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The Risk Structure of Land Markets*

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Real property as an asset class represents over half of the wealth in the United States. Nevertheless, the structure of risk in real property markets is poorly understood. This paper develops a model of urban and agricultural land prices that integrates spatial and asset pricing theories and characterizes the spatial and temporal risk structure of the land market. Urban land is priced by a CAPM and agricultural land is priced by a real option to convert into urban land. We show that the price of land awaiting conversion increases with the growth rate of urban rents and unsystematic risk but decreases with risk aversion. However, it may be increasing or decreasing in systematic risk. The free boundary for exercise determines city size, which increases with the growth rate of urban rents but decreases with systematic and unsystematic risk.

Real property as an asset class represents over half of the wealth in the United States. It is also the single most important cause of the worst financial crisis in the past 60 years. Despite the importance of this asset to investor portfolios and to the health of the entire financial sector of the economy, the structure of risk in real property markets is poorly understood.

In this paper we characterize the spatial and temporal risk structure of land markets by integrating modern theories of risk into spatial models of land markets. While the finance literature has developed asset pricing models with systematic risk and real options,[1] it has ignored spatial

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[1]Real options are options on real assets as opposed to financial assets.

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features of land markets. The urban land literature has addressed the spatial aspects of land pricing extensively but largely disregards modern theories of risk. Integrating the two approaches provides many new insights into both finance and urban land theory.

Raw or agricultural land is a real asset with an attached perpetual American option to convert to urban uses. In our model, urban land is priced by a CAPM valuation of urban rents, and agricultural land is priced by a real option to convert into urban land.[2] Real option theory stems from the perpetual American option models of Samuelson [21] and Merton [18], but recent developments have been applied to irreversible investment decisions. This literature has shown that the ability to delay an irreversible investment expenditure can profoundly affect the decision to invest. In particular it invalidates the usual net present value rule to invest when the present value of expected cash flows exceed the cost of the investment. An excellent review article of this literature has been provided by Pindyck [19].[3]

The spatial side of our model is in the spirit of durable capital, perfect foresight models of urban areas like those of Arnott and Lewis [1] and Wheaton [27]. Capozza and Helsley [6] extend these models to include uncertainty and examine the spatial implications of the option to develop using a hitting time approach but ignoring systematic risk.

This paper extends the financial economics literature in the following ways. First, by starting from a model of the market value of fundamental cash flows, we are able to examine the effect of changes in systematic risk on option value. An increase in systematic risk can increase or decrease option value. Second, we exploit the fact that the real underlying asset may be non-traded (or notional), which allows it to have a negative price. Both notional and actual cash flows on land may be negative and we use an additive diffusion to capture this feature. Third, we find that the additive diffusion on cash flows results in a log-normal or displaced log-normal diffusion for the option value.

This paper also introduces the following financial concepts into the urban land economics literature. First, we model the role of systematic and unsystematic risk on price and rent gradients and on city size. Second, we model the spatial variation of capitalization rates (or rent/price ratios). Third, we examine the interaction between the option to develop and land use.

[2]Models of real asset valuation using a CAPM applied to fundamental cash flows without option pricing include those by Constantinides [4] and Sick [23]. Ross [20] values risky cashflow streams by replicating them with a stream of trades in priced assets.

[3]Recent applications of the real option approach to land markets include Titman [25], Clarke and Reed [12], Capozza and Helsley [6], Capozza and Li [7], Capozza and Sick [11], and Williams [28].

The paper is organized into four sections in addition to this introduction. The next section describes and justifies the additive process for urban rents and values land with a CAPM. The second section solves for agricultural land value using option pricing methods. The third examines comparative statics for prices, city size, hurdle rent, and rent/price ratios with respect to growth rate, systematic risk, and unsystematic risk. The final section provides concluding remarks. Two appendices provide technical details for the solution of the model and for an alternative log-normal rent model, which generates the same comparative statics, but requires numerical solution.

I. THE MODEL

Our point of departure is a simple dynamic model of a monocentric urban area described in Capozza and Helsley [5] but with uncertain growth and simplified production and consumption decisions. Our treatment of the land conversion decision under uncertainty is based on the real option approach to the timing of investment decisions. We generate the differential equation for option value from a CAPM because short sales restrictions make the arbitrage approach less appropriate in a real asset context. However, our solution is the same as that which arises from the arbitrage approach.

A. Bid-Rent Model of Urban Land

Urban land at the center of the city or CBD (Central Business District) earns rent at the rate $R = R(0)$. We regard R as an imputed rental rate on owner-occupied land, or the net rental rate on leased land. Land at distance z from the CBD earns rent at the rate

$$R(z) = R - z. \tag{1}$$

We can think of z as the cost of commuting the distance z to work at the CBD, if all employment is at the CBD and workers live in the suburbs. Alternatively, z may represent the premium accorded to centrally located commercial property. Note that we are scaling the measurement of z so that the rental rate declines at the rate of \$1 per unit distance. Land in agricultural use earns rent A at all locations. This is a simplified monocentric model of an urban area (see [26]).

B. Stochastic Assumptions

For simplicity, we assume that the commuting costs that determine the rate at which urban rents decline with distance in (1) is constant, and that the agricultural rental rate A is also constant. Later in the paper we outline how to generalize the analysis to stochastic commuting costs and agricultural rents.

We study the model in detail under an additive diffusion process for the CBD rent R. We choose this process for three reasons. First, for urban areas the additive diffusion is consistent with the observed empirical regularities.[4] Second, it permits a simple treatment of negative notional and actual cash flows which are common in real estate. In addition, it illustrates general results with analytical rather than numerical solution techniques. Third, this process results in log-normal or displaced log-normal diffusions for the price of land with an imbedded option.[5,6] We discuss this last point further below.

