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Local limiting behavior of the zeros of approximating polynomials

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Abstract

Let f be a piecewise analytic (but not analytic) function in $C^k[a, b]$, $k \geqslant 0$, and let p_n^* be the sequence of polynomials of best uniform approximation to f on $[a, b]$. It is well known that every point of $[a, b]$ is a limit point of the zeros of the p_n^*. Let $x \in [a, b]$, and suppose that f is analytic at x and $f(x) \neq 0$. The main purpose of this paper is to show that there exists a constant γ (which depends only on x) such that there is no zero of p_n^* within the circle of radius $(\gamma/n)\log n$ centered at x, for all sufficiently large values of n.

Key words: Piecewise analytic functions; Polynomials of best uniform approximation; Zeros; Points of singularity; Points of analyticity

1. Introduction

Let E be a compact boundary set ($int(E) = \emptyset$) in the complex plane whose complement is connected and regular in the sense that the complement has Green's function $G(z)$ with pole at ∞ (Walsh [8, p. 65]). Let f be a continuous function on E, and for each positive integer n let p_n^* be the polynomial of degree at most n of best uniform approximation to f on E:

$$\| f - p_n^* \|_E < \| f - p \|_E,$$

for every polynomial $p \neq p_n^*$ of degree at most n. It is well known by Mergelyan's theorem [6, p. 423] that the left member of the above inequality tends to 0 as $n \to \infty$.

Now, consider a theorem of Blatt and Saff [2]:

Theorem 1.1. *If f has at least one point of singularity in E (which means that if there is a point in E at which f is not analytic), then every point of E is a limit point of the zeros of the p_n^*.*

The converse of Theorem 1.1 is a consequence of the following well-known result:

Theorem 1.2. *If there is a limit point $x \in E$ of the zeros of the p_n^* and $f(x) \neq 0$, then f has at least one point of singularity in E.*

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The above theorems imply the following statement: When and only when there is a point of singularity (somewhere) in E, *all points* of E (the points of analyticity as well as the points of singularity) attract zeros. This statement characterizes the existence of at least one point of singularity in terms of a global limiting behavior of the zeros.

In the present paper, we investigate the truth of the following hypothesis, which characterizes the points of singularity in terms of a *local limiting behavior of the zeros*:

Hypothesis. *The points of singularity attract zeros faster than the points of analyticity.*

The main results are stated in Section 2, and the proofs are given in Section 3. We conclude this introduction with the following definitions:

Set

$$E_\rho := \{z\colon G(z) < \log \rho\},$$

and $\Gamma_\rho := \partial E_\rho$, for $\rho > 1$. Any open neighborhood of E contains some E_ρ, and according to Walsh [8, p. 65], Γ_ρ "either consists of a finite number of finite mutually exterior analytic Jordan curves or consists of a finite number of contours which are mutually exterior except that each of a finite number of points may belong to several contours."

It is also known that if f is analytic on E_ρ, then p_n^* converges uniformly to f on every compact subset of E_ρ at a geometric rate. In fact, this can easily be used to prove Theorem 1.2.

For $E = [\alpha, \beta]$, Γ_ρ is the ellipse with foci α and β, whose major and minor semi-axes have lengths $\frac{1}{4}(\beta - \alpha)(\rho + 1/\rho)$ and $\frac{1}{4}(\beta - \alpha)(\rho - 1/\rho)$ respectively.

2. Main results

We begin this section with a theorem, which shows that the points of analyticity impose a certain speed limit on the approaching zeros:

Theorem 2.1. *Let f be a continuous (real or complex) function on E that does not vanish at any point of E, and let p_n be a sequence of polynomials (of respective degrees at most n) that converges to f uniformly on E. Let ρ_n be a decreasing sequence of real numbers approaching* 1, *such that*

$$\rho_n^n \| f - p_n \|_E \to 0,$$

as $n \to \infty$. If f is analytic on E, then p_n does not vanish at any point of E_{ρ_n}, for all sufficiently large values of n.

As a consequence of Theorem 2.1, consider a result on piecewise analytic functions:

Corollary 2.2. *Let $t_0 < t_1 < \cdots < t_m$, and let f be a function k times continuously differentiable on $[t_0, t_m]$, such that f is analytic on (t_{i-1}, t_i) for every $i = 1, \ldots, m$. Let p_n^* be the sequence of polynomials (of respective degrees at most n) of best uniform approximation to f on $[t_0, t_m]$. For some $j = 1, \ldots, m$, let $x \in (t_{j-1}, t_j)$, $c_x = \min\{x - t_{j-1}, t_j - x\}$, and $0 < \gamma < c_x(k+1)$. Suppose that f does not vanish in (t_{j-1}, t_j), and let $\Delta_n(x)$ be the disk of radius $(\gamma/n) \log n$ centered at x.*

