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Brief Paper

Optimal Controller Synthesis with $\mathscr{D}$ Stability*†

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Key Words—Control system synthesis; linear optimal control; algebraic Riccati equations.

Abstract—In this paper, we consider the problem of finding controllers which place the eigenvalues of the closed-loop system matrix in a prespecified circular region in the left-half plane and minimize an associated quadratic cost function. We give solutions to both state-feedback and output-feedback synthesis problems.

1. *Introduction*

ONE OF THE important objectives in the design of feedback controllers is the placement of the closed-loop poles in a desired region. The pole assignment problem is a classical one and has received a great deal of attention in the control literature. The location of the poles determines the performance of the feedback system to a certain extent. In particular, the pole location is related to the transient response of the system. From an applications viewpoint, the exact placement of the poles is not as important as their placement in a given region. Some of the well studied pole placement regions include horizontal strips, vertical strips, circles and sectors [see Anderson and Moore (1989), Furuta and Kim (1987), Gutman (1990), Haddad and Bernstein (1992), Kawasaki and Shimemura (1988), Kim and Furuta (1988), Liu and Yedavalli (1992), Saeki (1992), Shieh *et al.* (1988)]. These papers consider pole placement coupled with a linear quadratic regulator design with the exception of Liu and Yedavalli (1993), Saeki (1992) where $\mathscr{H}_\infty$ control with pole placement has been studied.

This paper considers the problem of designing controllers which minimize a cost functional while placing the eigenvalues of the closed-loop system matrix in a circular region of the left-half plane. As noted in Haddad and Bernstein (1992), the circular pole constraint region has practical significance since it places bounds on the damping ratio, the natural frequency and the damped natural frequency of the closed-loop poles. The problem of finding static state feedback controllers which minimize an LQ type cost functional and place the closed-loop poles in a circular region in the left-half plane has been considered in Furuta and Kim (1987), Kim and Furuta (1988), where a solution to this problem is given in terms of the solution to a discrete-time algebraic Riccati equation. In these papers, the authors also observe that controllers thus obtained are optimal for a certain discrete-time optimal control problem. In Haddad and Bernstein (1992), the authors proposed the problem of designing controllers for continuous-time systems which place the closed-loop poles in a circular region in the left-half plane and in addition minimize an "auxiliary" quadratic cost function. This cost function is characterized in terms of the solution to a modified Lyapunov equation and is an upper bound on the $\mathscr{H}_2$ norm of the feedback system. In Haddad and Bernstein (1992), the authors derive necessary and sufficient conditions for the existence of solution to the auxiliary cost minimization problem for the case of static output feedback and necessary conditions for the case of the dynamic fixed order (full and reduced) output feedback.

In this paper we provide a complete solution to the auxiliary cost minimization problem proposed in Haddad and Bernstein (1992). It is shown that the "auxiliary cost" introduced in Haddad and Bernstein (1992) is precisely (up to a scale factor) the integral of the square of the transfer function on the boundary of the circular pole constraint region. Thus, the auxiliary cost admits a natural discrete-time $\mathscr{H}_2$ type of interpretation. Recall that the $\mathscr{H}_2$ norm of a standard discrete-time linear time-invariant system is defined in terms of an integral on the unit circle. It also gives an intuitive reason as to why the solution to this problem involves discrete-time algebraic Riccati equations.

In the synthesis part, without making any assumptions on the order of the controller, we first solve the auxiliary cost minimization problem for the output feedback case. We show that this problem is equivalent to a discrete-time $\mathscr{H}_2$ optimal control problem subject to the constraint that the controller be strictly proper. {Note that unlike the continuous-time case, the $\mathscr{H}_2$ optimal controller for a discrete-time plant is not necessarily strictly proper [see Chen and Francis (1992)]}. Using standard results on discrete-time optimal control, it is shown that the optimal controller for the auxiliary cost minimization problem is an observer based controller, and it has a Linear Quadratic Gaussian (LQG)-type separation property. Such an observation has also been made in Haddad and Bernstein (1992) from the necessary conditions. An important consequence of this separation property is that the order of the optimal controller is no greater than that of the generalized plant (plant with weightings) in the output feedback case. We also give a solution to the auxiliary cost minimization problem for the full-information and state-feedback cases. Recall (Doyle *et al.*, 1989), full-information means both plant states and exogenous inputs are available for feedback. We show that dynamic full-information controllers do no better than static state-feedback gains. The optimal static state-feedback controller is given in terms of a standard discrete-time Linear Quadratic Regulator (LQR) gain.

The optimal controller can be obtained by solving one discrete-time algebraic Riccati equation for the state feedback case and two discrete-time algebraic Riccati equations for the output feedback case. Numerous numerical software packages are available to solve Riccati equations, which makes this synthesis technique easily implementable.