Thus, we consider the following normal process for rent at the CBD,

$$dR = g\,dt + \sigma\,dB. \tag{2}$$

Here, g is the growth of rents and B is a Weiner process with no drift and unit variance per unit time, t. We assume that the growth and variance,

[4]See Capozza and Schwann [9], Appendix A. There is evidence that urban areas become more stable as they grow because of the diversification of their economy. For example, the three largest cities (New York, Los Angeles, Chicago) have an average standard deviation of population growth of 0.2%, while the 3 smallest urban areas (Chattanooga, Lansing, Des Moines) in our sample of 64 areas have a standard deviation of 0.6%. The normal distribution is consistent with this decline in the standard deviation of growth as city size increases while the log-normal is not.

Capozza and Schwann [9] study the distribution of quarterly housing returns from 1979–1988 for 64 U.S. urban areas (SMAs). Using a χ^2 test they find that 10 urban areas fail the log-normal, but only 7 fail the normal specification at the 10% level. Thus there is weak empirical evidence which favors a normal diffusion even for prices.

[5]It is common to use a log-normal diffusion for the price of a financial asset, because this process guarantees that the price will remain positive. In fact, financial asset prices are non-negative because of limited liability, which is an option feature itself as pointed out by Black and Scholes [2]. Ownership of an all-equity firm is equivalent to ownership of the underlying real asset plus an American put option to dispose of the asset for an exercise price of zero. Thus financial options on such a stock are compound options as in Geske [13]. The real asset underlying the firm may have a negative value, but will cease to be a traded asset in that case. It will merely be a notional asset. In the urban setting, if net rents become sufficiently negative, the property is abandoned.

Optimal abandonment policy is discussed by Williams [28]. In this paper we do not analyze abandonment options, but we do not feel compelled to restrict ourselves to processes that ensure that real urban asset prices remain positive. The abandonment option is important in declining markets. We are mainly interested in growing markets with upward conversion in this paper. In growing markets the abandonment option is deep out of the money and property values are almost identical to the underlying real asset values.

[6]Outside the urban boundary, land is primarily priced as agricultural land, rather than urban land. There is nothing to prevent the notional urban land price from being negative since a short sale would require the sale of agricultural rather than urban property. Thus abandonment is not the only reason why a real asset can have a negative value.

σ^2, are constant over time. In Appendix B we also consider a model with the popular log-normal process. Numerical analysis of the comparative statics shows that the qualitative results are the same as those in the normal model and suggests that our analysis is not sensitive to this stochastic process assumption.

C. Urban Land Prices

Suppose that asset prices follow a single-factor asset pricing model, such as the capital asset pricing model (CAPM) or the consumption CAPM (Breeden [3]) of the form

$$E\left(\frac{Rdt + dP}{P}\right) = r\,dt + \lambda \operatorname{cov}\left[df, \frac{dP}{P}\right], \tag{3}$$

where P is the price of an asset paying a dividend R, r is the riskless real rate of return, f is the priced market factor, and λ is the market price of risk. Suppose that the process σB has systematic risk b defined by $b\,dt = \operatorname{cov}[\sigma dB, df]$. Then, the price P of a perpetual real asset at the CBD paying rent according to the normal model (2) is

$$P(0) = \frac{R(0)}{r} + \frac{g - \lambda b}{r^2}. \tag{4}$$

One can easily verify that (2) and (4) satisfy (3).

Equation (4) gives the price of urban land at the CBD, $P(0) = P$. Capitalizing the difference in rent z between land at the CBD and land at the distance z, from (1), we have the urban price at distance z,

$$P(z) = P(0) - \frac{z}{r}. \tag{5}$$

Similarly, capitalizing the agricultural rent A, we have the value of pure agricultural land, P^{pa}, which cannot be converted to urban land,

$$P^{\mathrm{pa}} = \frac{A}{r}. \tag{6}$$

This ignores the value of the option to convert agricultural land to urban land, which is discussed next.

D. Valuation of Convertible Agricultural Land

Agricultural land can be converted to urban land by paying the development cost, C, which is the cost of servicing the land.[7] Conversion involves the exchange of two assets (agricultural land with value A/r and conversion cost C for the urban asset with value $P(z)$.[8]

We define $W(P)$ to be the value of the option to convert agricultural land to urban land at time t when the urban land price is $P = P(z)$. The price of convertible agricultural land is

$$P^{a}(z) = \frac{A}{r} + W(P(z)), \tag{7}$$

where $P^{a}(z)$ is the price of convertible agricultural land.

Note that the value of agricultural land depends on the distance z from the CBD only through the dependence of the notional urban land value $P(z)$ on z. Thus, agricultural land consists of a dividend-paying asset of value A/r and a non-dividend-paying option of value $W(P(z))$. Since ownership of land has no time limit, the option to convert land is perpetual, and the option value is a function of W only, i.e., it is time invariant.

II. SOLUTION

We can derive a differential equation relating the option value to urban land value either by standard arbitrage arguments (Ingersoll [16, p. 371];

[7]In this paper we focus on the price of land rather than the price of developed property (land plus structure). The decision to construct a building also involves the choice of optimal density. The optimal-density decision is studied in Capozza and Li [7] and is not addressed directly here. However, for any given urban housing price, one can solve for the optimal building density conditional on conversion taking place at that price. The solution to this problem is time invariant. That is, development always occurs at the same density. Therefore we can treat the builder's problem as exogenous to our model.

[8]Alternatively one can think of conversion as the exchange of three assets of value A/r, C, and z/r for urban land at the CBD with value P. That is, conversion involves the surrender of agricultural land, conversion cost C, and a consol bond yielding a cash flow sufficient to pay the annual transportation cost to the CBD. This characterization is useful if we want to extend the analysis to allow C, A, and z to be stochastic. The analysis is straightforward as long as $(C + A/r + z/r)$ follows a joint normal process with P. Margrabe [17] has studied the option to exchange one risky asset for another. Briefly, the idea is to assume the assets follow a joint log-normal distribution and study the relative price of one asset in terms of the other as numeraire. In our model, since asset prices are normally distributed, the variable to study is the difference $P - (z/r + C + A)$ between the two asset values. The analysis is discussed in an earlier version of this paper (Capozza and Sick [10]).

