor in all fields excluding the Brans–Dicke field and $T^{\Phi}_{\mu\nu}$ is the energy–momentum tensor in the Φ field. In a FRW cosmology (10) gives the equation of motion for the scale factor $R(t)$,

$$H^2 + \frac{\kappa}{R^2} = \frac{8\pi\rho}{3\Phi} - \frac{\dot{\Phi}}{\Phi}H + \frac{\omega}{6}\left(\frac{\dot{\Phi}}{\Phi}\right)^2, \tag{11}$$

where $\kappa = 0$, $+1$, or -1 while ρ is the energy density and p is the pressure in all fields excluding the ψ field. The principle of stationary action with respect to the coordinate Φ gives

$$\ddot{\Phi} + 3H\dot{\Phi} = \frac{8\pi}{3+2\omega}(\rho - 3p). \tag{12}$$

Conservation of energy–momentum in the Φ sector, $-8\pi T^{\mu\nu}_{\Phi;\mu} = (\mathscr{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathscr{R})\Phi_{;\mu}$, is equivalent to the equation of motion of (12). Conservation of energy–momentum in the matter sector can be satisfied independently, $T^{\mu\nu}_{\mathrm{m};\nu} = 0$. In an isotropic and homogeneous universe the $\mu = 0$ component of the matter conservation equation gives

$$\dot{\rho} = -(\rho + p)3H. \tag{13}$$

Consider the radiation dominated era where $\rho = \frac{1}{30}\pi^2 g_* T^4$, $p = \frac{1}{3}\rho$, and g_* is the number of relativistic degrees of freedom in equilibrium. Since conservation of energy–momentum in ordinary matter does not involve Φ, we can deduce that the entropy per comoving volume in ordinary matter, $S = (\rho + p)V/T$, is conserved. We use the definition

$$\bar{S} = R^3T^3, \tag{14}$$

where $S \simeq \bar{S}(\frac{4}{3}) \cdot \frac{1}{30}\pi^2 g_{\mathrm{s}}$. For practical purposes we can take $g_* = g_{\mathrm{s}}$. Since we are interested in the early universe we treat the radiation dominated era. Solutions for the evolution of a Brans–Dicke universe during the matter dominated era are well known [12].

We present here the solutions to the equations of motion during a radiation dominated era for a Brans–Dicke theory with general κ. The flat ($\kappa = 0$) cosmology was described in detail in ref. [4] while the $\kappa \neq 0$ cases have been briefly described in the appendices of refs. [4,7] as well as in ref. [8]. Here the curved cosmologies are considered in detail. A flat cosmology is included as a particular case of these solutions. We parametrize R,T, and thus H by the Brans–Dicke field Φ.

The first integral of the Φ motion gives

$$\dot{\Phi} R^3 = -C, \quad \text{also} \quad H = -\frac{\ddot{\Phi}}{3\dot{\Phi}}. \tag{15}$$

C is an arbitrary constant of integration which can be positive, negative, or zero. The value of C is determined by the initial conditions for $\dot{\Phi}$. Consider the case of C identically zero. Then the Planck mass is constant and the cosmology imitates the usual standard cosmology described by Einstein gravity. However, we allow the value of the Planck mass to be $\tilde{m}_{\rm pl} \neq M_0$. In this case, Eq. (11) becomes familiar, $H^2 + \kappa/R^2 = \frac{8}{3}\pi\rho/\tilde{m}_{\rm pl}^2$.For $C = 0$, the curved cosmology is easy to understand: if $\kappa = +1$ (closed universe), the expansion will eventually cease and contraction will begin. If $\kappa = -1$ (open universe), the universe expands forever and is infinitely large. If $\kappa = 0$ (flat universe), the expansion will slow asymptotically to zero. If $C \neq 0$, the description of the universe's evolution is more complicated. Still, we expect that adding some energy density in a scalar field to the total energy density should not alter the rough behavior of the universe with κ. We verify that in fact the evolution of the Brans–Dicke universe is determined by the three values of κ in the usual way.

Solving the quadratic equation (11) for H with $C \neq 0$ and $\kappa \neq 0$ gives

$$H = -\frac{\dot{\Phi}}{2\Phi} \pm \sqrt{\frac{\left(1 + \frac{2}{3}\omega\right)}{4}\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{8\pi}{3\Phi}\rho - \frac{\kappa}{R^2}}\,. \tag{16}$$

Notice that the $\pm$ here refers to the two solutions of the quadratic Eq. (11) for H. We need to decipher which solution in Eq. (16) corresponds to a growing solution; that is, a positive Hubble expansion.

In the case of a flat universe, with $\kappa = 0$, the square root in Eq. (16) is necessarily larger than the first term. Thus, if we intend to study the expanding phase ($H > 0$), then we must choose the solution with the positive square root and so choose the $+$ sign. Eq. (16) becomes

$$H = -\frac{\dot{\Phi}}{2\Phi} + \sqrt{\frac{\left(1 + \frac{2}{3}\omega\right)}{4}\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{8\pi}{3\Phi}\rho - \frac{\kappa}{R^2}}\,. \tag{17}$$

Since we are studying the radiation dominated era, we use $\rho = \gamma T^4 = \bar{S}^{4/3}\gamma/R^4$, where $\bar{S}$ is defined in Eq. (14). Also, we pull a factor of $\frac{1}{4}(\dot{\Phi}/\Phi)^2\ (1+\frac{2}{3}\omega)$ out of the square root in Eq. (17) to write

$$H = -\frac{\dot{\Phi}}{2\Phi}\left[1+\epsilon\sqrt{1+\frac{\bar{S}^{4/3}\gamma}{\epsilon^2 R^4}\frac{\Phi}{\dot{\Phi}^2}-\frac{\kappa}{\epsilon^2 R^2}\left(\frac{\Phi}{\dot{\Phi}}\right)^2}\,\right], \tag{18}$$

where we define

$$\epsilon = \pm\tfrac{1}{2}\left(1+\tfrac{2}{3}\omega\right). \tag{19}$$

The $\pm$ in the definition of ϵ is needed to ensure that $-(\dot{\Phi}/\Phi)\epsilon > 0$ so that only the growing solution for H with the positive square root is considered. Thus the upper sign corresponds to $\dot{\Phi}/\Phi < 0$ and the lower sign corresponds to $\dot{\Phi}/\Phi > 0$. There are therefore two distinct sets of $\pm$. The first appears in Eq. (16) and distinguishes the two solutions for H which solves the quadratic Eq. (11). The second set of $\pm$ in the definition of ϵ are needed to ensure that only the solution for H with positive square root is considered.

If $\kappa = -1$, H is again positive only if the $+$ sign is chosen in Eq. (16). However, if $\kappa = +1$, it is possible that the square root is not larger than $\dot{\Phi}/2\Phi$. If this is the case, then the negative square root can yield a positive Hubble expansion. We will show in subsect. 3.3 that the growing solutions with negative square roots evolve from solutions which at earlier times obeyed Eq. (16) with a positive square root. This will be analyzed in detail when we study the overall behavior of a $\kappa = +1$ cosmology. In the end we will find that we can begin with the positive square root in (16) for growing solutions. We can proceed to solve for $R(\Phi)$ from Eq. (18).