The auxiliary cost is an upper bound on the $\mathscr{H}_2$ norm of the closed-loop system. As stated above, this upper bound is

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the $\mathcal{H}_2$ norm evaluated on the boundary of the stability region. The problem of minimizing the actual $\mathcal{H}_2$ norm (instead of the auxiliary cost) with pole constraints is a challenging open problem. Nevertheless, our synthesis problem and its solution are additional tools that an engineer can use during the design process. Initially, the designer forms the desired pole constraint region with the given closed-loop transient response requirements. Since the solution guarantees the placement of the eigenvalues of the system matrix in a prespecified disk in the left-half plane, the closed-loop system automatically satisfies the pole constraint requirements. The solution allows the designer additional freedom to concentrate on satisfying other design requirements by iterating over the plant weightings. Therefore, this procedure can be used as an effective design methodology like the well known LQG design methodology.

This paper is organized as follows. In Section 2, we introduce and characterize the auxiliary cost and in Section 3, we analyze the auxiliary cost for feedback systems. This is followed by the control problem formulation, the output-feedback and state-feedback solutions in Section 4 and some concluding remarks in Section 5.

2. *Characterization of the auxiliary cost*

In this section, we define the circular pole placement region and the associated auxiliary cost. This cost will then be used in setting up the controller synthesis problem in Section 4. We characterize this auxiliary cost by Lyapunov type equations and show that the auxiliary cost for a given feedback system is the integral of the square of the transfer function on the boundary of the pole constraint region.

Consider the region $\mathcal{D}$ defined by

$$\mathcal{D} := \{z : |z + q| < r,\ q \geq r > 0\}. \tag{1}$$

It is a disk in the left-half plane with center $(-q,\ 0)$ and radius r. Let $\alpha := q - r$. We are interested in placing the eigenvalues of the system matrix of the closed-loop feedback system in $\mathcal{D}$.

Let $\mathcal{T}$ be a finite-dimensional linear time-invariant system given by the following state-space equations:

$$\mathcal{T} : \begin{cases} \dot{x} = Fx + Gw \\ z = Hx + Ew, \end{cases} \tag{2}$$

where the matrices F, G, H and E are real and of compatible dimensions. Let T_{zw} denote the transfer matrix from w to z. The system $\mathcal{T}$ is called internally $\mathcal{D}$ stable if all the eigenvalues of the system matrix F are in $\mathcal{D}$. The matrix pair $(F,\ G)$ is said to be assignable with respect to the region $\mathcal{D}$ if there exists a matrix K such that $(F + GK)$ has all the eigenvalues in the region $\mathcal{D}$ [Haddad and Bernstein (1992)]. Note that if the uncontrollable modes of $(F,\ G)$ are in $\mathcal{D}$ then it is assignable with respect to the region $\mathcal{D}$.

Now we characterize the condition that F has all its eigenvalues in $\mathcal{D}$. Though there are various equivalent ways of doing this, we present characterizations in the following lemma which we will use later in the paper.

Lemma 2.1. Consider the region $\mathcal{D}$ described by (1). Then the following statements are equivalent:
(1) The matrix F has all the eigenvalues in $\mathcal{D}$.
(2) For a given $W > 0$ there exists a (unique) $Y > 0$ such that

$$(F + \alpha I)Y + Y(F + \alpha I)' + \frac{1}{r}(F + \alpha I)Y(F + \alpha I)' + W = 0. \tag{3}$$

(3) For any G such that $(F,\ G)$ is assignable with respect to $\mathcal{D}$, there exists a (unique) $Y \geq 0$ such that

$$(F + \alpha I)Y + Y(F + \alpha I)' + \frac{1}{r}(F + \alpha I)Y(F + \alpha I)' + \frac{GG'}{r} = 0. \tag{4}$$

The above lemma can be easily proved using standard Lyapunov arguments [see Haddad and Bernstein (1992) for some details]. For the sake of brevity, the proof is omitted.

Let $\mathcal{T}$ be internally $\mathcal{D}$ stable and let L_c denote its controllability gramian, i.e. L_c is the unique solution of the Lyapunov equation

$$FL_c + L_cF' + GG' = 0. \tag{5}$$

Then, as is well known, the square of the $\mathcal{H}_2$ norm of the transfer function from w to z is defined via

$$\begin{aligned} \|T_{zw}\|_2^2 &:= \frac{1}{2\pi}\int_0^{\infty} \operatorname{trace}(T_{zw}(j\omega)T'_{zw}(-j\omega))\,\mathrm{d}w \\ &= \begin{cases} \operatorname{trace}(HL_cH') & \text{if } E = 0 \\ \infty & \text{if } E \neq 0. \end{cases} \end{aligned} \tag{6}$$

Let $Y \geq 0$ be such that (4) holds. Then from (4) and (5), it follows that

$$0 \leq L_c \leq rY.$$

Thus,

$$\|T_{zw}\|_2^2 = \operatorname{trace}(HL_cH') \leq r\operatorname{trace}(HYH').$$

