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A Tower Construction for the Radical in Brauer's Centralizer Algebras

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In this paper we study the structure of the Brauer centralizer algebras in the case that the multiplication constant x is a rational integer. Several authors have studied the structure of these algebras for generic values of x. In particular, Hans Wenzl showed that the Brauer algebras can be obtained from Jones' Basic Construction and he used that fact to prove that the Brauer algebras are semisimple when x is not a rational integer. The Tower Construction is a method to study towers of semisimple algebras. Hence it does not apply in our case where the algebras in the tower eventually have radicals. Our first step in this paper is to modify the Tower Construction so that it does a simultaneous Tower Construction of the radicals and the semisimple quotients of a tower of algebras. The rest of the paper is devoted to describing these constructions explicitly in the Brauer algebra case. One surprising corollary of this method is a connection between two seemingly distinct criteria for the simplicity of certain subrings of the Brauer algebras. It is possible to identify explicitly certain subrings of the Brauer algebras which are the matrix rings corresponding to irreducible representations in the semisimple case. In previous work, these authors gave a combinatorial condition for simplicity of these individual matrix rings when x is a rational integer. The Tower Construction gives a second algebraic condition for simplicity. It is difficult to see why these two conditions are equivalent. However, the methods used in this paper make that clear.

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1. Introduction

Richard Brauer introduced the Brauer centralizer algebras $\mathscr{A}_f^{(x)}$ (see [Brr]) in order to study the centralizer algebras of orthogonal and symplectic groups on the tensor powers of their defining representations. Earlier Schur had used the group algebra of the symmetric group $\mathbb{C}S_f$ to study the corresponding centralizer algebras for the general linear groups. Brauer realized that the $\mathscr{A}_f^{(x)}$ have a richer structure than the $\mathbb{C}S_f$ (which are always semisimple). In many important cases the $\mathscr{A}_f^{(x)}$ are not semisimple and their algebra structure has been an interesting open problem for many years. Brauer [Brr], Brown [Brn], and Weyl [We] proved results about semisimplicity. Recently, we began an investigation of the radicals of the $\mathscr{A}_f^{(x)}$ and found many surprising combinatorial and algebraic properties [HW1–3]. This work led to a number of conjectures several of which are still open. One important conjecture was proven recently by Wenzl [W2]. In this work Wenzl made use of "the tower construction" pioneered by Vaughan Jones in his work on operator algebras (see [VJ, GHJ]). For related information see also [RaWe]. In the present paper we refine the tower construction to give a kind of tower construction for the radicals in the Brauer algebras.

The main step in the tower construction involves taking two semisimple algebras $\mathscr{U} \subseteq \mathscr{V}$ and constructing from them a third algebra $\mathscr{W}$ which contains $\mathscr{V}$. This is relevant to the Brauer algebras because it can be used in the case $\mathscr{U} = \mathscr{A}_{f-2}^{(x)}$ and $\mathscr{V} = \mathscr{A}_{f-1}^{(x)}$ to construct an ideal $\mathscr{W} = \mathscr{A}_f^{(x)}(1) \subseteq \mathscr{A}_f^{(x)}$ for which the quotient $\mathscr{A}_f^{(x)}/\mathscr{W}$ is just $\mathbb{R}S_f$ (see [W2]) where $\mathbb{R}$ denotes the reals. We consider the case where $\mathscr{A}_{f-2}^{(x)}$ is not semisimple. It is necessary to modify the above construction somewhat to handle this case. In the modified version $\mathscr{U}$ is the semisimple quotient of $\mathscr{A}_{f-2}^{(x)}$, $\mathscr{V}$ is an appropriately defined left $\mathscr{U}$-module, and the resulting algebra $\mathscr{W}$ is a homomorphic image of the ideal $\mathscr{A}_f^{(x)}(1)$. It turns out that $\mathscr{W} = \mathscr{A}_f^{(x)}(1)/N_f^{(1)}$ where $N_f^{(1)}$ is an ideal contained in the radical of $\mathscr{A}_f^{(x)}$. We are able to identify the ideal $N_f^{(1)}$ explicitly.

The above construction is interesting because it is one of the first cases where the tower construction has been used when the algebras in the tower are not semisimple. Recently Wenzl [W1] used different methods to analyze the Jones algebras and the Hecke algebras of type $\mathscr{A}_n$ in the non-semisimple case.

We will need some definitions and results from [HW1–3]. In all our work f is a positive integer and x is a real number. Suppose δ is a graph with $2f$ points and f lines in which every point has degree 1. Then δ is called a *1-factor on 2f points* and the set of these is denoted F_f. We view elements of F_f as having the $2f$ points arranged in two rows of f points, one above the other. For example,

$\delta =$ 

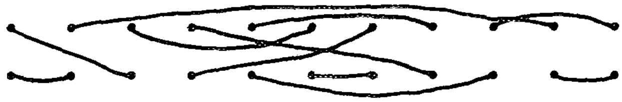
is a 1-factor with $f = 7$. When δ is arranged in this way we may talk about the top and bottom rows of δ. Lines joining points within a row are called *horizontal lines*; lines joining a point in the top row to one in the bottom row are called *vertical lines*.

The algebra $\mathscr{A}_f^{(x)}$ is an $\mathbb{R}$-algebra with F_f as basis. To describe the multiplication $*$ let δ_1 and δ_2 be elements of F_f. Place δ_1 above δ_2 and identify points in the bottom row of δ_1 with the corresponding points in the top row of δ_2. The resulting graph consists of f paths which start and finish in the top and bottom rows along with a certain number $\gamma(\delta_1, \delta_2)$ of cycles which use only points in the middle row. Form a new basis element δ by letting the edges of δ be the paths in the above diagram. The product $\delta_1 * \delta_2$ is $x^{\gamma(\delta_1, \delta_2)}\delta$.

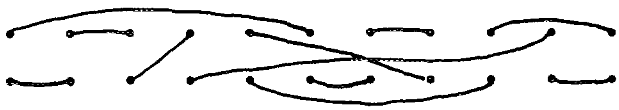
For example, if

$\delta_1 =$ 

and

$\delta_2 =$ 

then $\gamma(\delta_1, \delta_2) = 1$ so $\delta_1 * \delta_2 = x\delta$ where

$\delta =$  .

The algebra $\mathscr{A}_f^{(x)}$ is an associative algebra with identity which has dimension $|F_f| = (2f-1)!! = (2f-1)(2f-3)\cdots 3 \cdot 1$.

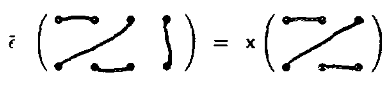
As discussed in [Brr], there is an important tower of ideals in $\mathscr{A}_f^{(x)}$. The span of all diagrams with at least k horizontal lines in the top row (and so in the bottom row) is an ideal in $\mathscr{A}_f^{(x)}$ denoted $\mathscr{A}_f^{(x)}(k)$. The quotients $\mathscr{A}_f^{(x)}(k)/\mathscr{A}_f^{(x)}(k+1)$, which we denote $\mathscr{A}_f^{(x)}[k]$, were studied extensively in [HW1]. More generally if A is any subspace of $\mathscr{A}_f^{(x)}(k)$ we let $A[k]$ denote the quotient

$$A[k] = (A + \mathscr{A}_f^{(x)}(k+1))/\mathscr{A}_f^{(x)}(k+1).$$

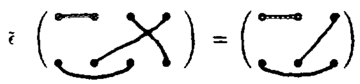
We will sometimes want to view $\mathscr{A}_{f-1}^{(x)}$ as a subalgebra of $\mathscr{A}_f^{(x)}$ via the obvious embedding. If δ^1 is a diagram in F_{f-1} there is a natural diagram

δ in F_f obtained from δ^1 by adding a new point on the right of the top row and a new point on the right of the bottom along with an edge between new points. Extend this map linearly to get an embedding of $\mathscr{A}_{f-1}^{(x)}$ in $\mathscr{A}_f^{(x)}$.

We also need a linear map $\tilde{\varepsilon}$ from $\mathscr{A}_f^{(x)}$ to $\mathscr{A}_{f-1}^{(x)}$ invented by Wenzl [W1]. The map is defined on basis diagrams as follows. If δ is a diagram in F_f which has a vertical line joining the fth point in the top row to the fth point in the bottom row, define $\tilde{\varepsilon}(\delta)=x\delta^1$ where $\delta^1\in F_{f-1}$ is obtained by deleting the last point in each row. Otherwise the fth points in the top and bottom row are joined to points u and v, respectively. In this case $\tilde{\varepsilon}(\delta)=\delta_1$, where δ_1 is obtained from δ by removing the fth points in each row and adding an edge between u and v. For example,



and

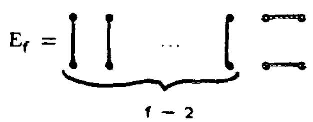


Extend $\tilde{\varepsilon}$ linearly to $\mathscr{A}_f^{(x)}$. This map $\tilde{\varepsilon}$ has the following important properties. Let u^1 be an element in $\mathscr{A}_{f-1}^{(x)}$ which is u when considered in $\mathscr{A}_f^{(x)}$. Then

$$\tilde{\varepsilon}(u * v) = u^1 * \tilde{\varepsilon}(v)$$
$$\tilde{\varepsilon}(v * u) = \tilde{\varepsilon}(v) * u^1$$

for any $v \in \mathscr{A}_f^{(x)}$.

We will also use the element E_f in $\mathscr{A}_f^{(x)}$ defined by the diagram



Note that $E_f * u = u * E_f$ for u in $\mathscr{A}_{f-2}^{(x)}$. We will also use the antiisomorphism of $\mathscr{A}_f^{(x)}$ to $\mathscr{A}_f^{(x)}$ which turns each diagram upsidedown. We denote this $\delta \to \delta'$. The map is obtained by extending linearly.

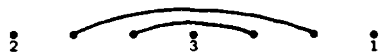
In [HW1] we determined the radicals of the $\mathscr{A}_f^{(x)}[k]$ in terms of the nullspaces of certain matrices $Z_\mu(x)$. We use these results later and so we briefly recall the definitions of the $Z_\mu(x)$ and the results relating them to the radicals of the $\mathscr{A}_f^{(x)}[k]$.

An *unlabelled* (m, k) partial 1-factor is a graph with $f=m+2k$ points, m of them isolated and $2k$ of them having degree 1. The set of unlabelled (m, k) partial 1-factors is denoted $\mathscr{B}_{m,k}$. A *labelled* (m, k) partial 1-factor is an (m, k) partial 1-factor in which the free points have been labelled with the integers $\{1, 2, ..., m\}$. The set of these is denoted $B_{m,k}$.

Let $W_{m,k}$ and $\mathscr{W}_{m,k}$ denote the vector spaces over $\mathbb{R}$ with bases $B_{m,k}$ and $\mathscr{B}_{m,k}$ respectively. As vector spaces we have $W_{m,k} = \mathscr{W}_{m,k} \otimes \mathbb{R}S_m$. If $\Delta \in \mathscr{B}_{m,k}$ and $\sigma \in S_m$ we let $\Delta \otimes \sigma$ denote the element in $B_{m,k}$ which is obtained from Δ by labelling the ith free point from the left by $\sigma(i)$. For example, if

$\Delta =$ 

and $\sigma = (1, 2, 3)$ then

$\Delta \otimes \sigma =$ 

Define a linear transformation $Z_{m,k}(x)$ on $W_{m,k}$ in the following way. For $\Delta_1, \Delta_2 \in B_{m,k}$, the Δ_1, Δ_2 entry of $Z_{m,k}(x)$ is determined by considering the graph $\Delta_1 \cup \Delta_2$. In this graph the free points of Δ_1 are joined to other free points of Δ_1 or Δ_2. We have $(Z_{m,k}(x))_{\Delta_1,\Delta_2} = 0$ unless the free point labelled i in Δ_1 is connected to the free point in Δ_2 also labelled i for $i = 1, 2, ..., m$. In this case,

$$(Z_{m,k}(x))_{\Delta_1,\Delta_2} = x^{\gamma(\Delta_1,\Delta_2)},$$

where $\gamma(\Delta_1, \Delta_2)$ is the number of cycles in $\Delta_1 \cup \Delta_2$. If

$\Delta_1 =$ 

$\Delta_2 =$ 

then

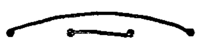
$\Delta_1 \cup \Delta_2 =$ 

The symmetric group S_m acts on $B_{m,k}$ by permuting the labels on the free points of each Δ. This action, extended to $W_{m,k}$, commutes with that of $Z_{m,k}(x)$. The irreducible representations of S_m are indexed by partitions μ of m. Let $Z_\mu(x)$ be the restriction of $Z_{m,k}(x)$ to the μ-isotypic component of $W_{m,k}$. Note that $Z_\mu(x)$ also depends on k.

We work out three examples, which combined appear in [HW1-3] to demonstrate these ideas. Suppose $f = 4$, $m = 0$, and $k = 2$. There are three labelled 0, 2 partial 1-factors

Δ_1

Δ_2 

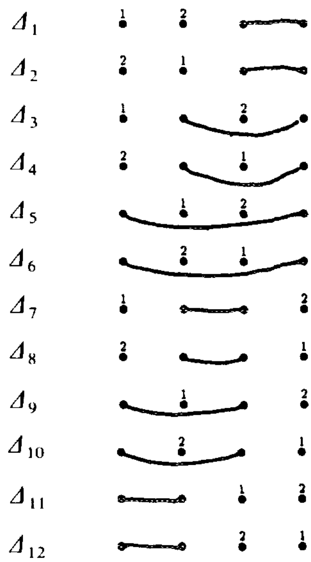
Δ_3  .

Now

$$Z_{0,2}(x)=\begin{pmatrix} x^2 & x & x \\ x & x^2 & x \\ x & x & x^2 \end{pmatrix}.$$

The only partition of 0 points is the empty partition, ϕ, and $Z_{\phi}(x)=Z_{0,2}(x)$.

Suppose $f=4$, $m=2$, and $k=1$. There are twelve $(2, 1)$ partial 1-factors.



