JOURNAL OF COMBINATORIAL THEORY, Series B 61, 71–82 (1994)

On the Decomposition of Cayley Color Graphs into Isomorphic Oriented Trees

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Received August 14, 1990

We prove that if Δ is a minimal generating set for a nontrivial group Γ and T is an oriented tree having $|\Delta|$ edges, then the Cayley color graph $D_\Delta(\Gamma)$ can be decomposed into $|\Gamma|$ edge-disjoint subgraphs, each of which is isomorphic to T; we say that $D_\Delta(\Gamma)$ is T-decomposable. This result is extended to obtain a result concerning H-decompositions of Cayley graphs for weakly connected oriented graphs H. The first result is then used to derive several theorems concerning decompositions of Cayley color graphs into prescribed families of oriented trees. Applications of some of these theorems to the verification of statements about decompositions of the n-dimensional hypercube Q_n are also discussed.

INTRODUCTION

Except where stated otherwise, the graphs and directed graphs considered in this paper may be finite or infinite, but will contain neither multiple edges nor loops. Where necessary, we have assumed the *axiom of choice*. An *oriented graph* is a directed graph that contains no symmetric pair of edges; that is, if (u, v) is an edge of the digraph D, then (v, u) is not an edge in D. Throughout the paper, we use the notation $V(D)$ and $E(D)$ to denote respectively the vertex set and edge set of a digraph D; the *size* of D is $|E(D)|$, the cardinality of $E(D)$. A $u-v$ *semipath* P in a digraph D is a finite sequence $u_0, e_1, u_1, e_2, u_2, ..., e_n, u_n$ such that each $u_i \in V(D)$, each $e_j \in E(D)$, $u = u_0$, $v = u_n$, and for each $i = 1, ..., n$, either $e_i = (u_{i-1}, u_i)$ or $e_i = (u_i, u_{i-1})$; the edge e_i is called a *forward edge* of P if $e_i = (u_{i-1}, u_i)$ and a *backward edge* of P if $e_i = (u_i, u_{i-1})$. A digraph D is *weakly connected* if it contains a $u-v$ semipath for every pair of distinct vertices u and v. A *weak component* of D is a subdigraph of D that is maximal with respect to the property of being weakly connected.

An *oriented tree* is an oriented graph whose underlying graph is a tree (finite or infinite). Since a tree contains a unique $u-v$ path for every pair

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of its vertices u and v, it follows that if T is an oriented tree, then T contains a unique $u-v$ semipath for every pair of vertices u and v in T. This property of oriented trees will be used extensively.

A *decomposition* of a directed graph D with at least one edge is a set $\mathscr{P}$ of pairwise edge-disjoint subdigraphs of D such that the set $\{E(P) \mid P \in \mathscr{P}\}$ is a partition of the edge set of D. Each member subdigraph of a decomposition of D is called a *part* of the decomposition. A decomposition in which all of the parts are isomorphic to a particular digraph H is called an *H-decomposition* of D. Furthermore, a digraph D that admits an H-decomposition is said to be *H-decomposable*. If each part of a decomposition of D is isomorphic to an element of a specified set $\mathscr{S}$ of digraphs and each element of $\mathscr{S}$ is isomorphic to a part of the decomposition, we call the decomposition an *$\mathscr{S}$-decomposition* and say that D is *$\mathscr{S}$-decomposable*. The obvious analogous definitions apply to graphs (rather than digraphs), and much work has been done relating to decompositions of graphs (see, e.g., [4, 6, 7, 9, 10, 13, 15]).

Let Γ be a nontrivial group and let Δ be a generating set for Γ. The *Cayley color graph of Γ with respect to Δ*, denoted $D_\Delta(\Gamma)$, is the digraph whose vertex set is Γ and whose edge set is defined as follows: for $\alpha, \beta \in \Gamma$ the (directed) edge (α, β) is in $D_\Delta(\Gamma)$ if and only if $\alpha g = \beta$ for some generator $g \in \Delta$. We often regard Δ as a set of colors and say that edge (α, β) has color g $(\in \Delta)$ if and only if $\alpha g = \beta$. If K is a subgroup of Γ we use the standard notation $[\Gamma : K]$ to denote the index of the subgroup in Γ. If $A \subseteq \Gamma$, the notation $\langle A \rangle$ denotes the subgroup of Γ generated by A. Note that we shall also use $\langle A \rangle$ to denote the subdigraph induced by a subset A of either the edge set or vertex set of the digraph under discussion; the meaning will be clear from the context.

For the meanings of graph theoretic terms not defined here and for basic graph-theoretic results used herein, we refer the reader to [2, 12, 14]. For basic group theoretic terms and results we suggest [5, 8, 11].

As indicated by the title of this paper, we shall be interested in T-decompositions of Cayley color graphs where T is an oriented tree. This line of investigation seemed natural in light of the following theorem which was obtained independently and nearly simultaneously by Fink [6] and Ramras [13].

