Release Control Policy for a Production System with Random Yield

Walid R. Abillama

Department of Industrial and Operations Engineering
University of Michigan
Ann Arbor, Michigan 48109-2117

December 1993

Technical Report 93-34

Release Control Policy for a Production System with Random Yield

Walid R. Abillama
Department of Industrial and Operations Engineering
University of Michigan
Ann Arbor, MI 48109

December 2, 1993

Abstract

We consider a single product, single stage, multiple periods production system with random yield that minimizes the total quantity released over the planning horizon while keeping a high probability of meeting the demand in each period. We present the optimal finite horizon policy, discuss the single assumption underlying it and present some examples using specific yield distributions.

1 Introduction

We consider the problem of releasing a quantity of a certain kind of raw material at the beginning of each of n future periods in a single stage production system. We assume that demand is deterministic and known for the entire planning horizon, and that the production outcome is a random fraction of the input release quantity. This random fraction, better known as yield rate, has a very large variability and upper management is having great difficulty estimating an accurate measure for the unit shortage cost per period. However, upper management does require that the demand be met in each period with a prespecified confidence level and that any unmet demand be backlogged and become due the next period. The objective therefore is to minimize the total quantity released over the planning horizon subject to meeting the demand in each period with a high probability. The tradeoff between acheiving a high service level and minimizing the amount of raw material released is clear. Releasing huge quantities of raw material in a particular line increases the chances of starving other lines in the plant which use the same raw material and will therefore hinder the overall production capacity of the plant. Quantities released must be kept at a minimum, only to satisfy in each period the service level requirement set by upper management. One possibility would be to decide at the beginning of the planning horizon on all the release quantities that should be made in each period, without waiting to see subsequent levels of production yield. However a clearly better choice would be to postpone the release of the kth quantity until the beginning of period k, when the inventory level at the beginning of that period would be known. This mode of operation involves information gathering and sequential decision making based on information as it becomes available and lends itself to Dynamic Programming techniques. It implies that we are interested in finding an optimal rule for choosing at the beginning of each period a release quantity for each possible value of inventory level that can occur. Mathematically the problem is one of finding a sequence of functions, which will be referred to as a control law or policy, mapping the inventory level at the beginning of each period into the release quantity so as to minimize the total quantity released in all future periods while keeping a high probability of meeting the demand in each period. We refer to such a problem as the multiple periods service level random yield model. The practical importance of taking into account yield randomness in analyzing production and inventory models was addressed by, among others, Karlin (1958 Sections 4-8) who considered a multilple periods problem where ordering is restricted to a fixed amount. Lee and

Yano (1988) considered a single period serial system, Sepehri, Silver and New (1986) considered a single period model with multiple setups, Yao (1988) considered a single period assembly system, Gerchak, Vickson and Parlar (1988) considered a multiple periods cost based model and showed that the solution is neither myopic nor order-up-to. Multiple products models were considered by Singh, Abraham and Akella (1990) and Tang (1990) in a single period setting. Yano and Lee (1989) reviewed the lot-sizing problem when the yields are random and reported finding little research done on multiple periods problem. This paper analyzes a multiple periods service level model and is organized as follows. In section 2, we define our notation and formulate the problem mathematically. In section 3 we study the structure of the optimal policy by considering the first and the second period problem separately, and then by showing the optimal release policy for a more general n periods problem. We show that this policy holds under one particular assumption that we will define later in the section. In section 4, we discuss our assumption and show that it is valid for all practical yield rate distributions and service level values. In section 5 we present some examples using specific yield rate distributions. Finally, we conclude in section 6 by suggesting some new directions for research.

2 Terminology and Formulation

In this section we define the terminology that we use throughout the paper and present the dynamic programming formulation. We use backward recursion to explore the structure of the optimal policy. Therefore the last period, n, becomes the first period to be analyzed, period $n-1$ becomes the second period and so on. Let d be the demand that needs to be met at the end of each period with a prespecified level of confidence γ. More specifically, the service level constraint in any period k is defined as:

$$P\left[I_k + UQ_k - d \geq 0\right] \geq \gamma \tag{1}$$

where I_k is the on-hand inventory at the beginning of period k, U is the yield rate, a continuous random variable between 0 and 1 with stationary cumulative distribution F ($F_k = F \; \forall k$), and Q_k is the quantity released in period k. Equation (1) can be rewritten as:

$$
\begin{aligned}
& P\left[U \geq \tfrac{d-I_k}{Q_k}\right] \geq \gamma \\
\Rightarrow \quad & Q_k \geq \tfrac{d-I_k}{F^{-1}[1-\gamma]}
\end{aligned}
\qquad (2)
$$

Let $J_k^*(I_k)$ be the value function, that is the minimum total quantity released to meet the service level, given the inventory level I_k and assuming that the best decision is taken in the current as well as in all the future periods. Finally let $Q_k^*(I_k)$ be the optimal quantity to be released at the beginning of period k, given I_k. Denote the ith period problem by $\mathcal{P}(n-(i-1))$, that is the problem of solving an i periods problem. We want to solve $\mathcal{P}(1)$. For a general period k, $\mathcal{P}$(k) is formulated as follows:

$$
\mathcal{P}(k) \qquad
\begin{aligned}
J_k^*(I_k) = \; & \text{Min } Q_k + E\left[J_{k+1}^*(I_{k+1})\right] \\
& \text{s.t.} \begin{cases} Q_k \geq \frac{d-I_k}{F^{-1}[1-\gamma]} & \text{if } I_k \leq d \\ Q_k \geq 0 & \text{if } I_k \geq d \end{cases}
\end{aligned}
$$

An interpretation of this formulation is the following. To solve an $n-(k-1)$ periods problem, we minimize the sum of the amount of material released in the current period k, plus the expected amount of material that might be released in the following $n-k$ periods given the beginning of period on-hand inventory I_{k+1}, which in turn is related to I_k, the on-hand inventory at the beginning of period k, through the production outcome UQ_k and the demand d in the current period. In other words we are assuming that capacity is never binding but at the same time we are allowed exactly one trial per period. As a result, lead time is not more than one period and the relationship between the state variables I_k and I_{k+1} is nothing but:

$$I_{k+1} = I_k + UQ_k - d \tag{3}$$

The service level constraint defined in equation (2) becomes active only if the current on-hand inventory level is less than the demand. Otherwise, the release quantity should only satisfy the non-negativity constraint. Substituting equation (3) in $\mathcal{P}(k)$ we get:

$$\mathcal{P}(k) \qquad \begin{aligned} J_k^*(I_k) = \ & \text{Min } Q_k + \int_0^1 J_{k+1}^*(I_k + uQ_k - d)\, f(u)du \\ & \text{s.t.} \begin{cases} Q_k \geq \frac{d - I_k}{F^{-1}[1-\gamma]} & \text{if } I_k \leq d \\ Q_k \geq 0 & \text{if } I_k \geq d \end{cases} \end{aligned}$$

We analyze in the next section the optimal finite horizon policy.

3 Structure of the Optimal Policy

3.1 First Period Problem

With $J_{n+1}^*(I_{n+1}) = 0 \ \forall I_{n+1}$, the optimal decision in the first period is simply to release the minimum amount of material required to satisfy the service level constraint in that period. Clearly we do not release any material if the on-hand inventory at the beginning of that period exceeds the demand. The first period problem is defined as:

$$\mathcal{P}(\text{n}) \qquad \begin{aligned} J_n^*(I_n) = \ & \text{Min } Q_n \\ & \text{s.t.} \begin{cases} Q_n \geq \frac{d - I_n}{F^{-1}[1-\gamma]} & \text{if } I_n \leq d \\ Q_n \geq 0 & \text{if } I_n \geq d \end{cases} \end{aligned}$$

The first period policy is given by:

$$Q_n^*(I_n) = J_n^*(I_n) = \begin{cases} \frac{d - I_n}{F^{-1}[1-\gamma]} & \text{if } I_n \leq d \\ 0 & \text{if } I_n \geq d \end{cases} \tag{4}$$

3.2 Second Period Problem

The objective function of $(\mathcal{P}_{n-1})$, $J_{n-1}(I_{n-1}, Q_{n-1})$ is convex in Q_{n-1} since $J_n^*(I_n)$ is convex in I_n. $(\mathcal{P}_{n-1})$ first order condition is given by

$$\frac{dJ_{n-1}(I_{n-1}, Q_{n-1})}{dQ_{n-1}} = 1 - \frac{1}{F^{-1}[1-\gamma]} \int_0^{\frac{2d-I_{n-1}}{Q_{n-1}}} udF(u) = 0 \tag{5}$$

3.2.1 Second Period Policy

Proposition 1 *The second period policy is given by*

$$Q_{n-1}^*(I_{n-1}) = \begin{cases} \frac{d-I_{n-1}}{F^{-1}[1-\gamma]} & if\, I_{n-1} \le y_1 \\ \frac{d-I_{n-1}}{\eta_{n-1}^*} & if\, y_1 \le I_{n-1} \le 2d \\ 0 & if\, I_{n-1} \ge 2d \end{cases} \tag{6}$$

where $y_1 \le d$ *and* $F^{-1}[1-\gamma] \le \eta_{n-1}^* \le 1$.

Clearly for $I_{n-1} \ge 2d$, we have that $Q_{n-1}^*(I_{n-1}) = 0$ since the left hand side of (5) is positive. For $I_{n-1} \le 2d$ and assuming that $E[U] \ge F^{-1}[1-\gamma]$, we get from (5) that $Q_{n-1}^*(I_{n-1}) = (2d - I_{n-1})/\eta_{n-1}^*$ where

$$\int_0^{\eta_{n-1}^*} udF(u) = F^{-1}[1-\gamma] \tag{7}$$

and hence $\eta_{n-1}^* \le 1$. Furthermore from equation (7), we have that $F^{-1}[1-\gamma] \le \eta_{n-1}^*$ since

$$\int_0^{F^{-1}[1-\gamma]} uf(u)\,du = (1-\gamma)F^{-1}[1-\gamma] - \int_0^{F^{-1}[1-\gamma]} F(x)\,dx \le F^{-1}[1-\gamma] \tag{8}$$

As a result, there exists $I_{n-1} = y_1 \le d$ below which the service level constraint is binding and y_1 is obtained by solving

$$\frac{1}{\eta_{n-1}^*}(2d - y_1) = \frac{d - y_1}{F^{-1}[1-\gamma]} \tag{9}$$

Figure 1 shows a plot of $Q_{n-1}^{*}(I_{n-1})$ versus I_{n-1}. We leave the second period policy with three observations:

1) The production quantity required to meet the demand (set by the reorder point) is augmented due to random yield, we say we are 'overproducing' ($Q_{n-1}^{*}(I_{n-1}) \geq 2d - I_{n-1}$), which is a common feature of random yield models.