The matrix $Z_{2,1}(x)$ is given by

$$\begin{bmatrix}
x & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & x & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & x & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & x & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & x & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & x & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & x & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & x & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & x & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & x & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & x & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & x
\end{bmatrix}.$$

There are two partitions of 2, namely, 2 and 1^2. The 2 isotypic component of $W_{2,1}$ is spanned by $\Delta_1+\Delta_2$, $\Delta_3+\Delta_4$, $\Delta_5+\Delta_6$, $\Delta_7+\Delta_8$, $\Delta_9+\Delta_{10}$, $\Delta_{11}+\Delta_{12}$. The 1^2 isotypic component is spanned by $\Delta_1-\Delta_2$, $\Delta_3-\Delta_4$, $\Delta_5-\Delta_6$, $\Delta_7-\Delta_8$, $\Delta_9-\Delta_{10}$, $\Delta_{11}-\Delta_{12}$. The matrices $Z_2(x)$ and $Z_{1^2}(x)$ are

$$Z_2(x)\begin{bmatrix} x & 1 & 1 & 1 & 1 & 0 \\ 1 & x & 1 & 1 & 0 & 1 \\ 1 & 1 & x & 0 & 1 & 1 \\ 1 & 1 & 0 & x & 1 & 1 \\ 1 & 0 & 1 & 1 & x & 1 \\ 0 & 1 & 1 & 1 & 1 & x \end{bmatrix}$$

$$Z_{1^2}(x)\begin{bmatrix} x & 1 & -1 & 1 & -1 & 0 \\ 1 & x & 1 & 1 & 0 & 1 \\ -1 & 1 & x & 0 & 1 & 1 \\ 1 & 1 & 0 & x & 1 & -1 \\ -1 & 0 & 1 & 1 & x & -1 \\ 0 & 1 & 1 & -1 & -1 & x \end{bmatrix}.$$

Let δ be an element in F_f having k horizontal lines in its top row. Then δ determines a pair (Δ_1, Δ_2) of labelled (m, k) partial 1-factors as follows. The edges of Δ_1 are the horizontal edges on the top row of δ. The remaining points of Δ_1 are isolated. Similarly the edges of Δ_2 are the horizontal edges on the bottom row of δ. The vertical edges of δ give a pairing between the isolated points of Δ_1 and the isolated points of Δ_2. We label the ith point of Δ_1 with j if it is joined to the jth isolated point of Δ_2. In this case we label the jth isolated point of Δ_2 with i. Note that $(\Delta_1, \Delta_2)=(\eta_1\otimes\sigma, \eta_2\otimes\sigma^{-1})$ for some $\eta_1, \eta_2\in\mathscr{B}_{m,k}$ and some $\sigma\in S_m$.

Let $F_f[k]$ denote the set of elements in F_f which have k horizontal edges in each row. Note that $F_f[k]$ is a basis for $\mathscr{A}_f^{(x)}[k]$. The above correspondence $\delta\leftrightarrow(\Delta_1, \Delta_2)$ in terms of vector spaces gives the isomorphism

$$\mathscr{A}_f^{(x)}[k]\cong W_{m,k}\otimes_{S_m} W_{m,k}. \tag{1.1}$$

We can now state the main result from [HW1] that we need.

THEOREM 1.2. *Let $K_{m,k}$ be the nullspace of $Z_{m,k}(x)$. In terms of the isomorphism* (1.1) *we have*

$$\mathrm{Rad}(\mathscr{A}_f^{(x)}[k])=(K_{m,k}\otimes_{S_m} W_{m,k})+(W_{m,k}\otimes_{S_m} K_{m,k}).$$

We assume that x is not zero in our proofs. Many of the constructions and proofs apply when x is zero but we will not comment further on this.

2. The Tower Construction

The *tower construction* is a method for constructing a tower of semisimple algebras. This construction has the important feature that it gives precise information about the structure of the algebras in the tower. This method was invented by Vaughan Jones [VJ] and has been used subsequently by several authors to achieve a number of striking results in algebra, topology, and combinatorics [GHJ].

In this section we describe this method in a slightly more algebraic form. The point of this is to obtain a construction that we can use when the algebras in the tower are not semisimple.

Let R be a finite dimensional semisimple $\mathbb{R}$-algebra with matrix ring decomposition

$$R = \bigoplus_{\mu} R_{\mu}. \tag{2.1}$$

Here R_μ is a d_μ by d_μ matrix ring. We let V_μ denote the corresponding irreducible representation of R. Choose bases z^μ_{ij} and e^μ_i for R_μ and V_μ such that

$$\begin{aligned} z^\mu_{ij} z^\mu_{rs} &= \delta_{jr} z^\mu_{is} \\ z^\mu_{ij} e^\mu_r &= \delta_{jr} e^\mu_i. \end{aligned} \tag{2.2}$$

Finally, let $x \to x'$ be the anti-isomorphism of R which is transposition on each R_μ.

Next let M be a left R-module which decomposes into irreducibles as

$$M = \bigoplus_{\mu} g_\mu V_\mu.$$

For each μ choose g_μ copies of V_μ in M and denote them $V_\mu(1), ..., V_\mu(g_\mu)$. Let $V_\mu(l)$ have basis $\{e^\mu_i(l)\}$. According to this notation $V_\mu(l)$ is isomorphic to V_μ via the map which sends $e^\mu_i(l)$ to e^μ_i. Also let M_μ denote the V_μ-isotypic component of the left R-module M.

Finally, assume we are also given a right R-module M' and a linear isomorphism $m \to m'$ from M to M' which satisfies

$$(rm)' = m'r'.$$

If n is in M', define n' to be the m in M for which $m' = n$. In particular $(m')' = m$ and $(n')' = n$.

DEFINITION 2.3. A *J-map* ε is an $\mathbb{R}$ bilinear map from $M \times M'$ to R which satisfies

(A) $\varepsilon(rx, y) = r\varepsilon(x, y)$

(B) $\varepsilon(x, yr) = \varepsilon(x, y)\, r$

(C) $\varepsilon(x, y)' = \varepsilon(y', x')$.

Let $\Omega(M, R)$ denote the vector space of J-maps from $M \times M$ to R which we may identify with maps from $M \otimes_{\mathbb{R}} M$ to R.

THEOREM 2.4. *There is a natural identification between $\Omega(M, R)$ and the space of symmetric matrices in $\bigoplus_\mu \operatorname{End}(\mathbb{R}^{g_\mu})$. In particular,*

$$\dim(\Omega(M, R)) = \sum_\mu \binom{g_\mu + 1}{2}.$$

Proof. The proof proceeds by a series of simple claims. For the most part the proofs are left to the reader.

CLAIM 1. *Let $x \in M_\mu$ and $y \in M'_\lambda$ where λ is different from μ. Then*

$$\varepsilon(x, y) = 0.$$

CLAIM 2. *Fix μ. For $l, m \in \{1, 2, ..., g_\mu\}$ and $i, j \in \{1, 2, ..., \dim V_\mu\}$ we have*

$$\varepsilon(e_i^\mu(l), e_j^\mu(m)') = h^\mu_{l,m,i,j} z^\mu_{ij}$$

for some $h^\mu_{l,m,i,j} \in \mathbb{R}$.

The content of Claim 2 is that ε applied to the pair $e_i^\mu(l)$, $e_j^\mu(m)'$ is some multiple of z^μ_{ij}.

CLAIM 3. *$h^\mu_{l,m,i,j}$ is independent of i and j.*

Proof.

$$\begin{aligned} h^\mu_{l,m,i,j} z^\mu_{ij} &= \varepsilon(e_i^\mu(l), e_j^\mu(m)') \\ &= \varepsilon(z^\mu_{i1} e_1^\mu(l), e_1^\mu(m)'\, z^\mu_{1j}) \\ &= z^\mu_{i1} h_{l,m,1,1} z^\mu_{11} z^\mu_{1j} \\ &= h_{l,m,1,1} z^\mu_{ij}. \end{aligned}$$

So $h^\mu_{l,m,i,j} = h^\mu_{l,m,1,1}$ for all i, j which proves Claim 3.

In view of Claim 3 we can define a $g_\mu \times g_\mu$ matrix H_μ whose l, m entry is the common value of $h^\mu_{l,m,i,j}$.

CLAIM 4. *H_μ is symmetric.*

Claim 1–Claim 4 show that any J-map ε determines a sequence (H_μ) of symmetric matrices in $\oplus_\mu \operatorname{End}(\mathbb{R}^{g_\mu})$. Conversely any sequence (H_μ) determines a J-map ε by

$$\varepsilon(e_i^\mu(l), e_j^\mu(m)') = h_{l,m}^\mu z_{ij}^\mu.$$

It is straightforward to check that this correspondence between J-maps and sequences of symmetric matrices is an isomorphism. ∎

Given a J-map ε, we call the corresponding sequence (H_μ) the *characteristic sequence of* ε.

DEFINITION 2.5. Given a J-map $\varepsilon: M \otimes_{\mathbb{R}} M \to R$ define an algebra $\mathscr{A} = \mathscr{A}(R, M, \varepsilon)$ with $\mathbb{R}$-basis $M' \otimes_R M$ and multiplication $*$ given by

$$\begin{aligned}(u' \otimes v) * (a' \otimes b) &= u' \cdot \varepsilon(v, a') \otimes b \\ &= u' \otimes \varepsilon(v, a')\, b.\end{aligned}$$

Conditions (A) and (B) on ε imply that $*$ is well-defined on $M' \otimes_R M$. Also one can easily check that $*$ is associative.

In this paper we will be interested in the structure of the algebras $\mathscr{A}(R, M, \varepsilon)$. As we have seen, the map ε is determined by the sequence of symmetric matrices (H_μ). So we can ask how to determine the structure of the algebra $\mathscr{A}(R, M, \varepsilon)$ from the matrices (H_μ). The next two results explain how to do this.

DEFINITION 2.6. Let H be a symmetric d by d matrix over $\mathbb{R}$. Define an $\mathbb{R}$-algebra $A(H)$ with $\mathbb{R}$-basis $\mathbb{R}^d \otimes \mathbb{R}^d$ and multiplication $\circ$ by

$$(\mathbf{u} \otimes \mathbf{v}) \circ (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{v}' H \mathbf{a})(\mathbf{u} \otimes \mathbf{b}).$$

The following result is not difficult to prove (see, for example, [HW1]).

THEOREM 2.7. *Let N and I be the nullspace and range of H. Then*

(1) $\operatorname{Rad}(A(H)) = (N \otimes \mathbb{R}^d) + (\mathbb{R}^d \otimes N)$

(2) $A(H)/\operatorname{Rad}(A(H)) \cong \operatorname{End}(I)$.

Proof. The subspace $N \otimes \mathbb{R}^d$ is a left ideal in $A(H)$ for which all left products are zero. Consequently $N \otimes \mathbb{R}^d$ is a nilpotent left ideal and so in $\operatorname{Rad}(A(H))$. Similarly $\mathbb{R}^d \otimes N$ is in $\operatorname{Rad}(A(H))$. The dimension of $N \otimes \mathbb{R}^d + \mathbb{R}^d \otimes N$ is $2sd - s^2$ where s is the dimension of N. Let $v_1, \ldots, v_r$ be an orthogonal basis of eigenvectors for I where $H(v_i) = \lambda_i v_i$ with $\lambda_i \neq 0$. There is such a basis as H is symmetric. The subalgebra spanned by $\{v_i \otimes v_j\}$ is isomorphic to End(I). Consequently $A(H)/(N\otimes\mathbb{R}^d+\mathbb{R}^d\otimes N)\cong\mathrm{End}(I)$ and $\mathrm{Rad}(A(H))=N\otimes\mathbb{R}^d+\mathbb{R}^d\otimes N$. ∎

Using this it is a simple matter to determine the algebra structure of $A(H)$ given the eigenspaces of H. We now come to the result which describes the algebra structure of $\mathscr{A}(R, M, \varepsilon)$ in terms of the characteristic sequence (H_μ) of ε.

THEOREM 2.8. *Let $\varepsilon: M\otimes_{\mathbb{R}} M\to R$ be a J-map with characteristic sequence (H_μ). Then*

$$\mathscr{A}(R, M, \varepsilon)\cong\bigoplus_\mu A(H_\mu),$$

where the sum on the right is a direct sum of algebras. In particular, the algebra structure of $\mathscr{A}(R, M, \varepsilon)$ is completely determined by the nullspaces and ranges of the matrices H_μ.

Proof. It is straightforward to see that

$$M'\otimes_R M=\bigoplus_\mu (M'_\mu\otimes_R M_\mu) \tag{2.9}$$

and that each summand on the right is an ideal in $\mathscr{A}(R, M, \varepsilon)$. So the direct sum on the right side of (2.9) is a direct sum of algebras. Hence it is enough to show that $M'_\mu\otimes_R M_\mu$ is isomorphic to $A(H_\mu)$.