The above inequality motivates the following definition of the auxiliary cost $J(T_{zw})$ for the linear time-invariant system $\mathcal{T}$:

$$J(T_{zw}) := \begin{cases} \operatorname{trace}(HYH') & \text{if } E = 0 \\ \infty & \text{if } E \neq 0. \end{cases} \tag{7}$$

Thus $\|T_{zw}\|_2^2 \leq rJ(T_{zw})$. It is easily seen that $J(T_{zw})$ is only a function of the transfer matrix T_{zw}, and does not depend on the choice of realization, as long as such a realization is internally $\mathcal{D}$ stable. This auxiliary cost $J(T_{zw})$ is the auxiliary cost in Haddad and Bernstein (1992) scaled by a factor of r. Although this auxiliary cost has been introduced as an ad-hoc upper bound on the $\mathcal{H}_2$ norm of the system, it admits a very nice transfer function interpretation in terms of an $\mathcal{H}_2$ type of integral computed on the boundary of $\mathcal{D}$ as shown below.

Theorem 2.2. Consider the continuous-time system $\mathcal{T}$ in (2) and the region $\mathcal{D}$ described by (1). Let F have all its eigenvalues in $\mathcal{D}$. Then

$$J(T_{zw}) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{trace}(T_{zw}(-q + re^{j\theta})T'_{zw}(-q + re^{-j\theta}))\,\mathrm{d}\theta, \tag{8}$$

where $T_{zw}(-q + re^{j\theta}) := T_{zw}(s)\,|_{s=-q+re^{j\theta}}$.

Proof. Since F has all its eigenvalues in $\mathcal{D}$, using Lemma 2.1 it follows that there exists a unique solution $Y \geq 0$ to (4). Also, a simple modification of (4) gives

$$\left(\frac{F + qI}{r}\right)(rY)\left(\frac{F + qI}{r}\right)' - rY + \frac{GG'}{r} = 0$$

from which it follows that

$$Y = \frac{1}{r^2}\sum_0^{\infty}\left(\frac{F + qI}{r}\right)^k GG'\left(\frac{F + qI}{r}\right)'^k.$$

Now the auxiliary cost for the system $\mathcal{T}$ is given by

$$\begin{aligned} J(T_{zw}) &= \operatorname{trace}(HYH') \\ &= \sum_0^{\infty}\operatorname{trace}\left[\frac{H}{r}\left(\frac{F + qI}{r}\right)^k GG'\left(\frac{F + qI}{r}\right)'^k \frac{H'}{r}\right] \qquad (9) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{trace}\left[\frac{H}{r}\left[e^{j\theta}I - \left(\frac{F + qI}{r}\right)\right]^{-1}\right. \\ &\qquad \left.\times\, GG'\left[e^{-j\theta}I - \left(\frac{F + qI}{r}\right)'\right]^{-1}\frac{H'}{r}\right]\mathrm{d}\theta \qquad (10) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{trace}\,[H[(-q + re^{j\theta})I - F]^{-1} \\ &\qquad \times\, GG'[(-q + re^{-j\theta})I - F']^{-1}H']\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{trace}\,(T_{zw}(-q + re^{j\theta})T'_{zw}(-q + re^{-j\theta}))\,\mathrm{d}\theta \end{aligned}$$

where Parseval's theorem is used to obtain (10) from (9). □

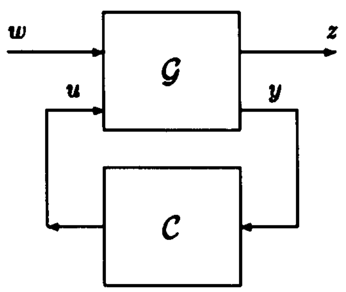

FIG. 1. Feedback interconnection of G and C.

3. *Analysis of the auxiliary cost for feedback systems*

In this section, we show that the auxiliary cost for a given continuous-time feedback system is equivalent to the square of the $\mathcal{H}_2$ norm of an associated discrete-time feedback system.

Consider the feedback system in Fig. 1, where $\mathcal{G}$ and $\mathcal{C}$ are finite dimensional linear time-invariant (FDLTI) causal plant and controller respectively. The signals w and u represent the exogenous and the control inputs, while the signals z and y represent the regulated and the measured outputs respectively. The closed-loop transfer matrix from the exogenous inputs w to the controlled outputs z is denoted as T_{zw} and the auxiliary cost defined in (7) is denoted as $J(T_{zw}) = J(\mathcal{G}, \mathcal{C})$.

