Optimal Starting Times and Suppliers Delivery Dates in a Stochastic Assembly System

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Abstract

We consider the problem of quoting delivery due dates to various suppliers in an assembly system with random processing times. Assume that an order for a project has been accepted and a due date for the completion of the project has been set in advance. Furthermore assume that the suppliers are perfectly reliable and that the suppliers delivery due dates must be quoted before any processing occurs in the system. Once the delivery due dates have been quoted and processing has begun in the system, it becomes necessary to determine the optimal starting time at every stage in the assembly system, due to the randomness in the processing times at the various stages. We show that the optimal starting policy at each stage calls for no intentional delay whenever outside supply parts arrive at that stage and that the optimal delivery due dates can be determined analytically. If the outside parts delivery dates were preset, the optimal starting time at each stage is described by a simple *wait-until* policy, where the manager waits until the greatest of the delivery

date and the beginning of the optimal cumulative planned processing time of all downstreams stages to begin processing. Thus the optimal starting policy at each stage is completely determined by a critical number, the optimal cumulative planned processing time of all downstream stages, showed to be the minimum of a convex function. We also consider the particular case when the ouside supply parts at each stage are available at no additional cost and characterize the *wait-until* policy that completely determines the optimal starting time at each stage. Finally we generalize by considering the case of unreliable outside suppliers.

1 The Model

Consider an assembly system with stochastic processing times as the one depicted in figure 1. The system consists of N production stages in series where outside supply parts are needed at each stage in order for processing to start at the following stage. Let stage 1 be the most downstream stage and let τ_i be the processing time at stage i with distribution function F_i, $i = 1, ..., N$. Assume that an order for a certain project has been accepted and a due date for the completion time of this project has been set at l_N time units from now. Naturally, the processing at each stage cannot start unless supply parts are delivered *and* processing at the prior stage is completed. There is a penalty p per unit of time for missing the project due date and a holding cost h_i per unit of time for holding the semi-finished project at the outlet of stage i. Outside supply parts needed for stage $(i-1)$ are held at cost $\overline{h}_i$ per unit of time at the outlet of stage i, $i = 2, ..., N+1$. We assume quite realistically that $h_i + \overline{h_i} \leq h_{(i-1)}$, with $h_{(N+1)}$ and $\overline{h}_{(N+1)} \geq 0$. We also assume that the suppliers are perfectly reliable, that the delivery due dates must be quoted before any processing occurs and that once processing occurs at a stage, it must be completed. However, due to the randomness in the processing time at the various stages in the system, once d_i^*, the optimal delivery due dates at stage $i = 1, ..., N$ have been quoted and processing has started in the system, it becomes necessary at the time processing is completed at each stage to determine the optimal starting time at the next stage, given the remaining time till the delivery due dates at the downstream stages and the project completion due date. Let $y_i^*(l_i, X_{i1}, ..., X_1)$ be the optimal waiting time between the time stage i is ready to be processed and its actual starting time, given that the project due date is l_i units of time away from now, X_{i1} units of time away from the delivery date of the outside supply parts needed for stage i, $X_{(i-1)1}$ units of time away from the delivery date of the outside supply parts needed for stage $(i-1)$ and so forth. Let $J_i^*(l_i, X_{i1}, ..., X_1)$ be the minimum cost of scheduling the processing at stages i through 1, given similar data as in $y_i^*(l_i, X_{i1}, ..., X_1)$. Also let y_N be the waiting time between the time stage N is ready to be processed and its actual starting time. Finally let $J_N^*(y_N, X_{N1}, ..., X_1)$ be the minimum cost of scheduling the processing and quoting the delivery due dates of the outside supply parts at stages N through 1, given that the project completion due date is l_N time units away from now.

2 Two-Stage Model for Determining Delivery Dates and Starting Times

Suppose that $N = 2$. We will use backward stochastic dynamic programming (SDP) to determine $y_1^*(l_1, X_1)$ in the first SDP stage, and y_2^*, X_{21}^* and X_1^* in the second SDP stage. The first SDP stage is triggered when job 2 is done processing. Figure 2 depicts the time advances in a two-job model. The first SDP stage problem is defined as following:

$$\begin{aligned} J_1^*(l_1, X_1) \; = \text{Min} \;\; & h_2 (l_1 - X_1)^+ + \left(h_2 + \overline{h}_2\right) y_1 + \\ & h_1 \int_0^{l_1-(l_1-X_1)^+-y_1} \left[\left(l_1 - (l_1 - X_1)^+ - y_1\right) - t\right] f_1(t)\, dt + \\ & p \int_{l_1-(l_1-X_1)^+-y_1}^{\infty} \left[t - \left(l_1 - (l_1 - X_1)^+ - y_1\right)\right] f_1(t)\, dt \qquad (1) \\ s.t. \;\; & y_1 \geq 0 \end{aligned}$$

It can be easily checked that $J_1(l_1, X_1)$ is convex in y_1 by differentiating it twice. Therefore, the optimal solution $y_1^*(l_1, X_1)$ to the first stage problem is obtained by differentiating equation (1) with respect to y_1 and setting to zero. Doing this we get the following *wait-until* policy, where we wait $l_1 - (l_1 - X_1)^+ - \overline{X}_1^*$ units of time before processing the job if $l_1 - (l_1 - X_1)^+ - \overline{X}_1^* \geq 0$, and process immediately otherwise:

$$y_1^*(l_1, X_1) = \begin{cases} l_1 - (l_1 - X_1)^+ - \overline{X}_1^* & \text{if } l_1 - (l_1 - X_1)^+ \geq \overline{X}_1^* \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $\overline{X}_1^* = F_1^{-1}\left[\left(p + h_2 + \overline{h}_2\right) / (p + h_1)\right]$ is called the optimal planned processing time for stage 1 given l_1 and X_1. Figure 2 shows that $l_1 = X_{21} - y_2 - \tau_2$. Hence the second stage problem is defined as following:

$$\begin{aligned} \text{Min } J_2(y_2, X_{21}, X_1) &= \overline{h}_2 \int_{X_{21}-y_2-X_1}^{\infty} [u - (X_{21} - y_2 - X_1)] f_2(u)\, du + \\ &\quad h_3(l_2 - X_{21}) + \left(h_3 + \overline{h}_3\right) y_2 + E\left[J_1^*(X_{21} - y_2 - \tau_2, X_1)\right] \\ &\quad \text{s.t. } y_2 \geq 0, X_{21} \geq X_1 \geq 0 \end{aligned} \tag{3}$$

Where the first term represents the cost of holding the outside supply parts needed at stage 1, given that they have arrived before processing at stage 2 has been completed. To find the optimal solution to the 2-stage problem, we will show that the point $X_0 = (y_2^*, X_{21}^*, X_1^*)$ that satisfies the necessary condition $\nabla J(X_0) = 0$ is feasible, satisfies $X_1^* \leq \overline{X}_1^*$, and that the Hessian of $J_2(y_2, X_{21}, X_1)$ is positive-definite for $X_1 \leq \overline{X}_1^*$. As a result, X_0 is the optimal solution to the 2-stage problem. To solve the set of first-order conditions, we substitute l_1 by $X_{21} - y_2 - \tau_2$ in $J_1^*(l_1, X_1)$, apply the expectation operator, differentiate (3) with respect to y_2, X_{21} and X_1 and set to zero. Substituting (2) in (1), we get

$$J_1^*(l_1, X_1) = \begin{cases} \left(h_2 + \overline{h}_2\right)\left(l_1 - \overline{X}_1^*\right) - \overline{h}_2 (l_1 - X_1)^+ + \\ h_1 \int_0^{\overline{X}_1^*} \left(\overline{X}_1^* - t\right) f_1(t)\, dt + \\ p \int_{\overline{X}_1^*}^{\infty} \left(t - \overline{X}_1^*\right) f_1(t)\, dt & \overline{X}_1^* \leq l_1 - (l_1 - X_1)^+ \\ \\ h_2 (l_1 - X_1)^+ + \\ h_1 \int_0^{l_1 - (l_1 - X_1)^+} \left[l_1 - (l_1 - X_1)^+ - t\right] f_1(t)\, dt + \\ p \int_{l_1 - (l_1 - X_1)^+}^{\infty} \left[t - l_1 + (l_1 - X_1)^+\right] f_1(t)\, dt & \overline{X}_1^* \geq l_1 - (l_1 - X_1)^+ \end{cases} \tag{4}$$

We differentiate between two cases: 1) $X_1 \geq \overline{X}_1^*$, 2) $X_1 \leq \overline{X}_1^*$.