THEOREM A. *The n-cube (i.e., hypercube) Q_n is T-decomposable for every (undirected) tree T of size n. Furthermore, there is a T-decomposition of Q_n each part of which is an induced subgraph of Q_n.*

One may see the connection between the n-cube Q_n and decompositions of Cayley color graphs by observing that Q_n is the underlying (simple) graph of the Cayley color graph $D_\Delta(\Gamma)$ when $\Gamma = (Z_2)^n$ and $\Delta = \{(1, 0,$

0, ..., 0), (0, 1, 0, ..., 0), (0, 0, 1, ..., 0), ..., (0, 0, 0, ..., 1)}. Note that the above theorem gives a best possible answer to one instance of the following question: If G is a specified graph that contains as a subgraph every tree T of size n, and n divides the size of G, is G T-decomposable for every tree T of size n? It is well known that any graph whose minimum degree is n contains every tree of size n. It is also true that if D is a digraph whose minimum indegree and minimum outdegree are each at least n, then D contains every oriented tree of size n. When Δ is an n-element generating set for a group Γ, the Cayley color graph $D_\Delta(\Gamma)$ is regular with indegree and outdegree both equal to n. Thus, it seems natural to ask whether $D_\Delta(\Gamma)$ is T-decomposable for every tree T of size n. We shall show that, regardless of whether $|\Gamma|$ is finite or infinite, the Cayley color graph $D_\Delta(\Gamma)$ is T-decomposable whenever Δ is a minimal generating set for Γ and T is any oriented tree having size $|\Delta|$. We shall then prove several related results.

Decomposing the Cayley Color Graphs

In the proofs of the decomposition theorems that follow, we will make use of the following two basic results about Cayley color graphs (see White [14] for similar statements).

LEMMA 1. *For any given element g of a group Γ with generating set Δ, the mapping $\varphi_g\colon V(D_\Delta(\Gamma)) \to V(D_\Delta(\Gamma))$ defined by $\varphi_g(v) = gv$ is an automorphism of $D_\Delta(\Gamma)$ that preserves edge colors. Moreover, the correspondence $g \leftrightarrow \varphi_g$ determines an isomorphism of Γ with the color-preserving automorphism group of $D_\Delta(\Gamma)$.*

LEMMA 2. *If A is a nonempty subset of a minimal generating set Δ for a nontrivial group Γ, and D_α is the subdigraph induced by those edges of $D_\Delta(\Gamma)$ whose colors belong to A, then D_α is a disconnected digraph with $[\Gamma : \langle A \rangle]$ weak components, each of which is isomorphic to the Cayley color graph $D_A(\langle A \rangle)$. Moreover, the vertex sets of these weak components are the left cosets of $\langle A \rangle$.*

The following theorem will serve as the foundation for all other results in this paper.

THEOREM 1. *If Γ is a nontrivial group with minimum generating set Δ, and T is an oriented tree of size $|\Delta|$, then the Cayley color graph $D_\Delta(\Gamma)$ is T-decomposable. Furthermore, a T-decomposition exists for which, if b is any specified vertex of T, then each vertex of $D_\Delta(\Gamma)$ assumes the role of b in exactly one part of the T-decomposition.*

Proof. Let $\chi: E(T) \to \Delta$ be a bijection; thus, χ colors the edges of T in such a way that each edge e of T receives a distinct color $\chi(e)$, and all colors of Δ are used. For each pair of vertices u and v, let $P_{u,v}$ denote the unique $u-v$ semipath in T. Let b be a fixed vertex of T. We define a vertex-labeling mapping $\lambda: V(T) \to \Gamma$ as follows. If $v \in V(T)$ and $P_{b,v}$ has the form $b = u_0, e_1, u_1, e_2, u_2, ..., e_n, u_n = v$, define $\lambda(v) = g_1 g_2 \cdots g_n$, where $g_i = \chi(e_i)$ if e_i is a forward edge of $P_{b,v}$ and $g_i = [\chi(e_i)]^{-1}$ if e_i is a backward edge of $P_{b,v}$; define $\lambda(b) = e$, where e denotes the identity element of Γ. Since the mapping χ is one-to-one and Δ is a minimal generating set for Γ, the mapping λ is one-to-one; i.e., each vertex v of T has a distinct label $\lambda(v)$. Note that we may interpret λ as a color-preserving mapping from the edge-colored digraph T into the edge-colored digraph $D_\Delta(\Gamma)$. Under this interpretation, $T \cong \lambda(T)$. Let T_e denote the subdigraph $\lambda(T)$.