Let the plant $\mathcal{G}$ in Fig. 1 be given by a finite-dimensional linear time-invariant state-space representation:

$$\mathcal{G}: \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_1 u \\ y = C_2 x + D_2 w. \end{cases} \tag{11}$$

We now introduce the following discrete-time system associated with the plant $\mathcal{G}$

$$\mathcal{G}_d: \begin{cases} x_d(k+1) = \dfrac{1}{r}(A + qI)x_d(k) + \dfrac{B_1}{r} w_d(k) + \dfrac{B_2}{r} u_d(k) \\ z_d(k) = C_1 x_d(k) + D_1 u_d(k) \\ y_d(k) = C_2 x_d(k) + D_2 w_d(k). \end{cases} \tag{12}$$

Here r and q are obtained from the description of the region $\mathcal{D}$. Let $\mathcal{C}_d$ be a finite-dimensional linear shift-invariant (FDLSI) discrete-time "controller" for $\mathcal{G}_d$ with the following state-space model

$$\mathcal{C}_d: \begin{cases} \xi_d(k+1) = \Phi \xi_d(k) + \Gamma y_d(k) \\ u_d(k) = \Theta \xi_d(k) + \Upsilon y_d(k). \end{cases} \tag{13}$$

Associated with the discrete-time system $\mathcal{C}_d$ introduce the continuous-time system

$$\mathcal{C}: \begin{cases} \dot{\xi} = r(\Phi - (q/r)I)\xi + r\Gamma y \\ u = \Theta \xi + \Upsilon y. \end{cases} \tag{14}$$

Let $T_{z_d w_d}$ denote the closed-loop transfer matrix from w_d to z_d for the feedback interconnection of $\mathcal{G}_d$ and $\mathcal{C}_d$ with the following state space representation

$$\eta(k+1) = F_d \eta(k) + G_d w_d(k)$$
$$z_d(k) = H_d \eta(k) + J_d w_d(k),$$

where $\eta(k) := [x_d(k)' \ \xi_d'(k)]'$. Let $T_{z_d w_d}$ be (discrete-time) internally stable. Let $L_d \geq 0$ be the unique solution to the Lyapunov equation

$$F_d L_d F_d' - L_d + G_d G_d' = 0.$$

Then, the square of the $\mathcal{H}_2$ norm of the discrete-time system $T_{z_d w_d}$ is defined (in the usual sense) as

$$\|T_{z_d w_d}\|_2^2 := \frac{1}{2\pi} \int_0^{2\pi} \operatorname{trace}\,(T_{z_d w_d}(e^{j\theta}) T'_{z_d w_d}(e^{-j\theta}))\, d\theta$$
$$= \operatorname{trace}\,(H_d L_d H_d') + J_d J_d'. \tag{15}$$

The following result is a key result of this paper and it establishes a connection between the continuous-time auxiliary cost control problem and $\mathcal{H}_2$ optimal control problem for an associated discrete-time system.

Theorem 3.1. Consider the plants $\mathcal{G}$ and $\mathcal{G}_d$ in (11) and (12) and the controllers $\mathcal{C}$ and $\mathcal{C}_d$ in (14) and (13) respectively. Let $\mathcal{D}$ be the desired pole constraint region as in (1). Then the feedback interconnection of $\mathcal{G}$ and $\mathcal{C}$ is internally $\mathcal{D}$ stable if and only if $\mathcal{C}_d$ internally stabilizes $\mathcal{G}_d$ in discrete-time. The auxiliary cost is finite if and only if $D_1 \Upsilon D_2 = 0$. Moreover, $J(\mathcal{G}, \mathcal{C}) = \|T_{z_d w_d}\|_2^2$.

Proof. We will first prove that if $\mathcal{C}_d$ internally stabilizes $\mathcal{G}_d$, then the feedback interconnection of $\mathcal{G}$ and $\mathcal{C}$ is internally $\mathcal{D}$ stable. The converse follows by a simple reversal of arguments. The feedback interconnection of $\mathcal{G}_d$ and $\mathcal{C}_d$ has a realization

$$\eta(k+1) = \begin{pmatrix} \frac{1}{r}[(A + qI) + B_2 \Upsilon C_2] & \frac{1}{r} B_2 \Theta \\ \Gamma C_2 & \Phi \end{pmatrix} \eta(k) + \begin{pmatrix} \frac{1}{r} B_1 \\ \Gamma D_2 \end{pmatrix} w_d(k)$$
$$z_d(k) = (C_1 + D_1 \Upsilon C_2 \ \ D_1 \Theta)\eta(k) + D_1 \Upsilon D_2 w_d(k)$$

with $\eta(k) := [x_d(k)' \ \xi_d'(k)]'$. Let

$$F_d := \begin{pmatrix} \frac{1}{r}[(A + qI) + B_2 \Upsilon C_2] & \frac{1}{r} B_2 \Theta \\ \Gamma C_2 & \Phi \end{pmatrix}.$$

Since the system $T_{z_d w_d}$ is stable, for a given $W > 0$, there exists a $Q > 0$ such that

$$\begin{pmatrix} \frac{1}{r}[(A + qI) + B_2 \Upsilon C_2] & \frac{1}{r} B_2 \Theta \\ \Gamma C_2 & \Phi \end{pmatrix} \times Q \begin{pmatrix} \frac{1}{r}[(A + qI) + B_2 \Upsilon C_2] & \frac{1}{r} B_2 \Theta \\ \Gamma C_2 & \Phi \end{pmatrix}' - Q + W = 0.$$