Then, p_n^* does not vanish at any point of $\Delta_n(x)$ for all sufficiently large values of n. Furthermore, let $x = t_j$, for some $j = 0, \dots, m$, such that $f(x) \neq 0$. Suppose that either the restriction of f on $(t_{j-1}, t_j]$ or the restriction of f on $[t_j, t_{j+1})$ is analytic at t_j, and let $c = \frac{1}{4}(t_j - t_{j-1})$ or $\frac{1}{4}(t_{j+1} - t_j)$, respectively. For $0 < \gamma < \sqrt{c}\,(k+1)$, let $\Delta_n(x)$ be the disk of radius $[(\gamma/n) \log n]^2$. Then, p_n^* does not vanish at any point of $\Delta_n(x)$ for all sufficiently large values of n.

The following two conjectures are based on some preliminary numerical computations.

Conjecture 2.3. *Let $f(x) = |x|$ (or $f(x) = |x| + 1$), and let p_n^* be the sequence of polynomials (of respective degrees at most n) of best uniform approximation to f on $[-1, 1]$. Then, there exists a constant $\delta < 1.18$ (or < 1.52), such that for all sufficiently large values of n, p_n^* vanishes in the disk of radius $(\delta/n) \log n$ centered at $x = \frac{1}{2}$.*

Notice that for the functions of Conjecture 2.3, in Corollary 2.2, if $x = \frac{1}{2}$, then $\gamma < 0.5$. This shows that the first part of Corollary 2.2 is sharp up to a constant.

The following conjecture supports our hypothesis that the points of singularity attract zeros faster than the points of analyticity.

Conjecture 2.4. *Let $f(x) = |x|$ (or $f(x) = |x| + 1$), and let p_n^* be the sequence of polynomials (of respective degrees at most n) of best uniform approximation to f on $[-1, 1]$. Then there exists a constant κ, such that p_n^* vanishes in Δ_n the disk of radius κ/n centered at 0, for all sufficiently large values of n.*

How sharp is Theorem 2.1?

Theorem 2.5. *Let $E = [-1, 1]$, and let ρ_n be a decreasing sequence of real numbers approaching 1, such that*

$$\rho_n^n (\rho_n - 1)^{2+\varepsilon} \to \infty, \tag{1}$$

for some $\varepsilon > 0$. Then, for every function f analytic on E, there exists a sequence p_n of polynomials (of respective degrees at most n) that converges to f uniformly on E, such that p_n vanishes in E_{ρ_n}, for all sufficiently large values of n.

We conclude this section by referring the readers to [1, p. 196; 2–5; 9], for related results on the distribution of zeros.

3. Proofs

Proof of Theorem 2.1. The function f is analytic and non-vanishing in E_ρ, for some $\rho > 1$. For each positive integer n, let p_n^* be the polynomial of degree at most n of best uniform approximation to f on E. Then

$$|p_n^*(z)| > \varepsilon,$$

in E_{ρ_n}, for n sufficiently large, and for some $\varepsilon > 0$.

By the Bernstein–Walsh Lemma [8, p. 77], for all $z \in E_{\rho_n}$, and for all sufficiently large values of n, we have

$$|p_n(z) - p_n^*(z)| \leqslant \rho_n^n \| p_n - p_n^* \|_E \leqslant 2\rho_n^n \| f - p_n \|_E < \tfrac{1}{2}\varepsilon,$$

which yields

$$|p_n(z)| \geqslant |p_n^*(z)| - |p_n(z) - p_n^*(z)| > \tfrac{1}{2}\varepsilon.$$

This completes the proof. □

Proof of Corollary 2.2. Let $t_{j-1} < x < t_j$, and $0 < \gamma < c_x(k+1)$. Choose $\varepsilon > 0$, such that

$$0 < \lambda := \frac{2\gamma}{(c_x - \varepsilon)(2 - \varepsilon)} < k + 1.$$

Then, let $E = [x - c_x + \varepsilon,\ x + c_x - \varepsilon]$, and

$$\rho_n = 1 + \frac{\lambda \log n}{n}.$$

It is easy to see that $\rho_n^n < n^\lambda$, and hence

$$\rho_n^n \| f - p_n^* \|_E \to 0,$$

since it is well known that $\| f - p_n^* \|_E < \text{const}/n^{k+1}$ [7, Chapter 7].

By Theorem 2.1, it remains to show that $\Delta_n(x) \subset E_{\rho_n}$. And for this, we may show that

$$\frac{\gamma \log n}{n} < \frac{c_x - \varepsilon}{2}\left(\rho_n - \frac{1}{\rho_n}\right),$$

by observing that

$$\rho_n - \frac{1}{\rho_n} = \rho_n - 1 + (\rho_n - 1) - (\rho_n - 1)^2 + \cdots$$
$$> (\rho_n - 1)(3 - \rho_n) > \frac{\lambda \log n}{n}(2 - \varepsilon),$$

for n large. This completes the first part of the proof.