Now, for each $g \in \Gamma$ define T_g to be the subdigraph that is the image of T_e under the automorphism φ_g defined by $\varphi_g(v) = gv$ for all $v \in \Gamma$. We claim that if $g \neq h$, then $E(T_g) \cap E(T_h)$ is empty. To see this, we note that for each $\delta \in \Delta$, each of T_g and T_h contains exactly one edge colored with the generator δ. If u is the terminal vertex of the edge labeled δ in T_e, then the terminal vertices of the edges colored δ in T_g and T_h are gu and hu, respectively. Thus, if T_g and T_h share an edge of color δ, we have $gu = hu$, whence $g = h$. Thus $\{T_g: g \in \Gamma\}$ is a set of edge-disjoint subgraphs of $D_\Delta(\Gamma)$ each isomorphic to the oriented tree T. Also, if (v_1, v_2) is an edge of $D_\Delta(\Gamma)$ that has color δ and (u_1, u_2) is the unique edge of T_e having color δ, then $(v_1, v_2) = \varphi_g(u_1, u_2)$, where $g = v_1 u_1^{-1}$; thus $\{E(T_g) \mid g \in \Gamma\}$ is a partition of $E(D_\Delta(\Gamma))$. We conclude that $\{T_g: g \in \Gamma\}$ is a T-decomposition of $D_\Delta(\Gamma)$. Note also that for each vertex $g \in \Gamma$ we have $g = \varphi_g(\lambda(b))$; i.e., g assumes the role of vertex b in exactly one part of the T-decomposition. ∎

The techniques used in proving Theorem 1 can be extended somewhat to answer a more general question. Several authors (see e.g., [6, 9, 14]) have asked whether, for a given graph (digraph) H, there exist regular graphs (digraphs) of a given type that are H-decomposable. The following result answers the question for weakly connected oriented graphs H and Cayley color graphs.

THEOREM 2. *If H is a weakly connected oriented graph of size σ, then there is a group Γ having a generating set Δ of cardinality σ such that the Cayley color graph $D_\Delta(\Gamma)$ is H-decomposable.*

Proof. Let T be a spanning oriented tree of H, and let Γ be a group having a minimal generating set Δ_0 such that $|\Delta_0| = |E(T)|$. Let b be a vertex of T, and for each pair of vertices u and v of T let $P_{u,v}$ be the unique

$u-v$ semipath in T. Define a bijection $\chi: E(T) \to \Delta_0$ and the one-to-one function $\lambda: V(T) \to \Gamma$ as in the proof of Theorem 1. We extend the mapping χ to the larger domain $E(H)$ and a larger range Δ as follows: For each pair of vertices u and v in T, if the semipath $P_{u,v}$ has the form $(u=)u_0, e_1, u_1, e_2, u_2, ..., e_n, u_n(=v)$, define $\rho(P_{u,v}) = g_1 g_2 \cdots g_n$, where $g_i = \chi(e_i)$ if e_i is a forward edge of $P_{u,v}$ and $g_i = [\chi(e_i)]^{-1}$ if e_i is a backward edge of $P_{u,v}$. For each edge $(u, v) \in E(H)$, define $\chi(u, v) = \rho(P_{u,v})$. Let $\Delta = \{\chi(u, v) \mid (u, v) \in E(H)\}$.

We claim that $\chi: E(H) \to \Delta$ is an injection. Let (u_1, v_1) and (u_2, v_2) be distinct edges in H and suppose that $\chi(u_1, v_1) = \chi(u_2, v_2)$. Letting P_1 and P_2 denote respectively the unique $u_1 - v_1$ and $u_2 - v_2$ semipaths in T, we then have $\rho(P_1) = \rho(P_2)$. Assume without loss of generality that P_2 contains an edge not on P_1. Since T is an oriented tree, either the first edge or the last edge of P_2 is not on P_1. If the first edge f_1 of P_2 is not on P_1 and $f_1 = (u_2, x)$, then $\rho(P_2) = \chi(f_1)\, \rho(P_{x,v_2})$ and $\rho(P_2) = \rho(P_1)$, so $\chi(f_1) = \rho(P_1)[\rho(P_{x,v_2})]^{-1}$; that is, $\chi(f_1) \in \langle \Delta_0 \setminus \{\chi(f_1)\} \rangle$, a contradiction to the minimality of Δ_0. If the first edge f_1 of P_2 is not on P_1 and $f_1 = (x, u_2)$, we obtain the same contradiction since, in this case, $\chi(f_1) = \rho(P_{x,v_2})[\rho(P_1)]^{-1}$. Similar contradictions arise in the cases when the last edge of P_2 is not on P_1. Thus, $\chi: E(H) \to \Delta$ is an injection.

Now, as in the proof of Theorem 1, we interpret the mapping λ as a color-preserving mapping of the oriented graph H into the Cayley color graph $D_\Delta(\Gamma)$, and we let H_e denote the subgraph that is the image of H. Now, we again consider the images of H_e under the automorphisms φ_g and show, as in the proof of Theorem 1, that the set of subgraphs $\{\varphi_g(H_e) \mid g \in \Gamma\}$ is a decomposition of $D_\Delta(\Gamma)$. Again we remark that each vertex of $D_\Delta(\Gamma)$ assumes the role of any specified vertex of H in exactly one part of the decomposition. ▮

We now turn to the question of $\mathscr{S}$-decompositions for various families $\mathscr{S}$ of trees. To simplify the discussion, we introduce a few new definitions.