Heaney and Jones [14]) or by the CAPM as

$$0 = \frac{\sigma^2}{2r^2}W'' + \frac{g - \lambda b}{r}W' - rW. \tag{8}$$

To verify that this satisfies the CAPM, note that by Itô's Lemma the expected dollar return per unit time on the option is

$$\frac{1}{2}\frac{\sigma^2}{r^2}W'' + \frac{g}{r}W'. \tag{9}$$

By (2), (4), and (5),

$$dP(z) = \frac{g}{r}\,dt + \frac{\sigma}{r}\,dB. \tag{10}$$

The systematic risk of the option is

$$\operatorname{cov}(\sigma df, dW) = W' \operatorname{cov}(\sigma df, dP(z)) = W'b/r;$$

therefore the required dollar rate of return on the option, by the CAPM, is

$$rW + \frac{\lambda b}{r}W'. \tag{11}$$

Equating (9) and (11) yields (8).

A conversion policy is characterized by a critical or hurdle value P^* such that the land is converted the first time the urban land price $P(z)$ reaches P^* from below.[9] The agricultural land value at the time of conversion equals the urban land value less the cost of conversion,

$$W(P^*) + \frac{A}{r} = P^* - C. \tag{12}$$

This is one boundary condition for the differential equation (8). Another boundary condition arises by considering land arbitrarily far from the CBD. As $z \to -\infty$, the notional urban rent $R(z) \to -\infty$ and the notional urban land value $P(z) \to -\infty$. At such distance, the prospect of converting agricultural land to urban land becomes remote, and the option value

[9] P^* is invariant to z because the mean and variance of the price process are the same for all z.

vanishes,

$$\lim_{P \to -\infty} W(P) = 0. \tag{13}$$

Solving (8) subject to (12) and (13) yields (see Appendix A)

$$W(P) = \left(P^* - \frac{A}{r} - C\right)e^{a(P-P^*)}, \tag{14}$$

where

$$a = \frac{r\left(-\hat{g} + \sqrt{\hat{g}^2 + 2r\sigma^2}\right)}{\sigma^2} \tag{15}$$

and $\hat{g} = g - \lambda b$.

We must choose a value of P^* to maximize the option value $W(P)$ in (14). In principle, this value could depend on P, but P enters the RHS of (14) only through the factor e^{aP}. Thus, we maximize W by performing

$$\max_{P^*}\left(P^* - \frac{A}{r} - C\right)e^{-aP^*}. \tag{16}$$

This occurs when

$$P^* = \frac{A}{r} + C + \frac{1}{a}. \tag{17}$$

This hurdle price is composed of the value of pure agricultural land, the cost of conversion, and an irreversibility premium,[10] $1/a$. In Appendix A, we show that this first-order condition for (16) is the smooth-pasting condition of Samuelson [21] and Merton [18].

Using (17), we can re-express (14) as

$$W(P(z)) = \frac{1}{a}e^{a(P(z)-P^*)}. \tag{18}$$

[10] The term $1/a > 0$ is the NPV arising from conversion. The conversion hurdle for NPV is not zero as in standard capital budgeting for two reasons. First, an adopted project exposes the owner to more downside risk than the option. Thus, $1/a$ declines as σ^2 declines. Second, the landowner must choose among a sequence of mutually exclusive projects indexed by the date of conversion. This is the classic timber-cutting problem in the certainty case. As $\sigma^2 \to 0$, $1/a \to \hat{g}/r^2$, which is the optimal NPV in the certainty case. Both of these premia arise because of the irreversibility of the project.

Substituting the hurdle price, P^*, into (5) and using (1) and (4) defines the hurdle rent, $R^* = rP^* - \hat{g}/r$. Define $z^*(R)$ as the location where $R(z) = R^*$ when the CBD rent is R. That is, by (1), $z^*(R) = R - R^*$. Therefore, when land is being converted at distance z^*,[11]

$$R^* - R(z) = z - z^* \tag{19}$$

or

$$P^* - P(z) = \frac{1}{r}(z - z^*). \tag{20}$$

Using (20) we can rewrite (18) as

$$W(P(z)) = \frac{1}{a}e^{-a(z-z^*)/r}. \tag{21}$$

Decomposing the right-hand side of (21) into two components and substituting them into (7) yields the value of agricultural land

$$P^a(z) = \frac{A}{r} + \frac{\hat{g}}{r^2}e^{-a(z-z^*)/r} + \frac{r^2 - a\hat{g}}{ar^2}e^{-a(z^*-z)/r}, \qquad z \geq z^{**}, \tag{22}$$

where z^{**} is defined in footnote 11. The first term is the value of pure agricultural land; the second and third are respectively the risk-adjusted growth and uncertainty premia.[12,13] From (17) and (20) we have the value of urban land

$$P(t) = \frac{A}{r} + C + \frac{\hat{g}}{r^2} + \frac{r^2 - a\hat{g}}{ar^2} + \frac{1}{r}(z^* - z), \qquad z \leq z^{**}. \tag{23}$$

The first term represents the value of pure agricultural land, the second is the cost of conversion, the third is the value of net or risk-adjusted growth (which may be negative), the fourth is the uncertainty premium (which is always positive), and the last is the accessibility premium.

[11]After land is converted to urban use, land rents could fall, but since conversion is irreversible the boundary would not shrink. At these times $P^* > P(z^*, t)$ and the actual boundary of the urban area is defined to be $z^{**}(t) = \sup\{z^*(\tau) | 0 \leq \tau \leq t\}$.

[12]Equation (22) reduces to the certainty case when the variance is zero. As $\sigma \to 0$, $a \to r^2/\hat{g}$, so that $(r^2 - a\hat{g})/(ar^2) \to 0$. Therefore the third term vanishes as $\sigma \to 0$. Moreover, $r^2 - a\hat{g} \geq 0$, so the third term is non-negative.