By a simple manipulation of the above equation we get

$$\begin{pmatrix} (A + B_2 \Upsilon C_2 + \alpha I) & B_2 \Theta \\ r\Gamma C_2 & A_c + \alpha I \end{pmatrix} \frac{Q}{r} + \frac{Q}{r} \begin{pmatrix} (A + B_2 \Upsilon C_2 + \alpha I) & B_2 \Theta \\ r\Gamma C_2 & A_c + \alpha I \end{pmatrix}'$$
$$+ \frac{1}{r} \begin{pmatrix} (A + B_2 \Upsilon C_2 + \alpha I) & B_2 \Theta \\ r\Gamma C_2 & A_c + \alpha I \end{pmatrix} \times \frac{Q}{r} \begin{pmatrix} (A + B_2 \Upsilon C_2 + \alpha I) & B_2 \Theta \\ r\Gamma C_2 & A_c + \alpha I \end{pmatrix}' + W = 0, \tag{16}$$

where $A_c = r(\Phi - (q/r)I)$. It is easy to verify that the transfer function from w to z in the feedback interconnection of $\mathcal{G}$ and $\mathcal{C}$ has the following state-space representation with $x_{cl} := (x' \ \ \xi')'$

$$\dot{x}_{cl} = \begin{pmatrix} A + B_2 \Upsilon C_2 & B_2 \Theta \\ r\Gamma C_2 & A_c \end{pmatrix} x_{cl} + \begin{pmatrix} B_1 \\ r\Gamma D_2 \end{pmatrix} w$$
$$z = (C_1 + D_1 \Upsilon C_2 \ \ D_1 \Theta) x_{cl} + D_1 \Upsilon D_2 w. \tag{17}$$

Let

$$F := \begin{pmatrix} A + B_2 \Upsilon C_2 & B_2 \Theta \\ r\Gamma C_2 & A_c \end{pmatrix}.$$

Now from (16) and Lemma 2.1 the feedback system of $\mathcal{G}$ and $\mathcal{C}$ is internally $\mathcal{D}$ stable. It is obvious that the converse of the first statement in the theorem follows by a simple reversal of the above arguments.

It is clear from (17) and the definition of the auxiliary cost that if the feedback system of $\mathcal{G}$ and $\mathcal{C}$ is internally $\mathcal{D}$ stable, the auxiliary cost $J(\mathcal{G}, \mathcal{C})$ is finite if and only if $D_1 \Upsilon D_2 = 0$. If $D_1 \Upsilon D_2 = 0$ and the feedback interconnection of $(\mathcal{G}_d, \mathcal{C}_d)$ is stable, then the square of the $\mathcal{H}_2$ norm from w_d to z_d is given by

$$\|T_{z_d w_d}\|_2^2 = \text{trace}\,[(C_1 + D_1 Y C_2 \;\; D_1\Theta)Q_d(C_1 + D_1 Y C_2 \;\; D_1\Theta)'] \tag{18}$$

where $Q_d \geq 0$ is the solution to the following Lyapunov equation

$$F_d Q_d F_d' - Q_d + \begin{pmatrix} \frac{1}{r}B_1 \\ \Gamma D_2 \end{pmatrix}\begin{pmatrix} \frac{1}{r}B_1' & D_2'\Gamma' \end{pmatrix} = 0.$$

Again by a simple manipulation of the above equation we get

$$(F + \alpha I)Q_d + Q_d(F + \alpha I)' + \frac{1}{r}(F + \alpha I)Q_d(F + \alpha I)'$$
$$+ \frac{1}{r}\begin{pmatrix} B_1 \\ r\Gamma D_2 \end{pmatrix}(B_1' \;\; rD_2'\Gamma') = 0.$$

Since the feedback interconnection of $\mathcal{G}$ and $\mathcal{C}$ is internally $\mathcal{D}$ stable, it follows from the last equation that

$$J(\mathcal{G}, \mathcal{C}) = \text{trace}\,[(C_1 + D_1 Y C_2 \;\; D_1\Theta)Q_d(C_1 + D_1 Y C_2 \;\; D_1\Theta)']. \tag{19}$$

The equivalence of the cost functions follows immediately from (18) and (19). ■

4. *The synthesis problem*

In this section, we address the controller synthesis problem. Specifically, given a plant, we give state-space formula for the controller (if one exists) that internally stabilizes the feedback system and minimizes the auxiliary cost defined in (7). Consider the feedback system in Fig. 1, where $\mathcal{G}$ is a FDLTI plant and $\mathcal{C}$ is a FDLTI controller. A controller $\mathcal{C}$ is called admissible (for the plant $\mathcal{G}$) if the closed-loop system is internally $\mathcal{D}$ stable. The set of all admissible controllers for the plant $\mathcal{G}$ is denoted as $\mathcal{A}(\mathcal{G})$.