Case 1: $X_1 \geq \overline{X}_1^*$

In this case, for $\tau_2 \leq X_{21} - y_2 - X_1$, the value function becomes

$$\begin{aligned} J_1^*(X_{21} - y_2 - \tau_2, X_1) &= h_2\left(X_{21} - y_2 - \tau_2 - \overline{X}_1^*\right) + \overline{h}_2\left(X_1 - \overline{X}_1^*\right) + \\ &\quad h_1 \int_0^{\overline{X}_1^*} \left(\overline{X}_1^* - t\right) f_1(t)\, dt + p \int_{\overline{X}_1^*}^{\infty} \left(t - \overline{X}_1^*\right) f_1(t)\, dt \end{aligned}$$

for $X_{21} - y_2 - X_1 \leq \tau_2 \leq X_{21} - y_2 - \overline{X}_1^*$ we get

$$\begin{aligned} J_1^*(X_{21} - y_2 - \tau_2, X_1) &= \left(h_2 + \overline{h}_2\right)\left(X_{21} - y_2 - \tau_2 - \overline{X}_1^*\right) \\ &\quad h_1 \int_0^{\overline{X}_1^*} \left(\overline{X}_1^* - t\right) f_1(t)\, dt + p \int_{\overline{X}_1^*}^{\infty} \left(t - \overline{X}_1^*\right) f_1(t)\, dt \end{aligned}$$

for $X_{21} - y_2 - \overline{X}_1^* \leq \tau_2 \leq X_{21} - y_2$ we get

$$\begin{aligned} J_1^*(X_{21} - y_2 - \tau_2, X_1) &= h_1 \int_0^{X_{21} - y_2 - \tau_2} (X_{21} - y_2 - \tau_2 - t) f_1(t)\, dt + \\ &\quad p \int_{X_{21} - y_2 - \tau_2}^{\infty} [t - (X_{21} - y_2 - \tau_2)] f_1(t)\, dt \end{aligned}$$

and finally for $\tau_2 \geq X_{21} - y_2$ we get

$$J_1^*(X_{21} - y_2 - \tau_2, X_1) = p\left[\mu_1 - (X_{21} - y_2 - \tau_2)\right]$$

hence in case 1, the 2-stage problem becomes:

$$
\begin{aligned}
\text{Min} J_2(y_2, X_{21}, X_1) \quad = \quad & \overline{h}_2 \int_{X_{21}-y_2-X_1}^{\infty} [u - (X_{21} - y_2 - X_1)] f_2(u)\, du + \\
& h_3 (l_2 - X_{21}) + \left(h_3 + \overline{h}_3\right) y_2 + \\
& \int_0^{X_{21}-y_2-X_1} \Big[h_2 \left(X_{21} - y_2 - \tau_2 - \overline{X}_1^*\right) + \overline{h}_2 \left(X_1 - \overline{X}_1^*\right) + \\
& h_1 \int_0^{\overline{X}_1^*} \left(\overline{X}_1^* - t\right) f_1(t)\, dt + p \int_{\overline{X}_1^*}^{\infty} \left(t - \overline{X}_1^*\right) f_1(t)\, dt \Big] f_2(u)\, du + \\
& \int_{X_{21}-y_2-X_1}^{X_{21}-y_2-\overline{X}_1^*} \Big[\left(h_2 + \overline{h}_2\right) \left(X_{21} - y_2 - u - \overline{X}_1^*\right) + \\
& h_1 \int_0^{\overline{X}_1^*} \left(\overline{X}_1^* - t\right) f_1(t)\, dt + p \int_{\overline{X}_1^*}^{\infty} \left(t - \overline{X}_1^*\right) f_1(t)\, dt \Big] f_2(u)\, du + \\
& \int_{X_{21}-y_2-\overline{X}_1^*}^{X_{21}-y_2} \Big[h_1 \int_0^{X_{21}-y_2-u} (X_{21} - y_2 - u - t) f_1(t)\, dt + \\
& p \int_{X_{21}-y_2-u}^{\infty} [t - (X_{21} - y_2 - u)] f_1(t)\, dt \Big] f_2(u)\, du + \\
& p \int_{X_{21}-y_2}^{\infty} (\mu_1 - X_{21} + y_2 + u) f_2(u)\, du \qquad (5) \\
\text{s.t.} \quad & y_2 \geq 0, l_2 \geq X_{21} \geq X_1 \geq 0
\end{aligned}
$$

Case 2: $X_1 \leq \overline{X}_1^*$

In this case, for $\tau_2 \leq X_{21} - y_2 - X_1$, the value function becomes

$$
\begin{aligned}
J_1^*(X_{21} - y_2 - \tau_2, X_1) \quad = \quad & h_2 (X_{21} - y_2 - \tau_2 - X_1) + \\
& h_1 \int_0^{\overline{X}_1^*} \left(\overline{X}_1^* - t\right) f_1(t)\, dt + p \int_{\overline{X}_1^*}^{\infty} \left(t - \overline{X}_1^*\right) f_1(t)\, dt
\end{aligned}
$$

for $X_{21} - y_2 - X_1 \leq \tau_2 \leq X_{21} - y_2$ we get

$$
\begin{aligned}
J_1^*(X_{21} - y_2 - \tau_2, X_1) \quad = \quad & h_1 \int_0^{X_{21}-y_2-\tau_2} (X_{21} - y_2 - \tau_2 - t) f_1(t)\, dt + \\
& p \int_{X_{21}-y_2-\tau_2}^{\infty} [t - (X_{21} - y_2 - \tau_2)] f_1(t)\, dt
\end{aligned}
$$

and finally for $\tau_2 \geq X_{21} - y_2$ we get

$$
J_1^*(X_{21} - y_2 - \tau_2, X_1) \quad = \quad p [\mu_1 - (X_{21} - y_2 - \tau_2)]
$$

hence in case 2 the 2-stage problem becomes:

$$\begin{aligned}
\text{Min} J_2(y_2, X_{21}, X_1) &= \overline{h}_2 \int_{X_{21}-y_2-X_1}^{\infty} [u - (X_{21} - y_2 - X_1)] f_2(u)\, du + \\
& h_3 (l_2 - X_{21}) + \left(h_3 + \overline{h}_3\right) y_2 + \\
& \int_0^{X_{21}-y_2-X_1} [h_2 (X_{21} - y_2 - X_1 - u) + \\
& h_1 \int_0^{X_1} (X_1 - t) f_1(t)\, dt + p \int_{X_1}^{\infty} (t - X_1) f_1(t)\, dt \Big] f_2(u)\, du + \\
& \int_{X_{21}-y_2-X_1}^{X_{21}-y_2} \left[h_1 \int_0^{X_{21}-y_2-u} (X_{21} - y_2 - u - t) f_1(t)\, dt + \right. \\
& \left. p \int_{X_{21}-y_2-u}^{\infty} [t - (X_{21} - y_2 - u)] f_1(t)\, dt \right] f_2(u)\, du + \\
& p \int_{X_{21}-y_2}^{\infty} (\mu_1 - X_{21} + y_2 + u) f_2(u)\, du \qquad (6) \\
\text{s.t.} \quad & y_2 \geq 0, l_2 \geq X_{21} \geq X_1 \geq 0
\end{aligned}$$

Differentiating with respect to y_2 we get $\forall\ X_1$:

$$\frac{\delta J_2(y_2, X_{21}, X_1)}{\delta y_2} = -\frac{\delta J_2(y_2, X_{21}, X_1)}{\delta X_{21}} + \overline{h}_3 = \overline{h}_3 \geq 0 \qquad (7)$$

hence $y_2^* = 0$ provided $\delta J_2(y_2, X_{21}, X_1)/\delta X_{21} = 0$ at $X_{21} = X_{21}^*$. Differentiating with respect to X_1 we get for $X_1 \geq \overline{X_1^*}$:

$$\frac{\delta J_2(y_2, X_{21}, X_1)}{\delta X_1} = \overline{h}_2 \int_{X_{21}-X_1}^{\infty} f_2(u)\, du + \overline{h}_2 \int_0^{X_{21}-X_1} f_2(u)\, du = \overline{h}_2 \geq 0 \qquad (8)$$

and for $X_1 \leq \overline{X_1^*}$:

$$\frac{\delta J_2(y_2, X_{21}, X_1)}{\delta X_1} = \overline{h}_2 - L(X_1) F_2[X_{21} - X_1] \qquad (9)$$

where $L(X_1) = \left[\left(\overline{h}_2 + h_2 + p\right) - (h_1 + p) F_1[X_1]\right] \geq 0$ for $X_1 \leq X_1^*$. Therefore if $X_{21}^* \geq X_1^*$, then X_1^* must be at most equal to $\overline{X_1^*}$. Differentiating with respect to X_{21} we get for

$X_1 \leq \overline{X}_1^*$:

$$\frac{\delta J_2(y_2, X_{21}, X_1)}{\delta X_{21}} = (h_1 + p)\int_{X_{21}-X_1}^{X_{21}} \int_0^{X_{21}-u} f_1(t) f_2(u)\, dt du + \left(h_2 + \overline{h}_2 + p\right)\int_0^{X_{21}-X_1} f_2(u)\, du - \left(h_3 + \overline{h}_2 + p\right) \quad (10)$$

Suppose $X_{21}^* \leq X_1^*$. Then (9) implies $X_{21}^* = X_1^* = 0$ and (10) implies $X_{21}^* = X_1^* = l_2$: contradiction, hence $X_{21}^* \geq X_1^*$. Similarly, suppose $X_1^* \leq 0$, then it must be from (9) that

$$\left(\overline{h}_2 + h_2 + p\right) F_2\left[X_{21}^*\right] \leq \overline{h}_2 \quad (11)$$