[13]We can interpret $e^{-a(z-z^*)/r}$ as a risk-neutral expected PV factor for the uncertain conversion time because the conversion NPV is $1/a$ and option value is $(1/a)e^{-a(z-z^*)/r}$.

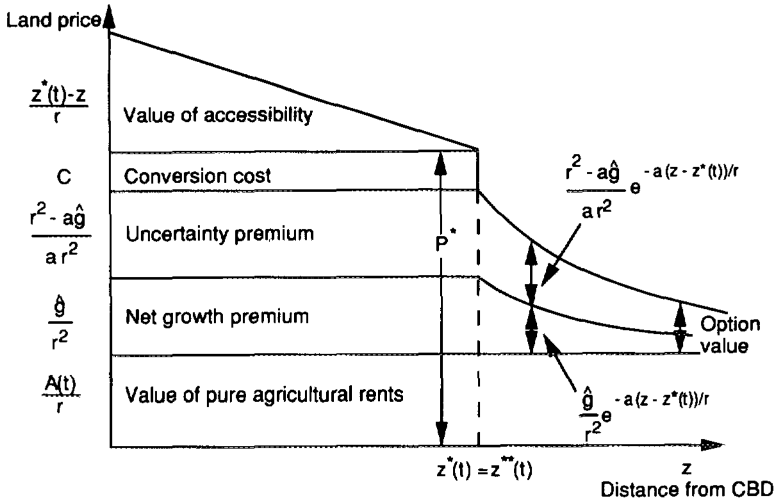

FIG. 1. Land prices inside and outside the urban area.

Equations (22) and (23) together define the pricing structure of an urban area and are illustrated in Fig. 1.

The hurdle rent is

$$R^* = rP^* - \frac{\hat{g}}{r}$$

$$= A + rC + \frac{r^2 - a\hat{g}}{ar}. \tag{24}$$

Thus, from (19) and (24) we can decompose urban rents

$$R(z) = A + rC + \frac{r^2 - a\hat{g}}{ar} + (z^* - z), \qquad z \leq z^{**}. \tag{25}$$

The first term in (25) is the agricultural rent, the second is the opportunity cost of the capital used to convert the land to urban use, the third is the uncertainty premium, and the last is the accessibility rent. Equation (25) defines the rent structure of the urban area and is illustrated in Fig. 2.

Finally, note that by (18) and (7)

$$P^{a}(z) = \frac{A}{r} + \frac{1}{a}e^{a(P(z)-P^*)}.$$

Since $P(z)$ follows an additive diffusion, the second term on the right-hand

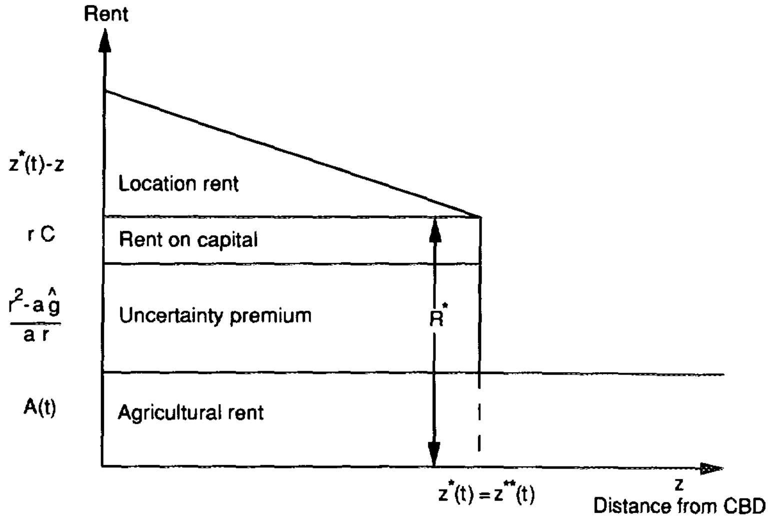

FIG. 2. Rents inside and outside the urban area.

side follows a log-normal diffusion. Thus agricultural land follows a log-normal diffusion displaced by pure agricultural land value, A/r. One might be interested in a financial option to acquire agricultural land for development purposes. Such a compound option can be analyzed with a standard log-normal diffusion on $W(P)$ by including A/r in the exercise price.

III. RISK AND RETURN

A. Land Prices, Growth, and Risk Aversion

Since $\hat{g} = g - \lambda b$, $\hat{g}$ aggregates the effects of growth, g, and risk aversion, λ, in the sense that these variables affect land prices only through $\hat{g}$. Systematic risk, b, affects land prices through $\hat{g}$ and through σ^2.

A technical result that will prove useful is

$$a_{\hat{g}} = \frac{\partial a}{\partial \hat{g}} < 0. \tag{27}$$

Then from (17)

$$\frac{\partial P^*}{\partial \hat{g}} = \frac{\partial P^*}{\partial a}\frac{\partial a}{\partial \hat{g}} = -\frac{1}{a^2}a_{\hat{g}} > 0. \tag{27a}$$

From (4) and (5)

$$\frac{\partial P}{\partial \hat{g}} = \frac{1}{r^2} > 0. \tag{27b}$$

With tedious calculation it can be shown that

$$\frac{\partial z^*}{\partial \hat{g}} = \frac{-\partial R^*}{\partial \hat{g}} > 0. \tag{27c}$$

Using (7) and (21),

$$\frac{\partial P^{a}}{\partial \hat{g}} = \frac{\partial W}{\partial \hat{g}}$$

$$= \left(-\left(\frac{1}{a^2} + \frac{1}{a}(z - z^*)\right)a_{\hat{g}} + \frac{1}{r}\frac{\partial z^*}{\partial \hat{g}}\right)e^{-a(z-z^*)/r} > 0, \qquad z \geq z^{**}. \tag{27d}$$

Thus if the market price of risk is lower or the urban growth rate is higher, the equilibrium market prices of urban and agricultural land will be higher, and the hurdle price for conversion will be larger. Intuitively, since rents are exogenous, less risk aversion or more expected growth will raise the prices investors are willing to pay for an asset. The notional urban boundary, z^*, where conversion takes place, moves away from the CBD when higher growth makes earlier conversion less risky. This occurs because for any given level of risk, high growth means rents are less likely to fall below agricultural rents in the future.