2) We are 'increasingly' overproducing with decreasing initial stock levels. This can be seen in proposition 1 where $dQ_{n-1}^{*}(I_{n-1})/dI_{n-1} \leq -1$, which is to be expected since there is no limit on backlogging and capacity is not binding.

3) The reorder point is equal to the total demand in all future periods, also expected since there is no penalty for early production.

Assumption 1 $E[U] \geq F^{-1}[1-\gamma]$

Note that this second period policy is valid only if $E[U] \geq F^{-1}[1-\gamma]$, which implies that there exists $\eta_{n-1}^{*} \leq 1$ that satisfies (7) and hence $Q_{n-1}^{*}(I_{n-1}) \geq 2d - I_{n-1}$ for $I_{n-1} \leq 2d$. For the time being we will neglect the case when $E[U] \leq F^{-1}[1-\gamma]$ since it is reasonable to assume that the service level will always be high enough so that $E[U] \geq F^{-1}[1-\gamma]$ will always be true. In a future section, we will show that this assumption holds for all practical yield distributions and service level values γ.

3.2.2 Second Period Value Function

Clearly $J_{n-1}^{*}(I_{n-1}) = 0$ for $I_{n-1} \geq 2d$. For $y_1 \leq I_{n-1} \leq 2d$, the value function is obtained by substituting $Q_{n-1}^{*}(I_{n-1})$ in the objective function of $(\mathcal{P}_{n-1})$. Doing this, we get that $J_{n-1}^{*}(I_{n-1})$ is given by

$$Q_{n-1}^{*}(I_{n-1}) - \frac{1}{F^{-1}[1-\gamma]} \int_{0}^{\eta_{n-1}^{*}} \left(I_{n-1} + uQ_{n-1}^{*}(I_{n-1}) - d\right) f(u)\, du \tag{10}$$

For $I_{n-1} \leq y_1$, the value function is obtained by substituting in (10) $Q_{n-1}^{*}(I_{n-1})$ by

$\frac{(d-I_{n-1})}{F^{-1}[1-\gamma]}$.

Proposition 2 $J_{n-1}^*(I_{n-1})$ *is convex.*

From (10), the value function is linear for $y_1 \leq I_{n-1} \leq 2d$ since

$$\begin{aligned}
\frac{dJ_{n-1}^*(I_{n-1})}{dI_{n-1}} &= \frac{dQ_{n-1}^*(I_{n-1})}{dI_{n-1}} - \frac{1}{F^{-1}[1-\gamma]} \int_0^{\eta_{n-1}^*} \left[1 + u\frac{dQ_{n-1}^*(I_{n-1})}{dI_{n-1}}\right] f(u)du \\
&= \frac{dQ_{n-1}^*(I_{n-1})}{dI_{n-1}} \left[1 - \int_0^{\eta_{n-1}^*} \frac{u}{F^{-1}[1-\gamma]} f(u)du\right] - \frac{F\left[\eta_{n-1}^*\right]}{F^{-1}[1-\gamma]} \\
&= -\frac{F\left[\eta_{n-1}^*\right]}{F^{-1}[1-\gamma]}
\end{aligned}$$

Differentiating $J_{n-1}^*(I_{n-1})$ for $I_{n-1} \leq y_1$ we get:

$$\frac{dJ_{n-1}^*(I_{n-1})}{dI_{n-1}} = -\frac{\left[1 + \int_0^{\frac{2d-I_{n-1}}{d-I_{n-1}}F^{-1}[1-\gamma]} \left(1 - \frac{u}{F^{-1}[1-\gamma]}\right) f(u)\, du\right]}{F^{-1}[1-\gamma]} \tag{11}$$

thus $J_{n-1}^*(I_{n-1})$ is differentiable at $I_{n-1} = y_1$ using (9). Furthermore $J_{n-1}^*(I_{n-1})$ is convex for $I_{n-1} \leq y_1$ by differentiating (11) one more time. As a result, $J_{n-1}^*(I_{n-1})$ is convex. Figure 2 shows a plot of $J_{n-1}^*(I_{n-1})$ versus I_{n-1}. Finally, we close this section by noting from (11) that $\lim_{I_{n-1}\to-\infty} dJ_{n-1}^*(I_{n-1})/dI_{n-1} = -(1+\rho)/F^{-1}[1-\gamma]$ where

$$\rho = \int_0^{F^{-1}[1-\gamma]} F(u)\, du / F^{-1}[1-\gamma]$$

3.3 General Period Policy

Proposition 3 $J_k(I_k, Q_k)$ *is convex in* Q_k, $k = 1, .., n$.

It is true for $k = n$ and $n-1$. We will show it for $1 \leq k \leq n-2$. To do that, we will assume that the value function for a $n-k$ periods problem is convex in I_{k+1}, show that the objective function for a $n-k+1$ periods problem is convex in Q_k, and then show that as a result, it implies that the value function for a $n-k+1$ periods problem is convex in I_k . Suppose that $J^*_{k+1}(I_{k+1})$, the value function for a $n-k$ periods problem, is convex in I_{k+1}. It implies that for a $n-k+1$ periods problem, $J_k(I_k, Q_k)$ is convex in Q_k since

$$J_k(I_k, Q_k) = Q_k + E\left[J^*_{k+1}(I_k + UQ_k - d)\right] \tag{12}$$

$$\frac{dJ_k(I_k, Q_k)}{dQ_k} = 1 + E\left[U\frac{dJ^*_{k+1}(I_{k+1})}{dI_{k+1}}\right] \tag{13}$$

$$\frac{d^2J_k(I_k, Q_k)}{dQ_k^2} = E\left[U^2\frac{d^2J^*_{k+1}(I_{k+1})}{dI_{k+1}^2}\right] \geq 0 \tag{14}$$

Therefore $\overline{Q}^*_k(I_k)$, the unconstrained optimal policy for a $n-k+1$ periods problem is obtained by setting (13) to zero and solving for Q_k. Now suppose that there exists a unique point $I_k = y_{n-k}$ below which the service level constraint is binding. That is suppose that the optimal policy for a $n-k+1$ periods problem is defined as follows:

$$Q^*_k(I_k) = \begin{cases} \overline{Q}^*_k(I_k) & \text{if } I_k \geq y_{n-k} \\ \frac{d-I_k}{F^{-1}[1-\gamma]} & \text{otherwise} \end{cases} \tag{15}$$

As a result, the value function in a $n-k+1$ periods problem is given by

$$J^*_k(I_k) = \begin{cases} \overline{Q}^*_k(I_k) + E\left[J^*_{k+1}\left(I_k + U\overline{Q}^*_k(I_k) - d\right)\right] & \text{if } I_k \geq y_{n-k} \\ \frac{d-I_k}{F^{-1}[1-\gamma]} + E\left[J^*_{k+1}\left((I_k - d)\left(1 - \frac{U}{F^{-1}[1-\gamma]}\right)\right)\right] & \text{otherwise} \end{cases} \tag{16}$$

The first derivative of the value function is given by

$$\frac{dJ^*_k(I_k)}{dI_k} = \begin{cases} \frac{d\overline{Q}^*_k(I_k)}{dI_k} + E\left[\left(1 + U\frac{d\overline{Q}^*_k(I_k)}{dI_k}\right)\frac{dJ^*_{k+1}(I_{k+1})}{dI_{k+1}}\right] & \text{if } I_k \geq y_{n-k} \\ \frac{-1}{F^{-1}[1-\gamma]} + E\left[\left(1 - \frac{U}{F^{-1}[1-\gamma]}\right)\frac{dJ^*_{k+1}(I_{k+1})}{dI_{k+1}}\right] & \text{otherwise} \end{cases} \tag{17}$$

Clearly the value function is differentiable at $I_k = y_{n-k}$. For $I_k \geq y_{n-k}$, the second derivative of the value function is given by

$$\frac{d^2J^*_k(I_k)}{dI_k^2} = \frac{d^2\overline{Q}^*_k(I_k)}{dI_k^2} + E\left[U\frac{d^2\overline{Q}^*_k(I_k)}{dI_k^2}\frac{dJ^*_{k+1}(I_{k+1})}{dI_{k+1}}\right] +$$

$$
\begin{aligned}
& E\left[\left(1+U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)^2 \frac{d^2 J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] \\
= \; & E\left[\left(1+U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)^2 \frac{d^2 J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] + \\
& \frac{d^2\overline{Q}_k^*(I_k)}{dI_k^2} \underbrace{\left(1+E\left[U\frac{dJ_{k+1}^*(I_{k+1})}{dI_{k+1}}\right]\right)}_{\text{zero by (13)}} \\
= \; & E\left[\left(1+U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)^2 \frac{d^2 J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] \geq 0 \qquad (18)
\end{aligned}
$$

Finally, if the service level constraint is binding, then from (17) the second derivative reduces to

$$
\frac{d^2 J_k^*(I_k)}{dI_k^2} = E\left[\left(1-\frac{U}{F^{-1}[1-\gamma]}\right)^2 \frac{d^2 J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] \geq 0 \qquad (19)
$$

Therefore $J_k^*(I_k)$ is convex and we are done. However, it still remains to be shown that $Q_k^*(I_k)$ is defined as in (15). To do this, it is sufficient to show that $\overline{Q}_k^*(I_k)$ is convex *and* $\lim_{I_k\to-\infty} d\overline{Q}_k^*(I_k)/dI_k \geq \frac{-1}{F^{-1}[1-\gamma]}$. The following is a series of propositions that prepare the ground for showing this and provide more insight on the structure of $Q_k^*(I_k)$.

Proposition 4 *$Q_k^*(I_k) = 0$ for $I_k \geq (n-k+1)d$, $k = 1,..n$.*

It is true for $k = n$ and $n-1$. We will show it for $1 \leq k \leq n-2$ using the same kind of inductive argument as in Proposistion 3. Suppose that for some $1 \leq k \leq n-2$, we have $J_{k+1}^*(I_{k+1}) = 0$ for $I_{k+1} \geq (n-k)d$. Then substituting for I_{k+1}, we get $J_{k+1}^*(I_k + UQ_k - d) = 0$ for $I_k \geq (n-k+1)d - UQ_k$ and hence for $I_k \geq (n-k+1)d$ w.p. 1. Therefore $E\left[J_{k+1}^*(I_k + UQ_k - d)\right] = 0$ for $I_k \geq (n-k+1)d$ and (12) becomes $J_k(I_k, Q_k) = Q_k$. As a result, $\overline{Q}_k^*(I_k) = Q_k^*(I_k) = J_k^*(I_k) = 0$ for $I_k \geq (n-k+1)d$ and we are done.