However, substituting X_1 by zero in (10) gives

$$\left(\overline{h}_2 + h_2 + p\right) F_2\left[X_{21}^*\right] = \left(\overline{h}_2 + h_3 + p\right)$$

contradicting (11), hence $X_1^* \geq 0$. It remains to show that the Hessian is positive-definite for $X_1 \leq \overline{X}_1^*$ and we are done. It fact, differentiating (9) and (10) we get

$$\frac{\delta^2 J_2(y_2, X_{21}, X_1)}{\delta X_1^2} = (h_1 + p) f_1(X_1)\int_0^{X_{21}-X_1} f_2(u)\, du + L(X_1) \quad (12)$$

$$\frac{\delta^2 J_2(y_2, X_{21}, X_1)}{\delta X_1 \delta X_{21}} = -L(X_1) f_2(X_{21} - X_1) \quad (13)$$

$$\frac{\delta^2 J_2(y_2, X_{21}, X_1)}{\delta X_{21}^2} = (h_1 + p)\int_{X_{21}-X_1}^{X_{21}} f_2(u) f_1(X_{21} - u)\, du + L(X_1) f_2(X_{21} - X_1) \quad (14)$$

where $L(X_1) \geq 0$ for $X_1 \leq \overline{X}_1^*$. Therefore the minor determinants are non-negative and thus the Hessian is positive-definite for $X_1 \leq \overline{X}_1^*$. As a result (y_2^*, X_{21}^*, X_1^*), where X_{21}^* vanishes (10), X_1^* vanishes (9) and $0 \leq X_1^* \leq \text{Min}\left\{X_{21}^*, \overline{X}_1^*\right\}$, is the optimal solution to the 2-stage problem. Since $X_1^* \leq \overline{X}_1^*$ then $y_1^* = 0$ w.p. 1 and the optimal policy calls for immediate processing whenever the outside supply parts needed at stage 2 are delivered, and for no intentional delay whenever the outside supply parts needed at stage 1 are delivered. To conclude this section, we must mention that the analysis presented here assumes that the project due date is sufficiently far in the future that there is enough time to plan for delivery of outside supply parts. However, there may be instances when the supplier at stage i requires that the order for the ouside parts be placed at least A_i units of time in advance. In such situations, schedule the delivery date for the outside supply parts needed at stage 1 at X_1^* if $X_1^* \leq l_2 - A_1$, and at $l_2 - A_1$ otherwise. Similarly, schedule the delivery date for the outside supply parts needed at stage 2 at X_{21}^* if $X_{21}^* \leq l_2 - A_2$, and at $l_2 - A_1$ otherwise. However, we showed that $X_{21}^* \geq X_1^*$. As a result, if $X_1^* \geq l_2 - A_2$, then schedule the delivery date for the outside supply parts needed at stage 2 at $l_2 - A_2$ since in this case $X_{21}^* \geq l_2 - A_2$.

2.1 Effect of the Processing Time Variance at Stage 1

In this section, we study the effect of the processing time variance at stage 1 on X_{21}^* and X_1^*, the optimal delivery dates of the outside supply parts needed at stage 2 and 1 respectively. To do this, we will use a simple mean-preserving transformation of a random variable. This transformation was first used by Baron [1], Rothschild and Stiglitz [3] and Sandmo [4] in Economic Theory, and was first used by Gerchak and Mossman [2] in Iventory Theory to show the effect of the demand variance on the optimal solution to the classical Newsvendor problem. With τ_1 as the processing time at stage 1, the transformation is

$$\tau_{1\alpha} = \alpha\left(\tau_1 - \mu_1\right) + \mu_1 \tag{15}$$

where μ_1 is the processing time mean at stage 1. It is clear that (15) implies $E\left[\tau_{1\alpha}\right] = E\left[\tau_1\right]$ and $Var\left[\tau_{1\alpha}\right] = \alpha^2 Var\left[\tau_1\right]$. Hence we increase or decrease the processing time variance at stage 1 by assigning values for α larger or smaller than 1 respectively. After making the substitution $X_2 = X_{21} - X_1$, equation (10) set to zero can be written as

$$\left(h_1 + p\right) Pr\left[\tau_2 \geq X_2, \tau_2 + \tau_1 \leq X_2 + X_1\right] + \left(h_2 + \overline{h}_2 + p\right) Pr\left[\tau_2 \leq X_2\right] = h_3 + \overline{h}_2 + p \tag{16}$$

and hence

$$(h_1+p)\,Pr\left[\tau_2 \geq X_{2\alpha}, \tau_2+\tau_{1\alpha} \leq X_{2\alpha}+X_{1\alpha}\right]+\left(h_2+\overline{h}_2+p\right)Pr\left[\tau_2 \leq X_{2\alpha}\right]=h_3+\overline{h}_2+p$$

which, after substituting for $\tau_{1\alpha}$ from (15), can be rewritten as

$$\begin{aligned}(h_1+p)\int_{X_{2\alpha}}^{X_{2\alpha}+X_{1\alpha}+\mu_1(\alpha-1)}\int_0^{\frac{X_{2\alpha}+X_{1\alpha}-\mu_1-u}{\alpha}+\mu_1} f_1(t)\,f_2(u)\,dt du+ \\ \left(h_2+\overline{h}_2+p\right)\int_0^{X_{2\alpha}} f_2(u)\,du \;=\; h_3+\overline{h}_2+p \quad (17)\end{aligned}$$

Similarly, equation (9) set to zero can be written as

$$\left[\left(\overline{h}_2+h_2+p\right)-(h_1+p)\int_0^{\frac{X_{1\alpha}-\mu_1}{\alpha}+\mu_1} f_1(t)\,dt\right]\int_0^{X_{2\alpha}} f_2(u)\,du=\overline{h}_2 \qquad (18)$$

Differentiating (17) with respect to α we get

$$\begin{aligned}\frac{(h_1+p)}{\alpha^2}\int_{X_{2\alpha}}^{X_{2\alpha}+X_{1\alpha}+\mu_1(\alpha-1)}\left[\alpha\left(\frac{dX_{2\alpha}}{d\alpha}+\frac{dX_{1\alpha}}{d\alpha}\right)-\right. \\ \left.(X_{2\alpha}+X_{1\alpha}-\mu_1-u)\right] f_1\left(\frac{X_{2\alpha}+X_{1\alpha}-\mu_1-u}{\alpha}+\mu_1\right) f_2(u)\,du \\ +\frac{dX_{2\alpha}}{d\alpha}f_2(X_2\alpha)\left[\left(\overline{h}_2+h_2+p\right)-(h_1+p)\int_0^{\frac{X_{1\alpha}-\mu_1}{\alpha}+\mu_1} f_1(t)\,dt\right]=0 \qquad (19)\end{aligned}$$

and differentiating (18) with respect to α we get

$$\begin{aligned}\frac{dX_{2\alpha}}{d\alpha}f_2(X_{2\alpha})\left[\left(\overline{h}_2+h_2+p\right)-(h_1+p)\int_0^{\frac{X_{1\alpha}-\mu_1}{\alpha}+\mu_1} f_1(t)\,dt\right] \\ -\frac{(h_1+p)}{\alpha^2}\left[\alpha\frac{dX_{1\alpha}}{d\alpha}-(X_{1\alpha}-\mu_1)\right] f_1\left(\frac{X_{1\alpha}-\mu_1}{\alpha}+\mu_1\right)\int_0^{X_{2\alpha}} f_2(u)\,du=0 \qquad (20)\end{aligned}$$

Suppose that for some α, we have $dX_{2\alpha}/d\alpha=0$. Equation (20) implies that either one of the following three statements are true:

$$
\begin{aligned}
&1) \quad X_{2\alpha} &&= 0 \\
&2) \quad X_{1\alpha} &&= \mu_1 (1-\alpha) \\
&3) \quad \tfrac{dX_{1\alpha}}{d\alpha} &&= \frac{(X_{1\alpha} - \mu_1)}{\alpha} \\
&4) \quad \alpha &&= \infty
\end{aligned}
$$

Equation (18) indicates that 1) leads to a contradiction. If 2) is true and $\alpha > 0$, then (18) and (17) imply

$$F_2 [X_{2\alpha}] = \frac{\overline{h}_2}{\overline{h}_2 + h_2 + p} \tag{21}$$

and

$$F_2 [X_{2\alpha}] = \frac{\overline{h}_2 + h_3 + p}{\overline{h}_2 + h_2 + p} \tag{22}$$

respectively: contradiction. If 2) is true and $\alpha = 0$, then (18) and (17) imply

$$X_{2\alpha}|_{\alpha=0} = F_2^{-1} \left[\frac{\overline{h}_2 + h_3 + p}{\overline{h}_2 + h_2 + p} \right] \tag{23}$$

and

$$\frac{dX_{1\alpha}}{d\alpha}|_{\alpha=0} = F_1^{-1} \left[\frac{\left(\overline{h}_2 + h_2 + p\right)(h_3 + p)}{(h_1 + p)\left(\overline{h}_2 + h_3 + p\right)} \right] - \mu_1 \tag{24}$$

respectively. If 3) is true, then (19) implies

$$\int_{X_{2\alpha}}^{X_{2\alpha}+X_{1\alpha}+\mu_1(\alpha-1)} (u - X_{2\alpha}) f_1 \left(\frac{X_{2\alpha} + X_{1\alpha} - \mu_1 - u}{\alpha} + \mu_1 \right) f_2 (u) \, du = 0 \tag{25}$$

which in turn implies either 2). Moreover, the fact that 2) is true and $\alpha = 0$ agrees with the fact that $X_1^* = \mu_1$ when the processing time at stage 1 is deterministic. As a result of this analysis, we conclude that $dX_{2\alpha}/d\alpha = 0$ only at $\alpha = 0$, which implies that (23) and (24) are true, and at $\alpha = \infty$ by 4). Suppose $dX_{2\alpha}/d\alpha > 0$ for $0 < \alpha < \infty$, then