B. Land Prices and Unsystematic Risk

In this subsection, we hold fixed the systematic risk b of urban rent, and consider the effect of changes in unsystematic risk, or equivalently, σ^2. We will denote the variance rate of unsystematic risk per unit time of B by σ_{h}^2. That is, $\sigma^2\,dt = \mathrm{var}(dB) = (b^2\sigma_{\mathrm{f}}^2 + \sigma_{\mathrm{h}}^2)\,dt$.

The effect of a change in σ^2 on a can be shown to be negative:

$$a_{\sigma^2} = \frac{\partial a}{\partial \sigma^2} = -\frac{a}{\sigma^2} + \frac{r^2}{\sigma^2\left(\hat{g}^2 + 2r\sigma^2\right)^{1/2}} < 0.$$

Then from (17)

$$\frac{\partial P^*}{\partial \sigma^2} = \frac{\partial P^*}{\partial a}\frac{\partial a}{\partial \sigma^2} = -\frac{1}{a^2}a_{\sigma^2} > 0, \tag{28a}$$

and from (4) and (5)

$$\frac{\partial P}{\partial \sigma^2} = 0. \tag{28b}$$

By (19) and (24)

$$\frac{\partial z^*}{\partial \sigma^2} = \frac{-\partial R^*}{\partial \sigma^2} = \frac{r}{a^2}\frac{\partial a}{\partial \sigma^2} < 0. \tag{28c}$$

Using (7), (21), and (28c),

$$\frac{\partial P^a}{\partial \sigma^2} = \frac{\partial W}{\partial \sigma^2} = -\frac{1}{ar}e^{-a(z-z^*)/r}(z-z^*)\frac{\partial a}{\partial \sigma^2} > 0, \qquad z \geq z^{**}. \tag{28d}$$

Thus an increase in unsystematic risk[14] leaves urban prices unchanged but increases agricultural land prices and the hurdle conversion price, while it shrinks the size of the urban area. This is an example of the important interaction between option value and land use.

The intuition for these effects is clear. As in other models that price financial assets, unsystematic risk is unpriced since it can be diversified away. Agricultural land prices, on the other hand, contain an option value component, which is positively related to total risk or volatility. Since the option in agricultural land is a claim on one tail of a price distribution on urban land, the higher volatility increases the size of the tail and the option value. In the discussion of the hurdle P^*, we noted that it includes an uncertainty premium required to compensate for the fact that urban land faces more downside risk than agricultural land. As the risk in urban land, σ^2, increases, so does the uncertainty premium and the hurdle price P^*.

C. *Land Prices and Systematic Risk*

An increase in systematic risk b will increase total risk but reduce the risk-adjusted growth rate $\hat{g}$, $\partial \sigma^2/\partial b > 0$ but $\partial \hat{g}/\partial b = -\lambda < 0$. This results in opposing effects so that the net effect of a change in systematic risk is often ambiguous. For example, $\partial P^a/\partial \hat{g} > 0$, and $\partial P^a/\partial \sigma^2 > 0$, so $\partial P^a/\partial b$ has an indeterminate sign. Similarly $\partial P^*/\partial b$ has an indeterminate sign, but both $\partial P/\partial b$ and $\partial z^*/\partial b$ are negative while $\partial R^*/\partial b > 0$.

Intuitively, an increase in systematic risk, b, has two effects: first, it increases total risk, σ^2, which tends to increase option value, and second, it reduces urban land value, P, which tends to reduce option value.

[14]The risk of zoning changes is one urban-related example.

Figure 3 illustrates how the net effects can be an increase or decrease in agricultural land values.

In Fig. 3a, the agricultural land is close to conversion, so that agricultural land prices closely track urban land prices downward as b increases. In Fig. 3b, there is less risk aversion and the market factor is riskier. For small b, the agricultural land is close to conversion and its price falls along with the urban land price as b increases. But for large b, the option value is a large proportion of the total agricultural value and the urban land price does not decrease rapidly because of the low risk aversion, so the option value increases with total risk.

In Fig. 3c, there is no risk aversion. An increase in b does not reduce urban prices and the value of agricultural land increases monotonically with b.[15]

The comparative statics of P^* with respect to systematic risk are similar to those of P^a. In summary, we have the comparative statics in Table 1.

D. Urban and Agricultural Rent Multipliers

The Urban Rent Multiplier (URM) is defined to be the ratio of the price of land to the rent. From (15) we have

$$\mathrm{URM} = \frac{P(z)}{R(z)} = \frac{1}{r} + \frac{g}{r^2 R(z)} - \frac{\lambda b}{r^2 R(z)}, \qquad z \le z^{**}. \tag{29}$$

The three terms in (29) are respectively the common rent multiplier, a growth premium, and a risk adjustment. The urban rent multiplier is decreasing (increasing) in distance from the center of the urban area if

[15] Figure 3 is derived from a basic case of $\sigma_f^2 = 0.04$ and $\lambda = 0.08$, which is consistent with historic CAPM parameters (Ibbotson and Sinquefield [15]). If the CBD urban rent is 40, the rent at distance $z = 10$ is 30, and a growth rate $g = 1$ is a 3.3% growth rate, initially. With a real interest rate $r = 3\%$ and systematic risk $b = 10$, $P = 1222$ and the beta of the rate of return on urban land is, by (7) and (13),

$$\frac{\operatorname{cov}(dP/P, df)}{\sigma^2\,dt} = \frac{b}{rP} = 0.27.$$

If the market explains 25% of the variation in land returns,

$$0.25 = \frac{b^2\sigma_f^2}{b^2\sigma_f^2 + \sigma_h^2} \Rightarrow \sigma_h^2 = 12.$$

This is the basis for Fig. 3a and is varied in Figs. 3b and 3c.