Lemma 1 *If* $E[U] \geq F^{-1}[1-\gamma]$, *then the set of equations* $\int_0^{\eta_k^*} uf(u)\,du = \Lambda_{k+1}$, $k = 1,...,n-1$, *where*

$$\Lambda_{k+1} = \frac{\int_0^{\eta_{k+1}^*} uf(u)\,du}{\int_0^{\eta_{k+1}^*} f(u)\,du} \quad and \quad \Lambda_n = F^{-1}[1-\gamma]$$

has for unique solution $F^{-1}[1-\gamma] \leq \eta_{n-1}^* \leq ... \leq \eta_1^* \leq 1$.

The proof to the lemma is very simple. It is true for $k = n-1$ from (7) and it is easy to see that for $1 \leq k \leq n-2$

$$\int_0^{\eta_{k+1}^*} f(u)\,du \leq \Lambda_{k+1} = \int_0^{\eta_k^*} f(u)\,du \Rightarrow \eta_{k+1}^* \leq \eta_k^*$$

and $\lim_{n\to\infty} \eta_1^* = 1$.

Proposition 5 $Q_k^*(I_k) = \frac{(n-k+1)d - I_k}{\eta_k^*}$ *for* $(n-k-1)d + y_1 \leq I_k \leq (n-k+1)d$, $k = 1,..,n-1$.

It is true for $k = n-1$. We will show it for $1 \leq k \leq n-2$ using the same kind of inductive argument as in Proposistion 3. Suppose that for some $(n-k-2)d + y_1 \leq I_{k+1} \leq (n-k)d$ we have

$$J_{k+1}^*(I_{k+1}) = \frac{\prod_{i=k+1}^{n-1} F[\eta_i^*]}{F^{-1}[1-\gamma]}[(n-k)d - I_{k+1}]$$

Then for $(n-k-1)d + y_1 \leq I_k \leq (n-k+1)d$, setting (13) to zero results in

$$\frac{dJ_k(I_k, Q_k)}{dQ_k} = 1 - \frac{\prod_{i=k+1}^{n-1} F[\eta_i^*]}{F^{-1}[1-\gamma]} \int_0^{\frac{(n-k+1)d-I_k}{Q_k}} uf(u)\,du = 0$$

Therefore, $\overline{Q_k^*}(I_k)\, Q_k^*(I_k) = [(n-k+1)d - I_k]/\eta_k^*$ where

$$\int_0^{\eta_k^*} uf(u)\,du = \frac{F^{-1}[1-\gamma]}{\prod_{i=k+1}^{n-1} F[\eta_i^*]} = \Lambda_{k+1}$$

if and only if there exists such $\eta_k^* \leq 1$. In other words, if and only if $E[U] \geq \Lambda_{k+1}$. The inequality is satisfied as equality for $\eta_{k+1}^* = 1$. Therefore it is sufficient to show that the first derivative of the right-hand side with respect to η_{k+1}^* is non-negative and we are done. In fact:

$$\frac{d\Lambda_{k+1}}{d\eta_{k+1}^*} = \frac{\eta_{k+1}^* f\left(\eta_{k+1}^*\right) \int_0^{\eta_{k+1}^*} f(u)du - f\left(\eta_{k+1}^*\right) \int_0^{\eta_{k+1}^*} uf(u)du}{\left[F\left[\eta_{k+1}^*\right]\right]^2}$$

$$= \frac{f\left(\eta_{k+1}^*\right) \int_0^{\eta_{k+1}^*} \left(\eta_{k+1}^* - u\right) f(u)du}{\left[F\left[\eta_{k+1}^*\right]\right]^2} \geq 0$$

Substituting Q_k in (12) by $Q_k^*(I_k) = [(n-k+1)d - I_k]/\eta_k^*$, we get for $(n-k-1)d + y_1 \leq I_k \leq (n-k+1)d$

$$J_k^*(I_k) = \frac{\prod_{i=k}^{n-1} F[\eta_i^*]}{F^{-1}[1-\gamma]} [(n-k+1)d - I_k] \tag{20}$$

and we are done.

Proposition 6 $d\overline{Q}_k^*(I_k)/dI_k \leq -1$ *for* $k = 1, .., n$.

It is true for $k = n$ and $n-1$. Recall that $\overline{Q}_k^*(I_k)$, the unconstrained optimal policy for a $n-k+1$ periods problem is obtained by setting (13) to zero and solving for Q_k. Hence the equalities

$$\frac{dJ_k\left(I_k, \overline{Q}_k^*\right)}{d\overline{Q}_k^*} = 0 \Rightarrow \frac{d^2 J_k\left(I_k, \overline{Q}_k^*\right)}{d\overline{Q}_k^* dI_k} + \frac{d\overline{Q}_k^*}{dI_k} \frac{d^2 J_k\left(I_k, \overline{Q}_k^*\right)}{d\overline{Q}_k^{2*}} = 0$$

$$\Rightarrow \frac{d\overline{Q}_k^*}{dI_k} = -\frac{\frac{d^2 J_k\left(I_k, \overline{Q}_k^*\right)}{d\overline{Q}_k^* dI_k}}{\frac{d^2 J_k\left(I_k, \overline{Q}_k^*\right)}{d\overline{Q}_k^{2*}}} = -\frac{E\left[U \frac{d^2 J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]}{E\left[U^2 \frac{d^2 J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]} \leq -1$$

Lemma 2 $F^{-1}[1-\gamma] \leq \beta_k^*$, $k = 1, .., n-1$ *where* β_k^* *is given by*

$$\int_0^{\beta_k^*} uf(u)du = F^{-1}[1-\gamma] \left[\sum_{i=0}^{n-k-1} \rho^i\right]^{-1}$$

The proof is the following. We can write

$$\begin{aligned}
\int_0^{\beta_k^*} uf(u)du &= F^{-1}[1-\gamma]\left[\sum_{i=0}^{n-k-1}\rho^i\right]^{-1} \\
&\geq F^{-1}[1-\gamma]\left[\sum_{i=0}^{\infty}\rho^i\right]^{-1} \\
&= F^{-1}[1-\gamma](1-\rho) \\
&= F^{-1}[1-\gamma] - \int_0^{F^{-1}[1-\gamma]} uf(u)\,du \geq F^{-1}[1-\gamma] \\
&\geq \int_0^{F^{-1}[1-\gamma]} uf(u)du \text{ by (8)}
\end{aligned}$$

and hence $\beta_k^* \geq -1/F^{-1}[1-\gamma]$.

Proposition 7 $\lim_{I_k \to -\infty} d\overline{Q}_k^*(I_k)/dI_k = \beta_k^*$ *for* $k = 1,..,n-1$.

It is true for $k = n-1$. Suppose that for some $1 \leq k \leq n-2$, we have that $\overline{Q}_{k+1}^*(I_{k+1})$ is convex and that $\lim_{I_{k+1} \to -\infty} d\overline{Q}_{k+1}^*(I_{k+1})/dI_{k+1} = \beta_{k+1}^*$. Since $\beta_{k+1}^* \geq \beta_k^*$ by definition, then by (2) these two assumptions imply that the optimal policy for a $n-k$ periods problem is defined as follows:

$$Q_{k+1}^*(I_{k+1}) = \begin{cases} \overline{Q}_{k+1}^*(I_{k+1}) & \text{if } I_{k+1} \geq y_{n-k-1} \\ \frac{d-I_{k+1}}{F^{-1}[1-\gamma]} & \text{otherwise} \end{cases} \tag{21}$$

Suppose furthermore that $y_{n-i} \leq d + y_{n-i-1}$, $i = k+1,..,n-1$ (where $y_0 = d$), then from (17) we get that in a $n-k$ periods problem, $\lim_{I_{k+1} \to -\infty} dJ_{k+1}^*(I_{k+1})/dI_{k+1}$ is equal to

$$-\frac{\left[1 + \sum_{i=0}^{n-k-2}\rho^i \lim_{I_{k+1} \to -\infty} \int_0^{\frac{d+y_{n-k-2}-I_{k+1}}{d-I_{k+1}}F^{-1}[1-\gamma]} \left(1 - \frac{u}{F^{-1}[1-\gamma]}\right) f(u)\,du\right]}{F^{-1}[1-\gamma]}$$

$$= -\frac{\left[1+\sum_{i=0}^{n-k-2}\rho^i \int\limits_0^{F^{-1}[1-\gamma]}\left(1-\frac{u}{F^{-1}[1-\gamma]}\right)f(u)\,du\right]}{F^{-1}[1-\gamma]}$$

$$= -\frac{1+\rho\sum_{i=0}^{n-k-2}\rho^i}{F^{-1}[1-\gamma]} = -\frac{\sum_{i=0}^{n-k-1}\rho^i}{F^{-1}[1-\gamma]} \tag{22}$$

Note that (22) is true for $k = n-2$, thus we are assuming that it is true for some $1 \le k \le n-3$. Therefore marching one period in time backwards (i.e considering a $n-k+1$ periods problem) and setting (13) to zero we get

$$\lim_{I_k \to -\infty} \frac{dJ_k(I_k, Q_k)}{dQ_k} = 1 - \frac{\sum_{i=0}^{n-k-1}\rho^i}{F^{-1}[1-\gamma]}\int_0^{\beta_k^*} uf(u)\,du = 0$$

where $\beta_k^* = \lim_{I_k \to -\infty}(d + y_{n-k-1} - I_k)/\overline{Q}_k^*(I_k)$. Therefore $\lim_{I_k \to -\infty} d\overline{Q}_k^*(I_k)/dI_k = 1/\beta_k^*$ because $\lim_{I_k \to -\infty} \overline{Q}_k^*(I_k) = \infty$ by proposition (6). Using lemma (2) and invoking the assumption that $\overline{Q}_k^*(I_k)$ is convex, it implies that there exists a unique $I_k = y_{n-k} \le d + y_{n-k-1}$ below which the service level constraint is binding for $k = 1,..,n-1$. y_{n-k} is obtained by solving

$$\overline{Q}_k^*(y_{n-k}) = \frac{d - y_{n-k}}{F^{-1}[1-\gamma]} \tag{23}$$

and thus

$$Q_k^*(I_k) = \begin{cases} \overline{Q}_k^*(I_k) & \text{if } I_k \ge y_{n-k} \\ \frac{d-I_k}{F^{-1}[1-\gamma]} & \text{otherwise} \end{cases}$$

Finally, as a check to (22), we get that in a $n-k+1$ periods problem $\lim_{I_k \to -\infty} dJ_k^*(I_k)/dI_k$ is equal to

$$-\frac{\left[1+\sum_{i=0}^{n-k-1}\rho^i \int_0^{\lim_{I_k\to-\infty}\frac{d+y_{n-k-1}-I_k}{d-I_k}F^{-1}[1-\gamma]} \left(1-\frac{u}{F^{-1}[1-\gamma]}\right) f(u)\,du\right]}{F^{-1}[1-\gamma]}$$

$$= -\frac{\left[1+\sum_{i=0}^{n-k-1}\rho^i \int_0^{F^{-1}[1-\gamma]} \left(1-\frac{u}{F^{-1}[1-\gamma]}\right) f(u)\,du\right]}{F^{-1}[1-\gamma]}$$

$$= -\frac{1+\rho\sum_{i=0}^{n-k-1}\rho^i}{F^{-1}[1-\gamma]} = -\frac{\sum_{i=0}^{n-k}\rho^i}{F^{-1}[1-\gamma]} \tag{24}$$

It still remains to be shown that $\overline{Q}_k^*(I_k)$ is convex and we are done. If this is true, then the optimal policy $Q_k^*(I_k)$ for a $n-k+1$ periods problem, $k=1,..,n$ is defined as in (15) and proposition (3) is true. We do this in the next proposition.