For a subdigraph H of a digraph D, we define the *boundary* of H, denoted $\partial(H)$, to be the set of all vertices in H that are adjacent with at least one vertex of D not in H. We define the *interior* of H to be the set $\text{int}(H) = V(H) - \partial H$. If F is an oriented tree and $\mathscr{S}$ is a family of oriented trees for which

(i) each element T_i of $\mathscr{S}$ has a subdigraph F_i isomorphic with F and

(ii) F has a vertex b, called the *bud*, such that if b_i is the vertex of F_i corresponding to b, then $\partial(F_i) = \{b_i\}$,

then we say that the oriented trees of $\mathscr{S}$ have *common rootstock* F. Figure 1 shows an oriented tree F of size three and two oriented trees T_1 and T_2 that have common rootstock F.

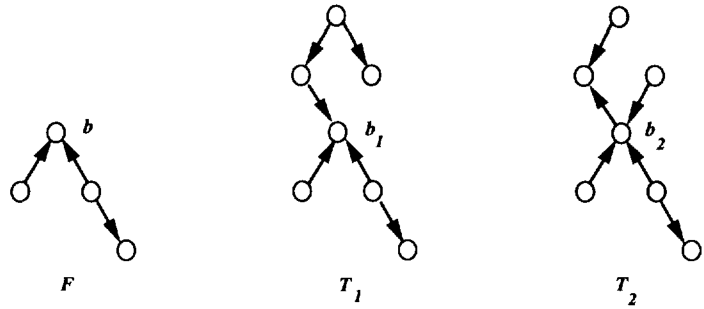

FIGURE 1

THEOREM 3. *Let Γ be a nontrivial group having a nonsingleton minimal generating set Δ and let $\{A, B\}$ be a partition of Δ. Let $\mathscr{S} = \{T_i \mid i \in I\}$ be a family of oriented trees, each of size $|\Delta|$, having a common rootstock F of size $|B|$, and assume further that the indexing set I of $\mathscr{S}$ has cardinality $|I| = [\Gamma : \langle A \rangle]$. Then $D_\Delta(\Gamma)$ is $\mathscr{S}$-decomposable.*

Proof. For each edge f of $D_\Delta(\Gamma)$ let $\chi(f)$ denote the color of f. Furthermore, define the subdigraphs $D_\alpha = \langle \{f \in E(D_\Delta(\Gamma)) \mid \chi(f) \in A\} \rangle$ and $D_\beta = \langle \{f \in E(D_\Delta(\Gamma)) \mid \chi(f) \in B\} \rangle$ of $D_\Delta(\Gamma)$. Note that $\{D_\alpha, D_\beta\}$ is a decomposition of $D_\Delta(\Gamma)$. For each $T_i \in \mathscr{S}$, let F_i denote the subgraph isomorphic to F, and let b_i denote the vertex of F_i corresponding to b; we will refer to F_i as the rootstock of T_i and to b_i as the bud of T_i. Define $T_i^* = T_i - \operatorname{int}(F_i)$ for each $i \in I$, and set $\mathscr{S}^* = \{T_i^* \mid i \in I\}$.

By Lemma 2, D_α is disconnected and has $[\Gamma : \langle A \rangle] = |I|$ weak components C_i, $i \in I$, each isomorphic to $D_A(\langle A \rangle)$. Thus, by Theorem 1, C_i is T_i^*-decomposable for each $i \in I$; furthermore, this can be done in such a way that each vertex of C_i assumes the role of b_i in exactly one part of the decomposition. Thus, D_α is $\mathscr{S}^*$-decomposable in such a way that each vertex assumes the role of the bud b_i of exactly one oriented tree T_i^* in $\mathscr{S}^*$.

Also by Lemma 2, D_β has $[\Gamma : \langle B \rangle]$ weak components, each isomorphic with $D_B(\langle B \rangle)$. Thus, by Theorem 1, D_β is F-decomposable in such a way that each vertex of D_β assumes the role of the bud b of exactly one part of the F-decomposition.