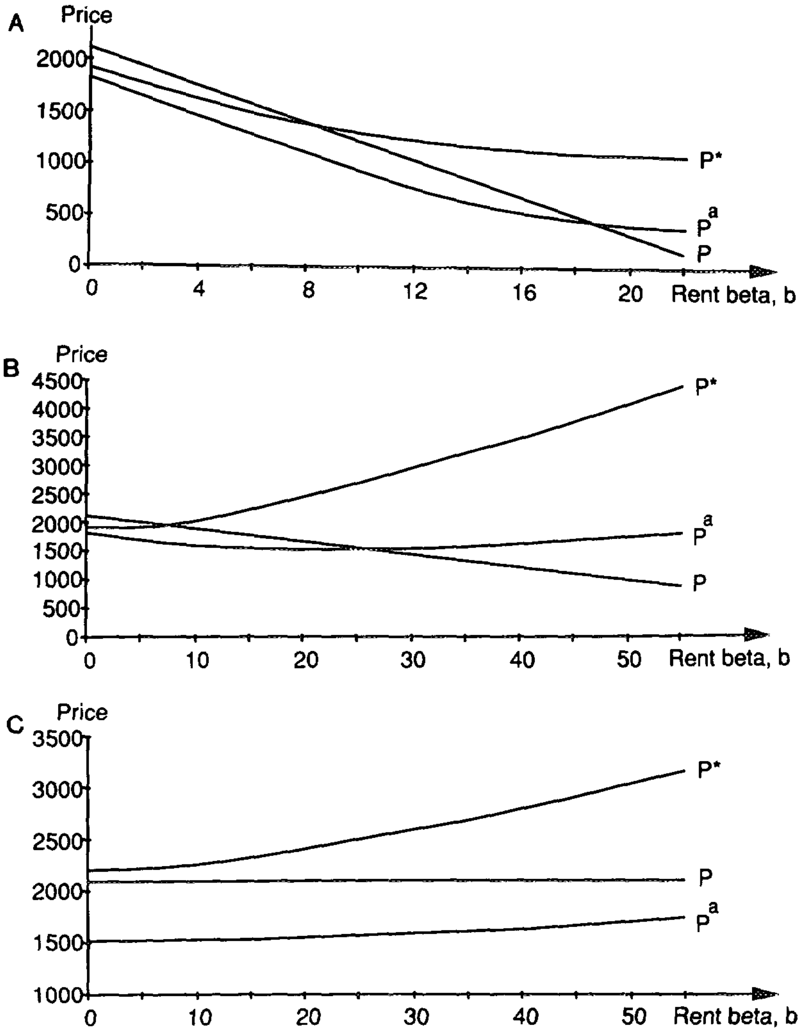

FIG. 3. Market prices and hurdle price as functions of systematic risk. This figure illustrates the ambiguous effect of systematic risk on the price of agricultural land. The base case has $A = 10$, $R = 40$, $r = 0.03$, $\sigma_h^2 = 12$, and $g = 1$. The remaining parameters vary as follows: (A) $C = 300$, $\lambda = 0.08$, and $\sigma_f^2 = 0.04$; (B) $C = 300$, $\lambda = 0.02$, and $\sigma_f^2 = 0.25$; and (C) $C = 600$, $\lambda = 0.0$, and $\sigma_f^2 = 0.04$. In (A) agricultural land is close to conversion so that agricultural land prices track urban land prices downward as b increases. In (B) risk aversion is low and the market factor is high. Agricultural land price falls then rises with b. In (C) risk aversion is zero so the option effect dominates and agricultural prices rise with rent beta, b.

TABLE 1
Comparative Statics

	Growth (g)	Price of risk (λ)	Unsystematic risk (σ_h^2)	Systematic risk (b)
Hurdle price, P^*	+	−	+	±
Urban price, P	+	−	0	−
Urban boundary, z^*	+	−	−	−
Agricultural price, P^a	+	−	+	±
Hurdle rent, R^*	−	+	+	+

Note. This table shows the qualitative impact of the parameters at the top of the table on the variables to the left of the table. For example, growth, g, has a positive impact on the hurdle price, P^*.

$(g - \lambda b)$ is positive (negative).[16] Among urban areas, urban rent multipliers will be higher in high-growth, low-systematic-risk areas.

We have the Agricultural Rent Multiplier (ARM) from (22),

$$\mathrm{ARM} = \frac{P^a}{A} = \frac{1}{r} + \frac{\hat{g}}{r^2 A} e^{-a(z-z^*)/r} + \frac{(r^2 - a\hat{g})}{ar^2 A} e^{-a(z-z^*)/r}, \qquad z > z^{**}. \tag{30}$$

Thus the agricultural rent multiplier can also be decomposed into three terms: the common multiplier, a net growth premium, and an uncertainty or option value premium. The ARM is always increasing in total risk, σ, and decreasing in distance from the center of the urban area.

Equations (29) and (30) define the structure of urban and agricultural multipliers and are illustrated in Figs. 4a–4c. In Fig. 4a, with $g > \lambda b$, the urban rent multiplier increases with distance z from the center of the urban area because the risk term is smaller than the growth term. Outside the urban area the agricultural rent multiplier declines with distance because of the declining option value.

[16]Note that with an arithmetic diffusion on rents, $g - \lambda b$ measures risk-adjusted growth in dollars per unit time, rather than in percent per unit time (as it would with a log-normal diffusion). Thus, the percentage risk-adjusted growth rate $(g - \lambda b)/R$ increases as one moves away from the CBD and R declines. This causes the multiplier to fall or rise according to the sign of $g - \lambda b$.