We define the quantity χ as

$$\chi(\Phi) = \frac{\bar{S}^{4/3}\gamma}{\epsilon^2 R^4}\frac{\Phi}{\dot{\Phi}^2}. \tag{20}$$

Using the first integral of motion (15) to eliminate $\dot{\Phi}$, we find that χ becomes

$$\chi(\Phi) = \bar{S}^{4/3}\gamma C^{-2}\epsilon^{-2}\Phi R^2, \tag{21}$$

and we note that χ is always a real positive quantity. We also define

$$Q^2 = \frac{\epsilon^2 C^2}{\gamma^2 \bar{S}^{8/3}}\kappa. \tag{22}$$

With this definition, $Q^2 > 0$ corresponds to $\kappa = +1$, $Q^2 < 0$ corresponds to $\kappa = -1$, and $Q^2 = 0$ corresponds to $\kappa = 0$. We rewrite (16) with these definitions,

$$H = -\frac{\dot{\Phi}}{2\Phi}\left(1+2\epsilon\sqrt{1+\chi-Q^2\chi^2}\,\right). \tag{23}$$

Using $H = \dot{R}/R$, the definition of χ, and rearranging, we are left with the integral

$$\int_{\chi_i}^{\chi}\frac{\mathrm{d}\chi'}{\chi'\sqrt{1+\chi'-Q^2\chi'^2}} = -2\epsilon\int_{\Phi_i}^{\Phi}\frac{\mathrm{d}\Phi'}{\Phi'}. \tag{24}$$

Integrating this equation we find

$$\chi = \frac{1}{\sinh^2\Theta + Q^2\,\mathrm{e}^{-2\Theta}}, \tag{25}$$

where we have absorbed the constants of integration into the constant $\tilde{\Phi}$,

$$\tilde{\Phi} = \Phi_i\left(\frac{2+\chi_i+2\sqrt{1+\chi_i-Q^2\chi_i^2}}{\chi_i}\right)^{1/2\epsilon}, \tag{26}$$

and we define

$$\Theta = \epsilon\,\ln\frac{\Phi}{\tilde{\Phi}}. \tag{27}$$

The relationship between Φ and $\tilde{\Phi}$ depends on the value of κ. For instance, if $\kappa = 0$ and the universe is flat, then Φ asymptotically approaches the value $\tilde{\Phi}$. For $\kappa = \pm 1$, on the other hand, $\tilde{\Phi}$ does not define an asymptotic value. For details see subsects. 3.2 and 3.3.

Using $R = (\epsilon C/\bar{S}^{2/3})(\gamma\Phi)^{-1/2}\chi^{1/2}$, from the definition of χ, we find

$$R = \frac{\epsilon C}{\bar{S}^{2/3}\gamma^{1/2}}\frac{1}{\Phi^{1/2}}\left(\frac{1}{\sinh^2\Theta + Q^2\,\mathrm{e}^{-2\Theta}}\right)^{1/2}. \tag{28}$$

The temperature of the universe is found from adiabaticity to be

$$T = \frac{\bar{S}\gamma^{1/2}}{\epsilon C}\Phi^{1/2}\left(\sinh^2\Theta + Q^2\,\mathrm{e}^{-2\Theta}\right)^{1/2}. \tag{29}$$

The Hubble constant in terms of Θ is

$$H = \gamma^{1/2}\frac{T^2}{\Phi^{1/2}}\frac{\sinh^2\Theta + 2\epsilon\sinh\Theta\cosh\Theta + Q^2(1-2\epsilon)\,\mathrm{e}^{-2\Theta}}{\left(\sinh^2\Theta + Q^2\,\mathrm{e}^{-2\Theta}\right)^{1/2}}. \tag{30}$$

Armed with these results we can now discuss the nature of the solutions to the equations of motion for the different values of κ.

3. Nature of solutions

3.1. A flat cosmology, $\kappa = 0$

In ref. [4] this example was worked out in detail. We provide only a brief description here for completeness. There are three possible initial conditions for $m_{\rm pl}$. The Planck mass could have the constant value denoted $\tilde{m}_{\rm pl}$ throughout the radiation dominated era. Alternatively, $m_{\rm pl}$ could start out initially small and grow. Lastly, $m_{\rm pl}$ could initially be large and drop. In both of these latter cases $m_{\rm pl}$ approaches the boundary value $\tilde{m}_{\rm pl}$ as the scale factor grows. As can be seen from

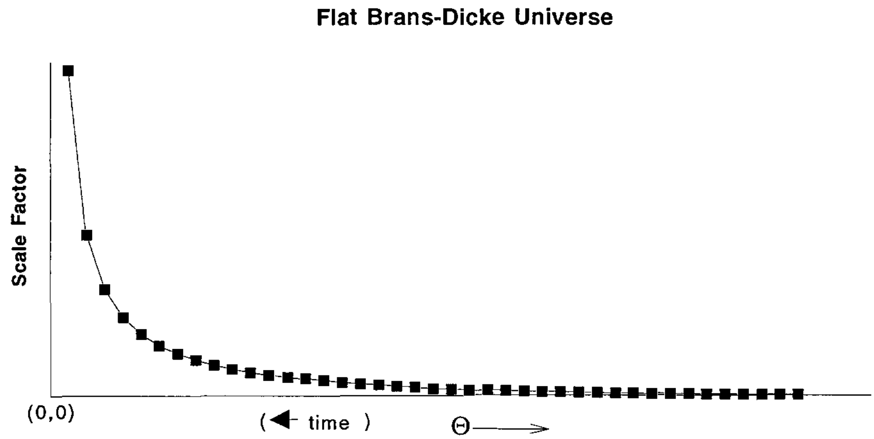

Fig. 1. A schematic picture of the scale factor as a function of Θ in a flat Brans-Dicke cosmology. Time increases along the horizontal axis from right to left. The scale factor grows infinitely large as Θ approaches zero.

Eq. (15), as the scale factor grows infinitely large, $\dot{\Phi} \to 0$ and the change in the Planck mass shuts off.

This general behavior is illustrated in Fig. 1 which shows schematically R as a function of Θ. Notice that time is increasing along the horizontal axis from right to left. As the Planck mass approaches the asymptotic value $\tilde{m}_{\rm pl}$ and thus Θ decreases toward zero, the scale factor grows.

While $\dot{\Phi}$ is significant, the scale factor and the temperature evolve with the changing $m_{\rm pl}$ in a complicated way. Once $m_{\rm pl}$ veers close to its asymptotic value $\tilde{m}_{\rm pl}$, then $\mathrm{d}m_{\rm pl}/\mathrm{d}t \approx 0$ and the universe evolves in a familiar way. For $m_{\rm pl} \approx \tilde{m}_{\rm pl}$ roughly constant, the equations of motion reduce to those of an ordinary radiation dominated Einstein cosmology with $M_{\rm o}$, the usual Planck mass of 10^{19} GeV, replaced with $\tilde{m}_{\rm pl}$. In particular, this means $R \propto t^{1/2}$, $T \propto t^{-1/2}$, and $H = 1/2t$. Despite the underlying structure of the theory, gravity appears to be described by a standard flat universe with a static gravitational constant. The universe will expand forever, slowing with age to almost a halt.

3.2. *An open cosmology, $\kappa = -1$*

If $\kappa = -1$ and so $Q^2 < 0$, the universe expands forever. This is confirmed by the expression for $R(\Theta)$. In Fig. 2 we plot R of Eq. (28) as a function of Θ. Time increases from right to left along the horizontal axis. According to our calculation, Θ is always positive and decreasing. Initially $R = 0$ at $\Theta = \infty$. As Θ drops R increases.

Let $\Theta_{\rm M}$ be the minimum value of Θ. At $\Theta = \Theta_{\rm M}$, the denominator in (28) vanishes and $R \to \infty$. According to the first integral of motion (15), $\dot{\Phi} \to 0$ when

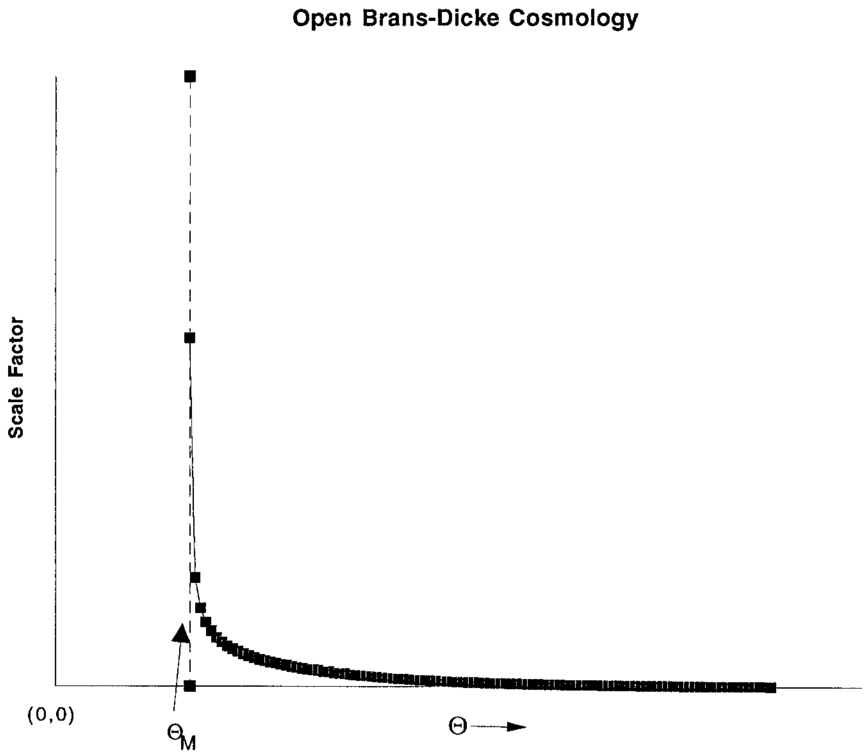

Fig. 2. A sketch of the scale factor as a function of Θ in an open Brans–Dicke universe. Time increases from right to left along the horizontal axis. The scale factor grows infinitely large as Θ approaches its minimum value denoted by Θ_M.