Proposition 8 *$\overline{Q}_k^*(I_k)$ is convex in I_k, $k=1,..,n$.*

It is true for $k=n$ and $n-1$. For $1\le k\le n-2$, the second derivative of $\overline{Q}_k^*$ with respect to I_k is given by

$$\frac{d\overline{Q}_k^2}{d^2I_k} = -\frac{\left[\frac{d^2J_k\left(I_k,\overline{Q}_k^*\right)}{d\overline{Q}_k^{2*}}\frac{d^3J_k\left(I_k,\overline{Q}_k^*\right)}{d\overline{Q}_k^*dI_k^2} - \frac{d^2J_k\left(I_k,\overline{Q}_k^*\right)}{d\overline{Q}_k^*dI_k}\frac{d^3J_k\left(I_k,\overline{Q}_k^*\right)}{d\overline{Q}_k^{2*}dI_k}\right]}{\left[\frac{d^2J_k\left(I_k,\overline{Q}_k^*\right)}{d\overline{Q}_k^{2*}}\right]^2}$$

$$= \frac{E\left[U\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]E\left[U^2\frac{d^3J_{k+1}^*(I_{k+1})}{dI_{k+1}^3}\right] - E\left[U^2\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]E\left[U\frac{d^3J_{k+1}^*(I_{k+1})}{dI_{k+1}^3}\right]}{\left[E\left[U^2\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]\right]^2}$$

Suppose that for a $n-k$ periods problem, we have that the second order condition is satisfied, i.e. that

$$E\left[U\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]E\left[U^2\frac{d^3J_{k+1}^*(I_{k+1})}{dI_{k+1}^3}\right] - E\left[U^2\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]E\left[U\frac{d^3J_{k+1}^*(I_{k+1})}{dI_{k+1}^3}\right] \ge 0 \tag{25}$$

and thus $d^2\overline{Q}_k^*/dI_k^2 \geq 0$. We want to obtain an expression for the third derivative of the value function for a $n-k+1$ periods problem. From (17),

$$\frac{dJ_k^*(I_k)}{dI_k} = \begin{cases} E\left[\frac{dJ_{k+1}^*(I_{k+1})}{dI_{k+1}}\right] + \frac{d\overline{Q}_k^*(I_k)}{dI_k}\left(1 + E\left[U\frac{dJ_{k+1}^*(I_{k+1})}{dI_{k+1}}\right]\right) & \text{if } I_k \geq y_{n-k} \\ E\left[\frac{dJ_{k+1}^*(I_{k+1})}{dI_{k+1}}\right] - \frac{1}{F^{-1}[1-\gamma]}\left(1 + E\left[U\frac{dJ_{k+1}^*(I_{k+1})}{dI_{k+1}}\right]\right) & \text{otherwise} \end{cases} \tag{26}$$

and

$$\frac{d^2J_k^*(I_k)}{dI_k^2} = \begin{cases} E\left[\left(1 + U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] & \text{if } I_k \geq y_{n-k} \\ E\left[\left(1 - \frac{U}{F^{-1}[1-\gamma]}\right)^2\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] & \text{otherwise} \end{cases} \tag{27}$$

$$\frac{d^3J_k^*(I_k)}{dI_k^3} = \begin{cases} E\left[\left(1 + U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)^2\frac{d^3J_{k+1}^*(I_{k+1})}{dI_{k+1}^3}\right] + & \\ E\left[U\frac{d^2\overline{Q}_k^*(I_k)}{dI_k^2}\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] & \text{if } I_k \geq y_{n-k} \\ E\left[\left(1 - \frac{U}{F^{-1}[1-\gamma]}\right)^3\frac{d^3J_{k+1}^*(I_{k+1})}{dI_{k+1}^3}\right] & \text{otherwise} \end{cases} \tag{28}$$

The second order condition expressed in a $n-k+1$ periods problem becomes

$$E\left[U\frac{d^2J_k^*(I_k)}{dI_k^2}\right]E\left[U^2\frac{d^3J_k^*(I_k)}{dI_k^3}\right] - E\left[U^2\frac{d^2J_k^*(I_k)}{dI_k^2}\right]E\left[U\frac{d^3J_k^*(I_k)}{dI_k^3}\right] \geq 0 \tag{29}$$

Substituting (18) and (28) in (29), we get for $I_k \geq y_{n-k}$

$$E\left[U\left(1 + U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)^2\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]E\left[U^3\frac{d^2\overline{Q}_k^*(I_k)}{dI_k^2}\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] +$$

$$E\left[U\left(1 + U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)^2\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]E\left[U^2\left(1 + U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)^2\frac{d^3J_{k+1}^*(I_{k+1})}{dI_{k+1}^3}\right] \geq$$

$$E\left[U^2\left(1 + U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)^2\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]E\left[U^2\frac{d^2\overline{Q}_k^*(I_k)}{dI_k^2}\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] +$$

$$E\left[U^2\left(1 + U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)^2\frac{d^2J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right]E\left[U\left(1 + U\frac{d\overline{Q}_k^*(I_k)}{dI_k}\right)^2\frac{d^3J_{k+1}^*(I_{k+1})}{dI_{k+1}^3}\right]$$

since $d^2\overline{Q}_k^*/dI_k^2 \geq 0$ and (25) is true. A similar argument is used to show that (29) is satisfied for $I_k \leq y_{n-k}$. Substituting (19) and (28) in (29), we get for $I_k \leq y_{n-k}$

$$E\left[U\left(1-\frac{U}{F^{-1}[1-\gamma]}\right)^2 \frac{d^2 J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] E\left[U^2\left(1-\frac{U}{F^{-1}[1-\gamma]}\right)^3 \frac{d^3 J_{k+1}^*(I_{k+1})}{dI_{k+1}^3}\right] -$$
$$E\left[U^2\left(1-\frac{U}{F^{-1}[1-\gamma]}\right)^2 \frac{d^2 J_{k+1}^*(I_{k+1})}{dI_{k+1}^2}\right] E\left[U\left(1-\frac{U}{F^{-1}[1-\gamma]}\right)^3 \frac{d^3 J_{k+1}^*(I_{k+1})}{dI_{k+1}^3}\right] \geq 0 \tag{30}$$

To show that (30) is true, it is sufficient to show that (28) is non-negative for $I_k \leq y_{n-k}$. By (22), $\lim_{I_k \to -\infty} d^3 J_k^*(I_k^3)/dI_k = 0$. Furthermore from (28) and the induction argument we have that

$$\frac{d^4 J_k^*(I_k)}{dI_k^4} = E\left[\left(1-\frac{U}{F^{-1}[1-\gamma]}\right)^4 \frac{d^4 J_{k+1}^*(I_{k+1})}{dI_{k+1}^4}\right] \geq 0 \tag{31}$$

hence (28) is non-negative for $I_k \leq y_{n-k}$ and we are done. We have shown that the unconstrained optimal solution $\overline{Q}_k^*(I_k)$ is convex in I_k, $k = 1, .., n-1$. Figure 3 shows a plot of the optimal policy $Q_k^*(I_k)$ in a $n-k+1$ periods problem versus I_k.

4 Justification of Assumption 1

At this juncture, we would like to digress in order to justify assumption (1). We claim that assumption (1) holds for all practical yield distributions and service level γ. Assumption (1) can be rewritten as $\gamma \geq 1 - F[E[U]]$. Consider the Beta distribution with parameters $a \geq 1$ and $b \geq 1$, whose density is given by:

$$f(u) = \begin{cases} \frac{u^{a-1}(1-u)^{b-1}}{B(a,b)} & \text{if } 0 < u < 1 \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

where $B(a,b)$ is the beta function. The Beta distribution is mainly used to model the distribution of random proportions such as the proportion of defective items in a shipment. Therefore it is the most appropriate among the standard probability distributions to use in conjonction to our model. Furthermore, it is a very general distribution that can take various shapes according to its parameters. Therefore showing that assumption (1) holds for all Beta distributions (with $a \geq 1$ and $b \geq 1$) and for all practical service level γ is sufficient for all practical reasons to justify the validity of assumption (1). Our goal is to find the highest value γ^* for which assumption (1) holds for $a \geq 1$ and $b \geq 1$, i.e to find

$$\gamma^* = \text{Max}_{\{a \geq 1, b \geq 1\}} \left\{ 1 - F\left[\frac{a}{a+b}\right] \right\}$$

If γ^* is reasonably high then we are done. In practice, typical values of γ range from 0.9 and above and it would be encouraging for the usefulness of the model to obtain γ^* below this range. γ^* occurs when $a = 1$ and b is very large, thus

$$\begin{aligned}
\lim_{b \to \infty} F\left[\frac{1}{1+b}\right] &= \lim_{b \to \infty} \frac{\int_0^{\frac{1}{1+b}} (1-u)^{b-1}\, du}{B(1,b)} \\
&= \lim_{b \to \infty} -\frac{b+1}{b} (1-u) \Big|_0^{\frac{1}{1+b}} \\
&= \lim_{b \to \infty} \frac{b+1}{b} \left[1 - \left(\frac{b}{1+b}\right)^b \right] \\
&= 1 - \lim_{b \to \infty} \left(\frac{b}{1+b}\right)^b = 1 - \exp^{-1} \approx 0.64
\end{aligned}$$

$\gamma^* \approx 0.64$ is by no means a restrictive value, hence assumption (1) is valid.