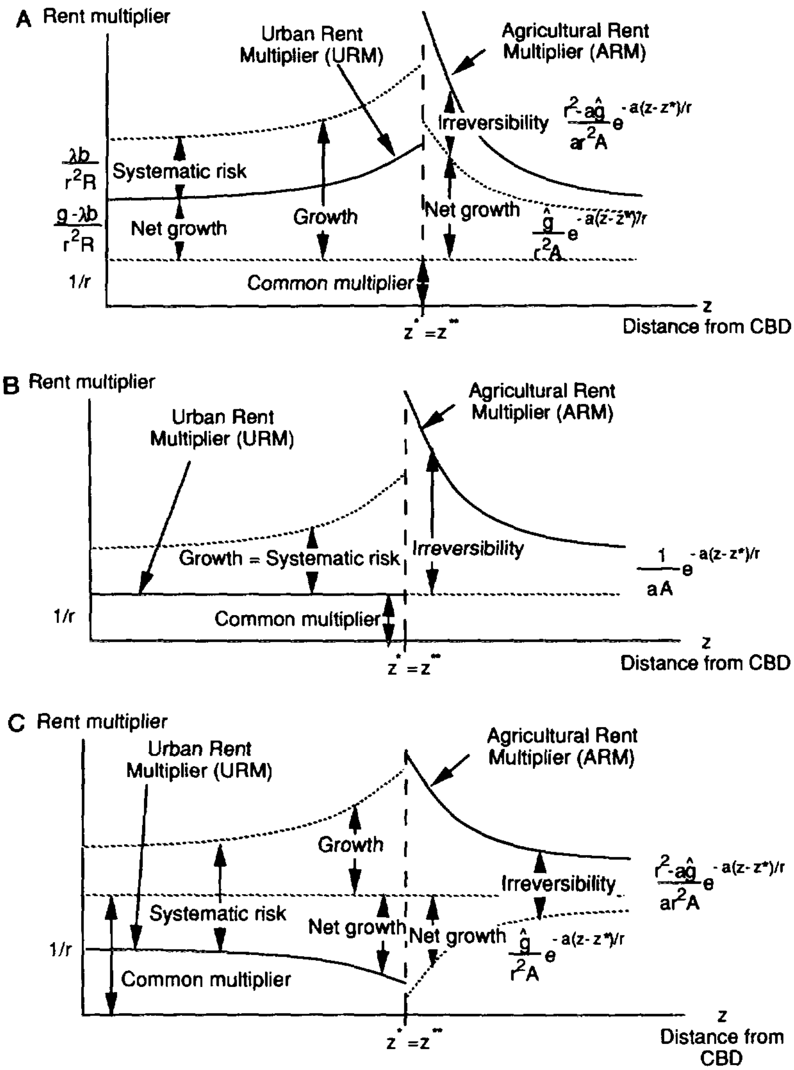

FIG. 4. Rent multipliers inside and outside the urban area. This figure illustrates that urban rent multipliers rise (remain constant, fall) with distance from the CBD when growth, g, exceeds (is equal to, is less than) the risk premium, λb. Agricultural rent multipliers always fall with distance. The solid lines in each panel are the rent multipliers. Dotted lines illustrate the components, viz. the common multiplier, growth, and systematic risk. (A) $g > \lambda b$; (B) $g = \lambda b$; and (C) $g < \lambda b$.

In Fig. 4b, with $g = \lambda b$, the growth term and risk term cancel each other out to result in a constant urban rent multiplier. The ARM declines as before. In Fig. 4c, with $g < \lambda b$, the risk term exceeds the growth term and the URM declines with distance. Capitalization rates, of course, have the inverse pattern to the rent multipliers.

IV. SUMMARY AND CONCLUSIONS

In this paper we have developed a model of land pricing in an urban area with risky growth and risk-averse investors. The model allows for negative cash flows and negative notional asset prices, which can occur in land markets. We obtain closed-form solutions for land prices and the size of the urban area. We also study the structure of rent multipliers outside and inside the urban area.

There are many insights from the model. First, since agricultural land price contains a real (as opposed to financial) option we can observe the effect of systematic rent risk on the option value. This effect is not readily apparent in a Black–Scholes model of financial options, where systematic risk enters only indirectly through stock prices. The price of land awaiting conversion increases with the growth rate of urban rents and unsystematic risk but decreases with risk aversion. However, it may be increasing or decreasing in systematic risk because on the one hand an increase in systematic risk increases total risk and hence increases option value, but on the other hand it also decreases the value of the underlying urban land.

Second, the real option affects not only land values but also land uses through the hurdle price P^*. When urban rents are riskier, the option to develop is more valuable and the hurdle price is higher. As a result land is developed later and city size is smaller for a given level of CBD rent.

Third, we generalize the structure of urban prices and rents to include the effects of irreversibility and uncertainty and show that urban land prices can be decomposed into five parts which include the value of pure agricultural land, the cost of conversion, the value of growth, an uncertainty premium, and an accessibility premium. Agricultural prices decompose into the value of pure agricultural land, a growth premium, and an uncertainty premium. Ubiquitous in this framework for urban structure is the irreversibility or option value term, $1/a$, which is the sum of the growth and uncertainty terms.

Fourth, agricultural prices decline with distance from the urban area as a function of the expected time to development. Therefore, prices outside the urban area will decline faster when growth is slower. Finally, we find that inside the urban area the urban rent multiplier is increasing or decreasing in distance from the center of the urban area depending on the sign of $(g - \lambda b)$. That is, the rent multiplier depends on the growth rate of the urban area relative to the systematic risk of rent and the price of risk.

The model has immediate empirical implications for studies of risk in urban and agricultural land.[17] First, holding city size constant, average urban prices will increase with the growth rate but decrease in systematic risk and increase in unsystematic risk. Second, the price of convertible agricultural land will increase with the growth rate and systematic risk, but may increase or decrease with systematic risk. Third, city size will fall with both types of risk. Fourth, among urban areas high-growth, low-risk cities will have higher rent multipliers.

The model also suggests a number of ways to improve the analysis of the risk of real property. Existing lender procedures focus primarily on loan to value and debt coverage ratios in credit (downside) risk decisions. Our model defines the roles of location, growth expectations, and systematic and unsystematic risk, which could all be incorporated into credit-risk evaluation procedures.