$R \to \infty$, and the change in Φ, and thus also in $\Theta(\Phi)$, turns off. The denominator in (28) vanishes at

$$\sinh^2\Theta_M + Q^2 \exp(-2\Theta_M) = 0, \tag{31}$$

which gives

$$\Theta_M = \tfrac{1}{2} \ln\left(1 + \sqrt{-4Q^2}\right). \tag{32}$$

For $\Theta \geqslant \Theta_M$, R is real. Notice that since $\Theta_M > 0$, we know that the Planck mass never reaches the value $\tilde{m}_{pl} = \tilde{\Phi}^{1/2}$.

A rough sketch of the history of an open Brans–Dicke cosmology begins with $\Theta = \infty$ and $R(\Theta) = 0$. As Θ drops toward Θ_M, R grows. The universe expands forever, growing infinitely large as Θ approaches Θ_M. There is no possibility for the Hubble expansion to vanish. Thus the gross behavior of this cosmology is similar to that of a standard open ($\kappa = -1$) cosmology.

3.3. A closed cosmology, $\kappa = +1$

The dynamical behavior of a closed universe ($\kappa = +1$) with Brans–Dicke gravity is very similar to that of a closed universe with canonical Einstein gravity: namely, the universe expands to some maximum size and then recontracts. If $\dot{\Phi}/\Phi = 0$, then the Planck mass is constant and it is simple to see that the standard behavior is reproduced with M_{o} replaced by $\tilde{m}_{\mathrm{pl}}$. If $\dot{\Phi}/\Phi \neq 0$, then the closed cosmology is a bit more subtle than in the standard model and it takes a bit of work to see this general behavior.

Firstly, solve (11) for H to rewrite Eq. (16) with the abbreviations $H_{\mathrm{R}}^2 \equiv (8\pi/3\Phi)\rho$ and $\mu \equiv \dot{\Phi}/\Phi$,

$$H = -\tfrac{1}{2}\mu \pm \sqrt{\epsilon^2\mu^2 + H_{\mathrm{R}}^2 - \frac{\kappa}{R^2}}\,, \tag{33}$$

where again the $\pm$ here refers to the two solutions of the equation of motion quadratic in H. If H is to reach zero at a finite temperature and reverse sign so the universe collapses, then it is critical that $-\frac{1}{2}\mu$ and the square root have opposite signs. It seems conceivable that, for instance, both $-\frac{1}{2}\mu$ and the square root will have the same sign and H will never vanish. We will find in the end that all is well; H will in fact reach zero and reverse course, but some effort will be required to demonstrate this fact. We will establish in the next subsections that the solution to Eq. (33) with negative square root evolves from the growing solution to Eq. (33) with the positive square root. We study the two possibilities, $\mu > 0$ and $\mu < 0$, separately.

For future reference, we write here the most pertinent results which will be derived below. In the end, it will be shown that the universe does stop expanding for $\kappa = +1$ and begins to collapse, regardless of μ, at a temperature of

$$T_{\mathrm{col}} = \frac{\Phi_{\mathrm{col}}^{1/2}}{\gamma^{1/2}\bar{S}^{1/3}}\,\frac{1}{|Q\chi_{\mathrm{col}}^{1/2}|}\,, \tag{34}$$

where

$$\chi_{\mathrm{col}} = \frac{1}{2Q^2}\left(1 + \sqrt{1 + Q^2\frac{(4\epsilon^2 - 1)}{\epsilon^2}}\right) \tag{35}$$

and

$$\frac{\Phi_{\mathrm{col}}}{\tilde{\Phi}} = \left(\frac{(2 + \chi_{\mathrm{col}})\epsilon + 1}{\chi_{\mathrm{col}}\epsilon}\right)^{1/2\epsilon}. \tag{36}$$

3.3.1. Closed Brans–Dicke cosmology with $\dot{\Phi}/\Phi > 0$

We show in this subsection that if $\kappa = +1$, the Hubble expansion will eventually end and the universe will ultimately collapse for the case of $\mu = \dot{\Phi}/\Phi > 0$. In the next subsection we repeat the analysis and verify that the same evolution is predicted for the case of $\mu < 0$. As well, we derive the results (34)–(36) here.

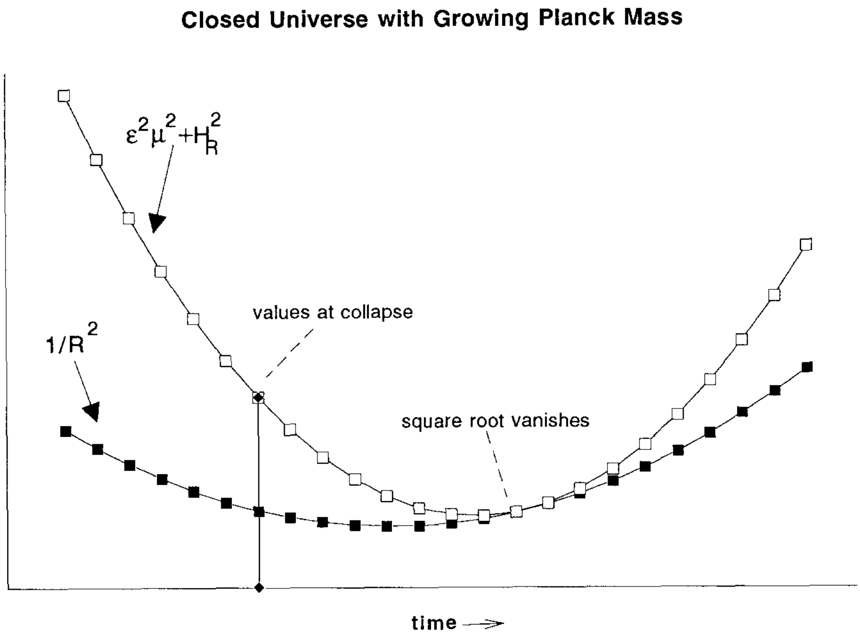

Fig. 3. A closed Brans–Dicke cosmology with a growing Planck mass. This figure shows the development of $\epsilon^2\mu^2 + H_R^2$ versus the development of the absolute value of the curvature term, $1/R^2$. For a growing Planck mass ($\mu > 0$), the universe collapses while the square root $\sqrt{\epsilon^2\mu^2 + H_R^2 - 1/R^2}$ is still positive.

In this subsection we take $\mu = \dot{\Phi}/\Phi > 0$ so that Φ grows with time. Consider the evolution of the three terms under the radical in Eq. (33): the kinetic term $\epsilon^2\mu^2$, the radiation term H_R^2, and the curvature term $-\kappa/R^2$. From the first integral of motion in Eq. (15), we can see that the kinetic term scales as $\epsilon^2\mu^2 \propto R^{-6}\Phi^{-2}$. From its definition we know that H_R^2 scales as $H_R^2 \propto R^{-4}\Phi^{-1}$ while the curvature term $\propto R^{-2}$. As we look back in time, Φ gets smaller with R. So, tracing back to $R \to 0$ for the sake of argument, we see that the kinetic term dominates over both the other terms initially and drops the most quickly. The next dominant term is H_R^2 which drops more slowly than the kinetic term but more quickly than the curvature. The curvature term is the least important of the three initially. Eventually, as R grows curvature gains importance.