5 Examples

Suppose that the density of the yield rate U is given by

$$f(u) = \begin{cases} au^{a-1} & \text{if } 0 < u < 1 \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

which is a special case of the Beta distribution for $b = 1$. We are interested in the cases when $a \geq 1$. Solving the set of equations in lemma (1), we get for $k = 1, .., n-1$

$$\eta_k^* = \left[\left(\frac{a+1}{a} \right) (1-\gamma)^{\frac{1}{a}} \right]^{\left(\frac{1}{1+a}\right)^{n-k}} \tag{34}$$

Solving the set of equations in lemma (2), we get for $k = 1, .., n-1$

$$\beta_k^* = \left\{ \frac{\left[\left(\frac{a+1}{a} \right) (1-\gamma)^{\frac{1}{a}} \right]}{\sum_{i=0}^{n-k-1} \left[\frac{1-\gamma}{1+a} \right]^i} \right\}^{\left(\frac{1}{1+a}\right)} \tag{35}$$

and

$$\lim_{n\to\infty} \beta_1^* = \left\{ \left(1 + \frac{\gamma}{a} \right) (1-\gamma)^{\frac{1}{a}} \right\}^{\left(\frac{1}{1+a}\right)} \tag{36}$$

Note that $\beta_{n-1}^* = \eta_{n-1}^*$. This is due to the fact that in a two periods problem, the unconstrained optimal policy is linear as can be seen in figure 1. Hence proposition (5) and (7) are identical in a two periods problem. However this will not be the case for $k \leq n-2$. Consider for example a three periods problem, that is $k = n-2$. Proposition (4) states that $Q_{n-2}^*(I_{n-2}) = 0$ for $I_{n-2} \geq 3d$ and proposition (5) states that $Q_{n-2}^*(I_{n-2}) = (3d - I_{n-2})/\eta_{n-2}^*$ for $d + y_1 \leq I_{n-2} \leq 3d$ where η_{n-2}^* is given by (34). Moreover, proposition (8) states that $\overline{Q}_{n-2}^*(I_{n-2})$ is convex and proposition (7) states that

$$Q_{n-2}^*(I_{n-2}) = \begin{cases} \overline{Q}_{n-2}^*(I_{n-2}) & \text{if } I_{n-2} \geq y_2 \\ \frac{d - I_{n-2}}{F^{-1}[1-\gamma]} & \text{otherwise} \end{cases}$$

where $y_2 \leq d + y_1$ is given by

$$\overline{Q}^*_{n-2}(y_2) = \frac{d - y_2}{F^{-1}[1-\gamma]} \tag{37}$$

Finally , by proposition (7) we have that $\lim_{I_{n-2}\to-\infty} d\overline{Q}^*_{n-2}(I_{n-2})/dI_{n-2} = \beta^*_{n-2}$ where β^*_{n-2} is given by (35). Figure 4 shows a plot of the optimal policy for a three periods problem. In a three periods problem, $-1/\eta^*_{n-2}$ is the 'limiting slope' of the portion of the optimal policy comprised between $d + y_1$ and $3d$, while $-1/\beta^*_{n-2}$ is the 'limiting slope' of the portion of the optimal policy comprised between y_2 and $d + y_1$. When the planning horizon is larger than three periods, say four periods, $-1/\eta^*_{n-3}$ becomes the 'limiting slope' of the portion of the optimal policy comprised between $2d + y_1$ and $4d$, and $-1/\beta^*_{n-3}$ is the 'limiting slope' of the portion of the optimal policy comprised between y_3 and $d + y_2$. We want to get an expression for the 'limiting slope' of the portion of the optimal policy comprised between $d + y_2$ and $2d + y_1$. We denote this 'limiting slope' by $-1/\eta^{*2}_{n-3}$. Consequently, $\eta^*_{n-3} \equiv \eta^{*1}_{n-3}$, $\beta^*_{n-3} \equiv \eta^{*3}_{n-3}$, $\eta^*_{n-2} \equiv \eta^{*1}_{n-2}$, $\beta^*_{n-2} \equiv \eta^{*2}_{n-2}$ and $\beta^*_{n-1} = \eta^*_{n-1} \equiv \eta^{*1}_{n-1}$. However, before doing that we would like to get an expression for $J^*_{n-2}(I_{n-2})$, the value function for a three periods problem. By proposition (4), equations (18), (19), (20) and (24), it is differentiable everywhere except at $I_{n-2} = 3d$ and convex. It is given by

$$J^*_{n-2}(I_{n-2}) = \begin{cases} 0 & I_{n-2} \geq 3d \\ J^{*1}_{n-2}(I_{n-2}) & d + y_1 \leq I_{n-2} \leq 3d \\ J^{*2}_{n-2}(I_{n-2}) & y_2 \leq I_{n-2} \leq d + y_1 \\ J^{*3}_{n-2}(I_{n-2}) & I_{n-2} \leq y_2 \end{cases} \tag{38}$$

where

$$\lim_{I_{n-2}\to-\infty} \frac{dJ^{*1}_{n-2}(I_{n-2})}{dI_{n-2}} = -\frac{F\left[\eta^{*1}_{n-1}\right] F\left[\eta^{*1}_{n-2}\right]}{F^{-1}[1-\gamma]}$$

$$\lim_{I_{n-2}\to-\infty}\frac{dJ^{*2}_{n-2}(I_{n-2})}{dI_{n-2}} = -\frac{F\left[\eta^{*2}_{n-2}\right](1+\rho)}{F^{-1}[1-\gamma]}$$
$$\lim_{I_{n-2}\to-\infty}\frac{dJ^{*3}_{n-2}(I_{n-2})}{dI_{n-2}} = -\frac{(1+\rho+\rho^2)}{F^{-1}[1-\gamma]}$$

and where $\lim_{I_{n-2}\to-\infty} dJ^{*2}_{n-2}(I_{n-2})/dI_{n-2}$ is obtained from (26) and is given by

$$\lim_{I_{n-2}\to-\infty}\frac{dJ^{*2}_{n-2}(I_{n-2})}{dI_{n-2}} = \int_0^{\lim_{I_{n-2}\to-\infty}\frac{d+y_1-I_{n-2}}{Q^*_{n-2}(I_{n-2})}}\lim_{I_{n-2}\to-\infty}\left[\frac{dJ^*_{n-1}(I_{n-1})}{dI_{n-1}}\right]f(u)\,du$$
$$= -\frac{(1+\rho)}{F^{-1}[1-\gamma]}\int_0^{\eta^{*2}_{n-2}} f(u)\,du = -\frac{F\left[\eta^{*2}_{n-2}\right](1+\rho)}{F^{-1}[1-\gamma]}$$

We now return to determining $-1/\eta^{*2}_{n-3}$, the 'limiting slope' of the portion of the four periods problem optimal policy comprised between $d+y_2$ and $2d+y_1$. Setting (13) to zero we get for $d+y_2 \le I_{n-3} \le 2d+y_1$

$$\frac{dJ_{n-3}(I_{n-3},Q_{n-3})}{dQ_{n-3}} = 1+\int_0^{\frac{2d+y_1-I_{n-3}}{Q_{n-3}}}\left[\frac{dJ^{*2}_{n-2}(I_{n-2})}{dI_{n-2}}\right]uf(u)\,du + \int_{\frac{2d+y_1-I_{n-3}}{Q_{n-3}}}^{\frac{4d-I_{n-3}}{Q_{n-3}}}\left[\frac{dJ^{*1}_{n-2}(I_{n-2})}{dI_{n-2}}\right]uf(u)\,du = 0 \qquad (39)$$

Therefore $Q^*_{n-3}(I_{n-3})$ for $d+y_2 \le I_{n-3} \le 2d+y_1$ solves (39). Hence

$$1+\int_0^{\lim_{I_{n-3}\to-\infty}\frac{2d+y_1-I_{n-3}}{Q^*_{n-3}(I_{n-3})}}\lim_{I_{n-3}\to-\infty}\left[\frac{dJ^{*2}_{n-2}(I_{n-2})}{dI_{n-2}}\right]uf(u)\,du = 0$$
$$\Rightarrow \int_0^{\eta^{*2}_{n-3}} uf(u)\,du = -\left\{\lim_{I_{n-2}\to-\infty}\left[\frac{dJ^{*2}_{n-2}(I_{n-2})}{dI_{n-2}}\right]\right\}^{-1}$$
$$\Rightarrow \int_0^{\eta^{*2}_{n-3}} uf(u)\,du = \frac{F^{-1}[1-\gamma]}{F[\eta^{*2}_{n-2}](1+\rho)}$$

For the specific yield rate distribution defined in (33), we get for a four periods problem, i.e. for $k=n-3$:

$$\eta^{*j}_k = \left\{\frac{\left[\left(\frac{a+1}{a}\right)(1-\gamma)^{\frac{1}{a}}\right]}{\sum_{i=0}^{j-1}\left[\frac{1-\gamma}{1+a}\right]^i}\right\}^{\left(\frac{1}{1+a}\right)^{(n-k-j+1)}} \qquad \text{for } j=1,..,n-k \qquad (40)$$

Figure 5 shows the optimal policy for a four periods problem. It can be shown that (40) is true for $1 \leq k \leq n-4$. Furthermore, it can be shown that for a general distribution we have for $n \geq 2$, $k = 1,..,n-1$ and $j = 1,..,n-k$:

$$\int_0^{\eta_k^{*j}} uf(u)\,du = \Lambda_{k+1}^j \tag{41}$$

where

$$\Lambda_{k+1}^j = \frac{\int_0^{\eta_{k+1}^{*j}} uf(u)\,du}{\int_0^{\eta_{k+1}^{*j}} f(u)\,du}$$

and

$$\Lambda_{(n-j+1)}^j = \frac{F^{-1}[1-\gamma]}{\sum_{i=0}^{j-1} \rho^i}$$

Naturally, (41) reduce to (40) when the yield rate is distributed as in (33). Table 1 provides a way to interpret the output of (41). The number of demand periods n increases vertically and I_k, the beginning of period k inventory level in a $n-k+1$ periods problem, increases from left to right. η_k^{*j} are the limiting multiplicative coefficients obtained after solving (41). Next we present numerical examples to illustrate (40) (example 1,2,7,8,9,10) and to illustrate (41) using a Beta distribution with more general parameters (example 3 to 7, 11 to 18) for $n \geq 2$, $k = 1,..,n-1$ and $j = 1,..,n-k$. These numerical examples will provide us insights on the impact of the two state variables, namely the number of demand periods and the beginning of period inventory, on the optimal release quantity that must be decided upon at the beginning of the planning horizon. The examples will be for $n = 8$ and $\gamma = 0.95, 0.99$. The following are tables that list η_k^{*j} for $k = 1$ to 7 and $j = 1,..,8-k$.