By Schur's Lemma we have

$$M'_\mu\otimes_R M_\mu\cong(\mathbb{R}^{g_\mu}\otimes_{\mathbb{R}}\mathbb{R}^{g_\mu}), \tag{2.10}$$

where (2.10) is an isomorphism of vector spaces. An explicit isomorphism φ from the right-hand side of (2.10) to the left-hand side is given by

$$\varphi(\boldsymbol{\alpha}\otimes\boldsymbol{\beta})=\sum_{l,m}\alpha_l\beta_m(e_1^\mu(l)'\otimes e_1^\mu(m)). \tag{2.11}$$

Recall that a vector space basis for $A(H_\mu)$ is $\mathbb{R}^{g_\mu}\otimes\mathbb{R}^{g_\mu}$. It is straightforward to check $e_1^\mu(l)'\otimes e_1^\mu(m)=e_i^\mu(l)'\otimes e_i^\mu(m)$ and $e_i^\mu(l)'\otimes e_j^\mu(m)=0$ if $i\neq j$. Consequently, φ is surjective. So we can consider φ to be a vector space isomorphism from $A(H_\mu)$ to $M'_\mu\otimes_R M_\mu$. It remains to show that φ is an algebra homomorphism. For $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}\in\mathbb{R}^{g_\mu}$ we have

$$\begin{aligned}\varphi((\boldsymbol{\alpha}\otimes\boldsymbol{\beta})\circ(\boldsymbol{\gamma}\otimes\boldsymbol{\delta}))&=\varphi((\boldsymbol{\beta}'H_\mu\boldsymbol{\gamma})(\boldsymbol{\alpha}\otimes\boldsymbol{\delta}))\\&=\left\{\sum_{l,m}(\beta_l h^\mu_{l,m}\gamma_m)\right\}\left\{\sum_{r,s}\alpha_r\delta_s e_1^\mu(r)'\otimes e_1^\mu(s)\right\}\\&=\sum_{l,m,r,s}\alpha_r\beta_l\gamma_m\delta_s\{(e_1^\mu(r)'\otimes e_1^\mu(l))*(e_1^\mu(m)'\otimes e_1^\mu(s))\}\end{aligned}$$

here using the observation that

$$(e_1^\mu(r)' \otimes e_1^\mu(l)) * (e_1^\mu(m)' \otimes e_1^\mu(s)) = e_1^\mu(r)' \, (h^\mu_{l,m} z^\mu_{11}) \otimes e_1^\mu(s)$$
$$= h^\mu_{l,m}(e_1^\mu(r)' \otimes e_1^\mu(s)).$$

So

$$\varphi((\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \circ (\boldsymbol{\gamma} \otimes \boldsymbol{\delta})) = \left\{ \sum_{r,l} \alpha_r \beta_l e_1^\mu(r)' \otimes e_1^\mu(l) \right\} * \left\{ \sum_{m,s} \gamma_m \delta_s e_1^\mu(m)' \otimes e_1^\mu(s) \right\}$$
$$= \varphi(\mathbf{a} \otimes \mathbf{b}) * \varphi(\boldsymbol{\gamma} \otimes \boldsymbol{\delta}). \quad \blacksquare$$

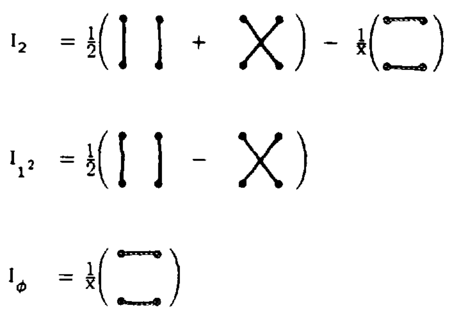
Theorems 2.7 and 2.8 together give a complete description of the algebra $\mathscr{A}(R, M, \varepsilon)$ in terms of the eigenspaces of the characteristic matrices H_μ. We end with an example which demonstrates how we use these results. See [Bour] for properties of Rad($A(H)$).

EXAMPLE 2.11. Let $R = \mathscr{A}_2^{(x)}$ and $M = \mathscr{A}_3^{(x)}$ where x is nonzero. Recall the linear map $\tilde{\varepsilon}: \mathscr{A}_3^{(x)} \to \mathscr{A}_2^{(x)}$ and the anti-isomorphism $\delta \to \delta'$ each of which is defined in Section 1. Let $\varepsilon: M \otimes_R M \to R$ be defined by

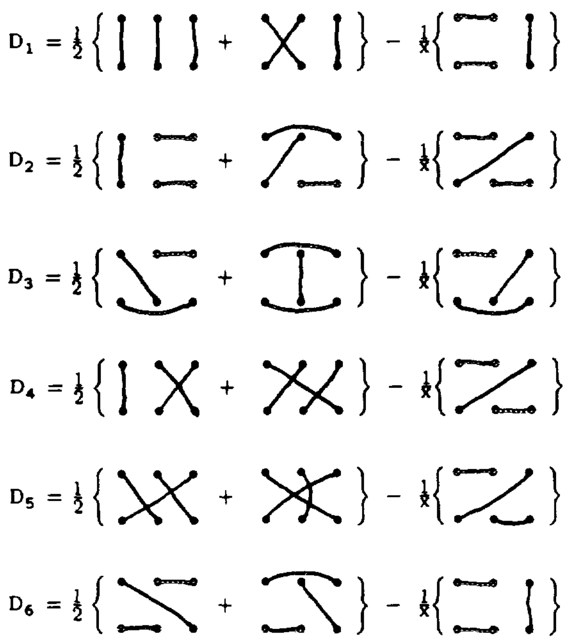
$$\varepsilon(a \otimes b) = \tilde{\varepsilon}(a * b).$$

It is easy to check that ε is a J-map. Wenzl [W2] showed that $\mathscr{A}(R, M, \varepsilon)$ is isomorphic to $\mathscr{A}_4^{(x)}(1)$ by using the fact that $\mathscr{A}_2^{(x)}$ is semisimple. So the algebra structure of $\mathscr{A}_4^{(x)}(1)$ can be determined from the characteristic sequence of ε via Theorems 2.7 and 2.8. We now briefly discuss the computation of the matrices in the characteristic sequence of ε.

R has three one-dimensional irreducible representations. The corresponding matrix rings in R are spanned by the idempotents:

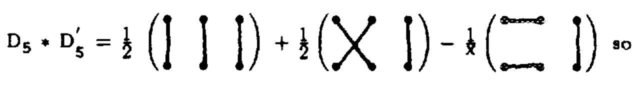


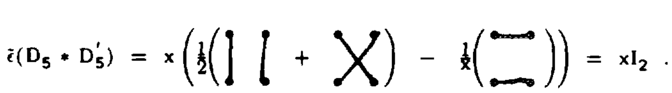
The 15-dimensional left module $\mathscr{A}_3^{(x)}$ splits into 6 copies of the first irreducible ($g_2 = 6$), 6 copies of the second irreducible ($g_{1^2} = 6$) and 3 copies of the third irreducible ($g_\phi = 3$). So matrices (H_2, H_{1^2}, H_ϕ) in the characteristic sequence are 6 by 6, 6 by 6, and 3 by 3 respectively. We compute some sample entries in the matrix H_2. To do so we need to determine the V_2-isotypic component in $\mathscr{A}_3^{(x)}$ explicitly. It is spanned by the following 6 vectors:



To compute the i, j entry in H_2 we apply ε to the pair $D_i \otimes D_j'$ (i.e., we compute $\tilde{\varepsilon}(D_i * D_j')$). This will yield a multiple of I_2 in $\mathscr{A}_2^{(x)}$. That multiple is $(H_2)_{i,j}$. We do three examples:

(1) Computation of $(\mathrm{H}_2)_{5,5}$. We need to compute $\varepsilon(D_5 \otimes D_5') = \tilde{\varepsilon}(D_5 * D_5')$.

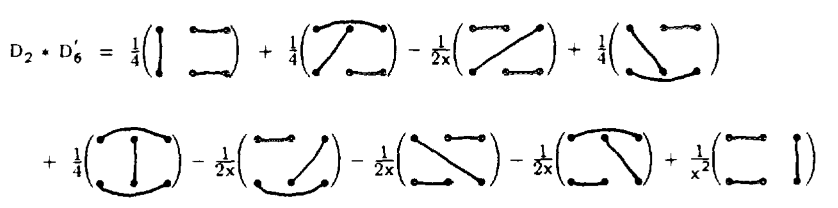


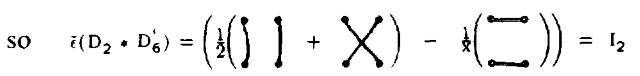


Thus $(H_2)_{5,5} = x$.

(2) Computation of $(H_2)_{3,4}$. We need to compute $\varepsilon(D_3 \otimes D_4') = \tilde{\varepsilon}(D_3 * D_4')$. But $D_3 * D_4' = 0$ so $(H_2)_{3,4} = 0$.

(3) Computation of $(H_2)_{2,6}$. We need to compute $\varepsilon(D_2, D_6') = \tilde{\varepsilon}(D_2 * D_6')$.





Thus $(H_2)_{2,6} = 1$.

These three computations are typical of those needed to compute the entries of H_2, H_{1^2}, and H_ϕ. The final matrices turn out to be

$$H_2 = \begin{bmatrix} x & 1 & 1 & 1 & 1 & 0 \\ 1 & x & 1 & 1 & 0 & 1 \\ 1 & 1 & x & 0 & 1 & 1 \\ 1 & 1 & 0 & x & 1 & 1 \\ 1 & 0 & 1 & 1 & x & 1 \\ 0 & 1 & 1 & 1 & 1 & x \end{bmatrix}$$

$$H_{1^2} = \begin{bmatrix} x & 1 & -1 & 1 & -1 & 0 \\ 1 & x & 1 & 1 & 0 & 1 \\ -1 & 1 & x & 0 & 1 & 1 \\ 1 & 1 & 0 & x & 1 & -1 \\ -1 & 0 & 1 & 1 & x & -1 \\ 0 & 1 & 1 & -1 & -1 & x \end{bmatrix}$$

$$H_\phi = \begin{bmatrix} x^2 & x & x \\ x & x^2 & x \\ x & x & x^2 \end{bmatrix}$$

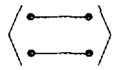
The reader should compare these matrices to the matrices $Z_2(x)$, $Z_{1^2}(x)$, and $Z_\phi(x)$ from Section 1.

3. The General Construction

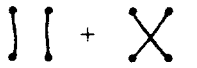
3.1. *An Example*

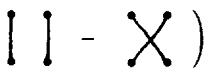
We begin by looking at a specific example which demonstrates our general procedure. Return to the situation considered in Example 2.11 but this time assume $x = 0$. So $R = \mathscr{A}_2^{(0)}$ is not semisimple. We want to see how the method used in Example 2.11 can be modified to analyze $\mathscr{A}_4^{(0)}(1)$ in this case.

The radical of R is

$N =$ .

R/N is a sum of two 1-by-1 matrix rings with idempotents

$I_2 = \frac{1}{2}\Big($  $\Big) + N$

$I_{1^2} = \frac{1}{2}\Big($  $\Big) + N$.

As explained in Section 1, consider N in $\mathscr{A}_3^{(0)}$.
Let $\bar{R} = R/N$ and let $\bar{M} = M/NM$ (where $M = \mathscr{A}_3^{(0)}$). It is straightforward to check that NM has basis

So $\bar{M}$ is 12-dimensional. As an $\bar{R}$-module it consists of 6 copies of the irreducible $\bar{V}_2$ and 6 copies of the irreducible $\bar{V}_{1^2}$.

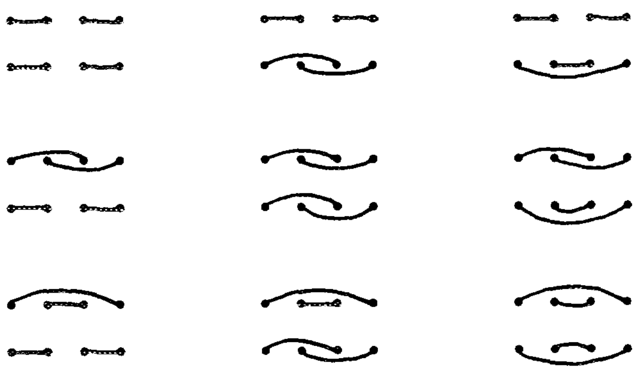
The map $\varepsilon: M \otimes_{\mathbb{R}} M \to R$ satisfies

$$\varepsilon(NM \otimes_{\mathbb{R}} M) \subseteq N$$
$$\varepsilon(M \otimes_{\mathbb{R}} (NM)') \subseteq N.$$

So ε induces a map $\bar{\varepsilon}: \bar{M} \otimes_R \bar{M}' \to \bar{R}$ which turns out to be a J-map.

There are two main ingredients to our general construction.

1. *Inheritance.* Let $J^{(1)} \subseteq \mathscr{A}_4^{(0)}(1)$ be the span of all vectors obtained by taking our vector  in N and adding a new horizontal edge in the top row and a new horizontal edge in the bottom row. So $N^{(1)}$ is the span of the nine vectors below:



We show that $N^{(1)}$ is a 2-sided ideal contained in the radical of $\mathscr{A}_4^{(0)}(1)$. We say that $N^{(1)}$ is the piece of the radical *inherited* from the radical of $\mathscr{A}_2^{(0)}$.

2. *Tower Construction.* We then show that

$$\mathscr{A}_4^{(0)}(1)/N^{(1)} \cong \mathscr{A}(\bar{R}, \bar{M}, \bar{\varepsilon}).$$

Recall that $\mathscr{A}(\bar{R}, \bar{M}, \bar{\varepsilon})$ may have a radical $\mathscr{J}$ of its own. That gives us the following picture of $\mathscr{A}_4^{(0)}(1)$:

$$\begin{array}{ccccc}
 & & \mathscr{A}_4^{(0)}(1) & \longleftrightarrow & \mathscr{A}(\bar{R}, \bar{M}, \bar{\varepsilon}) \\
 & & | & & | \\
\mathscr{A}_2^{(0)} & & N^{(0)} & \longleftrightarrow & \mathscr{J} \\
| & & | & & | \\
N & \xrightarrow{\text{inheritance procedure}} & N^{(1)} & \longleftrightarrow & 0
\end{array},$$

where $N^{(0)}$ is the true radical of $\mathscr{A}_4^{(0)}(1)$.

3.2. *The Inherited Piece of the Radical*

Assume we are in the situation where $\mathscr{A}_{f-2}^{(x)}$ has a non-trivial radical N_{f-2}. In this subsection we show how to construct, from N_{f-2}, an ideal $N_f^{(1)}$ in the radical of $\mathscr{A}_f^{(x)}$. We say that the ideal $N_f^{(1)}$ is *inherited* from the radical of $\mathscr{A}_{f-2}^{(x)}$. For convenience of notation we denote $\mathscr{A}_f^{(x)}$ by $\mathscr{A}_f$.

DEFINITION 3.2.1. Let δ be a diagram in F_{f-2} and let (r, s) and (a, b) be pairs of numbers with $1 \leqslant r < s \leqslant f$ and $1 \leqslant a < b \leqslant f$. Define $\delta_{a,b}^{r,s}$ to be the diagram in F_f obtained from δ by inserting a new horizontal edge in the top row joining points r and s and a new horizontal edge in the bottom row joining points a and b.