If Φ is growing then $H > 0$ only if we choose the positive square root in (33). With this choice of signs, Eq. (33) becomes $H = -|\frac{1}{2}\mu| + \sqrt{\epsilon^2\mu^2 + H_R^2 - \kappa/R^2}$. In the beginning, when the scale factor is quite small, the curvature term is much less important than the sum of the positive terms in the square root. This must be so for the square root to exceed $|\frac{1}{2}\mu|$ and thus lead to an expanding universe, at least initially. Note that H can vanish and will eventually do so. H will vanish and the expansion will cease when the square root equals $\frac{1}{2}\mu$.

In Fig. 3 we have a schematic picture of the development of the sum of positive terms versus the development of the absolute value of the curvature term, $1/R^2$.

For $\mu > 0$, the case we study here, the universe starts to collapse (i.e. $H = 0$) at the point indicated on the figure. Collapse begins before the sum of positive terms crosses the curvature term, i.e., before the square root vanishes.

The value of T at which H reaches zero can be found by setting $H = 0$ in (33) and solving for the temperature. Remember in Eq. (29) that the temperature is expressed completely in terms of the value of Φ, up to the constants ϵ, C, S, etc.. Instead of referring to the collapse temperature, we could equally well refer to the value of Φ at which $H = 0$, $\Phi_{\rm col}$. To find $\Phi_{\rm col}$ we first set $H = 0$ in (23) and solve for the maximum value of χ (see Eq. (21)), called $\chi_{\rm col}$,

$$\chi_{\rm col} = \frac{1}{2Q^2}\left(1 + \sqrt{1 + Q^2\frac{(4\epsilon^2 - 1)}{\epsilon^2}}\right). \tag{37}$$

This can then be used in the definition of the temperature in (29) to find the temperature at which the universe begins to collapse,

$$T_{\rm col} = \frac{\Phi_{\rm col}^{1/2}}{\gamma^{1/2}\bar{S}^{1/3}}\,\frac{1}{\left|Q\chi_{\rm col}^{1/2}\right|}. \tag{38}$$

From expression (21) for χ and the definition of Θ, we see that this corresponds to a maximum value of Φ for $\mu > 0$,

$$\frac{\Phi_{\rm col}}{\tilde{\Phi}} = \left(\frac{(2 + \chi_{\rm col})\epsilon + 1}{\chi_{\rm col}\epsilon}\right)^{1/2\epsilon}. \tag{39}$$

Recall that ϵ was defined in Eq. (19) so that the product $-\mu\epsilon > 0$. For the case of $\mu > 0$ treated here, $\epsilon < 0$ and (39) is less than 1. In terms of $\Theta \equiv \epsilon\ \ln(\Phi/\tilde{\Phi})$, Eq. (39) implies $\Theta_{\rm col} \geqslant 0$.

Once Φ reaches $\Phi_{\rm col}$, which is $\leqslant \tilde{\Phi}$, then the expansion ceases and the universe begins to collapse. Notice from the first integral of motion (15), that Φ continues

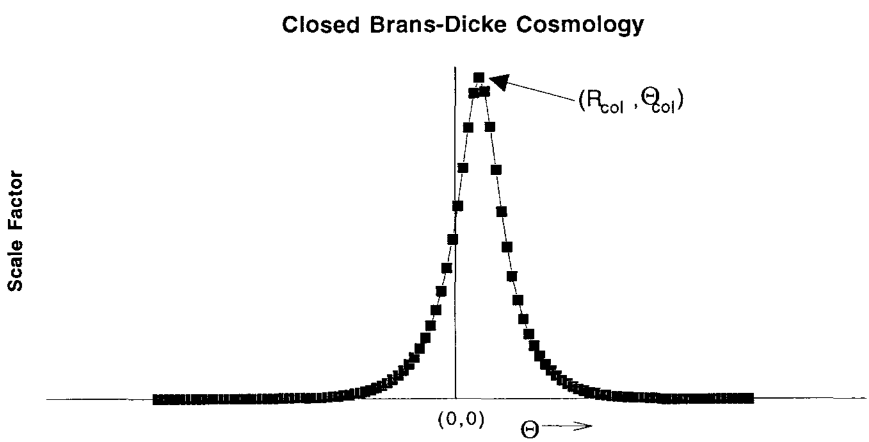

Fig. 4. The general behavior of the scale factor as a function of Θ in a closed Brans–Dicke universe. Time increases from right to left. The scale factor reaches its maximum extent at $\Theta_{\rm col}$. Subsequently the universe begins to shrink.

growing beyond $\Phi_{\rm col}$ as the universe contracts. In Fig. 4 we show the rough behavior of R with Θ where again time increases from right to left. The scale factor hits a maximum at $\Theta_{\rm col} \geqslant 0$ and begins contracting. Notice that as Φ continues to grow, Φ can exceed the value $\tilde{\Phi}$ in the definition of Θ and thus Θ can become negative.

We have not yet shown that the collapse temperature and collapse Φ defined here have relevance for $\mu < 0$ but we do so in the next subsection. For later reference we notice that if $\mu < 0$ and $\epsilon > 0$, then (39) is greater than 1 and again $\Theta_{\rm col} \geqslant 0$. An analogous picture to Fig. 4 applies for the closed universe with $\mu < 0$ discussed next.

3.3.2. Closed Brans–Dicke cosmology with $\dot{\Phi}/\Phi < 0$

If $\mu < 0$ and Φ is dropping then the analysis is a bit more complicated but the end result is very similar. We start with the assumption that initially the positive square root in (33) gives a real expanding cosmology and show that this is self-consistent. Consider again the three terms under the square root of (33): $\epsilon^2\mu^2 \propto R^{-6}\Phi^{-2}$, $H_{\rm R}^2 \propto R^{-4}\Phi^{-1}$, and κ/R^2. Since R grows while Φ drops, there is a competition in the denominator of the kinetic term. There is a similar competition in the denominator of the radiation term. We will show here that in fact ΦR^2 grows when the square root is positive and therefore establish that kinetic and radiation terms drop as the universe evolves. To get a handle on this, notice that the equation of motion (33), with the positive square root, can be rearranged to read

$$H + \frac{\mu}{2} = \frac{\dot{R}}{R} + \frac{\dot{\Phi}}{2\Phi} = \frac{1}{2}\frac{\mathrm{d}\ln(\Phi R^2)}{\mathrm{d}t} = +\sqrt{\epsilon^2\mu^2 + H_{\rm R}^2 - \frac{\kappa}{R^2}}\,. \tag{40}$$

This shows explicitly that ΦR^2 grows with time. Looking back in time, ΦR^2 drops and so $\epsilon^2\mu^2 \propto R^{-6}\Phi^{-2}$ grows as we go back in time. We also note that $\epsilon^2\mu^2 \propto H_{\rm R}^2(1/\Phi R^2)$. We can conclude then that if we trace back to $R \to 0$, that initially the kinetic term $\epsilon^2\mu^2$ dominates over $H_{\rm R}^2$ for very small values of R and loses its importance as ΦR^2 grows. Notice that $H_{\rm R}^2 \propto \kappa/R^2(1/\Phi R^2)$ and so, by pursuing the same reasoning, we see that in turn $H_{\rm R}^2$ dominates over the curvature. Again, the curvature term is the least important of the three initially. We then begin with the positive square root in (33).