7	6	5	4	3	2	1	
0.3122499	0.55879325	0.74752475	0.86459516	0.92983653	0.9642895	0.98217189	1
	0.3122499	0.55879325	0.74752479	0.86459598	0.92985425	0.96466162	2
		0.3122499	0.55879331	0.74752621	0.86462892	0.93057204	3
			0.31224996	0.55879544	0.74758317	0.86596432	4
				0.31225234	0.5588806	0.74989421	5
					0.31234752	0.56234133	6
						0.31622777	7

Table 1: $a = 1,\ b = 1\ \gamma = 0.95$

0.14106736	0.37558935	0.61285345	0.78284957	0.88478787	0.940632	0.96989964	1
	0.14106736	0.37558935	0.61285345	0.78284958	0.88478856	0.94070531	2
		0.14106736	0.37558935	0.61285346	0.78285079	0.88492647	3
			0.14106736	0.37558936	0.61285536	0.78309486	4
				0.14106737	0.3755917	0.61323756	5
					0.14106912	0.37606031	6
						0.14142136	7

Table 2: $a = 1,\ b = 1\ \gamma = 0.99$

0.467986	0.664428	0.786323	0.860675	0.907152	0.93708	0.957045	1
	0.467986	0.664428	0.786323	0.860675	0.907161	0.937404	2
		0.467986	0.664428	0.786323	0.860688	0.907658	3
			0.467986	0.664428	0.786345	0.861471	4
				0.467987	0.664463	0.787613	5
					0.468042	0.666552	6
						0.471280	7

Table 3: $a = 2,\ b = 2,\ \gamma = 0.95$

7	6	5	4	3	2	1	
0.340313	0.577729	0.733404	0.828249	0.886711	0.923812	0.948049	1
	0.340313	0.577729	0.733404	0.828249	0.886711	0.923859	2
		0.340313	0.577729	0.733404	0.82825	0.886784	3
			0.340313	0.577729	0.733405	0.828366	4
				0.340313	0.57773	0.733596	5
					0.340314	0.578041	6
						0.340745	7

Table 4: $a = 2,\ b = 2,\ \gamma = 0.99$

0.515613	0.679584	0.778068	0.840332	0.881982	0.911109	0.932367	1
	0.515613	0.679584	0.778068	0.840332	0.881988	0.911411	2
		0.515613	0.679584	0.778068	0.84034	0.882411	3
			0.515613	0.6795848	0.77808	0.840957	4
				0.515613	0.679604	0.779024	5
					0.515647	0.681138	6
						0.518224	7

Table 5: $a = 3,\ b = 3,\ \gamma = 0.95$

0.421832	0.623169	0.743767	0.818214	0.866944	0.900463	0.924418	1
	0.421832	0.623169	0.743767	0.818214	0.866944	0.900504	2
		0.421832	0.623169	0.743767	0.818214	0.867002	3
			0.421832	0.623169	0.743768	0.818299	4
				0.421832	0.623169	0.743901	5
					0.421833	0.623391	6
						0.422195	7

Table 6: $a = 3,\ b = 3,\ \gamma = 0.99$

7	6	5	4	3	2	1	
0.69091667	0.88404674	0.95975064	0.98639941	0.99544578	0.99847999	0.99950063	1
	0.69091667	0.88404674	0.95975064	0.98639946	0.9954469	0.99850263	2
		0.69091668	0.88404674	0.9597508	0.98640279	0.99551461	3
			0.69091669	0.88404719	0.95976051	0.9866041	4
				0.69091774	0.88407402	0.96034825	5
					0.69098066	0.8856992	6
						0.69479831	7

Table 7: $a = 2,\ b = 1\ \gamma = 0.95$

0.53073826	0.80964281	0.93203271	0.97681065	0.99220968	0.99739647	0.99913292	1
	0.53073826	0.80964281	0.93203271	0.97681065	0.99220973	0.99740102	2
		0.53073826	0.80964281	0.93203271	0.97681078	0.99222332	3
			0.53073826	0.80964282	0.9320331	0.97685091	4
				0.53073827	0.80964381	0.93214798	5
					0.53074023	0.80994324	6
						0.53132928	7

Table 8: $a = 2,\ b = 1\ \gamma = 0.99$

0.83454507	0.95578993	0.98875937	0.99717792	0.99929373	0.99982342	0.99995661	1
	0.83454507	0.95578993	0.98875937	0.99717793	0.99929388	0.99982646	2
		0.83454507	0.95578993	0.9887594	0.99717853	0.99930601	3
			0.83454507	0.95579004	0.98876178	0.99722692	4
				0.83454547	0.95579926	0.98895372	5
					0.83457767	0.9564164	6
						0.83717359	7

Table 9: $a = 3,\ b = 1\ \gamma = 0.95$

7	6	5	4	3	2	1	
0.73163798	0.92485588	0.98066012	0.99512956	0.99878016	0.9996949	0.99992387	1
	0.73163798	0.92485588	0.98066012	0.99512956	0.99878017	0.99969551	2
		0.73163798	0.92485588	0.98066012	0.99512959	0.9987826	3
			0.73163798	0.92485588	0.98066022	0.99513929	4
				0.73163799	0.92485624	0.98069848	5
					0.73163913	0.92500058	6
						0.73209597	7

Table 10: $a = 3,\ b = 1\ \gamma = 0.99$

0.166643	0.320559	0.466781	0.586955	0.680147	0.751112	0.805766	1
	0.166643	0.320559	0.466781	0.586956	0.680177	0.752055	2
		0.166643	0.320559	0.466782	0.586996	0.681419	3
			0.166643	0.32056	0.466833	0.588619	4
				0.166645	0.320619	0.468889	5
					0.1666995	0.32300	6
						0.16892	7

Table 11: $a = 1,\ b = 2,\ \gamma = 0.95$

0.0723905	0.20334	0.358563	0.499123	0.61235	0.699536	0.765922	1
	0.0723905	0.20334	0.358563	0.499123	0.612351	0.699676	2
		0.0723905	0.20334	0.358563	0.499124	0.612534	3
			0.0723905	0.20334	0.358565	0.499358	4
				0.0723905	0.203342	0.358844	5
					0.0723914	0.203621	6
						0.0725771	7

Table 12: $a = 1,\ b = 2,\ \gamma = 0.99$

7	6	5	4	3	2	1	
0.11351	0.223525	0.334649	0.432939	0.515482	0.583731	0.640885	1
	0.11351	0.223525	0.334649	0.43294	0.515510	0.584677	2
		0.11351	0.223525	0.33465	0.432974	0.516657	3
			0.113511	0.223526	0.33469	0.434356	4
				0.113511	0.223569	0.33631	5
					0.113549	0.225319	6
						0.115094	7

Table 13: $a = 1,\ b = 3,\ \gamma = 0.95$

0.0486773	0.139141	0.251609	0.360300	0.454725	0.533533	0.598713	1
	0.0486773	0.139141	0.251609	0.360300	0.454726	0.533664	2
		0.0486773	0.139141	0.251609	0.360301	0.454884	3
			0.0486773	0.139141	0.25161	0.360488	4
				0.0486773	0.139142	0.251818	5
					0.048678	0.139339	6
						0.0488038	7

Table 14: $a = 1,\ b = 3,\ \gamma = 0.99$

0.639767	0.806498	0.891131	0.936216	0.961579	0.97644	0.985472	1
	0.639767	0.806498	0.891132	0.936216	0.961582	0.97658	2
		0.639767	0.806498	0.891132	0.936222	0.961817	3
			0.639767	0.806498	0.891141	0.936629	4
				0.639767	0.806516	0.806516	5
					0.639804	0.807943	6
						0.64263	7

Table 15: $a = 3,\ b = 2,\ \gamma = 0.95$

7	6	5	4	3	2	1	
0.534911	0.752826	0.863585	0.921244	0.953021	0.971369	0.982324	1
	0.534911	0.752826	0.863585	0.921244	0.953021	0.971389	2
		0.534911	0.752826	0.863585	0.921244	0.953055	3
			0.534911	0.752826	0.863585	0.921304	4
				0.534911	0.752827	0.863697	5
					0.534912	0.753046	6
						0.535333	7

Table 16: $a = 3, \; b = 2, \; \gamma = 0.99$

0.352983	0.522535	0.642144	0.725568	0.785254	0.829266	0.862966	1
	0.352983	0.522535	0.642144	0.725568	0.785265	0.829778	2
		0.352983	0.522535	0.642144	0.725584	0.78595	3
			0.352983	0.522536	0.642167	0.726525	4
				0.352984	0.522569	0.643501	5
					0.353029	0.524492	6
						0.355664	7

Table 17: $a = 2, \; b = 3, \; \gamma = 0.95$

0.251065	0.444668	0.587941	0.687579	0.757849	0.808896	0.847109	1
	0.251065	0.444668	0.587941	0.687579	0.757849	0.808966	2
		0.251065	0.444668	0.587941	0.68758	0.757944	3
			0.251065	0.444668	0.587942	0.687711	4
				0.251065	0.444669	0.588131	5
					0.251066	0.444939	6
						0.251399	7

Table 18: $a = 2, \; b = 3, \; \gamma = 0.99$

All examples show that η_k^{*j} are converging rapidly with the planning horizon. Using this and the fact that $\lim_{n\to\infty} \eta_1^{*1} = 1$, then one might suspect that the long run effect of adding one more period is quickly reached especially in the vicinity of low values of beginning of period inventory. Therefore the effect on the optimal policy of adding one more period when inventory is low is simply increasing the release quantity by an amount equal to that period demand. True, this is a result of no holding cost considerations. However, the model is not intended to be used in infinite horizon situations. We are modelling finite horizon situations and holding cost considerations are not important in such cases, especially when the optimal policy is converging rapidly as suggested by the numerical examples. To get a feel how fast the optimal policy is converging, consider equations (40) for $k = 1$. We have

$$\eta_1^{*j} = \left\{ \frac{\left[\left(\frac{a+1}{a}\right)(1-\gamma)^{\frac{1}{a}}\right]}{\sum_{i=0}^{j-1}\left[\frac{1-\gamma}{1+a}\right]^i} \right\}^{\left(\frac{1}{1+a}\right)^{(n-j)}} \quad \text{for } j = 1,..,n-1 \tag{42}$$

As n increases, the effect of adding one more power dominates the effect of adding one more term to the sum in the denominator that converges rapidly especially for large values of α. Thus η_1^{*j} approximates

$$\eta_1^{*j} \approx \left\{ \left(1 + \frac{\gamma}{a}\right)(1-\gamma)^{\frac{1}{a}} \right\}^{\left(\frac{1}{1+a}\right)^{(n-j)}} \quad \text{for } j = 1,..,n-1 \tag{43}$$

This is shown in tables 1, 7 and 9 where each row becomes identical to the row below it except for one additional term close to 1. How large is the error in the optimal values of the release quantities when holding cost is considered and exactly how long should the planning horizon be for the holding cost effect to 'kick in' and truncate the forecast horizon are the topics of an ongoing research. Other useful observations in the numerical examples are that the optimal release quantities increase with the service level, increase as the distribution is skewed to the left as seen in tables 9, 15, 5,17 and 13, and increase with the variance of the yield rate as seen in tables 1 through 6.