For example, if $\delta =$  $\in F_5$ then

$\delta_{2,3}^{5,7} =$ 

The stars in the diagram $\delta_{2,3}^{5,7}$ indicate the new edges that were added to δ.

Extend this notation linearly, i.e., given

$$v = \sum_{\delta} a_\delta \delta \in \mathscr{A}_{f-2}$$

define $v^{r,s}_{a,b} = \sum_\delta a_\delta \delta^{r,s}_{a,b} \in \mathscr{A}_f$. Note that if v is in $\mathscr{A}_{f-2}(t)$ then $v^{r,s}_{a,b}$ is in $\mathscr{A}_f(t+1)$.

We begin with a technical lemma.

LEMMA 3.2.2. *Let (a, b) and (u, v) be pairs satisfying $1 \leqslant a < b \leqslant f$ and $1 \leqslant u < v \leqslant f$. Then there exists $\rho = \rho(a, b, u, v) \in \mathscr{A}_{f-2}$ such that for any $\delta, \pi \in F_{f-2}$ and any pairs (r, s), (y, z) we have*

$$\delta^{r,s}_{a,b} * \pi^{u,v}_{y,z} = (\delta * \rho * \pi)^{r,s}_{y,z}.$$

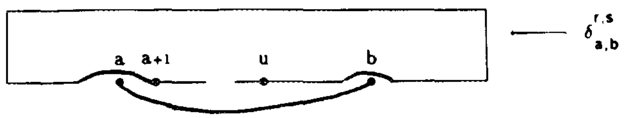
Proof. We define ρ according to the following three cases:

Case 1. $a = u$, $b = v$. In this case we have

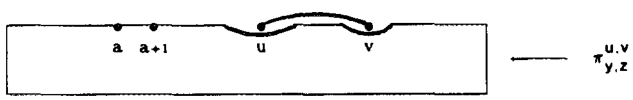
$$\delta^{r,s}_{a,b} * \pi^{u,v}_{y,z} = x(\delta * \pi)^{r,s}_{y,z},$$

so we can take ρ to be x times the identity.

Case 2. Exactly one of the equalities $a = u$, $a = v$, $b = u$, $b = v$ holds. We assume that $a < u < b = v$ (the other possibilities are handled in a similar way). Pictorially we have



(3.2.3)



Consider the points $\{a, a+1, ..., u\}$. In $\delta^{r,s}_{a,b}$ the points $\{a+1, ..., u\}$ are incident to the edges that were incident to $\{a, a+1, ..., u-1\}$ in δ. In $\pi^{u,v}_{y,z}$ the points $\{a, a+1, ..., u-1\}$ are incident to the same edges they were incident to in π. So when the product $\delta^{r,s}_{a,b} * \pi^{u,v}_{y,z}$ is formed one goes from the edge that was incident to p in δ to the edge that was incident to $\hat{p}$ in π where $\hat{p}$ is defined by

$$\hat{p} = \begin{cases} p+1 & \text{if } p \in \{a, a+1, ..., u-1\} \\ a & \text{if } p = u \\ p & \text{otherwise.} \end{cases} \tag{3.2.4}$$

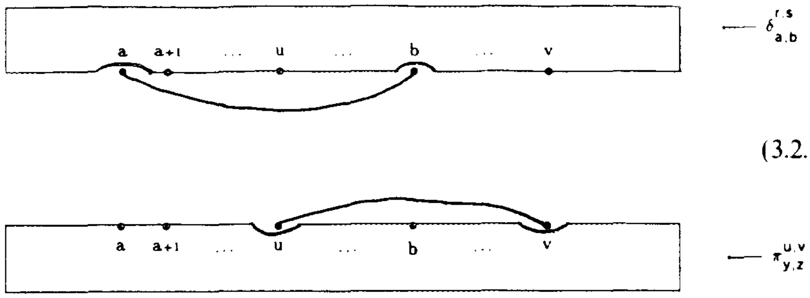
The case $p = u$ in (3.2.4) can be seen by considering the diagram (3.2.3).

It follows that we can take ρ to be the permutation

$$\rho = (a, a+1, ..., u)$$

in this case.

Case 3. None of the equalities $a=u$, $a=v$, $b=u$, $b=v$ hold. We assume $a<u<b<v$ (the other possibilities are handled in a similar way). Pictorially we have

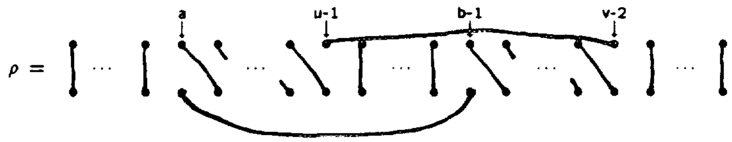


(3.2.5)

When the product $\delta^{r,s}_{a,b} * \pi^{u,v}_{y,z}$ is formed, one goes from the edge that was incident to p in the bottom row of δ to the edge that was incident to $\hat{p}$ in the top row of π where

$$\hat{p} = \begin{cases} p+1 & \text{if } p \in \{a, a+1, ..., u-2\} \cup \{b-1, ..., v-3\} \\ p & \text{if } p \in \{1, 2, ..., a-1\} \cup \{u, ..., b-2\} \cup \{v-1, ..., f\}. \end{cases}$$

Also one goes from the edge that was incident to a in the top row of π to the edge that was incident to $b-1$ in the top row of π and from the edge that was incident to $u-1$ in the bottom row of δ to the edge that was incident to $v-2$ in the bottom row of δ. It follows that we can take ρ to be



This completes the proof of the lemma. ∎

Definition 3.2.6. Let $\mathscr{I}$ be a linear subspace of $\mathscr{A}_{f-2}$. Define $\mathscr{I}^{(1)}$ to be the linear subspace of $\mathscr{A}_f$ spanned by the set of all $v^{r,s}_{a,b}$ such that $v \in \mathscr{I}$, $1 \leqslant r < s \leqslant f$ and $1 \leqslant a < b \leqslant f$.

LEMMA 3.2.7. *Let $\mathcal{I}$ and $\mathcal{J}$ be subspaces of $\mathcal{A}_{f-2}$.*

(a) *If $\mathcal{I}$ is a left (respectively right) ideal in $\mathcal{A}_{f-2}$ then $\mathcal{I}^{(1)}$ is a left (respectively right) ideal in $\mathcal{A}_f$.*

(b) *If $\mathcal{J}$ is a left ideal or $\mathcal{I}$ is a right ideal then*

$$\mathcal{I}^{(1)} * \mathcal{J}^{(1)} \subseteq (\mathcal{I} * \mathcal{J})^{(1)}.$$

Proof. We first prove (a). We assume that $\mathcal{I}$ is a left ideal in $\mathcal{A}_{f-2}$. It is enough to show that

$$\delta * v_{y,z}^{r,s} \in \mathcal{I}^{(1)}$$

for all diagrams δ in F_f and v in $\mathcal{I}$.

Case 1. δ is a permutation. One easily sees that

$$\delta * v_{y,z}^{r,s} = (\hat{\delta} * v)_{y,z}^{\delta^{-1}r,\, \delta^{-1}s} \qquad \text{for some} \quad \hat{\delta} \in F_{f-2}$$

Since $\mathcal{I}$ is a left deal, $\hat{\delta} * v$ is in $\mathcal{I}$ which completes this case.

Case 2. δ is not a permutation. Then we can write $\delta = \hat{\delta}_{a,b}^{p,q}$ for some pairs (p, q) and (a, b) and some $\hat{\delta} \in F_{f-2}$. By Lemma 3.2.2 we have

$$\delta * v_{y,z}^{r,s} = (\hat{\delta} * \rho(a, b, r, s) * v)_{y,z}^{p,q} \in \mathcal{I}^{(1)}$$

The last inclusion holds because $\mathcal{I}$ is a left ideal containing v.
This proves part (a).

To prove (b), suppose $u \in \mathcal{I}$ and $v \in \mathcal{J}$. Then

$$u_{a,b}^{r,s} * v_{y,z}^{p,q} = (u * \rho(a, b, p, q) * v)_{y,z}^{r,s} \in (\mathcal{I} * \mathcal{J})^{(1)}$$

The latter inclusion holds because $u * \rho(a, b, p, q) \in \mathcal{I}$ or $\rho(a, b, p, q) * v \in \mathcal{J}$ depending on whether $\mathcal{I}$ is a right ideal or $\mathcal{J}$ is a left ideal. This proves part (b). ∎

DEFINITION 3.2.8. For each f let N_f denote the radical of $\mathcal{A}_f$ and let $N_f^{(1)}$ denote $(N_{f-2})^{(1)}$.

THEOREM 3.2.9. *For any f, $N_f^{(1)}$ is a two-sided ideal of $\mathcal{A}_f$ contained in the radical N_f.*

Proof. It follows immediately from Lemma 3.2.7(a) that $N_f^{(1)}$ is a two-sided ideal of $\mathcal{A}_f$. Repeated use of Lemma 3.2.7(b) shows it is nilpotent. ∎

Theorem 3.2.9 shows how to construct a piece of the radical of $\mathcal{A}_f$ from the radical of $\mathcal{A}_{f-2}$. We call this piece of the radical, $N_f^{(1)}$, the *hereditary component* and we say that this component was *inherited* from $\mathcal{A}_{f-2}$.

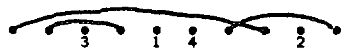
In Section 4 we show how to analyze the quotient $\mathscr{A}_f/N_f^{(1)}$ using the semisimple quotient $\mathscr{A}_{f-2}/N_{f-2}$.

3.3. *Heredity for the Matrices* $Z_{m,k}(x)$.

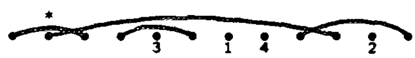
Recall the matrices $Z_{m,k}(x)$ defined in Section 1. The rows and columns of these matrices are indexed by the set $B_{m,k}$ of labelled (m, k) partial 1-factors. So $Z_{m,k}(x)$ can be considered to be a linear transformation of the vector space $W_{m,k}$ which has basis $B_{m,k}$. Let $K_{m,k} \subseteq W_{m,k}$ denote the kernel of $Z_{m,k}(x)$.

Let $f-2=m+2k$.

DEFINITION 3.3.1. Let $\rho \in B_{m,k}$ and let (a, b) be a pair with $1 \leqslant a < b \leqslant f$. Define $\rho^{a,b}$ to be the labelled $(m, k+1)$ partial 1-factor which has the same edges, free points and free point labels as ρ but which has a new edge from a to b.

For example, if ρ = 

then

$\rho^{1,3}$ = 

and $\rho^{6,9}$ = 

In these drawings the newly added edge is labelled with a star.

Extend this notation linearly. So if $v = \sum_\rho c_\rho \rho$ is in $V_{m,k}$ then

$$v^{a,b} = \sum_\rho c_\rho \rho^{a,b}.$$

DEFINITION 3.3.2. If $\mathscr{U}$ is a subspace of $W_{m,k}$ define $\mathscr{U}^{(1)}$ to be the subspace of $W_{m,k+1}$ spanned by all $v^{a,b}$ such that $v \in \mathscr{U}$ and $1 \leqslant a < b \leqslant f$,

$$\mathscr{U}^{(1)} = \langle v^{a,b} : v \in \mathscr{U},\ 1 \leqslant a < b \leqslant f \rangle.$$

The main result of this section is

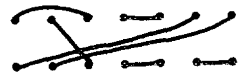
THEOREM 3.3.3. *For all* m, k *we have* $K_{m,k}^{(1)} \subseteq K_{m,k+1}$. *We call* $K_{m,k}^{(1)}$ *the inherited component of the nullspace of* $Z_{m,k+1}(x)$.

It is possible that Theorem 3.3.3 can be deduced using the description of the radical of $\mathscr{A}_{f-2}^{(x)}(k)$ and the results in Section 3.2. However, we give a purely combinatorial proof because we feel that this method of proof may shed light on the problem of determining the roots of the $\det(Z_\mu(x))$.

Recall that $\mathcal{A}_{f-2}^{(x)}[k]$ denotes the quotient algebra

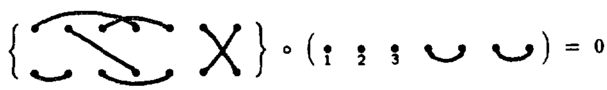
$$\mathcal{A}_{f-2}^{(x)}[k] = \mathcal{A}_{f-2}^{(x)}(k)/\mathcal{A}_{f-2}^{(x)}(k+1).$$

Let $\mathcal{W}_{m,k}$ denote the subspace of $\mathcal{A}_{f-2}^{(x)}[k]$ spanned by all diagrams for which the bottom row, δ^* has horizontal edges from $m+(2i-1)$ to $m+2i$ for $i=1, 2, ..., k$. It is easy to see that $\mathcal{W}_{m,k}$ is a left ideal of $\mathcal{A}_{f-2}^{(x)}[k]$ and is naturally isomorphic as a vector space to $W_{m,k}$. For example an isomorphism α takes the vector  in $W_{3,2}$ to the diagram

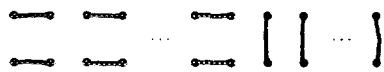
 $\in \mathcal{W}_{3,2}$

Since $\mathcal{A}_{f-2}^{(x)}[k]$ acts on $\mathcal{W}_{m,k}$ it acts (via α) on $W_{m,k}$. Let $\circ$ denote this action. For example we have

and



LEMMA 3.3.4. *Let δ_0 be the diagram in $\mathcal{A}_{f-2}^{(x)}[k]$ given by*

$\delta_0 =$ 

Define $z_{m,k}$ to be the element of $\mathcal{A}_{f-2}^{(x)}[k]$ given by

$$z_{m,k} = (2^k k!\, m!)^{-1}\left(\sum_{\sigma \in S_{f-2}} \sigma^{-1} * \delta_0 * \sigma\right).$$

Then for $v \in W_{m,k}$ we have

$$Z_{m,k}(x)\, v = z_{m,k} \circ v.$$

Proof. Let $T_{m,k}$ be the set of diagrams in $\mathcal{A}_{f-2}^{(x)}[k]$ which have the same k horizontal lines in the top row and bottom row and which have the property that each vertical line is incident to the same points in the top row and bottom row. In terms of our previous notation,

$$T_{m,k} = \{\delta \otimes \delta \otimes \mathrm{id} : \delta \in \mathcal{B}_{m,k}\},$$

where id denotes the identity permutation in S_m. An alternate description of $z_{m,k}$ is

$$z_{m,k} = \sum_{\Delta \in T_{m,k}} \Delta. \tag{3.3.5}$$

Return now to the proof of Lemma 3.3.4. It is enough to prove the lemma in the case that $v = \rho_0$ is an element $B_{m,k}$. Let ρ_1 be another element of $B_{m,k}$. We show that the coefficient of ρ_1 in $z_{m,k} \circ \rho_0$ equals the ρ_1, ρ_0 entry of $Z_{m,k}(x)$.