Now, each vertex $v \in \Gamma$ assumes the role of a bud b_i in exactly one part of the $\mathscr{S}^*$-decomposition of D_α (and for exactly one $i \in I$); call this part T_v^*. Also, v assumes the role of the bud b of F in exactly one part, say F_v, of the F-decomposition of D_β. Thus, we can adjoin the (necessarily edge-disjoint) parts T_v^* and F_v at vertex v to obtain a subdigraph T_v of size $|\Delta|$ in $D_\Delta(\Gamma)$. Since Δ is a minimal generating set for Γ and since T_v is weakly connected and has exactly one edge labeled δ for each $\delta \in \Delta$, T_v is an oriented tree. Furthermore, T_v is isomorphic with a tree in $\mathscr{S}$. Also, for each $T_i \in \mathscr{S}$, there is a vertex $v \in \Gamma$ such that T_i is isomorphic with T_v. Since T_u^*, T_v^*, F_u, and F_v are edge-disjoint for all vertices $u \neq v$, we see that T_u and T_v are edge-disjoint. Thus, since $\{D_\alpha, D_\beta\}$ is a decomposition of $D_\Delta(\Gamma)$, we conclude that $\{T_v \mid v \in \Gamma\}$ is an $\mathscr{S}$-decomposition of $D_\Delta(\Gamma)$. ∎

Although the hypothesis of Theorem 3 seems somewhat restrictive, we note that in many cases it actually allows a good bit of freedom in our choice of a family $\mathscr{S}$. A particular case that illustrates this is the following: Let $\mathscr{S}$ be a countably infinite family of oriented trees, each of size $\aleph_0$, and assume only that each of these oriented trees has an end-vertex of indegree one (so that F is an orientation of K_2). Let Γ be the group $\mathbb{Q}^+$ of positive rationals under multiplication, and let Δ be the set of all primes. If we take $B = \{2\}$ and $A = \Delta \backslash \{2\}$ and then apply Theorem 3, we see that $D_\Delta(\Gamma)$ is $\mathscr{S}$-decomposable. In fact; since $D_\Delta(\Gamma)$ is itself an oriented graph in this case, we see that its underlying undirected graph G is $\mathscr{S}$-decomposable for any countable family $\mathscr{S}$ of trees of size $\aleph_0$, so long as each tree has an end-vertex.

The hypothesis of Theorem 3 required that each tree T_i in the family $\mathscr{S}$ have a common weakly connected subdigraph F that is "attached to T_i" in a particular way. In the hypothesis of Theorem 4 we will relax the restrictions on the "mode of attachment" of certain subdigraphs, but we will impose other restrictions on the structural complexity of the subdigraph being attached. Let $\mathscr{S}$ be a family of oriented trees, each of size n, and let s_1 and s_2 be nonnegative integers such that $1 \leqslant s_1 + s_2 \leqslant n$. We say that $\mathscr{S}$ is an (s_1, s_2)-*similar family* if from each oriented tree T in $\mathscr{S}$ we can identify a set U consisting of s_1 end-vertices of indegree one (each such is called a *sink-leaf*) and s_2 end-vertices with outdegree one (each of these being a *source-leaf*); each vertex of $V(T) \backslash U$ will be called a *core vertex* of T.

THEOREM 4. *Let Δ be a minimal generating set for a finite group Γ, let B be a nonempty proper subset of Δ, let $A = \Delta \backslash B$, and let s_1 and s_2 be nonnegative integers such that $s_1 + s_2 = |B|$. Then, if $\mathscr{S} = \{T_i \mid i = 1, ..., [\Gamma : \langle A \rangle]\}$ is an (s_1, s_2)-similar family of oriented trees, each of size $|\Delta|$, the digraph $D_\Delta(\Gamma)$ is $\mathscr{S}$-decomposable.*

Proof. We proceed by induction on $|B|$, proving the statement $Q(k)$: For any choices of Δ, Γ, A, B, s_1, s_2, and $\mathscr{S}$ satisfying the above hypotheses and the condition that $|B| = k$, there is an $\mathscr{S}$-decomposition of $D_\Delta(\Gamma)$ satisfying the condition "K" below:

K. Corresponding to each vertex u of the digraph, there is a unique value of i such that, for each core vertex x of T_i, u assumes the role of x

in exactly one part of the decomposition and, furthermore, if $j \neq i$, then u assumes the role of no core vertex of any part isomorphic to T_j.

If $|B| = 1$, then the oriented trees in $\mathscr{S}$ have a common rootstock F of size one (namely an oriented K_2). Thus, by Theorem 3, $D_\Delta(\Gamma)$ is $\mathscr{S}$-decomposable. A review of the proof of Theorem 3 will convince the reader that the $\mathscr{S}$-decomposition constructed in that proof satisfies condition K. Thus, $Q(1)$ is true.

Assume now that $Q(n-1)$ is true $(n \geq 2)$, and let Δ, F, A, B, s_1, s_2, and $\mathscr{S}$ be chosen to satisfy both the required hypotheses and the condition that $|B| = n$.