The controller synthesis problem considered in this paper is defined as follows:

Compute the performance measure

$$v(\mathcal{G}) := \inf\{J(\mathcal{G}, \mathcal{C}) : \mathcal{C} \in \mathcal{A}(\mathcal{G})\}, \tag{20}$$

and find a controller (if it exists) $\mathcal{C} \in \mathcal{A}(\mathcal{G})$ such that $v(\mathcal{G}) = J(\mathcal{G}, \mathcal{C})$.

It should be noted that the optimization problem considered here is precisely the same as the one considered in Haddad and Bernstein (1992). We first state the main synthesis result in the general output feedback case and then give some interesting auxiliary results in the state-feedback and full-information cases.

4.1. *Output feedback problem.* In this section we solve the controller synthesis problem posed in Section 4 for the output feedback case. The solution to this case follows immediately from Theorem 3.1.

Theorem 4.1. Consider the plants $\mathcal{G}$ and $\mathcal{G}_d$ in (11) and (12) respectively. Let D_1 have full column rank and D_2 have full row rank. Then the controller $\mathcal{C}_d$ [as in (13) with $Y = 0$] internally stabilizes $\mathcal{G}_d$ and minimizes the (discrete-time) $\mathcal{H}_2$ norm of the feedback system if and only if $\mathcal{C}$ internally stabilizes $\mathcal{G}$ and minimizes the auxiliary cost $J(T_{zw})$ associated with the feedback system. Here $\mathcal{C}$ is the continuous-time system associated with $\mathcal{C}_d$ as in (14) with $Y = 0$.

Note that the Assumptions—D_1 has full column rank and D_2 has full row rank—are quite standard and it is to guarantee a nonsingular control problem. In this case, the condition (in Theorem 3.1) $D_1 Y D_2 = 0$ reduces to $Y = 0$. Thus, when only noisy output measurements are available for feedback, the controller synthesis problem can be converted to a discrete-time $\mathcal{H}_2$ optimal control problem over strictly proper controllers. The solution to this discrete-time problem is the classical LQG controller. As is well known the discrete-time optimal controller, if constrained to be strictly proper, has a nice separation property—it is a Kalman filter followed by an LQR gain [Kwakernaak and Sivan (1972)]. This controller structure has also been observed in Haddad and Bernstein (1992) for a fixed order controller derived from the necessary conditions. In this paper, this result is obtained without any *a priori* assumptions on the controller other than it being a finite dimensional causal linear time-invariant controller. Thus, the controller that minimizes the auxiliary cost can be expressed in terms of solutions to two algebraic Riccati equations. This leads to the following conceptual method to solve the output feedback optimal control problem:

Step 1. Given the continuous-time plant $\mathcal{G}$, form the equivalent discrete-time plant $\mathcal{G}_d$ as in (12).

Step 2. Solve the discrete-time $\mathcal{H}_2$ optimal control problem for $\mathcal{G}_d$ over strictly proper controllers and get the optimal controller $\mathcal{C}_d$.

Step 3. Form the controller $\mathcal{C}$ as in (14) (with $Y = 0$) and this compensator internally $\mathcal{D}$ stabilizes $\mathcal{G}$ and minimizes the auxiliary cost.

To state the precise formula for the optimal output feedback controller, we need to make the following assumptions about the plant $\mathcal{G}$.

Assumption 1. The matrix pair (A, B_2) is assignable with respect to the region $\mathcal{D}$.

Assumption 2. The matrix $\begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_1 \end{bmatrix}$ has full rank $\forall \lambda \in \partial\mathcal{D}$.

Assumption 3. D_1 has full column rank.

Assumption 4. The matrix pair (A', C_2') is assignable with respect to the region $\mathcal{D}$.

Assumption 5. The matrix $\begin{bmatrix} A - \lambda I & B_1 \\ C_2 & D_2 \end{bmatrix}$ has full rank $\forall \lambda \in \partial\mathcal{D}$.

Assumption 6. D_2 has full row rank.

Assumptions 1 and 4 guarantee the existence of a controller that internally $\mathcal{D}$ stabilizes the plant $\mathcal{G}$. The existence of stabilizing solutions to the control and filtering Riccati equations is guaranteed by Assumptions 2 and 5. As is well known in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control literature [see Doyle *et al.* (1989)], the Assumptions 3 and 6 are made to guarantee the nonsingularity of the optimal control problem.