Write $\rho_1 = \delta \otimes \sigma$ where $\delta \in \mathscr{B}_{m,k}$ and $\sigma \in S_m$. Define $\Delta_1 = \delta \otimes \delta \otimes \mathrm{id}$. Suppose that $\Delta \circ \rho_0$ is a nonzero multiple of ρ_1 for $\Delta \in T_{m,k}$. Then the horizontal lines on the top row of Δ must equal the edges of ρ_1. But each element of $T_{m,k}$ is completely determined by the horizontal lines in its top row. So if $\Delta \circ \rho_0$ is a multiple of ρ_1 then $\Delta = \Delta_1$. So it suffices to prove that

$$\Delta_1 \circ \rho_0 = (Z_{m,k}(x))_{\rho_1, \rho_0}\, \rho_1 \tag{3.3.6}$$

Case 1. In $\rho_0 \cup \rho_1$ there is a path joining two free points of ρ_0. In this case we have $(Z_{m,k}(x))_{\rho_1, \rho_0} = 0$.

Let $a = v_0, v_1, ..., v_{2l} = b$ be a path in $\rho_0 \cup \rho_1$ joining the free points a and b of ρ_0. Then there is an edge of ρ_1 from v_{2i} to v_{2i+1} for all i. Hence there is an edge in the bottom row of Δ_1 from v_{2i} to v_{2i+1} for all i. So the above path is also a path in the union of ρ_0 with the bottom row of Δ_1. Thus $\Delta_1 \circ \rho_0 = 0$ which proves (3.3.6) in this case.

Case 2. In $\rho_1 \cup \rho_0$ all m paths have one endpoint in ρ_0 and one endpoint in ρ_1. This case uses similar sorts of arguments to those used in Case 1. We leave details to the reader. ∎

We now come to the crucial computation.

LEMMA 3.3.7. *Let $v \in W_{m,k}$ and let (a, b) be a pair with $1 \leqslant a < b \leqslant f$. Then*

$$z_{m,k+1} \circ v^{a,b} = x((z_{m,k} \circ v)^{a,b}) + \sum_{u \neq a,b} \left\{((a, b, u) + (b, a, u)) * ((V'_u + \tfrac{1}{2}H'_u)(v))^{a,b}\right\},$$

where V'_u is the sum of all diagrams in $T_{m,k}$ which have a vertical edge at u' and H'_u is the sum of all diagrams in $T_{m,k}$ which have a horizontal edge at u' where

$$u' = \begin{cases} u & \text{if } u < a \\ u-1 & \text{if } a < u < b \\ u-2 & \text{if } u > b. \end{cases}$$

Here the cycle (a, b, u) refers to the diagram for the permutation corresponding to the 3-cycle (a, b, u).

Proof. We partition the set $T_{m,k+1}$ into disjoint subsets

$T_{m,k+1} = T_x \cup T_a \cup T_b \cup T_{ab} \cup T_0 \cup T_1$ where

$T_x = \{\Delta \in T_{m,k+1} : \Delta$ has horizontal edges joining a and $b\}$

$T_a = \{\Delta \in T_{m,k+1} : \Delta$ has a vertical edge at a but not $b\}$

$T_b = \{\Delta \in T_{m,k+1} : \Delta$ has a vertical edge at b but not $a\}$

$T_{ab} = \{\Delta \in T_{m,k+1} : \Delta$ has vertical edges at both a and $b\}$

$T_0 =$ The set of $\Delta \in T_{m,k+1}$ which have horizontal edges joining a to u and b to v where $u < v$.

$T_1 =$ The set of $\Delta \in T_{m,k+1}$ which have horizontal edges joining a to u and b to v where $u > v$.

We write $z_{m,k+1}$ as

$$z_{m,k+1} = z_x + z_a + z_b + z_{ab} + z_0 + z_1$$

where $z_* = \sum_{\Delta \in T_*} \Delta$.

We return now to the proof of the lemma. By linearity we may assume that $v = \rho \in B_{m,k}$. We calculate the contribution made to $z_{m,k+1} \circ \rho^{ab}$ by each summand $z_* \circ \rho^{ab}$.

$$z_x \circ \rho^{ab} = x(z_{m,k} \circ \rho)^{a,b}. \tag{3.3.8}$$

This is clear since the diagrams in T_x consist of the diagrams in $T_{m,k}$ but with an extra edge inserted between a and b in both the top and bottom rows:

$$z_a \circ \rho^{ab} = \sum_{u \neq a,b} (a, b, u) * ((V'_u \circ \rho)^{a,b}). \tag{3.3.9}$$

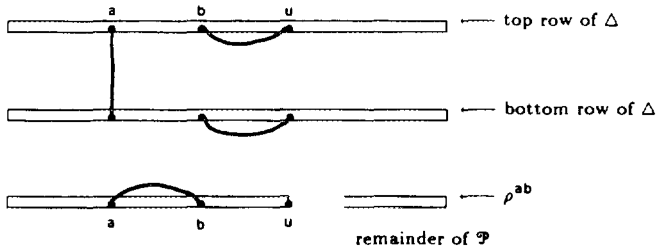
To see (3.3.9) consider a diagram $\Delta \in T_a$ which has horizontal edges from b to u. Let Δ' be the diagram in $T_{m,k}$ which is identical to Δ except that the points a and b have been removed from each row and a vertical edge has been inserted joining the uth point in the top row to the uth point in the bottom row. We claim that

$$\Delta \circ \rho^{ab} = (a, b, u)((\Delta' \circ \rho)^{ab}). \tag{3.3.10}$$

To see (3.3.10) consider what happens when we form the product $\Delta \circ \rho^{ab}$ by superimposing ρ^{ab} on the bottom row of Δ. Note that there is a path $\mathscr{P}$

which originates at the ath point in the top row of Δ, proceeds to the ath point in the bottom row of Δ, then goes to the bth point in the bottom row of Δ (along ρ^{ab}), and then on to the uth point in the bottom row of Δ (along Δ). In the top row we see the originating point of $\mathscr{P}$ at a and an edge from b to u.

$\Delta \circ \rho^{ab}$



In $(\Delta' \circ \rho)^{a,b}$ there is a path $\mathscr{P}'$ originating at the (u)th point in the top row and continuing on to the (u)th point in the bottom row. The crucial observation is the remainder of the path $\mathscr{P}'$ is the same as the remainder of the path $\mathscr{P}$. So the path $\mathscr{P}$ terminates with a point in the top row of Δ iff the path $\mathscr{P}'$ terminates with a point in the top row of Δ'. In that case we have

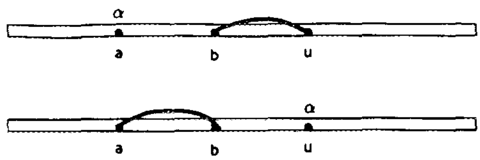
$$\Delta \circ \rho^{ab} = 0 \qquad \text{and} \qquad \Delta' \circ \rho = 0$$

so (3.3.10) holds.

Assume that $\mathscr{P}$ and $\mathscr{P}'$ terminate with a free point of ρ labelled α. Then the only differences in $\Delta \circ \rho^{a,b}$ and $(\Delta' \circ \rho)^{a,b}$ concern the points a, b, and u. What we see in each case is

$\Delta \circ \rho^{a,b}$

$(\Delta' \circ \rho)^{a,b}$



Equation (3.3.10) follows immediately. Note that as Δ runs over T_a we end up with exactly the Δ' which contribute to $\sum_{u \neq a,b} V'_u$ and (3.3.9) follows immediately.

By the same kinds of arguments we deduce that

$$z_b \circ \rho^{a,b} = \sum_{u \neq a,b} (b, a, u) * ((V'_u \circ \rho)^{a,b}) \tag{3.3.11}$$

$$(z_0 + z_1) \circ \rho^{a,b} = \tfrac{1}{2} \sum_{u \neq a,b} ((a, b, u) + (b, a, u)) * ((H'_u \circ \rho)^{a,b}). \tag{3.3.12}$$

This last equation comes from an equation

$$\Delta \circ \rho^{ab} = \tfrac{1}{2}((a, b, u) + (b, a, v))((\Delta' \circ \rho)^{ab}) \tag{3.3.13}$$

where Δ is in T_0 or T_1, Δ' is Δ with a, b deleted and u joined to v in the top and bottom.

Lastly note that $z_{ab} \circ \rho^{a,b} = 0$ because if $\Delta \in T_{ab}$ then $\Delta \circ \rho^{a,b}$ ends up with an edge from a to b in addition to the $(k+1)$ horizontal edges that occur in the top row of Δ. Lemma 3.3.7 follows from this observation and Eqs. (3.3.8), (3.3.9), (3.3.11), (3.3.12), and (3.3.13).

We can now return to the proof of Theorem 3.3.3. Let v be an element of $K_{m,k}$. Pick (a, b) with $1 \leqslant a < b \leqslant f$. We need to show that

$$z_{m,k+1} \circ v^{a,b} = 0.$$

We can compute $z_{m,k+1} \circ v^{a,b}$ using Lemma 3.3.7. By our choice of v we have $z_{m,k} \circ v = 0$. Let $u \in \{1, 2, ..., f\}$ with u not equal to a or b. Note that V'_u is the sum of all diagrams in $T_{m,k}$ which have a vertical edge at u'. Hence $V'_u \circ v$ is the projection of $z_{m,k} \circ v$ onto the subspace of $W_{m,k}$ spanned by all basis elements which have a free point at u'. Since $z_{m,k} \circ v = 0$ it follows that $V'_u \circ v = 0$. Similarly $H'_u \circ v$ is the projection of $z_{m,k} \circ v$ onto the subspace spanned by all basis elements in which u' is not a free point. So $H'_u \circ v = 0$. From Lemma 3.3.7 we have $z_{m,k+1} \circ v^{a,b} = 0$ as desired.

4. Application to $\mathscr{A}_f^{(x)}(1)$

In this section we use the above constructions to obtain information about a certain homomorphic image of $\mathscr{A}_f^{(x)}(1)$. Assume x is fixed. As in the previous section we denote $\mathscr{A}_f^{(x)}$ by $\mathscr{A}_f$. Let N_{f-2} be the Jacobson radical of $\mathscr{A}_{f-2}$ and $R = \mathscr{A}_{f-2}/N_{f-2}$. Here we consider $\mathscr{A}_{f-2} \subseteq \mathscr{A}_{f-1} \subseteq \mathscr{A}_f$ by adding vertices with one or two vertical lines to the right of the diagrams. Let $M = \mathscr{A}_{f-1}/N_{f-2}\mathscr{A}_{f-1}$. Note M is a left R module because $\mathscr{A}_{f-2}N_{f-2}\mathscr{A}_{f-1} \subseteq (\mathscr{A}_{f-2}N_{f-2})\mathscr{A}_{f-1} \subseteq N_{f-2}\mathscr{A}_{f-1}$, and so $\mathscr{A}_{f-2}/N_{f-2}$ acts on M. Of course R is semisimple.

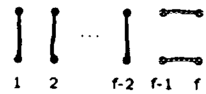
In order to apply the construction of $\mathscr{A}(R, M, \varepsilon)$ we need a map ε and an involution on M. For each diagram δ in $\mathscr{A}_{f-1}$ let δ' be the diagram turned upside down. Note $(N_{f-2}\mathscr{A}_{f-1})' = \mathscr{A}_{f-1}N_{f-2}$. Let $M' = \mathscr{A}_{f-1}N_{f-2}$ and note M' is a right R module.

In order to define a J-map from $M \otimes_{\mathbb{R}} M'$ to R recall the map $\tilde{\varepsilon}$ from Section 1. Define first $\varepsilon: \mathscr{A}_{f-1} \otimes_{\mathbb{R}} \mathscr{A}'_{f-1} \to \mathscr{A}_{f-2}$ by $\varepsilon(u \otimes v) = \tilde{\varepsilon}(uv)$. If w is in $\mathscr{A}_{f-2}$ and u in $\mathscr{A}_{f-1}$, recall $\tilde{\varepsilon}(wu) = w\tilde{\varepsilon}(u)$ and $\tilde{\varepsilon}(uw) = \tilde{\varepsilon}(u)\, w$. Now if w is in $\mathscr{A}_{f-2}$, $\varepsilon(wu \otimes v) = w\varepsilon(u \otimes v)$ and $\varepsilon(u \otimes vw) = \varepsilon(u \otimes v)\, w$. If w is in

N_{f-2}, $w\varepsilon(u \otimes v)$ and $\varepsilon(u \otimes v)\,w$ are in N_{f-2} and so ε can be defined from $M \otimes_{\mathbb{R}} M' \to R$.

Recall $(\tilde{\varepsilon}(uv))' = \tilde{\varepsilon}(v'u')$ and so $\varepsilon(u \otimes v)' = \varepsilon(v' \otimes u')$. Thus ε is a J-map and we may form the algebra $\mathscr{A}(R, M, \varepsilon)$.