Let $g \in B$ and define $B' = B \backslash \{g\}$, $\Delta' = \Delta \backslash \{g\}$, and $\Gamma' = \langle \Delta' \rangle$. Also, let D be the subdigraph of $D_\Delta(\Gamma)$ induced by those edges whose color is in Δ'. By Lemma 2, D is disconnected and has $\gamma = [\Gamma : \Gamma']$ weak components $C_1, ..., C_\gamma$, each isomorphic with $D_{\Delta'}(\Gamma')$. For each T_i in $\mathscr{S}$, let U_i be a collection of s_1 sink-leaves and s_2 source-leaves. Without loss of generality, assume that each oriented tree T_i in $\mathscr{S}$ has a sink-leaf v_i in U_i; let u_i be the unique vertex of T_i adjacent to v_i. Define $T_i^* = T_i - v_i$ and let $\mathscr{S}^* = \{T_i^* \mid i = 1, ..., [\Gamma : \langle A \rangle]\}$. Then $\mathscr{S}^*$ is an $(s_1 - 1, s_2)$-similar family of oriented trees, each of size $|\Delta'|$. Since $[\Gamma : \langle A \rangle] = [\Gamma : \Gamma'] \cdot [\Gamma' : \langle A \rangle]$, there is a partition $\{\mathscr{S}_j^* \mid j = 1, ..., [\Gamma : \Gamma']\}$ of $\mathscr{S}^*$, where $|\mathscr{S}_j^*| = [\Gamma' : \langle a \rangle]$ for each j. Since $|B'| = n - 1$, the induction hypothesis guarantees that there is, for each j, an $\mathscr{S}_j^*$-decomposition of component C_j of D that satisfies condition K (with Δ and Γ replaced by Δ' and Γ'). Thus, there is an $\mathscr{S}^*$-decomposition of D that satisfies condition *K*.

Now, for each vertex u of D, let i be the unique corresponding value discussed in condition K and let $P^*(u)$ be the part of the $\mathscr{S}^*$-decomposition in which u assumes the role of u_i. Also, let v be the unique vertex of $D_\Delta(\Gamma)$ such that edge (u, v) has color g. Since Δ is a minimal generating set for Γ, the vertices u and v are in different components of D. Thus, we may append the edge (u, v) and the vertex v to $P^*(u)$ to obtain a subdigraph $P(u)$ isomorphic to T_i. Now, the set $\{P(u) \mid u \in \Gamma\}$ is an $\mathscr{S}$-decomposition of $D_\Delta(\Gamma)$ satisfying condition K. Thus, $Q(n)$ is true and the theorem follows by the principle of mathematical induction. ∎

In each of Theorems 3 and 4, the decompositions obtained satisfied the property that, for each member T of $\mathscr{S}$, there were at least two (and for Theorem 3, perhaps infinitely many) parts isomorphic to T. In our final theorem we look at a situation where we conjecture that, under suitable cardinality conditions, each pair of parts could be nonisomorphic; counting arguments might show this is not possible if the Cayley color graph in question is finite.

Recall that the (external) weak direct product of a family $\mathscr{F} = \{\Gamma_k \mid k \in \mathrm{I}\}$ of nontrivial groups, denoted $\Pi^w \mathscr{F}$, is the set of all functions

$f: \mathbf{I} \to \bigcup \mathscr{F}$ such that (i) $f(k) \in \Gamma_k$ and (ii) $f(k) = e_k$ (the identity element of Γ_k) for all but finitely many $k \in \mathbf{I}$. If the indexing set $\mathbf{I}$ is finite, the weak direct product and the Cartesian product are identical. Also, if $\mathscr{D} = \{\Delta_k \mid k \in \mathbf{I}\}$ is a family of minimal generating sets for the groups in $\mathscr{F}$, then the set $\Delta = \{f \in \Pi^w \mathscr{F} \mid f(k) \in \Delta_k$ for exactly one value $k \in I$ and $f(j) = e_j$ for all $j \neq k\}$ is a minimal generating set for $\Pi^w \mathscr{F}$; we shall call Δ the *standard minimal generating set for* $\Pi^w \mathscr{F}$ *determined by* $\mathscr{D}$.

For each $k \in \mathbf{I}$ define the relation $\sim_k$ on $\Pi^w \mathscr{F}$ as follows: for all $f, g \in \Pi^w \mathscr{F}$, we write $f \sim_k g$ if and only if $f(i) = g(i)$ for all $i \in \mathbf{I} \setminus \{k\}$. Then $\sim_k$ is an equivalence relation on $\Pi^w \mathscr{F}$. Furthermore, the set $A_k = \{g \mid g(k) = e_k\}$ is a system of distinct representatives of the set of equivalence classes determined by $\sim_k$. For each $f \in \Pi^w \mathscr{F}$ and each $k \in \mathbf{I}$, let $[f]_k$ denote the equivalence class of f determined by $\sim_k$ and let $A_k(f)$ be the unique element of $[f]_k$ that is in A_k. Note that the subdigraph $\langle [f]_k \rangle$ of $D_\Delta(\Pi^w \mathscr{F})$ is isomorphic to $D_{\Delta_k}(\Gamma_k)$. Moreover, the subdigraph $G_k = \bigcup_{g \in A_k} \langle [g]_k \rangle$ is a spanning subdigraph of $D_\Delta(\Pi^w \mathscr{F})$, each component of which is isomorphic to $D_{\Delta_k}(\Gamma_k)$.