$\lim_{\alpha\to\infty} X_{2\alpha} = \infty$ since $X_{2\alpha}$ is continuous in α. This leads to a contradiction in (17) since $h_2 \geq h_3$. Therefore $dX_{2\alpha}/d\alpha < 0$ for $0 < \alpha < \infty$. Equation (18) implies that $X_{1\alpha} \leq \alpha\left(\overline{X}_1^* - \mu_1\right) + \mu_1$ since equation (9) implies that

$$\frac{\delta J_2\left(y_2, X_{2\alpha}, X_{1\alpha}\right)}{\delta X_{1\alpha}}|_{X_{1\alpha}=\alpha\left(\overline{X}_1^*-\mu_1\right)+\mu_1} = \overline{h}_2 - L\left(\overline{X}_1^*\right) F_2\left[X_{2\alpha}\right] = \overline{h}_2 \geq 0 \qquad (26)$$

Therefore, having shown that $\overline{X}_1^* \geq \left(X_{1\alpha} - \mu_1\right)/\alpha + \mu_1$, we rewrite (20) as

$$\begin{aligned} &\left(h_1 + p\right)\frac{dX_{2\alpha}}{d\alpha} f_2\left(X_{2\alpha}\right)\left[F_1\left[\overline{X}_1^*\right] - F_1\left[\frac{X_{1\alpha} - \mu_1}{\alpha} + \mu_1\right]\right] \\ &- \frac{\left(h_1 + p\right)}{\alpha^2}\left[\alpha\frac{dX_{1\alpha}}{d\alpha} - \left(X_{1\alpha} - \mu_1\right)\right] f_1\left(\frac{X_{1\alpha} - \mu_1}{\alpha} + \mu_1\right)\int_0^{X_{2\alpha}} f_2\left(u\right) du = 0 \qquad (27)\end{aligned}$$

As a result, for equation (27) to be true, it must be that

$$\frac{dX_{1\alpha}}{d\alpha} \leq \frac{X_{1\alpha} - \mu_1}{\alpha} \qquad (28)$$

Suppose that for some α, we have $dX_{1\alpha}/d\alpha = 0$. Equation (28) implies that $X_{1\alpha} \geq \mu_1$. Therefore, $X_{1\alpha}$ is strictly decreasing in the region $X_{1\alpha} < \mu_1$. Finally, as α approaches ∞, equations (17) and (18) become

$$\left(h_1 + p\right)\int_{\tilde{X}_{2\alpha}}^{\infty}\int_0^{\tilde{X}_{1\alpha}+\mu_1} f_1\left(t\right) f_2\left(u\right) dt du + \left(h_2 + \overline{h}_2 + p\right)\int_0^{\tilde{X}_{2\alpha}} f_2\left(u\right) du = \left(h_3 + \overline{h}_2 + p\right) \qquad (29)$$

and

$$\left[\left(\overline{h}_2 + h_2 + p\right) - \left(h_1 + p\right)\int_0^{\tilde{X}_{1\alpha}+\mu_1} f_1\left(t\right) dt\right]\int_0^{\tilde{X}_{2\alpha}} f_2\left(u\right) du = \overline{h}_2 \qquad (30)$$

where $\tilde{X}_{2\alpha} = lim_{\alpha\to\infty} X_{2\alpha}$ and $\tilde{X}_{1\alpha} = lim_{\alpha\to\infty} dX_{1\alpha}/d\alpha$. After solving equations (29) and (30), we get

$$lim_{\alpha\to\infty} X_{2\alpha} \quad = \quad F_2^{-1}\left[\frac{\overline{h}_2}{\overline{h}_2 + h_2 - h_3}\right] \qquad (31)$$

$$lim_{\alpha \to \infty} \frac{dX_{1\alpha}}{d\alpha} = F_1^{-1}\left[\frac{h_3+p}{h_1+p}\right] - \mu_1 \tag{32}$$

If (24) is negative, then $X_{1\alpha}$ is strictly decreasing in α. If (24) is positive and (32) is negative, then $X_{1\alpha}$ increases as uncertainty is introduced, only to decreases towards $X_{1\alpha} = \mu_1$ as α keeps on increasing. After hitting $X_{1\alpha} = \mu_1$, $X_{1\alpha}$ strictly decreases as $\alpha \to \infty$. If (32) is positive, then $X_{1\alpha}$ is increasing with α, and $lim_{\alpha \to \infty} dX_{1\alpha}/d\alpha$ is given by (32). In conclusion, $X_{2\alpha}$ is decreasing with α with $X_{2\alpha}|_{\alpha=0}$ given by (23) and $lim_{\alpha \to \infty} X_{2\alpha}$ given by (31), while $X_{1\alpha}$ is increasing (assuming (24) is positive) in α with $dX_{1\alpha}/d\alpha|_{\alpha=0}$ given by (24) and $lim_{\alpha \to \infty} dX_{1\alpha}/d\alpha$ given by (32).

3 Two-Stage Model for Determining Starting Times with Preset Delivery Dates

Suppose that the delivery dates of the outside parts needed at stage 2 and stage 1 have been preset at X_{21} and X_1 respectively, together with the project due date at l_2. In this case it is necessary to determine $y_2^*(l_2, X_{21}, X_1)$ as defined in section 1. However, with this information structure in mind, the relationship between the state variables is now

$$l_1 = l_2 - (l_2 - X_{21})^+ - y_2 - \tau_2 \tag{33}$$

depending on whether the present is located before or after X_{21}, and $y_2^*(l_2, X_{21}, X_1)$ is obtained by solving the following problem:

$$\begin{aligned} J_2^*(l_2, X_{21}, X_1) = \text{Min} \quad & \overline{h}_2 \int_{l_2-(l_2-X_{21})^+-y_2-X_1}^{\infty} \left[u - \left(l_2 - (l_2 - X_{21})^+ - y_2 - X_1\right)\right] f_2(u)\, du + \\ & h_3 (l_2 - X_{21})^+ + \left(h_3 + \overline{h}_3\right) y_2 + \\ & E\left[J_1^*\left(l_2 - (l_2 - X_{21})^+ - y_2 - \tau_2, X_1\right)\right] \\ \text{s.t.} \quad & y_2 \geq 0 \end{aligned} \tag{34}$$

Now that X_{21} and X_1 are data, we may have either $X_1 \leq \overline{X}_1^*$ or $X_1 \leq \overline{X}_1^*$. If $X_1 \leq \overline{X}_1^*$, then the last term in (34) is expressed as in (6) (but with $l_2 - (l_2 - X_{21})^+$ instead of X_{21}), and as in (5) otherwise. Our goal is to show that in both cases, (34) is convex in y_2. If $X_1 \leq \overline{X}_1^*$, then differentiating (34) twice with respect to y_2 gives

$$\begin{aligned}\frac{\delta^2 J_2(l_2, X_{21}, X_1)}{\delta y_2^2} &= (h_1 + p) \int_{l_2-(l_2-X_{21})^+-y_2-X_1}^{l_2-(l_2-X_{21})^+-y_2} f_2(u) f_1\left(l_2 - (l_2 - X_{21})^+ - y_2 - u\right) du \\ &\quad + L(X_1) f_2\left(l_2 - (l_2 - X_{21})^+ - y_2 - X_1\right) \geq 0 \qquad (35)\end{aligned}$$

since $L(X_1) \geq 0$ for $X_1 \leq \overline{X}_1^*$. On the other hand if $X_1 \geq \overline{X}_1^*$, then differentiating (34) twice with respect to y_2 gives

$$\begin{aligned}\frac{\delta^2 J_2(l_2, X_{21}, X_1)}{\delta y_2^2} &= (h_1 + p) \int_{l_2-(l_2-X_{21})^+-y_2-\overline{X}_1^*}^{l_2-(l_2-X_{21})^+-y_2} f_2(u) f_1\left(l_2 - (l_2 - X_{21})^+ - y_2 u\right) du \\ &\quad + L\left(\overline{X}_1^*\right) f_2\left(l_2 - (l_2 - X_{21})^+ - y_2 - \overline{X}_1^*\right) \\ &= (h_1 + p) \int_{l_2-(l_2-X_{21})^+-y_2-\overline{X}_1^*}^{l_2-(l_2-X_{21})^+-y_2} f_2(u) f_1\left(l_2 - (l_2 - X_{21})^+ - y_2 - u\right) du\end{aligned}$$

since $L\left(\overline{X}_1^*\right) = 0$. Therefore making the substitution $\overline{X}_{21} = l_2 - (l_2 - X_{21})^+ - y_2$, we get that $y_2^*(l_2, X_{21}, X_1)$ is defined by a *wait-until* policy where we wait $l_2 - (l_2 - X_{21})^+ - \overline{X}_{21}^*$ units of time before processing the job at stage 2 if $l_2 - (l_2 - X_{21})^+ - \overline{X}_{21}^* \geq 0$, and process immediately otherwise:

$$y_2^*(l_2, X_{21}, X_1) = \begin{cases} l_2 - (l_2 - X_{21})^+ - \overline{X}_{21}^* & \text{if } l_2 - (l_2 - X_{21})^+ \geq \overline{X}_{21}^* \\ 0 & \text{otherwise} \end{cases} \qquad (36)$$