We now come to an important construction due to Wenal. Define ψ mapping $\mathscr{A}'_{f-1} \otimes_{\mathscr{A}_{f-2}} \mathscr{A}_{f-1} \to \mathscr{A}_f(1)$ by $\psi(u \otimes v) = uE_f v$ where

$E_f =$ 

Note that if w is in $\mathscr{A}_{f-2}$, $wE_f = E_f w$ and so $\psi(uw \otimes v) = \psi(u \otimes wv)$. Therefore ψ is defined on $\mathscr{A}'_{f-1} \otimes_{\mathscr{A}_{f-2}} \mathscr{A}_{f-1}$. One can check that $\psi(\mathscr{A}_{f-1} N_{f-2} \otimes \mathscr{A}_{f-1}) \subseteq N_f^{(1)}$ and $\psi(\mathscr{A}_{f-1} \otimes N_{f-2} \mathscr{A}_{f-1}) \subseteq N_f^{(1)}$ as $N_{f-2} E_f$ is in $N_f^{(1)}$ and $N_f^{(1)}$ is an ideal in $\mathscr{A}_f(1)$. We may now define $\tilde{\psi}$ on quotients $\tilde{\psi}: M' \otimes_R M \to \mathscr{A}_f(1)/N_f^{(1)}$. We show both ψ and $\tilde{\psi}$ are isomorphisms.

THEOREM 4.1. *The map ψ above is an isomorphism from $\mathscr{A}_{f-1} \otimes_{\mathscr{A}_{f-2}} \mathscr{A}_{f-1}$ onto $\mathscr{A}_f(1)$.*

THEOREM 4.2. *The map $\tilde{\psi}$ above is an isomorphism from $M' \otimes_R M$ onto $\mathscr{A}_f(1)/N_f^{(1)}$.*

Proof of Theorem 4.1. To simplify notation let $B = \mathscr{A}_{f-1}$ and $A = \mathscr{A}_{f-2}$. It is straightforward to check that $E_f w E_f = \tilde{\varepsilon}(w)\, E_f$ for w in $\mathscr{A}_{f-1}$. This means

$$(b_1 \otimes b_2)(c_1 \otimes c_2) = b_1 \tilde{\varepsilon}(b_2 c_1) \otimes c_2$$

and so

$$\begin{aligned} \psi((b_1 \otimes b_2)(c_1 \otimes c_2)) &= \psi(b_1 \tilde{\varepsilon}(b_2 c_1) \otimes c_2) \\ &= b_1 \tilde{\varepsilon}(b_2 c_1)\, E_f c_2 \\ &= b_1 E_f b_2 c_1 E_f c_2 \\ &= \psi(b_1 \otimes b_2)\, \psi(c_1 \otimes c_2). \end{aligned}$$

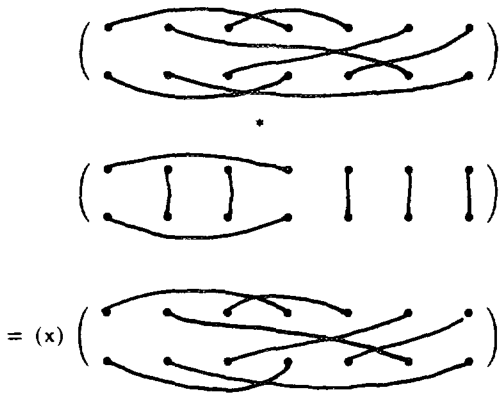
Thus ψ is a homomorphism.

It is shown in [W2] that $\mathscr{A}_f(1)$ is spanned by $\mathscr{A}_{f-1} E_f \mathscr{A}_{f-1}$. As ψ is clearly onto $\mathscr{A}_{f-1} E_f \mathscr{A}_{f-1}$, ψ is onto $\mathscr{A}_f(1)$.

To show that ψ is $1:1$ we show that each element $b_1 \otimes b_2$ can be put into a standard form, using the tensor product relations and then that each diagram in $\mathscr{A}_f(1)$ is obtained uniquely up to powers of x. This shows again that ψ is onto and shows it is an isomorphism.

We note as in Section 1 that a diagram, δ, in F_f can be viewed as a triple $(\Delta_1, \Delta_2, \sigma)$ where Δ_1, Δ_2 are unlabelled partial 1-factors and σ is a permutation in S_m where m is the number of free points in Δ_1 and Δ_2. Here Δ_1 has as lines the horizontal lines in the top of δ and free points the nodes in the top of δ which are in vertical lines. The partial 1-factor Δ_2 is the same as Δ_1 except the bottom of δ is used. The permutation σ describes the vertical lines in δ. In particular we take σ to be the permutation mapping i to $\sigma(i)$ where the ith free point from the left of the top of δ is joined to the $\sigma(i)$-th free point from the left of the bottom of δ. We denote Δ_1 by top(δ) and Δ_2 by bot(δ). For convenience we denote by the "end" points of Δ_1, Δ_2, or δ the nodes on the right. Lines containing an end point are end point lines. If Δ is an unlabelled partial 1-factor on f points, we denote by $h(\Delta)$ the diagram corresponding to $(\Delta, \Delta, \mathrm{id})$ in F_f. This is of course the diagram whose top and bottom is Δ and whose vertical lines map each free point on the top to the one immediately below it.

Suppose b_1 and b_2 are two diagrams in B. The first step is to use the relations to obtain c_1 and c_2 for which $b_1 \otimes b_2 = x^n c_1 \otimes c_2$ and for which top(c_2) and bot(c_1) are almost the same. Let Δ be bot(b_1), if the endpoint of bot(b_1) is isolated. If the end point is joined to the jth point, let Δ be bot(b_1) with this line removed. In this case the jth and end nodes are isolated points. Notice $b_1 h(\Delta) = x^{n_1} b_1$ where n_1 is the number of lines in Δ. For example,



Now

$$x^{n_1} b_1 \otimes b_2 = b_1 h(\Delta) \otimes b_2$$
$$= b_1 \otimes h(\Delta)\, b_2.$$

Note that every horizontal line in bot(b_1), is a horizontal line in top($h(\Delta)\, b_2$) except possibly an end point line if there is one.

Now let Δ' be the partial 1-factor determined by $\mathrm{top}(h(\Delta)\, b_2)$ again deleting the end point line if there is one.

$$h(\Delta')\, h(\Delta)\, b_2 = x^{n_2} h(\Delta)\, b_2,$$

where n_2 is the number of horizontal lines in Δ'. Note that

$$x^{n_1+n_2} b_1 \otimes b_2 = b_1(\Delta') \otimes h(\Delta)\, b_2$$

Let $c_1 = b_1 h(\Delta')$ and $c_2 = h(\Delta)\, b_2$. With this choice of $x^n b_1 \otimes b_2 = c_1 \otimes c_2$. Note that $\mathrm{bot}(c_1)$ and $\mathrm{top}(c_2)$ have lines in the same places except for end point lines.

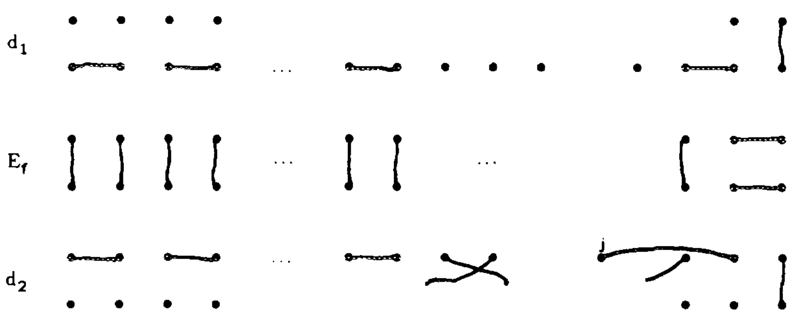
The next step is to use permutations in A, considered in B. In particular,

$$c_1 \otimes c_2 = c_1 \sigma \otimes \sigma^{-1} c_2.$$

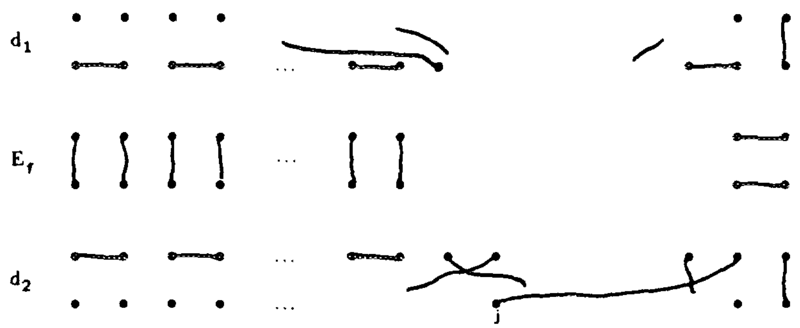
The effect of such a σ is to permute the first $f-2$ nodes in $\mathrm{bot}(c_1)$ according to any permutation in $\mathrm{Sym}(f-2)$ and simultaneously permute the first $n-2$ nodes in $\mathrm{top}(c_2)$ by the same permutation. We do this in such a way that all of the lines except the end point lines in $\mathrm{bot}(c_1)$ and $\mathrm{top}(c_2)$ are to the left. We then permute the free points in $\mathrm{bot}(c_1)$ except for the end point. If the end point is in a line we permute the other end to position $f-1$ and arrange so the ith free point in $\mathrm{bot}(c_1)$ is joined to the ith free point in $\mathrm{top}(c_1)$. Otherwise the end point is joined to a free point j of $\mathrm{top}(c_1)$. Permute the remaining free points in $\mathrm{bot}(c_1)$ so that the ith free point is joined to the ith free point of $\mathrm{top}(c_1)$ after removing j.

This is the standard form for $b_1 \otimes b_2$. We have shown $x^n b_1 \otimes b_2 = d_1 \otimes d_2$ where d_1, d_2 are as described. We wish to show that the diagram $e = b_1 E_f b_2$ arises as a multiple of $d_1 E_f d_2$ and for no other $d_1' E_f d_2'$ with $d_1' \otimes d_2'$ in standard form. We distinguish the four possible cases for the lines containing the end points of $\mathrm{bot}(d_1)$ and $\mathrm{top}(d_2)$:

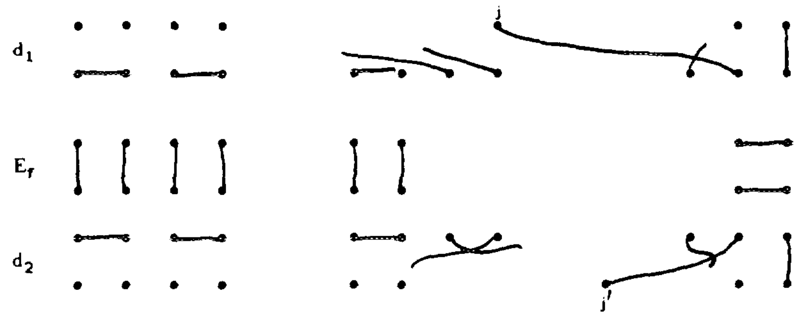
1.



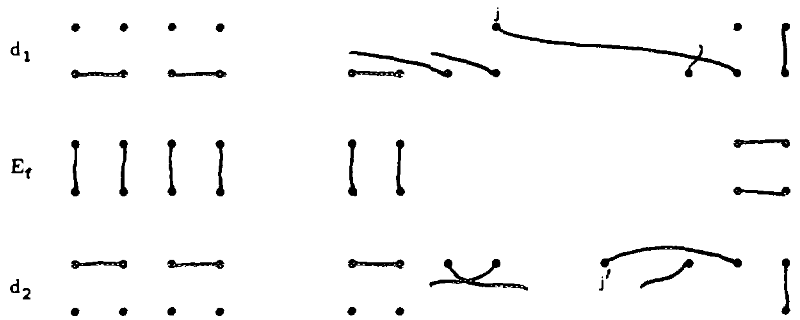
2.



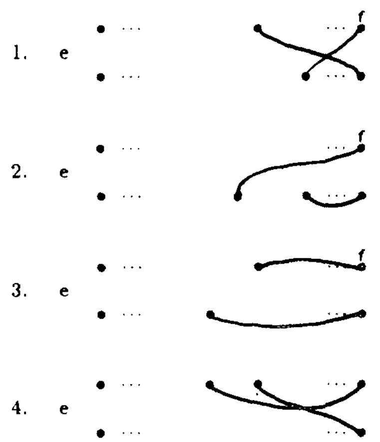
3.



4.



These correspond to the following possibilities for the lines in $e = d_1 E_f d_2$ containing end points:



We must show that given e, the choices of d_1 and d_2 are unique. Straightforward computations show that the four possibilities for the lines containing the end points of e correspond to the configurations shown. The variables for d_1 are top(d_1), k, and line containing the end point of bot(d_1) if it is isolated where k is the number of non-end-point lines in bot(d_1). The variables for d_2 are bot(d_2), k, the line containing the end point of top(d_2), and the permutation corresponding to the vertical lines.

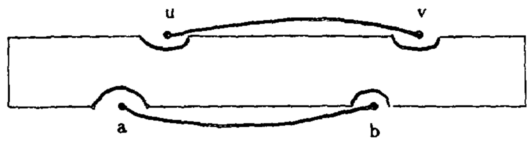
We will do Case 1 in detail. The remaining ones are similar. In this case k is the number of lines in top(e). The j is determined by the line containing the end point of bot(e). The line containing the end point of top(e) determines the line containing the $(f-2)$ node in top(d_2). The lines in top(e) determine the lines in top(d_1) and those in bot(e) determine the lines in bot(d_2). The permutation is determined by the permutation in the representation of e as (top(e), bot(e), σ). This shows d_1 and d_2 are unique. It is clear any e of this type can be obtained in this way. The only complication in the other types is working out how the horizontal lines containing the end points arise.