Now, for each $k \in \mathbf{I}$, let $\mathscr{T}_k = \{T_{k,g} \mid g \in A_k\}$ be a family of oriented trees, each of size $|\Delta_k|$, and let $\mathscr{T} = \bigcup_{k \in \mathbf{I}} \mathscr{T}_k$. Also, for each $k \in \mathbf{I}$ and each $g \in A_k$, choose a vertex of $T_{k,g}$ and label it $u_k(g)$. Having done this, we construct an oriented tree T_f for each $f \in \Pi^w \mathscr{F}$ by taking the threes in the subset $\{T_{k, A_k(f)} \mid k \in \mathbf{I}\}$ of $\mathscr{T}$ and identifying the vertices $\{u_k(A_k(f)) \mid k \in \mathbf{I}\}$ to form a single vertex u_f which we shall call the *articulation point* of T_f. The family $\mathscr{S} = \{T_f \mid f \in \Pi^w \mathscr{F}\}$ of oriented trees constructed in this way will be called an $(\mathscr{F}, \mathscr{D})$-product family.

THEOREM 5. *Let* $\mathscr{F} = \{\Gamma_k \mid k \in \mathbf{I}\}$ *be a family of nontrivial groups, let* $\mathscr{D} = \{\Delta_k \mid k \in \mathbf{I}\}$ *be a family of minimal generating sets for the groups in* $\mathscr{F}$*, and let* Δ *be the standard minimal generating set for* $\Pi^w \mathscr{F}$ *determined by* $\mathscr{D}$*. Then* $D_\Delta(\Pi^w \mathscr{F})$ *is* $\mathscr{S}$*-decomposable for every* $(\mathscr{F}, \mathscr{D})$*-product family* $\mathscr{S}$.

Proof. We use the notation introduced prior to the statement of the theorem. As noted earlier, for each $k \in \mathbf{I}$, the subdigraph $G_k = \bigcup_{g \in A_k} \langle [g]_k \rangle$ is a spanning subdigraph of $D_\Delta(\Pi^w \mathscr{F})$, each component of which is isomorphic to $D_{\Delta_k}(\Gamma_k)$. Furthermore, if $i, j \in \mathbf{I}$ and $i \neq j$, then G_i and G_j are edge-disjoint. Thus $\mathscr{G} = \{G_k \mid k \in \mathbf{I}\}$ is a decomposition of $D_\Delta(\Pi^w \mathscr{F})$. We now use $\mathscr{G}$ to obtain an $\mathscr{S}$-decomposition of $D_\Delta(\Pi^w \mathscr{F})$.

By Theorem 1, there is, for each $k \in \mathbf{I}$ and each $g \in A_k$, a $T_{k,g}$-decomposition of the subdigraph $\langle [g]_k \rangle$ in which each vertex assumes the role of $u_k(g)$ in exactly one part of the decomposition. Recall also that the $T_{k,g}$-decomposition of $\langle [g]_k \rangle$ has the property that the edge labels of each part are in one-to-one correspondence with the elements of Δ_k. Consider the $\mathscr{T}_k$-decompositions of the G_k's determined by these componentwise decompositions.

Since Δ is a minimal generating set of $\Pi^w\mathscr{F}$, at each vertex f of $D_\Delta(\Pi^w\mathscr{F})$, the parts of the G_k-decompositions isomorphic to the trees $\{T_{k, A_k(f)} \mid k \in \mathbf{I}\}$ with each $u_k(A_k(f))$ located at f must have only the vertex f in common; call the union of these parts P_f. Then P_f is isomorphic to T_f for each $f \in \Pi^w\mathscr{F}$, and the family $\{P_f \mid f \in \Pi^w\mathscr{F}\}$ is an $\mathscr{S}$-decomposition of $D_\Delta(\Pi^w\mathscr{F})$. ∎

Applications to Q_n

Several results concerning decompositions of the n-cube Q_n can be had as consequences of work presented in the previous section. The central idea behind applying these theorems to tree-decompositions of the n-cubes is that of introducing a *bipartite orientation* to each tree; that is, orienting each tree so that all vertices in one partite set are sources and all vertices in the other partite set are sinks. One then exploits the facts that Q_n is a bipartite graph that exhibits a high degree of symmetry. We illustrate this below, and obtain a new proof of Theorem A.