6 Conclusion

We have analyzed a multiple periods service level constrained model of a production system with random yield in a finite horizon setting. The objective was to determine the optimal release quantity in the current period, given a certain inventory level, that minimizes the total input quantity throughout the planning horizon while keeping a high probability of meeting the demand in each period. We have showed that under the mild assumtions of the model, the reorder point in any period is equal to the total remaining demand in all future periods (including the current period) and that in any period, we release a quantity higher than the amount that brings our inventory up to the total remaining demand in all future periods. Furthermore, we showed that the optimal policy is convex with the initial inventory level and that there exists in each period a value of the initial inventory level below which the service level constraint is binding. Although there is no simple way to compute the optimal policy, we derived expressions for the limiting slopes of various portions of the optimal policy in each period. These limiting slopes provide us insights on the impact of the yield rate distribution, the service level, initial inventory level and the addition of another demand period as was shown in the numerical examples. Future research involve the study of the effect of holding cost on the optimal release quantity in a finite horizon setting, and determining infinite horizon policies in the presence of holding cost.

References

1. Bertsekas, D. (1987), *Dynamic Programming: Deterministic and Stochastic Models*, Prentice-Hall.

2. Gerchak, Y., M. Parlar and R. G. Vickson (1986) Periodic Review production Models

With Variable Yield and Uncertain Demand, *IIE Transactions*, **20**, 144-150.

3. Karlin, S. (1958) Steady State Solutions, In *Studies in the Mathematical Theory of Inventory and Production*, K. J. Arrow, S. Karlin and H. Scarf, Stanford University Press, Stanford, Ca.

4. Karlin, S. and H. M. Taylor (1975), *A First Course in Stochastic Processes*, Academic Press.

5. Lee, H. L. and C. A. Yano (1989) Lot Sizing With Random Yields: A Review, Technical Report 89-16, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor.

6. Lee, H. L. and C. A. Yano (1988) Production Control in Multistage Systems With Variable Yield losses, *Operations Research*, **36**, 269-278.

7. Sepehri, M., E. A. Silver and C. New (1986) A Heuristic for Multiple Lot Sizing for an Order Under Yield Variability, *IIE Transactions*, 63-69.

8. Singh, M., C. Abraham, R. Akella (1990) Planning for Production of a Set of Components when Yield is Random, *IEEE Transactions on Comp., Hybrids, and Manufacturing technology*, **36**, 359-367.

9. Tang, C. (1990) Composing Batches With Yield Uncertainty, Operations and Technology Management Working Paper #7-90, The John F. Anderson Graduate School of Management, University of California, Los angeles.

10. Yao, D. (1988) Optimal Run Quantities for an Assembly System With Random

Yields, *IIE Transactions*, **20**, 399-403.

$y_{n-1}+(1-k)d \le I_k \le y_{n-2}+(2-k)d$	$y_{n-2}+(2-k)d \le I_k \le y_{n-3}+(3-k)d$		$y_3+(n-k-3)d \le I_k \le y_2+(n-k-2)d$	$y_2+(n-k-2)d \le I_k \le y_1+(n-k-1)d$	$y_1+(n-k-1)d \le I_k \le (n-k+1)d$	k	# of demand periods
$\eta_1^{*(n-1)}$	$\eta_1^{*(n-2)}$		η_1^{*3}	η_1^{*2}	η_1^{*1}	1	n
	$\eta_2^{*(n-2)}$		η_2^{*3}	η_2^{*2}	η_2^{*1}	2	n-1
			η_{n-3}^{*3}	η_{n-3}^{*2}	η_{n-3}^{*1}	n-3	4
				η_{n-2}^{*2}	η_{n-2}^{*1}	n-2	3
					η_{n-1}^{*1}	n-1	2

Table 1: limiting multiplicative coefficients. for n > 1, k=1,..,n-1, j=1,..,n-k

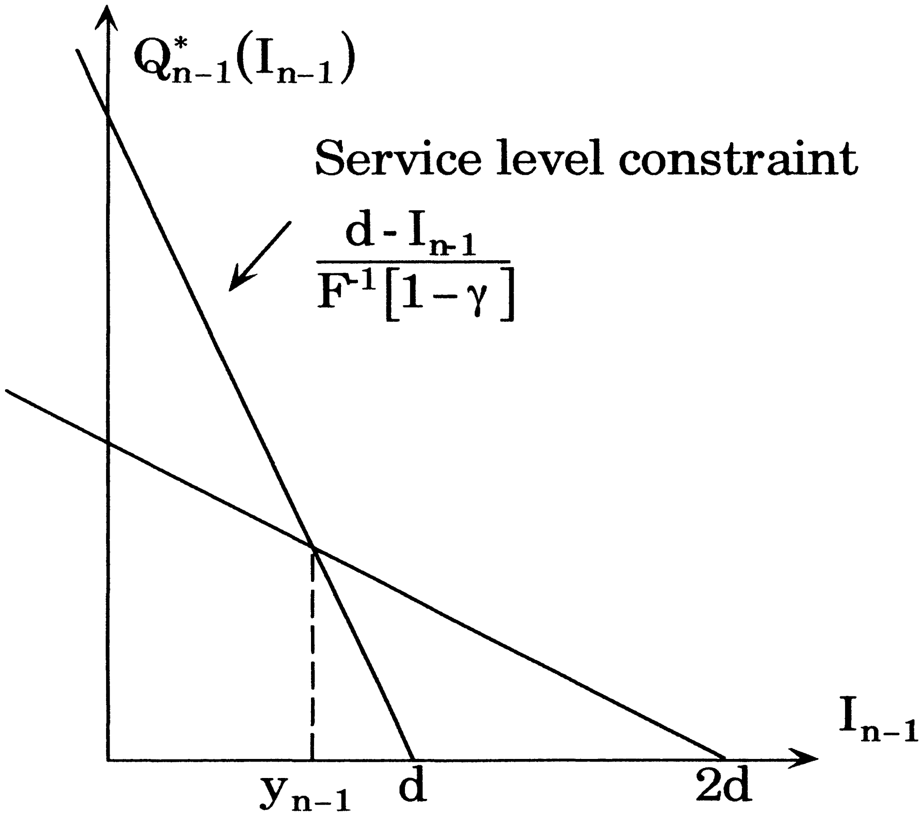

Figure 1: Optimal policy for a 2-periods problem when $E[U] \geq F^{-1}[1-\gamma]$

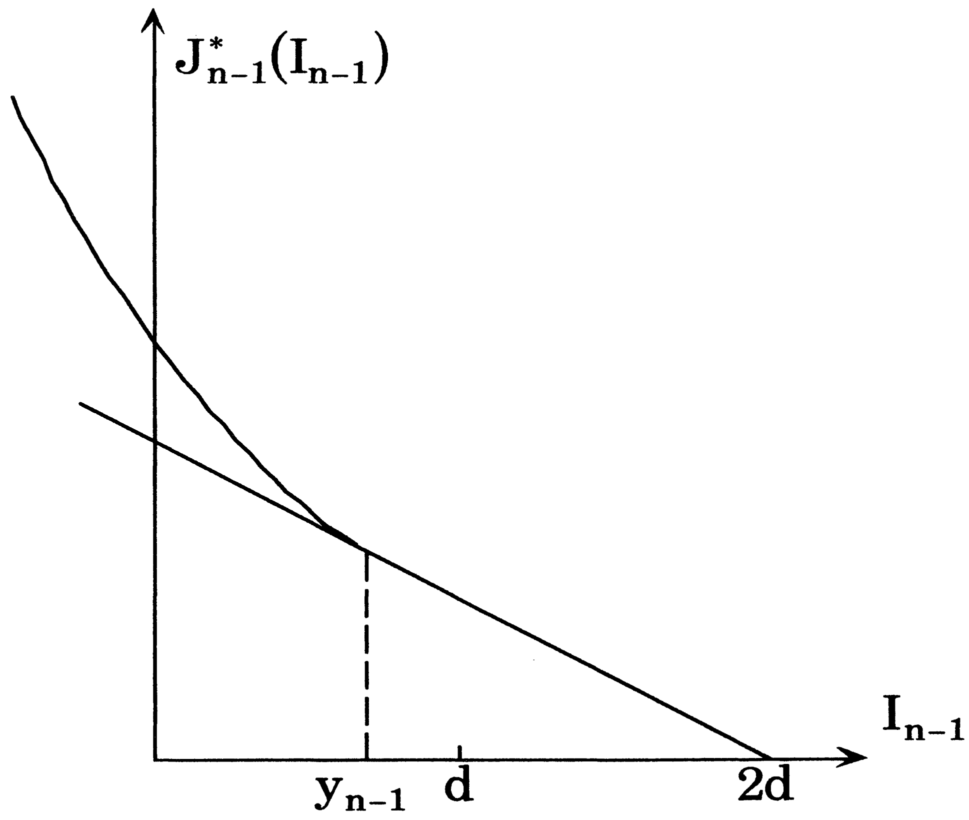

Figure 2: Value function for a 2-periods problem when $E[U] \geq F^{-1}[1-\gamma]$