Proof of Theorem 4.2. Let N be Rad(A). It follows from [Bour, Sect. 3.3, Coro. to Prop. 2 and Sect. 3.6 Coro. 1 to Prop. 6] that $B/BN \otimes_{A/N} B/NB \cong B \otimes_A B/(BN \otimes_A B + B \otimes_A NB)$. Recall $M = NB$ and $M' = NB$. By the tensor product relations $BN \otimes_A B = B \otimes_A NB$ (as N is in A.) Clearly

$$\psi(BN \otimes_A B) = \psi(B \otimes_A NB) = BNE_f B.$$

If r is in N, $rE_f = r^{f-1,f}_{f-1,f}$ so rE_f is in $N_f^{(1)}$. Consequently $\psi(BN \otimes B) \subseteq N_f^{(1)}$ as $N_f^{(1)}$ is an ideal. Also

$r_{a,b}^{u,v}$ =



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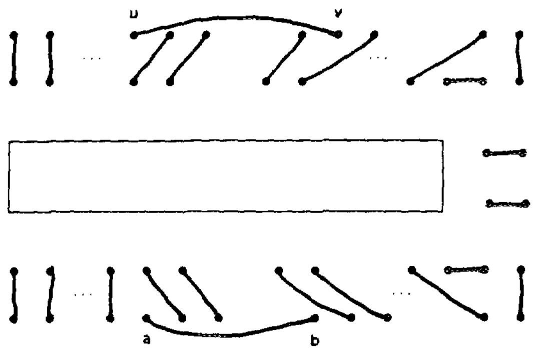


If r is in $N = \text{Rad}\, \mathscr{A}_{f-2}$, this shows $r_{a,b}^{u,v}$ is in $BNE_f B$ and shows

$$N_f^{(1)} = BNE_f B = \psi(BN \otimes B).$$

Taking quotients gives the result.

To get more information from this construction we must do two things. First we must identify the ideal $N_f^{(1)}$ in the radical of $\mathscr{A}_f$. Second, we must compute the matrices (H_μ) in the characteristic sequence of the map ε. We work towards that in the next sections.

5. The Idempotents of $\mathscr{A}_f^{(x)}$

In order to analyse $M' \otimes_R M$ we need to identify primitive idempotents affording the irreducibles. We may know a primitive idempotent of $\mathscr{A}_f^{(x)}[k]$ indexing an irreducible μ and wish to get a primitive idempotent of $\mathscr{A}_f^{(x)}(k)$ indexing the corresponding irreducible. This requires adding a "tail" in $\mathscr{A}_f^{(x)}(k+1)$ to give such an idempotent.

To see how to do this we suppose Δ is a diagram in F_f with k horizontal lines in the top and bottom (i.e., $\Delta \in F_f[k]$). There are a certain number of irreducible representations of $\mathscr{A}_f^{(x)}(k+1)$ labelled $\mathscr{F}_1, \mathscr{F}_2, ..., \mathscr{F}_t$. These act on modules $V_1, V_2, ..., V_t$, which are indeed modules for $\mathscr{A}_f^{(x)}$ as $\mathscr{A}_f^{(x)}(k+1)$ is an ideal in $\mathscr{A}_f^{(x)}(0)$. In particular, Δ acts on each V_i and this action is an endomorphism of V_i. As V_i is irreducible, there is an element η_i in $\mathscr{A}_f^{(x)}(k+1)$ for which $\mathscr{F}_i(\eta_i)$ acts the same on V_i as Δ and $\mathscr{F}_j(\eta_i)$ is the zero action, $\square$, on V_j for $j \neq i$. Define the tail of Δ by

$$T(\Delta) = -\sum_{i=1}^{t} \eta_i. \tag{5.1}$$

Now $\Delta + T(\Delta)$ acts on each V_i as $\square$ and $\Delta + T(\Delta) + \mathscr{A}_f^{(x)}(k+1)$ is the same as Δ in $\mathscr{A}_f^{(x)}[k]$. Of course $T(\Delta)$ is not unique but two different choices differ by an element of the radical of $\mathscr{A}_f^{(x)}(k+1)$. The property that $T(\Delta)$ satisfies is as follows:

$$(\Delta + T(\Delta))\,\Gamma \in \mathrm{Rad}(\mathscr{A}_f^{(x)}(k+1)). \tag{5.2}$$

Here $\Gamma \in \mathscr{A}_f^{(x)}(k+1)$. To see this note $\mathscr{F}_i((\Delta + T(\Delta))\,\Gamma) = \mathscr{F}_i(\Delta + T(\Delta))$ $\mathscr{F}_i(\Gamma) = \square$ and so $(\Delta + T(\Delta))\,\Gamma$ is in the kernel of all the irreducible representations of $\mathscr{A}_f^{(x)}(k+1)$ and so is in Rad $\mathscr{A}_f^{(x)}(k+1)$. Conversely any element $T(\Delta)$ in $\mathscr{A}_f^{(x)}(k+1)$ which satisfies (5.2) will serve as a tail of Δ.

In order to see how to compute $T(\Delta)$ we suppose $\Delta = (\Delta')_{ab}^{cd}$ and suppose we can compute $T(\Delta')$. Suppose $\Gamma = (\Gamma')_{a'b'}^{c'd'}$ with Γ' in $\mathscr{A}_f^{(x)}(k)$. Now $(\Delta' + T(\Delta'))_{ab}^{cd}\,(\Gamma')_{a'b'}^{c'd'}$ is $((\Delta' + T(\Delta'))\,\rho(a, b, c', d')\,\Gamma')_{a'b'}^{cd}$ which is in $\mathrm{Rad}(\mathscr{A}_f^{(x)}(k+1))$ by (5.2) and Theorem 3.2.9. This proves the following lemma.

LEMMA 5.3. $(T(\Delta'))_{ab}^{cd} = T((\Delta')_{ab}^{cd})$.

Suppose Δ is the identity in $\mathscr{A}_f^{(x)}$. Then $T(\Delta) = -\sum \varepsilon_i$ where ε_i acts as the identity on V_i and as $\square$ on V_j, $j \neq i$. This means $\mathscr{F}_i(\varepsilon_j) = \delta_{ij}$ (identity). If Δ' is any permutation

$$\begin{aligned}(\Delta' + \Delta' T(\Delta))\,\Gamma &= \Delta'(\Delta + T(\Delta))\,\Gamma \\ &\in \Delta'(\mathrm{Rad}\,\mathscr{A}_f^{(x)})\end{aligned}$$

and so

$$T(\Delta') = \Delta' T(\Delta).$$

LEMMA 5.4. *If Δ is the identity in $\mathscr{A}_f^{(x)}$ and Δ' is any permutation,* $T(\Delta') = \Delta' T(\Delta)$.

This shows that computations of $T(\Delta)$ can be read from $T(\Delta)$ for diagrams which are permutations with the appropriate number of vertical lines.

The introduction of $T(\Delta)$ also provides a direct proof for a result in [HW1].

LEMMA 5.5. $\mathrm{Rad}(\mathcal{A}_f[k]) \cong \mathrm{Rad}(\mathcal{A}_f(k))/\mathrm{Rad}(\mathcal{A}_f(k+1))$.

Proof. Choose $\sum c_i \bar{\Delta}_i$ in $\mathrm{Rad}\,\mathcal{A}_f[k]$, $\bar{\Delta}_i = (\Delta_i + \mathcal{A}_f(k+1))/\mathcal{A}_f(k+1)$, $\Delta_i \in F_f[k]$. Then $\sum c_i \Delta_i + \sum c_i T(\Delta_i)$ is an element in the kernel of all irreducible representations of $\mathcal{A}_f(k+1)$ and the irreducible representatives of $\mathcal{A}_f[k]$ and so of all irreducible representations of $\mathcal{A}_f(k)$. In particular, it is in $\mathrm{Rad}(\mathcal{A}_f(k))$. This provides a map from $\mathrm{Rad}(\mathcal{A}_f[k])$ to $\mathrm{Rad}(\mathcal{A}_f(k))/\mathrm{Rad}(\mathcal{A}_f(k+1))$ which is onto. This proves the lemma. ▌

We introduced $T(\Delta)$ in order to produce primitive idempotents for $\mathcal{A}_f^{(x)}(k)$ given one for $\mathcal{A}_f^{(x)}[k]$. Suppose that $\bar{\varepsilon}$ is a primitive idempotent for $\mathcal{A}_f^{(x)}[k]$ where $\bar{\varepsilon} = \varepsilon + \mathcal{A}_f^{(x)}(k+1)$. Consider $E = \varepsilon + T(\varepsilon)$. We know that $\mathcal{F}_i(E) = \square$ for irreducibles $\mathcal{F}_i$ of $\mathcal{A}_f^{(x)}(k+1)$. The remaining irreducibles of $\mathcal{A}_f^{(x)}(k)$ have $\mathcal{A}_f^{(x)}(k+1)$ in their kernel and can be considered irreducibles of $\mathcal{A}_f^{(x)}[k]$. As such they all represent E by $\square$ except the irreducible, $\mathcal{F}$, indexed by $\bar{\varepsilon}$ for which $\mathcal{F}(\varepsilon)$ has rank 1.

This shows that were E to be an idempotent, it would be a primitive idempotent indexing $\mathcal{F}$. Furthermore $\mathcal{F}(E^2) = \mathcal{F}(E)$ for all irreducibles of $\mathcal{A}_f(k)$ and $E^2 - E$ is in $\mathcal{A}_f(k+1)$ as $(\bar{\varepsilon})^2 = \bar{\varepsilon}$. Consequently E^2 is an idempotent modulo $\mathrm{Rad}(\mathcal{A}_f(k+1))$. By the lifting lemma for idempotents [ANT, Th. 9.3c] there is a choice of $T(\varepsilon)$ for which $\varepsilon + T(\varepsilon)$ is an idempotent. This will be a primitive idempotent affording $\mathcal{F}$.

6. CONNECTIONS BETWEEN H_μ AND $Z_\mu(x)$

Consider the problem of computing the matrices H_μ defined in Section 4. This would involve finding a primitive idempotent I for the matrix ring in $\mathcal{A}_{f-2}^{(x)}/N_{f-2}$ indexed by μ, finding a basis for $I\mathcal{A}_{f-1}^{(x)}/IN_{f-2}\mathcal{A}_{f-1}^{(x)}$, and then computing $\bar{\varepsilon}$ on products of this basis. This is a formidable computation. In this section we will show the amazing fact that the matrices H_μ are related to the matrices $Z_\mu(x)$ defined in Section 1.

6.1. If we apply the tower construction from Section 4 with $\mathcal{A}_{f-2}^{(x)}$ semisimple we obtain matrices (H_μ), one for each partition μ such that $|\mu| = f - 2l$ for some $l \geqslant 1$. The algebra $M' \otimes_R M$ is semisimple iff all the H_μ are non-degenerate and we have shown $M' \otimes_R M$ is isomorphic to $\mathcal{A}_f^{(x)}(1)$ in this case. In earlier work (see Section 1) these authors define matrices $Z_\mu(x) = \bigoplus_\lambda Z_{\lambda\mu}(x)$ which are also defined for partitions μ such that $|\mu| = f - 2l$ and which also have the property that $\mathcal{A}_f^{(x)}(1)$ is semisimple iff all the $Z_\mu(x)$ are non-degenerate. In this first section we will prove that $H_\mu = Z_\mu(x)$ for all μ when $\mathcal{A}_{f-2}^{(x)}$ is semisimple and will show that H_μ and $Z_\mu(x)$ are related when $\mathcal{A}_{f-2}^{(x)}$ is not semisimple.

The first difficulty we encounter proving that $H_\mu = Z_\mu(x)$ when $\mathcal{A}_{f-2}^{(x)}$ is semisimple is that the two matrices act on completely different spaces. The rows and columns of H_μ are indexed by occurrences of the irreducible $\mathscr{A}_{f-2}^{(x)}$ module indexed by μ in the left module $\mathscr{A}_{f-1}^{(x)}$. The rows and columns of $Z_\mu(x)$ are indexed by the occurrences of the S_m-irreducible indexed by μ in the space of (m, k) labelled partial 1-factors. Our first goal is to identify the two spaces upon which these matrices act.

We assume throughout that μ is a partition of $m \leqslant (f-2)$ and $f-2-m$ is even. Define $k = (f-2-m)/2$. We begin by identifying a basis for the space acted on by H_μ when $\mathscr{A}_{f-2}^{(x)}$ is semisimple and a spanning set when $\mathscr{A}_{f-2}^{(x)}$ is not semisimple.

DEFINITION 6.1.1. For $\sigma \in S_m$ define $X(\sigma) \in F_{f-1}$ by

(a) $X(\sigma)$ has vertical edges from i in row 1 to σi in row 2 for $1 \leqslant i \leqslant m$. $X(\sigma)$ also has a vertical edge from $(f-1)$ in row 1 to $(f-1)$ in row 2.

(b) $X(\sigma)$ has horizontal edges from $m+(2j-1)$ to $m+2j$ for $j = 1, 2, ..., k$ in both the top row and the bottom row.

Extend this notation linearly to $\mathbb{R}S_m$, i.e., if $\pi = \sum_i c_i \sigma_i$ is in $\mathbb{R}S_m$ then

$$X(\pi) = \sum_i c_i X(\sigma_i).$$

For example, if $f = 9$, $m = 3$ and $\sigma = (1, 2, 3)$ then

$X(\sigma) =$ 

DEFINITION 6.1.2. For each $(m-1, k+1)$ (unlabelled) partial 1-factor δ, let $Y(\delta)$ be the diagram in F_{f-1} satisfying:

(a) There are $k+1$ horizontal edges in the top row of δ. They join the points $(m-1)+(2i-1)$ and $(m-1)+2i$ for $i = 1, 2, ..., k+1$.

(b) The $k+1$ horizontal edges in the bottom row of $Y(\delta)$ are exactly the edges of δ.

(c) There is a vertical edge from the jth point on the top row of $Y(\delta)$ to the jth free point of δ on the bottom row of $Y(\delta)$.

For example if $m-1 = k+1 = 3$ and

$\delta =$ 

then $Y(\delta) =$  .

Definition 6.1.3. For each $(m+1, k)$ (unlabelled) partial 1-factor δ and each $j \in \{1, 2, ..., m+1\}$ define $Z(\delta, j)$ to be the diagram in F_{f-1} satisfying:

(a) There are k horizontal edges in the top row of $Z(\delta, j)$. They joint the points $(m+1)+(2i-1)$ and $(m+1)+2i$ for $i = 1, 2, ..., k$.

(b) The k horizontal edges in the bottom row of $Z(\delta, j)$ are exactly the edges of δ.