The model can be readily extended to allow for further stochastic values for agricultural rents, conversion costs, and commuting cost, using a technique that is similar in spirit to those employed by Margrabe [17] when analyzing the option to exchange one risky asset for another.

APPENDIX A

Proof of Eqs. (14) and (17)

The general solution to (8) is

$$W = k_+ e^{a_+ P} + k_- e^{a_- P}, \tag{A.1}$$

where k_+ and k_- are constants and $a_+ > 0$ and $a_- < 0$ are given by

$$\begin{aligned} a_{+,-} &= \frac{-\hat{g}/r \pm \sqrt{\hat{g}^2/r^2 + 2(\sigma^2/r)}}{\sigma^2/r^2} \\ &= \frac{r(-\hat{g}) \pm \sqrt{\hat{g}^2 + 2r\sigma^2}}{\sigma^2}, \end{aligned} \tag{A.2}$$

where $\hat{g} = g - \lambda b$. Note that $\hat{g}$ can have any sign.

By using (13) we have $k_- = 0$. Thus

$$W_P = k_+ a_+ e^{a_+ P}.$$

[17]See Capozza and Schwann [8] for a survey of this literature and Capozza and Schwann [9] for results that verify empirically the role of systematic and unsystematic risk.

Letting $P = P^*$ and using (12) yields

$$P^* - \frac{A}{r} - C = k_+ e^{a_+ P^*}$$

or

$$k_+ = e^{-a_+ P^*}\left(P^* - \frac{A}{r} - C\right),$$

so that

$$W = \left(P^* - \frac{A}{r} - C\right)e^{a_+(P - P^*)}.$$

Now for a given P, choose P^* to maximize W, and note that the choice of P^* is independent of P because P only enters W through the factor $e^{a_+ P}$. The first-order condition is

$$\frac{\partial W}{\partial P^*} = e^{a_+(P-P^*)}\left(1 - a_+\left(P^* - \frac{A}{r} - C\right)\right) = 0$$

or

$$P^* = \frac{A}{r} + C + \frac{1}{a_+}. \tag{A.3}$$

Note that

$$\frac{\partial W}{\partial P^*} \lesseqgtr 0 \quad \text{if} \quad P^* \lesseqgtr \left(\frac{A}{r} + C + \frac{1}{a_+}\right),$$

so (A.3) gives a global maximum. Also note that the first-order condition is equivalent to the high-contact condition $W'(P^*) = 1$.

Substituting (A.3) into (A.1) yields

$$W = \frac{1}{a_+} e^{-a_+(P^* - P)}. \tag{A.4}$$

Noting that $a_+ = a$, we can see that (A.4) and (A.3) give (14) and (17), respectively.

APPENDIX B

A Log-Normal Rent Model

In this paper, we used an arithmetic diffusion for rents at the CBD. Since rents decay as a linear non-stochastic function of distance from the CBD, urban rents at the other locations also follow an arithmetic diffusion. If we want to have a log-normal process for rents at all locations, the rent decay must be a multiplicative rather than an additive function of the rent at the CBD. That is, the dollar transportation cost must vary stochastically in exactly the same fashion as dollar rent at the CBD. This type of variation would occur only if caused by a general CPI price-level variation. By converting everything to real terms, the model would become non-stochastic and no interesting comparative statics would arise.

On the other hand, meaningful comparative statics do arise from a model in which only the rent at the CBD follows a log-normal process, and urban rent decays linearly and non-stochastically with distance. In effect, this gives operating leverage to suburban rents. The comparative statics of this log-normal model are basically the same as those of the normal model. Below, we sketch the details of the model.

Assume the rent at the CBD follows the log-normal diffusion

$$dR = gR\,dt + \sigma R\,dB, \tag{B.1}$$

while rents at distance z from the CBD are obtained from (1).

The urban price at distance z is then

$$P(z) = \frac{R(0)}{r - \hat{g}} - \frac{z}{r}, \tag{B.2}$$

where $\hat{g} = g - \lambda b$ and $b\,dt = \mathrm{cov}(\sigma dB(t), df)$.

The market price of risk λ and the systematic and unsystematic risk factors, $df(t)$ and $dh(t)$, respectively, are as in the paper. Note that b is now the traditional beta of land at the CBD in the sense that it is the slope coefficient in a regression of rates of return for land at the CBD on the market factor.

The urban rent multiplier is

$$\mathrm{URM} = \frac{P(z)}{R(z)} = \frac{R(0)/(r - \hat{g}) - z/r}{R(0) - z}. \tag{B.3}$$

One can check that

$$\frac{\partial\,\mathrm{URM}}{\partial z} > 0 \Leftrightarrow \hat{g} > 0. \tag{B.4}$$

To determine the value of the option to convert agricultural land to urban land, recall that by surrendering the conversion cost C, the right to perpetual rents with present value A/r and a perpetual bond of value z/r that pays perpetual transport costs, the agricultural land can be converted to a perfect substitute for land at the CBD. Thus, agricultural land value is

$$P^{\mathrm{a}}(z) = W(P(0), z) + \frac{A}{r}, \tag{B.5}$$

where $W(P(0), z)$ is the value of the option to convert agricultural land to urban land valued at $P(0)$ when the total exercise price is $C + A/r + z/r$. The "dividend yield" on land at the CBD is $R(0)/P(0) = r - \hat{g}$ by (B.2).

The value of this option is (see Samuelson [21], Merton [18], and Ingersoll [16])

$$W(P(0), z) = \frac{P^*(z)}{\gamma}\left(\frac{P(0)}{P^*(z)}\right)^{\gamma}, \tag{B.6}$$

where

$$P^*(z) = \frac{\gamma}{\gamma - 1}\left(C + \frac{A}{r} + \frac{z}{r}\right)$$

and

$$\gamma = \frac{1}{2} - \frac{\hat{g}}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\hat{g}}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}\,.$$

Here $P^*(z)$ is the CBD price for which conversion occurs at distance z.

The comparative statics of this model are the same as those for the arithmetic diffusion, although at times one must resort to numerical methods to verify them.

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