$\overline{X}_{21}^*$ is the optimal cumulative planned processing for stages 2 and 1. If $X_1 \leq \overline{X}_1^*$, $\overline{X}_{21}^*$ solves

$$\begin{aligned}\frac{\delta J_2(l_2, X_{21}, X_1)}{\delta \overline{X}_{21}} &= (h_1 + p) \int_{\overline{X}_{21}-X_1}^{\overline{X}_{21}} \int_0^{\overline{X}_{21}-u} f_1(t) f_2(u)\, dt du + \\ &\quad \left(h_2 + \overline{h}_2 + p\right) \int_0^{\overline{X}_{21}-X_1} f_2(u)\, du - \left(h_3 + \overline{h}_3 + \overline{h}_2 + p\right) = 0 \quad (37)\end{aligned}$$

and if $X_1 \geq \overline{X}_1^*$, $\overline{X}_{21}^*$ solves

$$\frac{\delta J_2(l_2, X_{21}, X_1)}{\delta \overline{X}_{21}} = (h_1 + p) \int_{\overline{X}_{21}-\overline{X}_1^*}^{\overline{X}_{21}} \int_0^{\overline{X}_{21}-u} f_1(t) f_2(u)\, dt du + \left(h_2 + \overline{h}_2 + p\right) \int_0^{\overline{X}_{21}-\overline{X}_1^*} f_2(u)\, du - \left(h_3 + \overline{h}_3 + \overline{h}_2 + p\right) = 0 \quad (38)$$

Note that we may have from (37) that $\overline{X}_{21}^* \leq X_1$, in which case it must have been that

$$\frac{\delta J_2(l_2, X_{21}, X_1)}{\delta \overline{X}_{21}}|_{\overline{X}_{21}=X_1} = (h_1 + p) \int_0^{X_1} \int_0^{X_1-u} f_1(t) f_2(u)\, dt du \geq \left(h_3 + \overline{h}_3 + \overline{h}_2 + p\right)$$

that is

$$\begin{aligned} X_1 &\geq F_{21}^{-1}\left[\frac{h_3 + \overline{h}_3 + \overline{h}_2 + p}{h_1 + p}\right] \\ \Rightarrow \overline{X}_{21}^* &= F_{21}^{-1}\left[\frac{h_3 + \overline{h}_3 + \overline{h}_2 + p}{h_1 + p}\right] \leq X_1 \end{aligned}$$

Similarly, we may have from (38) that $\overline{X}_{21}^* \leq \overline{X}_1^*$, in which case it must have been that

$$\begin{aligned} \overline{X}_1^* &= F_1^{-1}\left[\frac{h_2 + \overline{h}_2 + p}{h_1 + p}\right] \geq F_{21}^{-1}\left[\frac{h_3 + \overline{h}_3 + \overline{h}_2 + p}{h_1 + p}\right] \\ \Rightarrow \overline{X}_{21}^* &= F_{21}^{-1}\left[\frac{h_3 + \overline{h}_3 + \overline{h}_2 + p}{h_1 + p}\right] \leq \overline{X}_1^* \end{aligned}$$

hence $y_1^* = 0$ w.p.1 and stage 1 is processed immediately when processing at stage 2 is completed.

4 Case when $\overline{h}_3 = \overline{h}_2 = 0$

Suppose that the outside supply parts needed at each stage can be made available at no additional cost. Assume furthermore that these parts are actually available at the time the order is accepted. Then the problem reduces to the one considered in Yano [5]. In the framework of our paper, the *wait-until* policy at stage 1 is given by:

$$y_1^*(l_1) = \begin{cases} l_1 - \overline{X_1^*} & \text{if } l_1 \geq \overline{X_1^*} \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

where $\overline{X_1^*} = F_1^{-1}[(p+h_2)/(p+h_1)]$. To determine $y_2^*(l_2)$ we solve

$$\begin{aligned} J_2^*(l_2) &= \text{Min} \quad h_3 y_2 + E[J_1^*(l_2 - y_2 - \tau_2)] \\ &\quad \text{s.t.} \quad y_2 \geq 0 \end{aligned} \tag{40}$$

Our goal is to show that $J_2(l_2)$ is convex in y_2. The expectation operator conserves convexity. Thus suppose that $J_1^*(l_1)$ is convex, then we are done. Our goal is to show that the Hessian of $J_1^*(l_1, X_1)$ is positive-definite. Substituting (39) in (1) (with $\overline{h}_2 = 0$ and $X_1 \geq l_1$), we get

$$J_1^*(l_1) = \begin{cases} h_2\left(l_1 - \overline{X_1^*}\right) + h_1 \int_0^{\overline{X_1^*}} \left(\overline{X_1^*} - t\right) f_1(t)\,dt + \\ p\int_{\overline{X_1^*}}^{\infty} \left(t - \overline{X_1^*}\right) f_1(t)\,dt & \overline{X_1^*} \leq l_1 \\ h_1 \int_0^{l_1} (l_1 - t) f_1(t)\,dt + p\int_{l_1}^{\infty} (t - l_1) f_1(t)\,dt & \overline{X_1^*} \geq l_1 \end{cases} \tag{41}$$

Differentiating (41) a first time with respect to l_1 we get:

$$\frac{d[J_1^*(l_1)]}{dl_1} = \begin{cases} h_2 & l_1 \geq \overline{X_1^*} \\ (h_1 + p)\int_0^{l_1} f_1(t)\,dt - p & \overline{X_1^*} \geq l_1 \end{cases} \tag{42}$$

Differentiating (41) a second time with respect to l_1 we get:

$$\frac{d^2[J_1^*(l_1)]}{dl_1^2} = \begin{cases} (h_1 + p) f_1(l_1) & \overline{X_1^*} \geq l_1 \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

It is easy to see from (41) and (42) that (41) is continuous at $l_1 = \overline{X_1^*}$. (4) is also differentiable at $l_1 = \overline{X_1^*}$ by using the fact that $\overline{X_1^*} = F^{-1}[(p+h_2)/(p+h_1)]$. We have shown that $J_1^*(l_1)$ is convex in l_1 and thus $J_2(l_2)$ is convex in y_2. To determine $y_2^*(l_2)$, we substitute l_1 by $l_2 - y_2 - \tau_2$ in (40) and get:

$$\begin{aligned} J_2^*(l_2) = \mathrm{Min}_{y_2 \geq 0} \quad & h_3 y_2 + h_2 \int_0^{l_2 - y_2 - \overline{X}_1^*} \left(l_2 - y_2 - u - \overline{X}_1^*\right) f_2(u)\, du \\ + \; & h_1 \int_0^{l_2 - y_2 - \overline{X}_1^*} f_2(u)\, du \int_0^{\overline{X}_1^*} \left(\overline{X}_1^* - t\right) f_1(t)\, dt \\ + \; & h_1 \int_{l_2 - y_2 - \overline{X}_1^*}^{l_2 - y_2} \int_0^{l_2 - y_2 - u} (l_2 - y_2 - u - t)\, f_2(u)\, f_1(t)\, dt du \\ + \; & p \int_0^{l_2 - y_2 - \overline{X}_1^*} f_2(u)\, du \int_{\overline{X}_1^*}^{\infty} \left(t - \overline{X}_1^*\right) f_1(t)\, dt \\ + \; & p \int_{l_2 - y_2}^{\infty} (\mu_1 + u - l_2 + y_2)\, f_2(u)\, du \\ + \; & p \int_{l_2 - y_2 - \overline{X}_1^*}^{l_2 - y_2} \int_{l_2 - y_2 - u}^{\infty} (t + u - l_2 + y_2)\, f_2(u)\, f_1(t)\, dt du \end{aligned} \tag{44}$$

Therefore substituting in (44) $l_2 - y_2$ by $\overline{X}_{21}$, we obtain $y_2^*(l_2)$, the optimal waiting time before production is started at stage 2 given that we are l_2 periods away from the due date. $y_2^*(l_2)$ can be written as:

$$y_2^*(l_2) = \begin{cases} l_2 - \overline{X}_{21}^* & \text{if } l_2 \geq \overline{X}_{21}^* \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

To compute $\overline{X}_{21}^*$, we differentiate (44) after substituting $l_2 - y_2$ by $\overline{X}_{21}$ and set it to zero. Further manipulations result in the following first order condition:

$$(h_1 + p) \int_{\overline{X}_{21} - \overline{X}_1^*}^{\overline{X}_{21}} \int_0^{\overline{X}_{21} - u} f_1(t)\, f_2(u)\, dt du + (h_2 + p) \int_0^{\overline{X}_{21} - \overline{X}_1^*} f_2(u)\, du - (h_3 + p) = 0 \tag{46}$$

we may have from (46) that $\overline{X}_{21}^* \leq \overline{X}_1^*$, in which case it must have been that

$$\frac{dJ_2(l_2)}{dl_2}\Big|_{\overline{X}_{21} = \overline{X}_1^*} = (h_1 + p) \int_0^{\overline{X}_1^*} \int_0^{\overline{X}_1^* - u} f_1(t)\, f_2(u)\, dt du \geq (h_3 + p)$$

that is

$$\overline{X}_1^* = F_1^{-1}\left[\frac{h_2 + p}{h_1 + p}\right] \geq F_{21}^{-1}\left[\frac{h_3 + p}{h_1 + p}\right]$$

$$\Rightarrow \overline{X}_{21}^{*} \quad = \quad F_{21}^{-1}\left[\frac{h_3+p}{h_1+p}\right] \leq \overline{X}_1^{*}$$

hence $y_1^* = 0$ w.p.1 and stage 1 is processed immediately when processing at stage 2 is completed.