At first glance it seems that H will not go to zero at finite temperature, $H = |\frac{1}{2}\mu| + \sqrt{\epsilon^2\mu^2 + H_{\rm R}^2 - \kappa/R^2}$. However, as R grows, curvature eventually gains importance and the square root passes through zero. (Eq. (40) shows that $\ln(\Phi R^2)$ has an extremum when the square root vanishes. Taking the second derivative of $\ln(\Phi R^2)$, evaluated when the square root vanishes, we see that the extremum is a maximum of $\ln(\Phi R^2)$. In other words, the first derivative of $\ln(\Phi R^2)$ passes through zero and then becomes negative. We can make the connection that the square root in Eq. (40) is equivalent to the first derivative of $\ln(\Phi R^2)$ and so we know that the square root falls smoothly through zero, becoming negative.) The solution for H is then Eq. (16) with the negative square root, $H = |\frac{1}{2}\mu|$

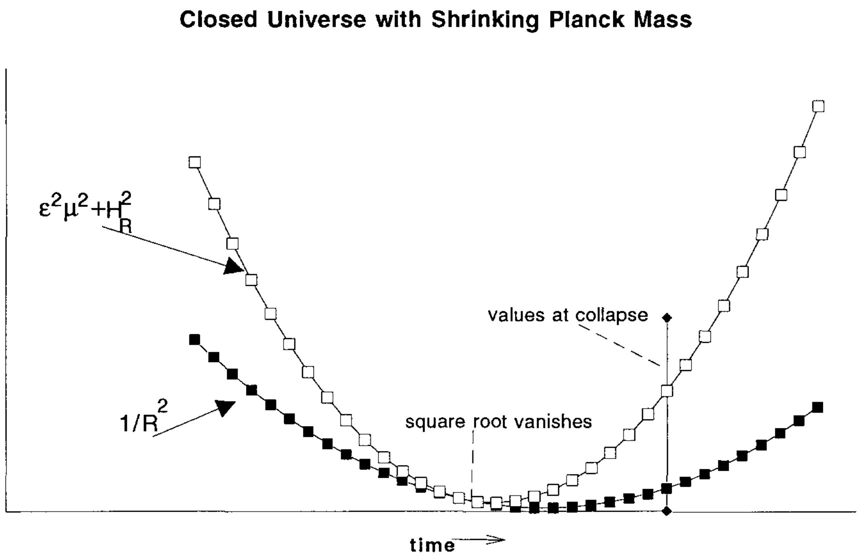

Fig. 5. A closed Brans–Dicke cosmology with a shrinking Planck mass. Here is shown a schematic picture of the sum of positive terms, $\epsilon^2\mu^2 + H_R^2$, versus the magnitude of the curvature term, $1/R^2$. As the figure demonstrates, the universe begins to contract after the sum of positive terms equals the curvature term and the square root vanishes. In other words, the universe begins collapse after the square root goes negative.

$-\sqrt{\epsilon^2\mu^2 + H_R^2 - \kappa/R^2}$. We find that solutions to H with negative square root grow out of solutions to H which began with positive square root.

As the magnitude of the square root grows, it eventually balances the $\frac{1}{2}\mu$ term until $H = 0$. The expanding phase ends and the universe begins to contract. This will happen at the same collapse temperature as defined in Eq. (38) for $\mu > 0$. Thus T_{col} of Eqs. (34) and (38) is the general expression defining the temperature at which a closed Brans–Dicke universe begins to contract. As Φ drops to the value Φ_{col}, which is $\geqslant \tilde{\Phi}$, the expansion ceases and reverses direction. As the universe collapses, Φ continues to drop.

Before we close this section, we note that we traced back to $R \to 0$ to draw conclusions. We cannot actually trace back all the way to $R \to 0$ since we would enter the epoch of quantum gravity at some finite R. If instead we start the evolution of the universe at finite R then the relative importance of the terms contributing to the square root depends on the relative amplitudes. For $\mu < 0$, in principle we could begin at finite R with a positive solution for H with a negative square root. What we have shown is that, in general, solutions to H with negative square root grow out of solutions to H which began with the positive square root. In Fig. 5 is drawn a schematic picture of the sum of the positive terms versus the magnitude of the curvature term. Along the horizontal axis time grows from left to right. This figure shows that the universe collapses after the sum of positive terms equals the curvature term and the square root vanishes, as we have argued above.

To the left of the crossing point, the square root is positive while to the right of the crossing point, the square root is negative. In principle, for $\mu < 0$, as this figure shows, one could begin with the universe at finite R between the points in Fig. 5 when the square root vanishes and the universe begins to collapse. So one could begin with growing solutions ($H > 0$) with negative square root, for some range of parameters.

Now that we have a picture of the large-scale behavior of a curved cosmology in a theory of modified gravity, we can discuss the flatness problem in these theories. We will work in analogy with the standard model and so build the framework for the standard flatness discussion here.

4. The flatness problem in the standard model

We argued in the introduction that, generically, adiabatic cosmologies will have to contend with a large $\bar{S}$ and so a flatness problem. In this section we interpret flatness for the standard cosmology in terms of the early cosmic dynamics and the energy density of the universe [13].

Consider a closed cosmology so that $\kappa = +1$. According to the standard Einstein equations, $H^2 + \kappa/R^2 = 8\pi\rho/3M_0^2$, for $\kappa = +1$ the expansion ceases and the universe then starts to collapse at a temperature of

$$T_{\mathrm{col}} = \frac{M_0}{\gamma^{1/2}\bar{S}^{1/3}}, \tag{41}$$

with $\gamma \equiv \frac{8}{90}\pi^3 g_*(t)$ and $g_*(t)$ is the number of relativistic degrees of freedom in equilibrium at time t. For ease of notation we again use the definition $\bar{S} = R^3T^3$, where $S \simeq \bar{S}(\frac{4}{3}) \cdot (\frac{1}{30}\pi^2)g_*$, and S is the constant entropy. For moderate values of $\bar{S}$, then $T_{\mathrm{col}} \sim M_0$. At a temperature of $\sim M_0$ the universe would reach its maximum extent and then implode. If we require that the universe continues to expand until today so that $T_{\mathrm{col}} < T_0$, then it must be that $\bar{S}^{1/3} \gtrsim M_0/T_0 \sim 10^{32}$. Thus the arbitrary constant entropy of the standard big bang model must be extraordinarly large if the universe is to survive until a temperature of $T_0 \sim 2.74$ K.

We can relate the flatness problem to the commonly used parameter $\Omega \equiv \rho/\rho_{\mathrm{cr}}$, where ρ is the total energy density of the universe and ρ_{cr} is the critical value required to just close the universe; that is, ρ_{cr} is that value of the energy density required to just balance H^2 if $\kappa = 0$,

$$\frac{8\pi}{3M_0^2}\rho_{\mathrm{cr}} = H^2. \tag{42}$$

Numerically, $\rho_{\mathrm{cr}} = 1.88 \times 10^{-29} h_0^2$ gm $\cdot$ cm^{-3}, where $h_0 = \frac{1}{100}H_0$ km $\cdot$ s^{-1} Mpc^{-1}.

According to the standard Einstein equations, we can write for general κ

$$\Omega - 1 = \frac{\kappa}{H^2R^2}. \tag{43}$$

If $\kappa/H^2R^2 \to 0$ and the cosmology is nearly flat, then $\Omega \to 1$. Written another way,

$$\Omega = \frac{1}{1-x(t)} \tag{44}$$

and

$$x = \frac{M_0^2\kappa}{\gamma \bar{S}^{2/3} T^2} \quad \text{or} \quad x = \left(\frac{T_{\mathrm{col}}}{T}\right)^2 \kappa. \tag{45}$$

For the closed cosmology of $\kappa = +1$, Eqs. (44) and (45) say that when $T = T_{\mathrm{col}}$, $x = 1$ and $\Omega \to \infty$. Thus $\Omega \sim 1$ is unstable and Ω will quickly diverge for temperatures below T_{col}. In terms of Ω, a large value for $\bar{S}$ means a small collapse temperature and so a small x. A small x in turn renders $\Omega \sim 1$, corresponding to a nearly flat universe.

So the flatness problem can be stated in terms of Ω. As $\Omega \sim 1$ is very unstable, it is unlikely and in some sense unnatural for it to be near 1 today. The observations that today $\Omega_0 \sim 1$ would require, for instance, at a temperature of the Planck scale, that $\Omega(T = M_0) - 1 \simeq O(10^{-60})$. In words, for Ω to be of order 1 today requires the universe to be created with the extreme condition that initially Ω be identical to 1 to better than one part in 10^{60}.

Similarly, in a standard open cosmology for which $\kappa = -1$, there is a flatness problem. Near a temperature of T_{col} given in (41), $x(T_{\mathrm{col}}) \sim -1$ and $\Omega \sim \frac{1}{2}$ which, astrophysically speaking, is on the order of 1. For temperatures $T < T_{\mathrm{col}}$, the universe will not collapse as in the closed case. However, x gets large and negative as the temperature drops below T_{col} and this drives $\Omega \to 0$. Thus, even for a standard open cosmology, the temperature defined as T_{col} represents the temperature at which $\Omega \sim 1$ becomes unstable. If today $\Omega_0 \sim 1$ then today $0 \geqslant x(T_0) \gtrsim -1$. The requirement that $\Omega_0 \sim 1$ today demands that $T_{\mathrm{col}} < T_0$ which in turn demands that $\bar{S}^{1/3} \gtrsim M_0/T_0$. If this were not the case, the universe would cool to the low temperatures of today in a Planck time, i.e., 10^{-43} s.