Let $P \geq 0$ be the unique stabilizing solution to

$$\frac{(A + qI - B_2(D_1'D_1)^{-1}D_1'C_1)'}{r} P\left(I + \frac{B_2}{r}(D_1'D_1)^{-1}\frac{B_2'}{r}P\right)^{-1}$$
$$\times \frac{(A + qI - B_2(D_1'D_1)^{-1}D_1'C_1)}{r}$$
$$- P + C_1'(I - D_1(D_1'D_1)^{-1}D_1')C_1 = 0 \tag{21}$$

and let $Q \geq 0$ be the unique stabilizing solution to

$$\frac{(A + qI - B_1D_2'(D_2D_2')^{-1}C_2)}{r} Q(I + C_2'(D_2D_2')^{-1}C_2Q)^{-1}$$
$$\times \frac{(A + qI - B_1D_2'(D_2D_2')^{-1}C_2')'}{r}$$
$$- Q + \frac{B_1}{r}(I - D_2'(D_2D_2')^{-1}D_2)\frac{B_1'}{r} = 0. \tag{22}$$

The existence of solution to (21) and (22) is guaranteed by Assumptions 1–6 stated above. Define

$$K := -\left(D_1'D_1 + \frac{1}{r^2}B_2'PB_2\right)^{-1}\left(\frac{B_2'}{r}P\frac{(A + qI)}{r} + D_1'C_1\right) \tag{23}$$

$$V := (D_2D_2' + C_2QC_2')^{-1} \quad L := -\frac{1}{r}((A + qI)QC_2' + B_1D_2')V^{-1} \tag{24}$$

and the discrete-time systems (in packed matrix notation)

$$g_c := \left[\begin{array}{c|c} \frac{1}{r}(A + qI + B_2K) & I \\ \hline C_1 + D_1K & 0 \end{array}\right] \text{ and }$$
$$g_f := \left[\begin{array}{c|c} \frac{1}{r}(A + qI) + LC_2 & \frac{B_1}{r} + LD_2 \\ \hline C_1 & 0 \end{array}\right]. \tag{25}$$

Using Theorem 4.1 and the standard discrete-time output feedback optimal controller result [Chen and Francis (1992), Kwakernaak and Sivan (1972)], we next give the optimal controller formula for the auxiliary cost minimization problem.

Theorem 4.2. Consider the plant $\mathcal{G}$ in (11) and the pole constraint region $\mathcal{D}$ in (1). Let the plant $\mathcal{G}$ satisfy Assumptions 1–6 stated above. Then the dynamic output feedback controller

$$\mathcal{C}_o: \begin{cases} \dot{\xi} = A\xi + B_2 u - rL(y - C_2\xi) \\ u = K\xi, \end{cases} \tag{26}$$

satisfies $\mathcal{C}_o \in \mathcal{A}(\mathcal{G})$ and

$$v(\mathcal{G}) = J(\mathcal{G}, \mathcal{C}_o) = \|g_c L V^{1/2}\|_2^2 + \|g_f\|_2^2 = \text{trace}\,(L'PLV) + \text{trace}\,(C_1 Q C_1'), \tag{27}$$

where the matrices K, V and L are given by (23) and (24) and the systems g_c and g_f are as given in (25).

Remarks. The theorem shows that the optimal cost is achieved by a standard observer based controller. The filter and control gains are obtained by solving discrete-time algebraic Riccati equations. The optimal controller exhibits a separation property similar to the LQG optimal controller. An immediate consequence of this solution is that the order of the optimal controller is no larger than that of the plant $\mathcal{G}$.

4.2. *State-feedback and full information problems.* In this section we give some auxiliary results on the controller synthesis problem posed in Section 4 for the full-information and state feedback cases. We show that the optimal full information (and state-feedback) controller which minimizes the auxiliary cost can be chosen to be a constant state-feedback gain and it can be found by solving a discrete-time algebraic Riccati equation. This result is significant because it shows that even if complete information about the exogenous inputs is available along with the plant states, we need to search only over the set of admissible static state-feedback controllers to find the optimal controller. It also shows that dynamic compensators offer no advantages over constant gains in the full-information and state-feedback cases. This result is analogous to the corresponding results for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problem (Doyle *et al.*, 1989; Khargonekar *et al.*, 1988), the generalized $\mathcal{H}_2$ control problem (Rotea, 1993) and for a class of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems (Khargonekar and Rotea, 1991).

Consider the feedback system in Fig. 1. For the state-feedback case, $y = x$ which corresponds to $C_2 = I$ and $D_2 = 0$ the state-space representation of the plant $\mathcal{G}$ in (11). Let $\mathcal{G}_{sf}$ denote this plant with all the states available for feedback. In the full-information case, as is well known, $y = [x' \ w']'$. This corresponds to $C_2 = [I \ 0]'$ and $D_2 = [0 \ I]'$ in the state-space representation of the plant $\mathcal{G}$ in (11). Let $\mathcal{G}_{fi}$ denote this plant with all the states and disturbances available for feedback. Let $\mathcal{G}_{dfi}$, $\mathcal{C}_{dfi}$ and C_{fi} denote the corresponding full-information systems as in (12), (13) and (14) respectively.

Let the plant $\mathcal{G}_{fi}$ satisfy Assumptions 1–3 stated in the previous section. We now give the main synthesis result of this section which shows that the standard state-feedback discrete-time LQR solution (Chen and Francis, 1992; Kwakernaak and Sivan, 1972) is a solution to the auxiliary cost minimization problem in the state-feedback and full-information cases.