4.1 Effect of the Processing Time Variance at Stage 1

In this section, we study the effect of the processing time variance at stage 1 on $\overline{X}_2^*$ and $\overline{X}_1^*$, the optimal planned lead times at stage 2 and 1 respectively. To do this, we will use the simple mean-preserving transformation of a random variable defined by (15). As a result, we get

$$\overline{X}_{1\alpha} = \alpha\left(\overline{X}_1^* - \mu_1\right) + \mu_1 \tag{47}$$

After making the substitution $\overline{X}_2 = \overline{X}_{21} - \overline{X}_1$, using the transformation defined by (15) and substituting $\overline{X}_{1\alpha}$ from (47), equation (46) becomes

$$(h_1+p)\int_{X_{2\alpha}}^{X_{2\alpha}+\alpha\overline{X}_1^*}\int_0^{\frac{X_{2\alpha}-u}{\alpha}+\overline{X}_1^*} f_1(t)\, f_2(u)\, dt du + (h_2+p)\int_0^{X_{2\alpha}} f_2(u)\, du = (h_3+p) \tag{48}$$

Equation (48) gives $\overline{X}_{2\alpha}|_{\alpha=0} = F_2^{-1}\left[(p+h_3)/(p+h_2)\right]$. Differentiating (48) with respect to α we get

$$\frac{(h_1+p)}{\alpha^2}\int_{X_{2\alpha}}^{X_{2\alpha}+\alpha\overline{X}_1^*}\left[\alpha\frac{dX_{2\alpha}}{d\alpha} - (X_{2\alpha}-u)\right] f_1\left(\frac{X_{2\alpha}-u}{\alpha}+\overline{X}_1^*\right) f_2(u)\, du$$
$$+\frac{dX_{2\alpha}}{d\alpha} f_2(X_2\alpha)\left[(h_2+p)-(h_1+p)\int_0^{\overline{X}_1^*} f_1(t)\, dt\right] = 0 \tag{49}$$

and hence

$$\frac{(h_1+p)}{\alpha^2}\int_{X_{2\alpha}}^{X_{2\alpha}+\alpha\overline{X}_1^*}\left[\alpha\frac{dX_{2\alpha}}{d\alpha} - (X_{2\alpha}-u)\right] f_1\left(\frac{X_{2\alpha}-u}{\alpha}+\overline{X}_1^*\right) f_2(u)\, du = 0 \tag{50}$$

Suppose there exists α for which $d\overline{X}_{2\alpha}/d\alpha = 0$. This implies that either $\alpha = 0$ or $\alpha \to \infty$. As $\alpha \to \infty$, (48) gives $h_2 = h_3$: contradiction. Therefore $d\overline{X}_{2\alpha}/d\alpha = 0$ only at $\alpha = 0$. For $\alpha > 0$, $d\overline{X}_{2\alpha}/d\alpha < 0$ for (50) to be true. Finally, $\overline{X}_{2\alpha} = 0$ implies from (48) that α^*, the amount of variance at stage 1 that is required to pool the two stages into a single stage whose processing time distribution is the convolution of the two stages, satisfies

$$(h_1 + p) \int_0^{\alpha^* \overline{X}_1^*} \int_0^{\overline{X}_1^* - \frac{u}{\alpha^*}} f_1(t) f_2(u)\, dtdu = h_3 + p \tag{51}$$

Recall that for $\alpha = 1$, $\overline{X}_2^* \geq 0$ only if $F_{21}\left[\overline{X}_1^*\right] \leq (p + h_3)/(p + h_1)$, hence the same is true for $\alpha^* \geq 1$. For $\alpha \geq \alpha^*$, $\overline{X}_{2\alpha} = 0$ and $\overline{X}_{1\alpha}$ is given by

$$(h_1 + p)\, Pr\left[\tau_{1\alpha} + \tau_2 \leq \overline{X}_{1\alpha}\right] = h_3 + p \tag{52}$$

Using (15), (52) becomes

$$(h_1 + p) \int_0^{\overline{X}_{1\alpha} + \mu_1(\alpha - 1)} \int_0^{\frac{\overline{X}_{1\alpha} - \mu_1 - u}{\alpha} + \mu_1} f_1(t) f_2(u)\, dtdu = h_3 + p \tag{53}$$

It is clear that (53) reduces to (51) at $\alpha = \alpha^*$, with $\overline{X}_{1\alpha} = \alpha\left(\overline{X}_1^* - \mu_1\right) + \mu_1$. Differentiating (53) with respect to α, we get

$$\frac{(h_1 + p)}{\alpha^2} \int_0^{\overline{X}_{1\alpha} + \mu_1(\alpha - 1)} \left[\alpha \frac{d\overline{X}_{1\alpha}}{d\alpha} - \left(\overline{X}_{1\alpha} - \mu_1 - u\right)\right] f_1\left(\frac{\overline{X}_{1\alpha} - \mu_1 - u}{\alpha} + \mu_1\right) f_2(u)\, du = 0 \tag{54}$$

Finally, as α approaches ∞, equations (53) gives

$$lim_{\alpha \to \infty} \frac{d\overline{X}_{1\alpha}}{d\alpha} = F_1^{-1}\left[\frac{h_3 + p}{h_1 + p}\right] - \mu_1 \tag{55}$$

For equation (54) to be true, it must be that

$$\frac{dX_{1\alpha}}{d\alpha} \leq \frac{X_{1\alpha} - \mu_1}{\alpha} \tag{56}$$

Suppose that for some $\alpha > \alpha^*$, we have $dX_{1\alpha}/d\alpha = 0$. Equation (56) implies that $X_{1\alpha} \geq \mu_1$. Therefore, $X_{1\alpha}$ is strictly decreasing in the region $X_{1\alpha} < \mu_1$. Therefore, if α is increased more than α^* and (55) is negative, $X_{1\alpha}$ increases first, only to decrease with higher α and to hit $X_{1\alpha} = \mu_1$ at some $\alpha > \alpha^*$. After that, it strictly decreases with a limiting slope given by (55). If α is increased more than α^* and (55) is positive, $X_{1\alpha}$ increases first, there does not exist $\alpha > \alpha^*$ such that $X_{1\alpha} = \mu_1$, and $lim_{\alpha \to \infty} d\overline{X}_{1\alpha}/d\alpha$ is given by by (55). We shall derive a quite restrictive sufficient condition for having $d\overline{X}_{1\alpha}/d\alpha \geq 0$ for $\alpha \geq \alpha^*$.

$$(h_1 + p)\,Pr\left[\alpha(\tau_1 - \mu_1) + \mu_1 + \tau_2 \leq \overline{X}_{1\alpha}\right] = h_3 + p \tag{57}$$

Define α' as

$$\tau_{21\alpha'} = \alpha'(\tau_{21} - \mu_{21}) + \mu_{21} = \alpha(\tau_1 - \mu_1) + \mu_1 + \tau_2 \tag{58}$$

where $\tau_{21} = \tau_2 + \tau_1$. As a result, (57) becomes

$$(h_1 + p)\,Pr\left[\alpha'(\tau_{21} - \mu_{21}) + \mu_{21} \leq \overline{X}_{1\alpha}\right] = h_3 + p \tag{59}$$

and hence

$$\overline{X}_{1\alpha} = \alpha'\left(G_{21}^{-1}\left[\frac{h_3 + p}{h_1 + p}\right] - \mu_{21}\right) + \mu_{21} \tag{60}$$

where G_{21} is the distribution of τ_{21}. Therefore $G_{21}^{-1}\left[(h_3 + p)/(h_1 + p)\right] \geq \mu_{21}$ implies that $d\overline{X}_{1\alpha}/d\alpha \geq 0$ since (58) gives

$$\alpha' = \sqrt{\frac{\alpha^2 Var(\tau_2) + Var(\tau_1)}{Var(\tau_2) + Var(\tau_1)}} \tag{61}$$

and hence

$$\frac{d\overline{X}_{1\alpha}}{d\alpha} = \frac{d\overline{X}_{1\alpha}}{d\alpha'}\frac{d\alpha'}{d\alpha} \geq 0 \tag{62}$$

if $G_{21}^{-1}\left[(h_3+p)/(h_1+p)\right] \geq \mu_{21}$. As a corollary to this result, we get

$$G_{21}^{-1}\left[\frac{h_3+p}{h_1+p}\right] \geq \mu_{21} \Rightarrow F_1^{-1}\left[\frac{h_3+p}{h_1+p}\right] \geq \mu_1 \tag{63}$$

since the opposite would contradict $d\overline{X}_{1\alpha}/d\alpha \geq 0$ for $\alpha \geq \alpha^*$.