New Proof of Theorem A. Let T be a tree of size n and let b be a vertex of T. Recall that every tree is bipartite. Orient the edges of T so that all edges are directed from the partite set of T which contains b to the other; call the resulting oriented tree T_0. By Theorem 1, there is a T_0-decomposition of $D_\Delta(\Gamma)$, with $\Gamma = (\mathbb{Z}_2)^n$ and $\Delta = \{(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), (0, 0, 1, ..., 0), ..., (0, 0, 0, ..., 1)\}$, such that every vertex of $D_\Delta(\Gamma)$ assumes the role of b in exactly one part of the decomposition. The subdigraph H of $D_\Delta(\Gamma)$ induced by all those edges whose initial vertex (n-tuple) has even Hamming weight (i.e., and even number of ones) is precisely the subdigraph induced by all the 2^{n-1} parts of the T_0-decomposition for which the role of b is assumed by a vertex of even Hamming weight. Since H is merely an orientation of Q_n with all edges directed from one partite set to the other, the theorem follows. ∎

It is easy to modify the above proof to obtain the following result concerning infinite trees and an infinite version of the cube. If we take Γ to be the group of all binary sequences that are finitely nonzero and define Q_∞ to be the graph having vertex set Γ and having two vertices adjacent if and only if they differ in exactly one coordinate, then Q_∞ is T-decomposable for every tree T having $\aleph_0$ edges.

As one would expect, there is also an undirected analogue to Theorem 3 that applies to $\mathscr{S}$-decompositions of n-cube. This analogue can be easily proved as a corollary to Theorem 3 using techniques similar to the above proof of Theorem A. We refer the reader to the second theorem in [6] for the statement of the result and an alternate proof.

Theorem 4 also has its analogue for n-cubes. We say that a family $\mathscr{S}$ of trees is *s-similar* if every tree in $\mathscr{S}$ has at least s end-vertices. Now, using bipartite orientations, we can show that if $\mathscr{S}$ is an s-similar family of 2^{s-1} trees, each of size n, where $2 \leqslant s \leqslant n$, then Q_n is $\mathscr{S}$-decomposable.

Finally, to apply Theorem 5 to Q_n, one selects a partition $n_1 + n_2 + \cdots + n_p = n$ of n, takes $\mathscr{F} = \{(\mathbb{Z}_2)^{n_1}, (\mathbb{Z}_2)^{n_2}, ..., (\mathbb{Z}_2)^{n_p}\}$, Δ_k as the standard basis for $(\mathbb{Z}_2)^{n_k}$, $\mathscr{T}_k$ to be family of trees with bipartite orientations, and for each $g \in \Delta_k$, takes vertex $u_k(g)$ to be a source vertex in $T_{k,g}$. Then each tree T_f has a bipartite orientation in which the articulation point u_f is a source. Now, taking $\mathscr{S}$ to be the family of trees underlying the oriented trees T_f such that f is a vertex of even Hamming weight, we see that Q_n is $\mathscr{S}$-decomposable. We encourage the reader to work through a nontrivial example of such a decomposition of Q_n in order to appreciate the diversity possible among the trees in sets $\mathscr{S}$ constructed in this manner.

Concluding Remarks

The thoughtful reader will find that many questions remain regarding $\mathscr{S}$-decompositions of Cayley color graphs and n-cubes. In this paper, we have restricted the investigation to decompositions in which each part was *fully chromatic* in the sense that the edge set of each part was in one-to-one correspondence with the set of colors. By doing this, we were able to exploit the power inherent in elementary group theory. One might reasonably ask questions about decompositions into parts of various types that are not required to be fully chromatic. In fact this has been done by several people who were interested in decomposing Cayley graphs into hamiltonian cycles; e.g., see [3, 16]. Such questions may be quite difficult to answer since some group-theoretic power may be sacrified. Also, in [1], Alspach asked whether every circulant graph, Cayley color graph, or vertex-transitive graph has an isomorphic decomposition into $t > 0$ parts whenever t divides the size of the graph. Perhaps the ideas used in this paper could help answer Alspach's question regarding the Cayley color graphs. We close with three questions concerning decompositions of Cayley color graphs into fully chromatic parts; the first is very general.

Question 1. Given a nontrivial group Γ and a minimal generating set Δ, what (nontrivial) conditions must be satisfied by a family $\mathscr{S}$ of oriented trees of size $|\Delta|$ in order that the Cayley color graph $D_\Delta(\Gamma)$ is $\mathscr{S}$-decomposable into fully chromatic parts?

Question 2. For a given group Γ and a minimal generating set Δ for Γ, what are the largest cardinals κ_a and κ_b such that $D_\Delta(\Gamma)$ is $\mathscr{S}$-decomposable into fully chromatic parts: (a) for every set $\mathscr{S}$ of κ_a pairwise nonisomorphic oriented trees of size $|\Delta|$? and (b) for some set $\mathscr{S}$ of κ_b pairwise nonisomorphic oriented trees of size $|\Delta|$?

Question 3. For the n-cube Q_n, what is the optimal way to partition n so that the number of trees in the decomposition obtained by the methods of Theorem 5 is as large as possible? Can the decomposition be done so that the parts of the decomposition are pairwise nonisomorphic?

The author wishes to thank the referees for their valuable suggestions and corrections that led to a much improved paper.

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