Theorem 4.3. Consider the plant $\mathcal{G}_{fi}$ in (11) (with $C_2 = [I \ 0]'$ and $D_2 = [0 \ I]'$) and the desired pole constraint region $\mathcal{D}$ in (1). Let Assumptions 1, 2 and 3 be satisfied. Then the optimal gain $[K \ 0]$ internally $\mathcal{D}$ stabilizes $\mathcal{G}_{fi}$ and achieves the infimum $v(\mathcal{G}_{fi})$. Here K is given by (23) and in this case

$$v(\mathcal{G}_{fi}) = v(\mathcal{G}_{sf}) = J(\mathcal{G}_{sf}, K) = \left\| g_c \left(\frac{B_1}{r}\right) \right\|_2^2 = \frac{1}{r^2}\,\text{trace}\,(B_1' P B_1), \tag{28}$$

where $P \geq 0$ is the unique stabilizing solution to (23) and the system g_c is as given in (25).

To prove the above theorem, we need some preliminary results. With the special structure of the measurement equation for $\mathcal{G}_{fi}$, the direct feedthrough term Y in (14) can be partitioned as $Y := [Y_1 \ Y_2]$ where the number of columns of Y_1 is equal to the state dimension of the system. Consider the plant $\mathcal{G}_{fi}$ in (11) and the associated discrete-time plant $\mathcal{G}_{dfi}$ in (12) (with $C_2 = [I \ 0]'$ and $D_2 = [0 \ I]'$) and the controllers $\mathcal{C}_{fi}$ and $\mathcal{C}_{dfi}$ in (14) and (13) (with $Y = [Y_1 \ Y_2]$) respectively. Using Theorem 3.1, it can be easily shown that the feedback interconnection of $\mathcal{G}_{fi}$ and $\mathcal{C}_{fi}$ is internally $\mathcal{D}$ stable if and only if $\mathcal{C}_{dfi}$ internally stabilizes $\mathcal{G}_{dfi}$ in discrete-time. Also, by observing that D_1 is full column rank and $D_1 Y D_2 = D_1 Y_2$, it is clear that the auxiliary cost is finite if and only if $Y_2 = 0$. Moreover, $J(\mathcal{G}_{fi}, \mathcal{C}_{fi}) = \|T_{z_d w_d}\|_2^2$ where $T_{z_d w_d}$ denotes the closed-loop transfer function from w_d to z_d of the feedback interconnection of $\mathcal{G}_{dfi}$ and $\mathcal{C}_{dfi}$.

So the auxiliary cost minimization problem in the full-information case is equivalent to a discrete-time $\mathcal{H}_2$ optimal control problem over all controllers which are strictly proper with respect to the input w_d. But it can be easily established that for the discrete-time $\mathcal{H}_2$ optimal control problem in the full-information case, if the controller is constrained to be strictly proper w.r.t. w_d, then dynamic full-information controller does no better than static state-feedback. This is stated in the next proposition.

Proposition 4.4. Consider the discrete-time full-information plant $\mathcal{G}_{dfi}$ as in (12) (with $C_2 = [I \ 0]'$ and $D_2 = [0 \ I]'$) and the discrete-time compensator $\mathcal{C}_{dfi}$ in (13) (with $Y = [Y_1 \ Y_2]$). Then

$$\begin{aligned} &\inf_{\mathcal{C}_{dfi}} \{\|T_{z_d w_d}\|_2 : \mathcal{C}_{dfi} \text{ as in (13) with } Y_2 = 0\} \\ &\quad = \inf_{K \in \Re^{m\times n}} \{\|T_{z_d w_d}\|_2 : \mathcal{C}_{dfi} = [K \ 0]\} \end{aligned} \tag{29}$$

where n is the state dimension of the plant $\mathcal{G}_{dfi}$ and m is the dimension of the control input u_d. The above proposition can be easily established using standard Lyapunov arguments as in Kaminer *et al.* (1993). Therefore, the proof is omitted for the sake of brevity.

It is clear from Theorem 3.1 and Proposition 4.4 that the static state-feedback gain which solves the full-information discrete-time $\mathcal{H}_2$ problem also solves the auxiliary cost minimization problem. As is well-known, this state-feedback gain is the classical discrete-time LQR compensator given by (23) and the associated optimal cost is given by (28). This proves Theorem 4.3.

5. *Conclusion*

We have shown that the auxiliary cost has a nice transfer function interpretation as an integral of the square of the transfer function over the boundary of the region $\mathcal{D}$. We have given a complete synthesis solution to the auxiliary cost minimization problem. In the output feedback case, the optimal controller is a Kalman filter followed by a gain which can be obtained by solving two discrete-time algebraic Riccati equations. In the state-feedback and the full-information cases, the optimal controller is a static state feedback gain which can be obtained by solving a discrete-time algebraic Riccati equation.

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