4.2 Case when $h_2 \geq h_3 \geq 0$ Revisited

We want to shown that in the case of $h_2 \geq h_3 \geq 0$, we also have the sufficient condition for $dX_{1\alpha}/d\alpha \geq 0$, $\alpha \geq 0$, that is $G_{21}^{-1}\left[(h_3+p)/(h_1+p)\right] \geq \mu_{21}$ implies $dX_{1\alpha}/d\alpha \geq 0$. Substituting in (17) from (18) we get

$$\int_{X_{2\alpha}}^{X_{2\alpha}+X_{1\alpha}+\mu_1(\alpha-1)} \int_0^{\frac{X_{2\alpha}+X_{1\alpha}-\mu_1-u}{\alpha}+\mu_1} f_1(t)\, f_2(u)\, dt du + \int_0^{\frac{X_{1\alpha}-\mu_1}{\alpha}+\mu_1} f_1(t)\, dt \int_0^{X_{2\alpha}} f_2(u)\, du = \frac{h_3+p}{h_1+p} \tag{64}$$

which can be rewritten as

$$\int_0^{\frac{X_{1\alpha}-\mu_1-u}{\alpha}+\mu_1} \int_0^{X_{2\alpha}+X_{1\alpha}-\alpha t+\mu_1(\alpha-1)} f_2(u)\, f_1(t)\, du dt = \frac{h_3+p}{h_1+p} \tag{65}$$

and equivalently as

$$\int_0^{\frac{X_{2\alpha}+X_{1\alpha}-\mu_1-u}{\alpha}+\mu_1} \int_0^{X_{2\alpha}+X_{1\alpha}-\alpha t+\mu_1(\alpha-1)} f_2(u)\, f_1(t)\, du dt = \int_{\frac{X_{1\alpha}-\mu_1-u}{\alpha}+\mu_1}^{\frac{X_{2\alpha}+X_{1\alpha}-\mu_1-u}{\alpha}+\mu_1} \int_0^{X_{2\alpha}+X_{1\alpha}-\alpha t+\mu_1(\alpha-1)} f_2(u)\, f_1(t)\, du dt + \frac{h_3+p}{h_1+p} \tag{66}$$

As a result, if $G_{21}^{-1}\left[(h_3+p)/(h_1+p)\right] \geq \mu_{21}$, then the right-hand side of (67) is also at least equal to μ_{21}. Therefore $dX_{1\alpha}/d\alpha \geq -dX_{2\alpha}/d\alpha \geq 0$.

5 Generalization with Uncertain Delivery Dates

Suppose that the outside suppliers are unreliable and it is required to quote, before any processing occurs, the delivery dates for the outside supply parts needed at stage 2 and 1. Suppliers unreliability is captured by defining η_2 and η_1 as the time elapsed between the quoted delivery date and the actual delivery dates at stage 2 and 1 respectively. We assume η_2 and η_1 to be continuous random variables with distributions G_2 and G_1 respectively. The first stage in the SDP is triggered whenever outside supply parts arrive at stage 1 or whenever processing at stage 2 is completed, whichever occurs last. The optimal starting policy at the first stage is still a *wait-until policy* given by

$$y_1^*(l_1) = \begin{cases} l_1 - \overline{X_1^*} & \text{if } l_1 \geq \overline{X_1^*} \\ 0 & \text{otherwise} \end{cases} \tag{67}$$

where $\overline{X_1^*} = F_1^{-1}\left[\left(p + h_2 + \overline{h}_2\right) / \left(p + h_1\right)\right]$. To determine y_2^*, X_{21}^* and X_1^* we solve

$$\begin{aligned} \text{Min } J_2(y_2, X_{21}, X_1) \; = \; & h_3\left(l_2 - X_{21} + E\left[\eta_2\right]\right) + \left(h_3 + \overline{h}_3\right) y_2 + \\ & \overline{h}_2 \int_0^\infty \int_{v+X_{21}-y_2-X_1}^\infty \left[w - v - \left(X_{21} - y_2 - X_1\right)\right] \overline{g}(w)\, g_1(v)\, dw dv + \\ & h_2 \int_0^\infty \int_0^{v+X_{21}-y_2-X_1} \left[\left(X_{21} - y_2 - X_1\right) + v - w\right] \overline{g}(w)\, g_1(v)\, dw dv + \\ & Pr\left[X_{21} - y_2 - \overline{\eta} \leq X_1 - \eta_1\right] E\left[J_1^*\left(X_{21} - y_2 - \overline{\eta}\right)\right] + \\ & Pr\left[X_{21} - y_2 - \overline{\eta} \geq X_1 - \eta_1\right] E\left[J_1^*\left(X_1 - \eta_1\right)\right] \end{aligned} \tag{68}$$

$$\text{s.t. } y_2 \geq 0, X_{21} \geq 0, X_1 \geq 0$$

where $\overline{\eta} \equiv \eta_2 + \tau_2$ with distribution $\overline{G}$. Differentiating with respect to y_2 we get $\forall\ X_1$:

$$\frac{\delta J_2(y_2, X_{21}, X_1)}{\delta y_2} = -\frac{\delta J_2(y_2, X_{21}, X_1)}{\delta X_{21}} + \overline{h}_3 = \overline{h}_3 \geq 0 \tag{69}$$

hence $y_2^* = 0$ provided $\delta J_2(y_2, X_{21}, X_1)/\delta X_{21} = 0$ at $X_{21} = X_{21}^*$. Equation (68) becomes

$$\begin{aligned}
\text{Min } J_2(X_{21}, X_1) &= h_3(l_2 - X_{21} + E[\eta_2]) + \left(h_3 + \overline{h}_3\right) y_2 + \qquad (70)\\
&\overline{h}_2 \int_0^\infty \int_{v+X_{21}-X_1}^\infty [w - v - (X_{21} - X_1)]\,\overline{g}(w)\,g_1(v)\,dwdv + \\
&h_2 \int_0^\infty \int_0^{v+X_{21}-X_1} [(X_{21} - X_1) + v - w]\,\overline{g}(w)\,g_1(v)\,dwdv + \\
&\int_{X_1-\overline{X}_1^*}^{X_1} \int_0^{v+X_2-X_1} \left[h_1 \int_0^{X_1-v} [(X_1 - v) - t]\,f_1(t)\,dt + \right.\\
&\left. p\int_{X_1-v}^\infty [t - (X_1 - v)]\,f_1(t)\,dt\right] \overline{g}(w)\,g_1(v)\,dwdv + \\
&p\int_{X_1}^\infty \int_0^{v+X_{21}-X_1} [\mu_1 - (X_1 - v)]\,\overline{g}(w)\,g_1(v)\,dwdv + \\
&\int_0^{X_1-\overline{X}_1^*} \int_0^{v+X_{21}-X_1} \left[\left(h_2 + \overline{h}_2\right)\left(X_1 - \overline{X}_1^* - v\right) + \right.\\
&\left. h_1\int_0^{\overline{X}_1^*} \left(\overline{X}_1^* - t\right) f_1(t)\,dt + p\int_{\overline{X}_1^*}^\infty \left(t - \overline{X}_1^*\right) f_1(t)\,dt\right] \overline{g}(w)\,g_1(v)\,dwdv + \\
&\int_{X_{21}-\overline{X}_1^*}^{X_{21}} \int_0^{w-(X_{21}-X_1)} \left[h_1 \int_0^{X_{21}-w} [(X_{21} - w) - t]\,f_1(t)\,dt + \right.\\
&\left. p\int_{X_{21}-w}^\infty [t - (X_{21} - w)]\,f_1(t)\,dt\right] g_1(v)\,\overline{g}(w)\,dvdw + \\
&p\int_{X_{21}}^\infty \int_0^{w-(X_{21}-X_1)} [\mu_1 - (X_{21} - w)]\,g_1(v)\,\overline{g}(w)\,dvdw + \\
&\int_{X_{21}-\overline{X}_1^*}^{X_{21}-X_1} \int_0^{w-(X_{21}-X_1)} \left[\left(h_2 + \overline{h}_2\right)\left(X_{21} - \overline{X}_1^* - w\right) + \right.\\
&\left. h_1\int_0^{\overline{X}_1^*} \left(\overline{X}_1^* - t\right) f_1(t)\,dt + p\int_{\overline{X}_1^*}^\infty \left(t - \overline{X}_1^*\right) f_1(t)\,dt\right] g_1(v)\,\overline{g}(w)\,dvdw
\end{aligned}$$

s.t. $X_{21} \geq 0, X_1 \geq 0$

After further manipilations, differentiating with respect to X_{21} and X_1 gives