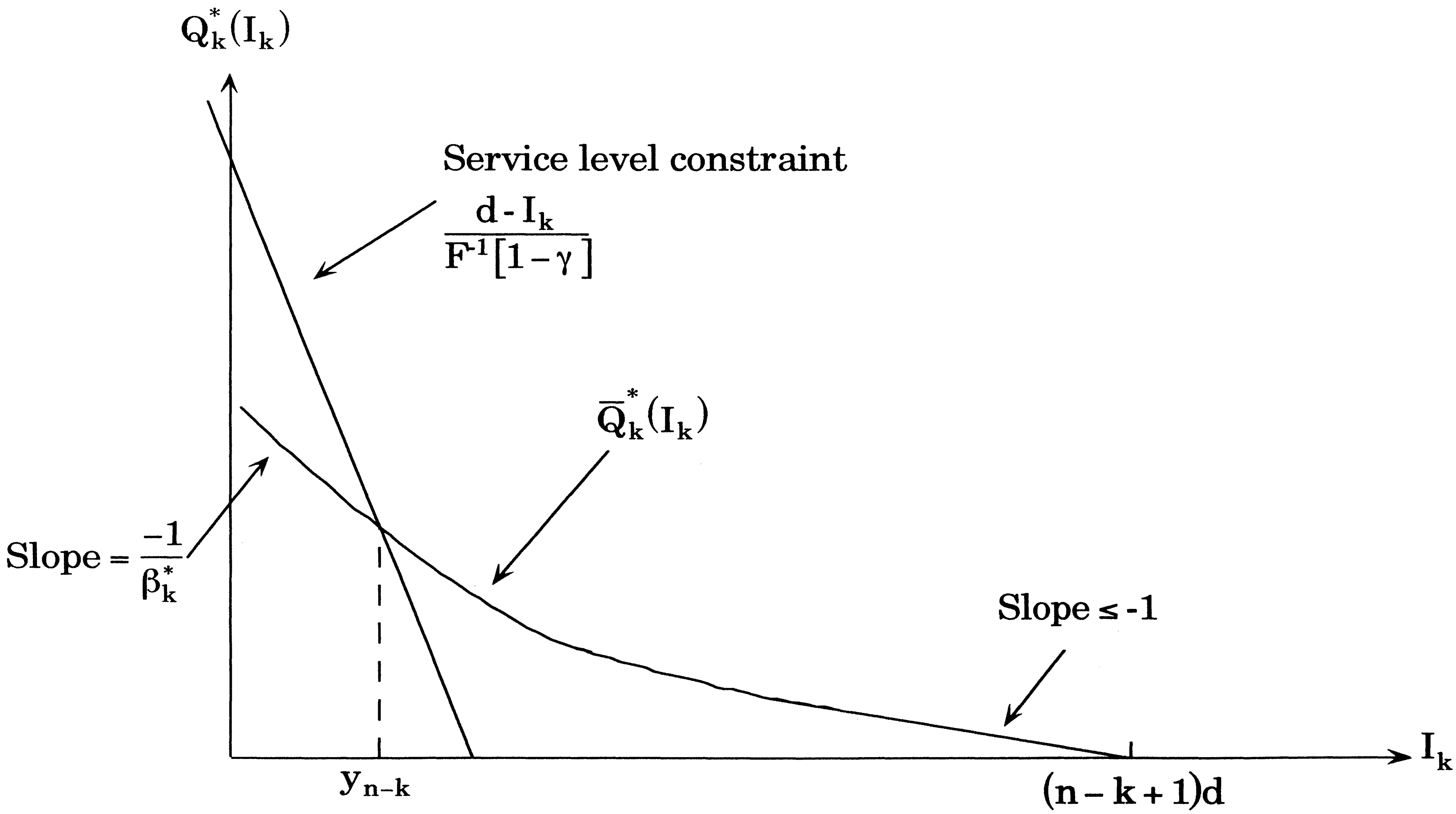

Figure 3: Optimal policy for a (n - k + 1) periods problem

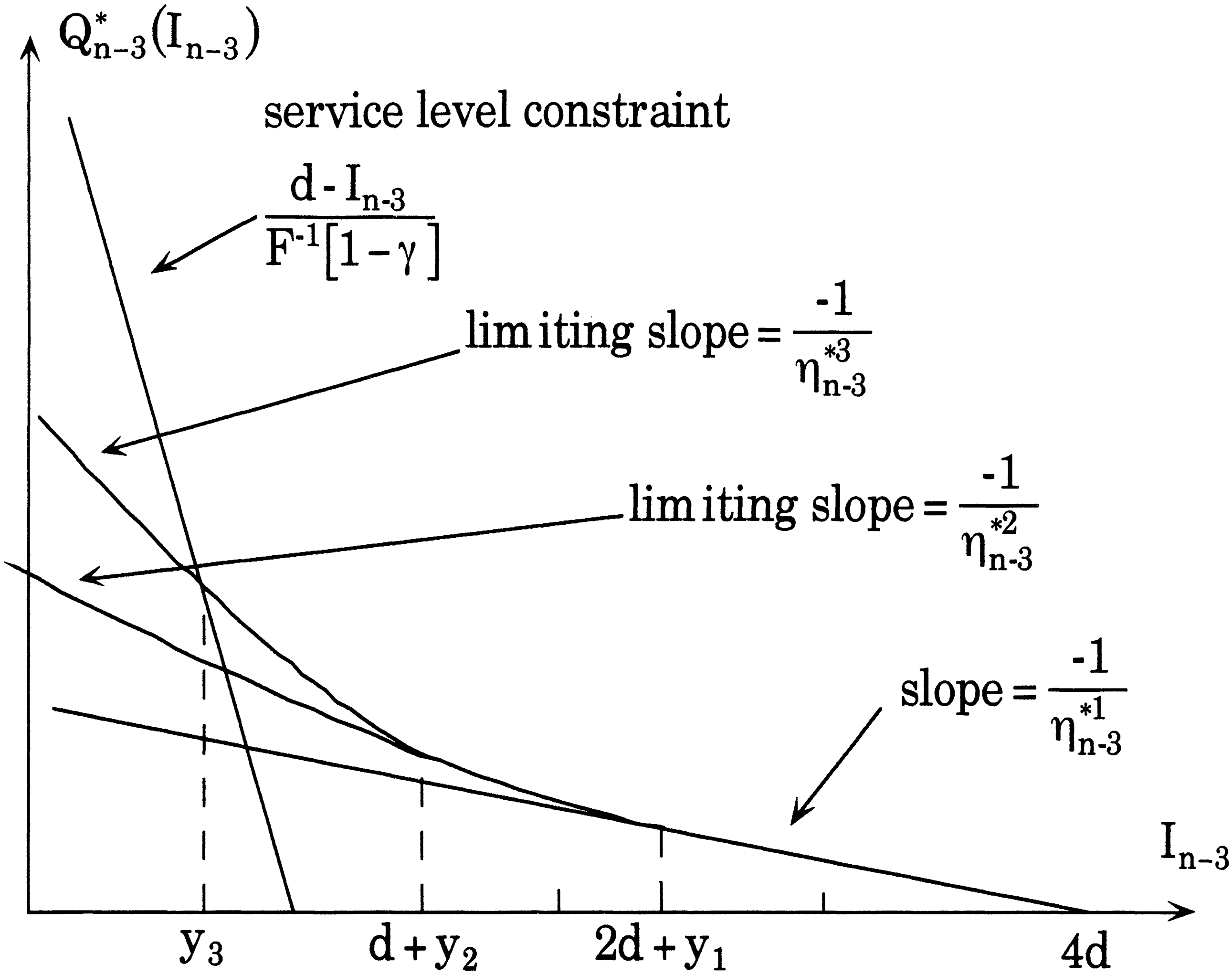

Figure 5: optimal policy in a 4 - periods problem

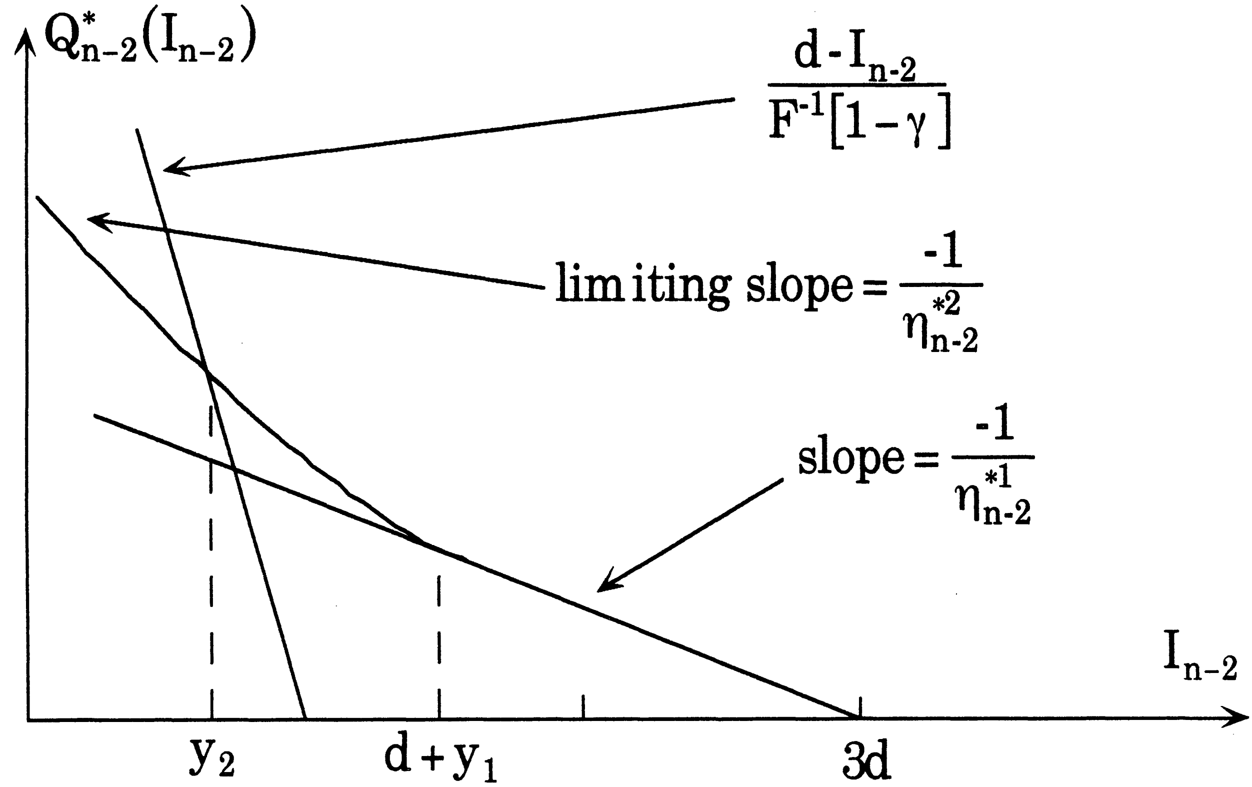

Figure 4: optimal policy in a 3 - periods problem