Now suppose that $x = t_j$, for some $j = 1, \ldots, m$ (the case where $j = 0$ is similar), $f(x) \neq 0$, and the restriction of f on $(t_{j-1}, t_j]$ is analytic at t_j (the case where the restriction of f on $[t_j, t_{j+1})$ is analytic at t_j is similar). Let $c = \frac{1}{4}(t_j - t_{j-1})$, $0 < \gamma < \sqrt{c}\,(k+1)$, and $\Delta_n(x)$ be the disk of radius $[(\gamma/n) \log n]^2$ centered at x.

Choose $\varepsilon > 0$, such that

$$\lambda := \frac{\gamma}{\sqrt{(c - \varepsilon)(1 - \varepsilon)}} < k + 1.$$

Then, let $E = [t_{j-1} + 4\varepsilon,\ t_j]$, and $\rho_n = 1 + (\lambda/n) \log n$. It is sufficient to show that

$$\left(\frac{\gamma \log n}{n}\right)^2 < (c - \varepsilon)\left(\rho_n + \frac{1}{\rho_n}\right) - 2(c - \varepsilon),$$

by using

$$\rho_n + \frac{1}{\rho_n} > 2 + (\rho_n - 1)^2(2 - \rho_n) > 2 + \left(\frac{\lambda \log n}{n}\right)^2(1 - \varepsilon),$$

for n large. This completes the proof. □

Proof of Theorem 2.5. Let

$$r_n = 1 + \frac{\rho_n - 1}{1 + \frac{1}{2}\varepsilon}, \qquad b_n = \tfrac{1}{2}(r_n - r_n^{-1}), \qquad \beta_n = \tfrac{1}{2}(\rho_n - \rho_n^{-1}),$$

$$y_n = \tfrac{1}{2}(b_n + \beta_n), \qquad f_n(z) = \frac{f(z)}{z - \mathrm{i}y_n}.$$

Each function f_n is analytic on the closure of E_{r_n}, for all sufficiently large values of n, and $\mathrm{i}y_n \in E_{\rho_n}$.

Let p_{n-1}^* be the polynomial of degree at most $n - 1$ of best uniform approximation to f_n on E, for each n. Then, by Bernstein's error bound [1, p. 82], we have

$$\| f_n - p_{n-1}^* \|_E < \frac{2M_n}{r_n^{n-1}(r_n - 1)}, \tag{2}$$

where

$$M_n = \max_{z \in \Gamma_{r_n}} | f_n(z) | \leqslant \frac{M}{y_n - b_n} \leqslant \frac{K}{\varepsilon(r_n - 1)},$$

for some constants M and K.

Condition (1) yields

$$\rho_n^{n/(1+\varepsilon/2)}(\rho_n - 1)^2 \to \infty,$$

which implies

$$r_n^n(r_n - 1)^2 \to \infty,$$

and hence the left member of (2) tends to zero.

Now, set

$$p_n(z) = (z - \mathrm{i}y_n)p_{n-1}^*(z),$$

which converges to f uniformly on E, and this completes the proof. □

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References

[1] S.N. Bernstein, *Collected Works, Vol. I: Constructive Theory of Functions* (1905–1930) (Academy of Sciences USSR Press, Moscow, 1952).

[2] H.-P. Blatt and E.B. Saff, Behavior of zeros of polynomials of near best approximation, *J. Approx. Theory* 46 (1986) 323–344.

[3] H.-P. Blatt, E.B. Saff and M. Simkani, Jentzsch–Szegö type theorems for the zeros of best approximants, *J. London Math. Soc. (2)* 38 (1988) 307–316.

[4] P.B. Borwein, The relationship between the zeros of best approximation and differentiability, *Proc. Amer. Math. Soc.* 92 (1984) 528–532.

[5] W. Pleśniak, On the distribution of the zeros of the polynomials of best L^2-approximation to holomorphic functions, *Zeszyty Nauk. Uniw. Jagiell. Prace Mat.* 22 (1981) 29–35.

[6] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 2nd ed., 1974).

[7] A.F. Timan, *Theory of Approximation of Functions of a Real Variable* (Pergamon Press, New York, 1963).

[8] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, AMS Colloquium Publications 20 (AMS, Providence, RI, 5th ed., 1969).

[9] A.P. Wójcik, On zeros of polynomials of best approximation to holomorphic and C^∞ functions, *Mh. Math.* 105 (1988) 75–81.