5. Defining $\bar{\Omega}$ for scalar gravity

In subsect. 3.3., the collapse temperature was defined in Eq. (34) for a closed Brans–Dicke cosmology. Before addressing the flatness problem in the Brans–Dicke model, we first develop the last tool needed and define here a new measure of the energy density of the universe, $\bar{\Omega}$.

We want to cast a flatness argument in analogy with the treatment for the standard cosmology. To do so, we here define a quantity $\bar{\Omega} \equiv \rho_{\mathrm{tot}}/\rho_{\mathrm{cr}}$ where ρ_{tot} is the sum of all energy densities, including the energy density in Φ, and where $(8\pi/3\Phi)\rho_{\mathrm{cr}} = H^2$ corresponds to the value of the total energy density required to just close the universe. Equivalently, $\bar{\Omega} \equiv (H^2 + \kappa/R^2)/H^2$ or, using Eq. (11),

$$\bar{\Omega} = \left[\frac{8\pi\rho}{3\Phi} - \frac{\dot{\Phi}}{\Phi}H + \frac{\omega}{6}\left(\frac{\dot{\Phi}}{\Phi}\right)^2\right]\left(\frac{8\pi\rho_{\mathrm{cr}}}{3\Phi^{-}}\right)^{-1}. \tag{46}$$

With this definition for $\bar{\Omega}$ we can write the equation of motion (11) as

$$\bar{\Omega}-1=\frac{\kappa}{H^2R^2}. \tag{47}$$

If $\kappa/H^2R^2 \to 0$ then $\bar{\Omega} \to 1$. Written another way,

$$\bar{\Omega}=\frac{1}{1-x(t)} \tag{48}$$

and

$$x=\frac{\kappa}{R^2}\left[\frac{8\pi\rho}{3\Phi}-\frac{\dot{\Phi}}{\Phi}H+\frac{\omega}{6}\left(\frac{\dot{\Phi}}{\Phi}\right)^2\right]^{-1}=\frac{\kappa/R^2}{H^2+\kappa/R^2}. \tag{49}$$

With some work we can rewrite x as

$$x=\frac{\kappa}{(T/T_{\rm col})^2\left[1+(T^2/\Phi)\chi_{\rm col}^{-1}\left(1+2\epsilon\sqrt{1+\chi-Q^2\chi^2}\right)\right]}. \tag{50}$$

There are several things to notice about these expressions for x. Firstly, for $\kappa=+1$, the far right hand side of Eq. (49) makes clear that $x=1$ at $H=0$ and so $x=1$ when $T=T_{\rm col}$. At $x=1$, $\bar{\Omega}\to\infty$ according to (48). This adheres to our expectations. At the collapse temperature, $\bar{\Omega}\sim 1$ becomes unstable.

In an open cosmology ($\kappa=-1$), the universe does not collapse as it does in the closed ($\kappa=+1$) universe. Still, near a temperature of $T_{\rm col}$ given in (38), $x(T_{\rm col})\sim -1$ and $\bar{\Omega}\sim\frac{1}{2}$. For temperatures $T\ll T_{\rm col}$, $x\to-\infty$ and $\Omega\to 0$. If $\bar{\Omega}_0\sim\mathcal{O}(1)$ today, then $0\geqslant x(T_0)\gtrsim\mathcal{O}(-1)$ today. Thus, the requirement that $\bar{\Omega}_0\sim 1$ today brings the same conclusion as that of the $\kappa=+1$ case.

6. The flatness problem in Brans–Dicke cosmology

The flatness problem in a Brans–Dicke cosmology can be quite complicated. Here we take $m_{\rm pl}\approx\tilde{m}_{\rm pl}$ to move slowly and to be near the value $M_0=10^{19}$ GeV, so that there is little deviation from standard Einstein gravity. These assumptions greatly simplify the discussion. We will discuss in the next sections a Planck mass far from the value M_0.

As for the standard model, we discuss the huge entropy condition in terms of a collapse temperature and consider first the closed cosmology ($\kappa=+1$). With $\dot{\Phi}/\Phi\approx 0$ the universe evolves as in a standard cosmology with M_0 replaced by $\tilde{m}_{\rm pl}$. If we study our example of the closed cosmology again, we find that the collapse temperature of Eq. (41) reduces to

$$T_{\rm col}=\frac{\tilde{m}_{\rm pl}}{\gamma^{1/2}\bar{S}^{1/3}}. \tag{51}$$

For moderate values of $\bar{S}$, the universe would contract at a temperature near the Planck scale.

Although Eq. (51) only holds true during radiation domination, we can easily correct for the era of matter domination to have a rough indication of the condition through to today. If we want the universe to survive until today, then $T_{\rm col} \lesssim T_0$ with $\tilde{m}_{\rm pl} \sim M_0$ and it must be that $\bar{S} \gtrsim 10^{90}$. Of course, we should guess that the standard model flatness problem surfaces here since we assumed that the cosmology looks like standard Einstein gravity.

7. MAD gravity and the horizon, monopole, and flatness problems

A more complicated situation arises if we allow for a large deviation from Einstein gravity. In particular, a conflict arises for MAD gravity which tries to exploit modified gravity to address the horizon and monopole problems. Although the universe is in principle made flatter in MAD gravity, we show here that the flatness problem is not solved if an assumption of adiabaticity is made.

The standard cosmology does not explain the remarkable smoothness and flatness of the observed universe. We can presently see across many regions which were not in causal contact at earlier times. All the same, today the universe does seem to be largely homogeneous and isotropic. This apparent smoothness seems to violate causality. As well, the universe appears to be roughly flat today; that is, matter continues to be important in determining the cosmic evolution so it must be that curvature does not completely dominate. In the standard cosmology, a universe which began with arbitrary initial conditions would quickly veer away from a flat appearance. In the absence of a dynamical explanation, a nearly flat universe today requires extraordinary initial conditions which render the universe extremely close to flat at early times. In addition, the inclusion of grand unified theories into the standard cosmology gives rise to a cosmologically disastrous abundance of monopoles. Although the monopole problem has a very different source from the horizon problem and flatness problem, solutions to one are often intimately connected with solutions to the others.

The inflationary model proposed by Guth addresses the horizon, flatness, and monopole problems. In the inflationary scenario, a potential energy density drives a period of accelerated growth of the scale factor. During this period, a causally connected region that was small at the beginning of inflation grows large enough to contain our observed universe. Then the homogeneity of the observed universe can be explained by a common history. As the universe inflates, the monopole abundance is diluted, as is everything else. Subsequent to this era of supercooling, entropy is produced as the potential energy is converted to radiation and the universe resumes an ordinary evolution. The generous entropy production reheats the universe to some high temperature, arranged to be below the temperature at which monopoles form. Thus inflation explains the present homogeneity and lack of monopoles. In addition, an inflationary epoch also allows the universe to begin with moderate initial values for the entropy. The enormous value of the entropy needed to explain the cosmic flatness today is generated dynamically.

In refs. [4,5] we suggested that a cosmology with a dynamical Planck mass, such as the Brans–Dicke model studied in this paper, can provide an alternative resolution to the horizon and monopole problems, though not the flatness problem. The horizon problem is resolved by slowing the evolution of the universe during the era of radiation domination. Early in the universe's history the structure of gravity slows the Hubble expansion, thus slowing the cosmological evolution. As a result, the universe at a given temperature is much older than in the standard model. Thus enough time elapses for the entire observable universe to be in causal contact. Large regions could thereby become smooth without violating causality. Expanding the horizon can also dilute the monopole density. As well, the slow Hubble expansion keeps monopole–antimonopole annihilations in equilibrium longer, allowing for a very low relic monopole abundance at the end of the day.

Adiabaticity was assumed in the original formulations of MAD gravity to make clear the role of the dynamical Planck mass. In refs. [9,10] obstacles to completing the adiabatic MAD picture are discussed. Some of these obstacles could be circumvented if the assumption of adiabaticity is removed or if higher-order theories of gravity are considered. Regardless of the troubles the MAD model faces, it is always the case that adiabatic MAD gravity will not address the flatness problem. The persistence of a flatness problem in the Brans–Dicke model is a direct consequence of the assumption of adiabaticity.