$$\begin{aligned}
\frac{\delta J_2(X_{21}, X_1)}{\delta X_{21}} &= (h_1 + p) \int_{X_1-\overline{X}_1^*}^{X_1} \int_{v+X_{21}-X_1}^{X_{21}} \int_0^{X_{21}-w} f_1(t)\,\overline{g}(w)\,g_1(v)\,dtdwdv + \qquad (71)\\
&(h_1 + p) \int_0^{X_1-\overline{X}_1^*} \int_{X_{21}-\overline{X}_1^*}^{X_{21}} \int_0^{X_{21}-w} f_1(t)\,\overline{g}(w)\,g_1(v)\,dtdwdv +
\end{aligned}$$

$$\left(h_2 + \overline{h}_2 + p\right) \int_{X_1 - \overline{X}_1^*}^{\infty} \int_0^{v + X_{21} - X_1} \overline{g}(w)\, g_1(v)\, dwdv +$$
$$\left(h_2 + \overline{h}_2 + p\right) \int_0^{X_1 - \overline{X}_1^*} \int_0^{X_{21} - \overline{X}_1^*} \overline{g}(w)\, g_1(v)\, dwdv - \left(h_3 + \overline{h}_2 + p\right) = 0$$

and

$$\begin{aligned} \frac{\delta J_2(X_{21}, X_1)}{\delta X_1} &= (h_1 + p) \int_{X_1 - \overline{X}_1^*}^{X_1} \int_0^{v + X_{21} - X_1} \int_0^{X_1 - v} f_1(t)\, \overline{g}(w)\, g_1(v)\, dtdwdv \\ &\quad - \left(h_2 + \overline{h}_2 + p\right) \int_{X_1 - \overline{X}_1^*}^{\infty} \int_0^{v + X_{21} - X_1} \overline{g}(w)\, g_1(v)\, dwdv + \overline{h}_2 = 0 \end{aligned} \tag{72}$$

It can be shown that the Hessian of $J_2(X_{21}, X_1)$ is positive-definite by differentiating (70) twice, hence X_{21}^* and X_1^* are indeed given by (71) and (72). Note that in the case of unreliable outside suppliers, there are instances where $X_1^* \geq \overline{X}_1^*$. Therefore, the realization of the delivery date for the outside supply parts needed at stage 1 may be such that parts arrive while the remaining time till the due date is still larger than the optimal planned lead time at stage 1. And if, in addition to this, processing at stage 2 has already been completed, then some intentional waiting time at stage 1 is induced until the remaining time till the due date becomes equal to the optimal planned lead time at stage 1 for processing at stage 1 to start. This extra waiting time represents the additional cost due to quoting outside supply parts delivery dates earlier than the beginning of the optimal planned lead time at stage 1, as a protection against suppliers uncertainty.

6 Conclusion

We considered the problem of quoting delivery due dates to various suppliers in an assembly system with random processing times. We assumed that an order for a project has been accepted and a due date for the completion of the project has been set in advance. We also assumed that the suppliers are perfectly reliable and that the suppliers delivery due dates must be quoted before any processing occurs in the system. Once the delivery due dates have been quoted and processing has begun in the system, it was necessary to determine the optimal starting time at every stage in the assembly system, due to the randomness in the processing times at the various stages. We showed that the optimal starting policy at each stage calls for no intentional delay whenever outside supply parts arrive at that stage and that the optimal delivery due dates can be determined analytically. We also showed that in the case of the system consisting of two stages in series, the difference between the optimal delivery date of outside supply parts needed at stage 2 and the optimal delivery date of outside supply parts needed at stage 1 decreases with increasing processing time variance at stage 1, while the optimal delivery date for outside supply parts needed at stage 1 is advanced under mild conditions with increasing processing time variance at stage 1. If the outside parts delivery dates were preset, the optimal starting time at each stage is described by a simple *wait-until* policy, where the manager waits until the greatest of the delivery date and the beginning of the optimal cumulative planned processing time of all downstreams stages to begin processing. Thus the optimal starting policy at each stage is completely determined by a critical number, the optimal cumulative planned processing time of all downstream stages, showed to be the minimum of a convex function. With increasing processing time variance at stage 1, the optimal planned lead time at stage 2 decreases and the optimal planned lead time at stage 1 increases under mild conditions. We also consider the particular case when the ouside supply parts at each stage are available at no additional cost and characterize the *wait-until* policy that completely determines the optimal starting time at each stage. Finally we consider the case of unreliable outside suppliers and show that there are instances where the optimal delivery date for outside supply parts needed at stage 1 is quoted earlier than the planned lead time for stage 1 due the uncertainty in the actual delivery date, hence inducing some intentional waiting time in case processing is completed at stage 2, the ouside supply parts have been delivered and the remaining time until the due date is still larger than the planned lead time at stage 1.

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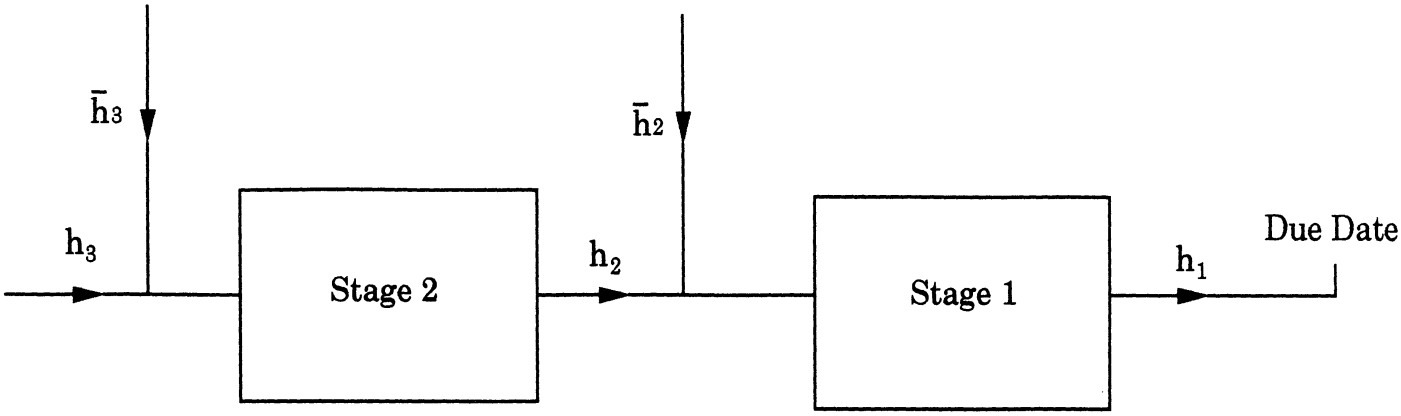

Figure 1

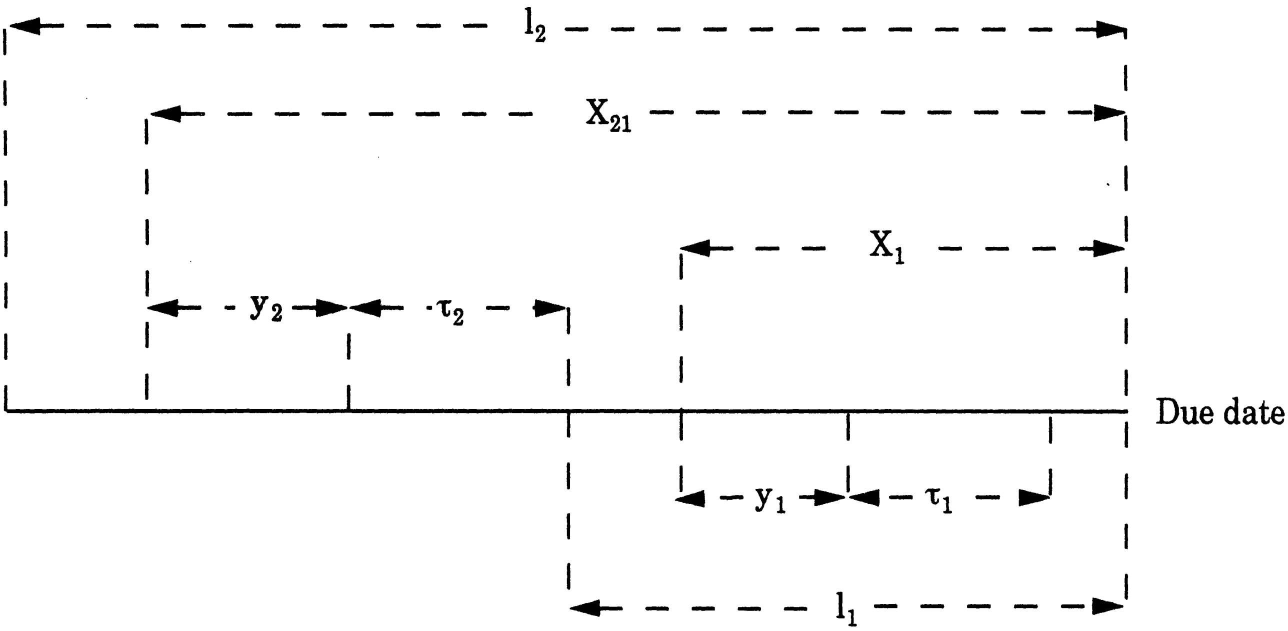

Figure 2