Since the flatness problem and the horizon problem are related, we first introduce the horizon problem and sketch the MAD prescription. A causal explanation of the homogenity of our observable universe could exist if a region causally connected at some high temperature grows big enough to encompass everything we can see. Since we can see back to the time of decoupling, the size of the observable universe is roughly the distance light could have traveled since that time, $\Delta t_0 \sim H_0^{-1}$, where H_0 is the Hubble constant today. Thus we can take the present comoving Hubble radius, $1/H_0 R_0$, as a measure of the comoving radius of the observable universe. The particle horizon defines the extent of a causally connected region. In the standard model the horizon $\sim H^{-1}$, so that the causality condition can be written as

$$\frac{1}{H_c R_c} > \frac{1}{H_0 R_0}. \tag{52}$$

The subscript c denotes values at an early time and subscript 0 denotes values today. (This equation only holds if the horizon size, d_{horiz}, obeys $d_{\mathrm{horiz}} \sim H^{-1}$. More generally the causality condition is $d_{\mathrm{horiz}}(t_c) R_c^{-1} \gtrsim d_{\mathrm{horiz}}(t_0) R_0^{-1}$.) The observable universe today fits inside a region causally connected at time t_c if Eq. (52) is satisfied. Then the horizon size at t_c before nucleosynthesis is large enough to allow for a causal explanation for the smoothness of the universe today. Since $H = \dot{R}/R$, Eq. (52) is equivalent to the requirement $\dot{R}_0 \gtrsim \dot{R}_c$; that is, the scale factor grows faster today than at earlier times and thus there must have been a period of acceleration between t_c and today.

For Brans–Dicke gravity with general curvature, the causality condition (52) would require

$$\frac{\Phi_c^{1/2}}{T_c}2\epsilon\frac{\left[\sinh^2\Theta_c+Q^2\exp(-2\Theta_c)\right]^{1/2}}{\sinh^2\Theta_c+2\epsilon\sinh\Theta_c\cosh\Theta_c+Q^2(1-2\epsilon)\exp(-2\Theta_c)}\gtrsim\beta\frac{M_0}{T_0}. \tag{53}$$

Notice that as $Q^2 \to 0$, (28), (29), and (53) reduce to the corresponding results for a flat universe, as it must. Similarly, for large Θ, $e^{-2\Theta} \to 0$, and we have the same causality condition as in the case of the flat universe.

7.1. The MAD slow roll limit

In comparison to the complicated constraint Eq. (53), consider the simplifying assumptions of a slowly rolling Planck mass and a flat cosmology. In the slow roll limit the condition (53) becomes much more simple,

$$\frac{m_{\rm pl}(t_c)}{M_0}\geqslant\beta\frac{T_c}{T_0}. \tag{54}$$

If the Planck mass were this large during an early hot epoch and thus the strength of gravity was weak, then a causally connected region would have time to grow large enough to encompass everything we can see. Subsequent to t_c, the strength of gravity must grow as the Planck mass drops. In the absence of all entropy production, it is difficult to drive the Planck mass from the large value indicated in Eq. (54) to the value M_0 [9,10].

In principle, the disparity between the large early value of the Planck mass needed to resolve the horizon problem in the slow roll limit and the Planck mass today leads to a flatter universe. As the Planck mass drops after time t_c and the strength of G increases, the universe becomes flatter; that is, since G describes the strength with which matter affects the cosmic development, curvature becomes less important than matter as the coupling strength increases. Still, the flatness problem is not removed entirely in a MAD era. Instead it is pushed to a higher energy scale. We study this question in detail here for a MAD Brans–Dicke theory.

First notice that in terms of $\overline{\Omega} = 1/(1-x)$ for μ identically zero, x reduces to

$$x=\frac{\Phi}{\gamma S^{2/3}T^2}\kappa \quad \text{or} \quad x=\left(\frac{T_{\rm col}}{T}\right)^2\kappa. \tag{55}$$

Since $\mu = \dot{\Phi}/\Phi$, the above expression can be taken as an approximation in the slow roll limit. Between t_c and today, x changes by

$$\frac{x_0}{x_c}\sim\left(\frac{M_0}{m_{\rm pl}(t_c)}\right)^2\left(\frac{T_c}{T_0}\right)^2\lesssim 1, \tag{56}$$

where the second relation follows from the horizon condition for a nearly constant Planck mass, Eq. (54). In other words, once the Planck mass reaches the value M_0, there is a new collapse temperature, $T_{\mathrm{col}}(M_0) = [M_0/m_{\mathrm{pl}}(t_c)]^2 T_{\mathrm{col}}(t_c)$. In the standard model, on the other hand, x would have grown by a factor of $(T_c/T_0)^2$. For example, if $T_c = 10^{16}$ GeV, x would have grown by a monstrous factor of 10^{55}. Thus, MAD assists the approach to flatness.

Although the universe gets flatter, there is still a flatness problem. Consider a closed cosmology. If the universe does not survive until the temperature drops to T_c, then the MAD model does not have the opportunity to address even the horizon problem. We will therefore always require that the universe continues to expand until a temperature below T_c. By the way, it is also true in an inflationary cosmology that the temperature at which the universe begins to collapse must also be below the temperature at which inflation begins.

In the slow roll limit, the collapse temperature is given roughly by

$$T_{\mathrm{col}} = \frac{m_{\mathrm{pl}}}{\gamma^{1/2}\bar{S}^{1/2}}. \tag{57}$$

For the slow roll MAD model, m_{pl} is many orders of magnitude larger than in the standard model. As a result of the huge Planck scale the temperature at which the universe begins to collapse is correspondingly larger. Given $T_{\mathrm{col}} < T_c$, Eq. (57) can be expressed as a condition on the entropy

$$\bar{S}^{1/3} \gtrsim \frac{m_{\mathrm{pl}}(t_c)}{\gamma^{1/2} T_c}. \tag{58}$$

The constraint on the Planck mass in the slow roll limit for a MAD model which addresses the horizon problem in Eq. (54) can be used to fix the constraint on the entropy. We find

$$\bar{S}^{1/3} \gtrsim \beta \frac{M_0}{T_0}; \tag{59}$$

that is to say, $\bar{S} \gtrsim 10^{90}$. A large entropy is needed if the curvature of the universe is not to take over just below the very large Planck scale. Although the universe gets flatter, the initial requirement of Eq. (59) that $\bar{S}^{1/3} \gtrsim \beta M_0/T_0$ is not alleviated. For $\bar{S} \sim 1$, the huge Planck scale and thus early Planck time leads to the instability of $\bar{\Omega} \sim 1$ well above T_c. Thus there is a flatness problem.

In an inflationary model where inflation begins at $T_c = M_0$, the flatness problem is solved. On the other hand, if $T_c < M_0$, then an inflationary model may require $S \gg 1$ in order for the universe to be able to reach the temperature at which inflation begins. In particular, in a closed universe, inflation requires that $\bar{S}^{1/3} \gtrsim M_0/T_{\mathrm{col}}$, where T_{col} must be less than the temperature at which inflation ensues. Here we can take T_c to mean the temperature at which an inflationary epoch begins. For example, if inflation begins at $T_c \sim 10^{14}$ GeV, then $\bar{S} \gtrsim 10^{15}$ is needed for the universe to survive to T_c. Although the numerical value of $\bar{S}$ will be smaller in an inflationary universe than in a MAD universe, the numerical value of $x(T)$ at a given temperature above T_c will be similar. Comparing $x_{\text{inflation}}$ before an inflationary epoch begins to x_{mad} above temperature T_c, shows that $x_{\text{inflation}} = x_{\text{mad}} \sim (T_{\text{col}}/T)^2$, where T_{col} is chosen less than T_c; that is, $\bar{\Omega}(T)$ is the same at a given temperature above T_c in a MAD world as it is before inflation. The distinction is that the Planck scale in inflation is only 10^{19} GeV while in MAD it can be many orders of magnitude larger. Thus, for $\bar{S} \sim 1$ the Planck time at which $\bar{\Omega} \sim 1$ would become unstable is much smaller in a MAD universe than in inflation. As a result, larger values of the constant of motion $\bar{S}$ are required in the MAD model to ensure that the universe survives until T_c.

More generally, if the Planck mass is moving rapidly, the flatness problem is a bit stickier to discuss although in the end the conclusions are much the same. The interested reader is referred to the appendix.

The flatness problem in an open MAD model has not been discussed here. We state without proof that the flatness problem persists in the open model as well. The reason is that the MAD prescription requires an old universe at a high temperature. From our experience with the standard model we learned that if the entropy is of order one, then the universe would cool below 2.74 K in 10^{-11} s. Similarly, in MAD gravity, if $\bar{S} \sim 1$, the universe would rapidly grow cold while the universe was still quite young.

8. Conclusions

We presented a detailed description of the Brans–Dicke early universe. For a homogeneous and isotropic cosmology, the three values of the curvature $\kappa = +1, -1$, and 0 separate the Brans–Dicke universe into expanding and recontracting, expanding forever, and the critical case between the two extremes, just as it does with standard cosmology.

In the Brans–Dicke action there is no coupling of the Planck mass directly to matter. As a result, no energy is transfered from the Planck sector into radiation and the cosmic evolution is adiabatic. As a direct result of this assumption of adiabaticity, the Brans–Dicke universe has the usual standard model flatness problem. An enormous value of the constant entropy $\bar{S}$ is required for the universe to survive until today. However, if a direct coupling of the Planck mass to matter is considered, then it could be that energy is transferred from the Planck sector into radiation and entropy is produced. In the spirit of inflation, a large entropy production could explain the present cosmic flatness.

Any dynamical model which solves the horizon problem automatically makes the universe flatter. For instance in the MAD model, Brans–Dicke gravity can be used to allow our present cosmology to be in causal contact during our earliest history. In the limit of a slowly rolling Brans–Dicke field, this is accomplished with a large early value for the Planck mass and thus a weak strength of gravity. As the strength of gravity increases, curvature becomes less and less important. Thus the universe does become flatter. However, because of the large early Planck mass and thus small Planck time, the universe quickly becomes curvature dominated, before the strength of gravity increases, unless the universe is very nearly flat at the Planck scale. As it stands, this generates the same flatness problem as in the standard model. Again, this is a direct consequence of the assumption of adiabaticity in Brans–Dicke gravity. The tenacious flatness problem may encourage us to move away from the adiabatic assumption and allow for the possibility of entropy production in a MAD cosmology [1].

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Appendix A. MAD flatness problem with a variable Planck mass

In this appendix we will study in some detail the flatness problem in a closed ($\kappa = +1$) MAD cosmology with a variable Planck mass.

If $\mu = \dot{\Phi}/\Phi \neq 0$ the flatness problem in a closed cosmology is a bit stickier although in the end the conclusions are much the same. We work with the more general collapse temperature of Eq. (34). We purposely wrote $T_{\rm col}$ to look similar to the collapse temperature in a standard model. To ensure that the universe survives at least until $T = T_{\rm c}$, the temperature at which the causality condition is met, we can require that the temperature at which the universe starts to collapse is less than $T_{\rm c}$. Subsequent to time $t_{\rm c}$ the universe will become flatter so we only have to worry about the very high-temperature behavior.

The collapse temperature is clearly more involved than if the Planck mass is constant. We will study loosely the imposed requirement that $T_{\rm col} < T_{\rm c}$ for different ranges of the constants of integration $S, C, \tilde{m}_{\rm pl}$, etc.. (We will restrict ourselves to $\omega \gtrsim 1$ since we are using Brans–Dicke gravity for which the observations have constrained $\omega > 500$ [14].) Demanding that $T_{\rm col} < T_{\rm c}$ gives the requirement

$$\bar{S}^{1/3}|Q\chi_{\rm col}^{1/2}| = \bar{S}^{1/3}\left(1 + \sqrt{1 + Q^2\frac{(4\epsilon^2 - 1)}{\epsilon^2}}\right)^{1/2} \gtrsim \frac{\Phi_{\rm col}^{1/2}}{T_{\rm c}\gamma^{1/2}}, \tag{60}$$

where $\Phi_{\rm col}$ is defined in Eq. (39).

If Q^2 is small to moderate, say $Q^2 \lesssim$ few, then Eq. (60) reduces to roughly

$$\bar{S}^{1/3} \gtrsim \frac{\Phi_{\rm col}^{1/2}}{T_{\rm c}}. \tag{61}$$

[1] The flatness problem is rooted in the assumption that an entropy of $\bar{S} > 10^{90}$ is unnatural while $S = O(1)$ is preferred; this assumption could someday be found to be incorrect.

Such a large $\bar{S}$ is consistent with a small Q^2 as can be seen from the definition for Q^2 in Eq. (22). For a small Q^2, Eq. (39) shows that $\Phi_{\rm col} \sim \tilde{\Phi}$. Therefore the universe will begin to collapse when Φ nears $\tilde{\Phi}$. We can use the causality condition to constrain $\Phi_{\rm col}$ and then make the bound on $\bar{S}$ more specific. The weakest requirement on $\tilde{\Phi}$ from the causality condition came from the slow roll limit of $\Phi(T_c) \sim \tilde{\Phi}$, for which

$$\frac{\tilde{\Phi}^{1/2}}{T_c} \gtrsim \beta \frac{M_0}{T_0}. \tag{62}$$

If the Planck mass had not entered the slow roll limit then $\tilde{\Phi}$ would only have been driven to even large values than Eq. (62) demands. Since $\Phi_{\rm col} \sim \tilde{\Phi}$ here, we have the bound on $\Phi_{\rm col}$ of

$$\frac{\Phi_{\rm col}^{1/2}}{T_c} \gtrsim \beta \frac{M_0}{T_0}. \tag{63}$$

Finally then (63) in (61) gives

$$\bar{S}^{1/3} \gtrsim \beta \frac{M_0}{T_0}. \tag{64}$$

This is similar to the standard model of cosmology which needs a very large $\bar{S}$, corresponding to a nearly flat universe, to avoid the immediate collapse of the universe.

If instead Q^2 is large, then Eq. (60) becomes roughly

$$\bar{S}^{1/3} Q \gtrsim \frac{\Phi_{\rm col}^{1/2}}{T_c}. \tag{65}$$

Also, we see from Eq. (39), that

$$\frac{\Phi_{\rm col}}{\tilde{\Phi}} \sim Q^{1/2\epsilon}, \tag{66}$$

and $\Phi_{\rm col}$ is far from $\tilde{\Phi}$. If $Q \to \infty$, then the curvature dependence is substantial. A dominant curvature drives the Planck scale at which collapse ensues further and further from the value $\tilde{\Phi}$.

The causality condition becomes difficult to satisfy if Q^2 is large. Notice, at high temperatures and values of Φ far from $\tilde{\Phi}$, that $\Theta \gg 1$. Both a huge Q^2 and a huge Θ suppress the left hand side of Eq. (53) driving $\Phi_c^{1/2} = m_{\rm pl}(T_c)$ to higher and higher scales to reach the demands of this condition. Using the causality condition (53) in the constraint Eq. (65) gives

$$\bar{S}^{1/3} \gtrsim \frac{1}{Q} 2\epsilon \frac{\sinh^2\Theta_c + 2\epsilon \sinh \Theta_c \cosh \Theta_c + Q^2(1-2\epsilon)\exp(-2\Theta_c)}{\left[\sinh^2\Theta_c + Q^2 \exp(-2\Theta_c)\right]^{1/2}} \beta \frac{M_0}{T_0}. \tag{67}$$

From expression (66) we can identify $e^{\Theta} \sim Q^{1/2}$, and since this is large we can also approximate $\sinh \Theta \sim \frac{1}{2}e^{\Theta}$. Putting this information together in (67) gives crudely

$$\bar{S} \gtrsim \frac{\epsilon}{Q} \frac{Q^2(1+2\epsilon) + Q(1-2\epsilon)}{(Q^2+Q)^{1/2}} \beta \frac{M_0}{T_0} \gtrsim \beta \frac{M_0}{T_0} \tag{68}$$

in the limit of large Q. We find in fact that unless $\bar{S}$ is large it is impossible to both satisfy the causality condition and fix $T_{col} < T_c$.

We conclude in general that although a MAD world gets flatter below a temperature of T_c, the flatness of the early universe is not explained. The huge Planck scale and so very early Planck time would quickly lead to a curvature dominated cosmology unless the otherwise arbitrary constant entropy is quite huge.

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