(c) There is a vertical edge from the $(m+1)$st point in the top row to the jth free point of δ in the bottom row. The other vertical edges join the first m points in the top row to the other free points in δ in order. In other words, the lth point in the top row is joined by a vertical edge to the $\hat{l}$th free point of δ in the bottom row where

$$\hat{l} = \begin{cases} l & \text{if } l < j \\ l+1 & \text{if } l > j. \end{cases}$$

For example, if $j = 3$ and

$\delta =$ 

then

$Z(\delta, 3) =$ 

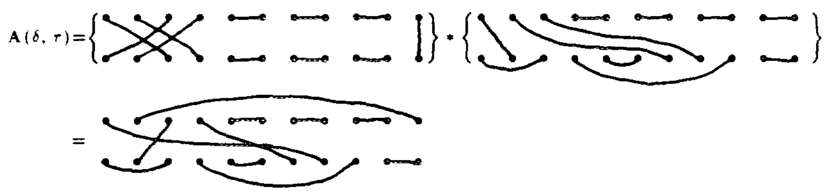
Note that the vertical edge joining the starred points is the one vertical edge which is out of order.

At this point we have developed notation for a number of diagrams in F_{f-1}. We use these diagrams to build an indexing set for the rows and columns of the matrix H_μ.

Definition 6.1.4. For $\delta \in \mathscr{B}_{m-1,k+1}$ and $r \in \mathbb{R}S_m$ define $A(\delta, r)$ to be

$$A(\delta, \tau) = X(\tau) * Y(\delta).$$

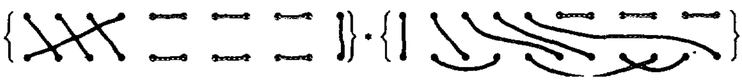
For example if $\delta =$  and $\tau = (1, 3)(2, 4)$ then

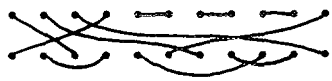


DEFINITION 6.1.5. For $\delta \in \mathscr{B}_{m+1,k}$, $j \in \{1, 2, ..., m+1\}$ and $\tau \in \mathbb{R}S_m$ define

$$B(\delta, j, \tau) = X(\tau) * Z(\delta, j).$$

For example if $j = 3$, $\tau = (1, 2, 3, 4)$ and $\delta =$ , then

$B(\delta, j, \tau) =$ 

$=$ 

In these definitions note that $f - 1 = m - 1 + 2(k+1)$, $A(\delta, \tau)$ has $k+1$ horizontal lines in the top and bottom and $B(\delta, j, \tau)$ has k horizontal lines in the top and bottom.

LEMMA 6.1.6. *Suppose that $\tau_1, ..., \tau_p$ are linearly independent elements of $\mathbb{R}S_m$. Define sets A and B by*

$$A = \{A(\delta, \tau_l): l = 1, 2, ..., p, \delta \in \mathscr{B}_{m-1,k+1}\}$$
$$B = \{B(\delta, j, \tau_l): j = 1, 2, ..., m+1, l = 1, 2, ..., p, \delta \in \mathscr{B}_{m+1,k}\}.$$

Then $A \cup B$ is a linearly independent set of elements in $\mathscr{A}_{f-1}$.

Proof. By considering horizontal lines in the bottom row it is enough to show that

(i) $\{A(\delta, \tau_l): l = 1, 2, ..., p\}$ is linearly independent for each $\delta \in \mathscr{B}_{m-1,k+1}$ and

(ii) $\{B(\delta, j, \tau_l): l = 1, 2, ..., p, j = 1, 2, ..., m+1\}$ is linearly independent for each $\delta \in \mathscr{B}_{m+1,k}$.

First fix $\delta \in \mathscr{B}_{m-1,k+1}$ and consider the set $\{A(\delta, \tau_l): l = 1, 2, ..., p\}$. By the linear independence of the τ_l it is enough to show that σ can be recovered from $A(\delta, \sigma)$ for each $\sigma \in S_m$. In $A(\delta, \sigma)$ there is an edge from $f-1$ to $\sigma^{-1}m$ in the top which can be used to determine $\sigma^{-1}m$. Also there is a vertical edge from the jth free point of δ in the bottom row to the $(\sigma^{-1}j)$th point in the top row (for $j = 1, 2, ..., m-1$). These edges can be used to recover the rest of σ^{-1}. Hence σ can be determined from $A(\delta, \sigma)$ so $\{A(\delta, \tau_l): l = 1, 2, ..., p\}$ is linearly independent.

Next fix $\delta \in \mathscr{B}_{m+1,k}$ and consider

$$\{B(\delta, j, \tau_l): j = 1, 2, ..., m+1, l = 1, 2, ..., p\}.$$

Again it is sufficient to show that j and σ can be recovered from $B(\delta, j, \sigma)$ for all j, σ. In $B(\delta, j, \sigma)$ there is a vertical edge from $f-1$ in the top row to the jth free point of δ in the bottom row. This vertical edge determines j. Also there is a vertical edge from the ith point in the top row to the $\hat{\sigma}i$th free point of δ in the bottom row where

$$\hat{\sigma}i = \begin{cases} \sigma i & \text{if } \sigma i < j \\ (\sigma i) + 1 & \text{if } \sigma i > j. \end{cases}$$

So one can recover each value σi. This proves the lemma. ∎

Let f_μ denote the dimension of the S_m-irreducible indexed by μ. Fix a matrix ring decomposition of $\mathbb{R}S_m$ and let $\pi_1, ..., \pi_{f_\mu}$ be the $(1,1), (1,2), ..., (1, f_\mu)$ entries in the matrix ring indexed by μ.

Definition 6.1.7. Define S to be the set

$$\begin{aligned} S = &\{A(\delta, \pi_l): l = 1, 2, ..., f_\mu, \delta \in \mathscr{B}_{m-1,k+1}\} \\ &\cup \{B(\delta, j, \pi_l): l = 1, 2, ..., f_\mu, j = 1, 2, ..., m+1, \delta \in \mathscr{B}_{m+1,k}\}. \end{aligned}$$

By Lemma 6.1.6, S is a set of linearly independent elements of $\mathscr{A}_{f-1}$. A straightforward computation shows that

$$X(\pi_1) * \Sigma = x^k \Sigma$$

for all $\Sigma \in S$. So, if $x \neq 0$, then the elements of S are contained in $X(\pi_1)\mathscr{A}^{(x)}_{f-1}$. Also note that

$$\begin{aligned} |S| &= |\mathscr{B}_{m-1,k+1}| f_\mu + |\mathscr{B}_{m+1,k}| (m+1) f_\mu \\ &= f_\mu \left\{ \binom{f-1}{m-1} (2k+1)!! + \binom{f-1}{m+1} (m+1)(2k-1)!! \right\}. \end{aligned}$$

It follows from the construction in [HW1] that $(1/x^k) X(\pi_1)$ considered in $\mathscr{A}^{(x)}_{f-2}[k]$ is a primitive idempotent affording the irreducible indexed by μ. Let $E = E(\pi_1) = X(\pi_1) + T(X(\pi_1))$ be the x^k multiple of a primitive idempotent of $\mathscr{A}^{(x)}_{f-2}(k)$ indexing the irreducible of $\mathscr{A}^{(x)}_{f-2}(k)$ indexed by μ (see Section 5). We have

$$(E)^2 = x^k E$$

We need a spanning set for $E\mathscr{A}^{(x)}_{f-1} \bmod N_{f-2}\mathscr{A}^{(x)}_{f-1}$. In this section we denote the image of elements Γ in $\mathscr{A}^{(x)}_{f-1}$ modulo $(N_{f-2}\mathscr{A}^{(x)}_{f-1})$ by $\bar{\Gamma}$.

Definition 6.1.8. Define $\bar{S}$ to be the set $\overline{ES}$.

LEMMA 6.1.9. *With notation as above,* $\overline{E\mathcal{A}^{(x)}_{f-1}}$ *is spanned by* $\bar{S}$. *If* $N_{f-2}=0$, *ES is a basis for* $E\mathcal{A}^{(x)}_{f-1}$. *In particular* $\dim \bar{M} \leqslant |S|$.

Proof. Recall for this that $f-1=(m+1)+2(k-1)$, $A(\delta, \pi_l) \in \mathcal{A}^{(x)}_{f-1}(k+1)$, and $B(\delta, j, \pi_l) \in \mathcal{A}^{(x)}_{f-1}(k)$.

As $(\bar{E})^2 = x^k \bar{E}$ and $E \in \mathcal{A}_{f-2}(k)$ we need only consider the images $\overline{E\Delta}$ for diagrams in $\mathcal{A}^{(x)}_{f-1}(k)$ (i.e., at least k horizontal lines in the top and bottom).

Suppose first Δ is a diagram in $\mathcal{A}^{(x)}_{f-1}(k+2)$. We wish to show $E\Delta$ is in $N_{f-2}\mathcal{A}^{(x)}_{f-1}$. We know that if Δ' is a diagram in $\mathcal{A}^{(x)}_{f-2}(k+1)$, $E\Delta'$ is in N_{f-2} here considering E in $\mathcal{A}^{(x)}_{f-2}$ in the usual way. (see Section 1). This is because Δ' is in the kernel of the irreducible indexed by μ. Showing $E\Delta$ is in $N_{f-2}\mathcal{A}^{(x)}_{f-1}$ is equivalent to showing $E\Delta\sigma$ is in $N_{f-2}\mathcal{A}^{(x)}_{f-1}$ for a permutation σ and conversely. This means we can permute the bottom row of Δ any way we wish. We now divide into two cases.

Case 1. The $(f-1)$st top node of Δ is part of a vertical line. (6.1.10)

In this case permute the bottom row of Δ so that a vertical line joins the $(f-1)$st top and bottom nodes. The resulting Δ considered in $\mathcal{A}^{(x)}_{f-1}$ is in $\mathcal{A}^{(x)}_{f-2}(k+2)$ and so $E\Delta$ is in $N_{f-2}\mathcal{A}^{(x)}_{f-1}$. Note this argument works even if $\Delta \in \mathcal{A}^{(x)}_{f-1}(k+1)$ for Case 1.

Case 2. The $(f-1)$st top node is part of a horizontal line. (6.1.11)

Suppose the $(f-1)$st top node is joined to c on the top. Pick a horizontal line on the bottom, say (a, b). Let Δ_1 be Δ except the horizontal lines $(c, f-1)$ and (a, b) are replaced by vertical lines (c, a) and $(f-1, b)$. Now $E\Delta_1$ is in $N_{f-2}\mathcal{A}^{(x)}_{f-1}$ by Case 1 (see last sentence in Case 1 if Δ is in $F_{f-1}[k+2]$). Now let Δ_2 be in $\mathcal{A}^{(x)}_{f-1}$ with horizontal lines (a, b) on the top and bottom and all other top nodes i joined to the bottom node i. Now $\Delta = \Delta_1\Delta_2$ and $E\Delta = E\Delta_1\Delta_2$ and $E\Delta$ is in $N_{f-2}\mathcal{A}^{(x)}_{f-1}$ also.

This shows we need only consider diagrams in which there are either k or $k+1$ horizontal lines in the top or bottom. By the remark at the end of Case 1, if there are $k+1$ horizontal lines we may assume the $(f-1)$st top node is part of a horizontal line. Note this is the situation for the diagrams in S.

Suppose, then, that Δ is in $F_f[k+1]$ and the $(f-1)$st node on the top is joined to the cth node on the top. If $E\Delta$ were in $\mathcal{A}^{(x)}_{f-1}(k+2)$, $EE\Delta = x^k E\Delta$ would be in $N_{f-2}\mathcal{A}^{(x)}_{f-1}$. Consequently we may assume that $E\Delta$ has terms in $F_{f-1}[k+1]$ which means this is the case for $X(\pi_1)\Delta$. In particular, all diagrams appearing in $X(\pi_1)\Delta$ have horizontal lines in the top in the same positions as those of $X(\pi_1)$ and have horizontal lines in the bottom in the same positions as those of Δ. Each must have one remaining line on the top. The line containing the $(f-1)$st top node for a permutation in S_m will be a vertical line depending on the location of c and the top of Δ. It will be vertical for all permutations in S_m. In this case, applying E again will give an element in $N_{f-2}\mathscr{A}_{f-1}^{(x)}$ by Case 1 above. Otherwise, it will be a horizontal line joined to one of the first m top nodes. The remaining first top m nodes must become vertical lines joined to points on the bottom of Δ which are on vertical lines. For a given σ occuring in π_1, $X(\sigma)\Delta = A(\delta, \sigma\tau)$ where δ is the unlabelled partial 1-factor whose lines are the horizontal lines in the bottom of Δ. The permutation τ can be obtained considering $X(e)\Delta$. If in $X(e)\Delta$, the top node b is joined to the top $(f-1)$st, $\tau(m)=b$. For $1 \leqslant i \leqslant m-1$, the top ith node is joined to the jth node of δ where $\tau(j)=i$. Now $X(e)\Delta = A(\delta, \tau)$ and $X(\sigma)\Delta = (1/x^k)X(\sigma)X(e)\Delta = A(\delta, \sigma\tau)$. In this case $X(\pi_1)\Delta = X(\delta, \pi_1\tau)$. As π_1 represents the $(1, 1)$ entry in the representation indexed by μ, $\pi_1\tau$ represents an entry in the top row and so is a linear combination of $\pi_1, \pi_2, ..., \pi_{f_\mu}$. Now $X(\pi_1)\Delta$ is a linear combination of $A(\delta, \pi_1), ..., A(\delta, \pi_{f_\mu})$ and $E\Delta$ minus the same linear combination of $EA(\delta, \pi_1), ..., EA(\delta, \pi_{f_\mu})$ is in $\mathscr{A}_{f-1}^{(x)}(k+2)$ and so in $N_{f-2}\mathscr{A}_{f-1}^{(x)}$.

In the final case, Δ is in $F_{f-1}[k]$. If $E\Delta$ is in $\mathscr{A}_{f-1}(k+1)$, the argument above with $E \cdot E \cdot \Delta$ handles this case. We may assume then that some terms in $E\Delta$ are in $F_{f-1}[k]$ and again the same is true in $X(\pi_1)\Delta$. The points in the top on vertical lines must be on vertical lines in $X(\pi_1)\Delta$. Arguing as above, $X(\pi_1)\Delta = B(\delta, j, \pi_1\tau)$ where the $(f-1)$st top node is joined to the jth node on the bottom on a vertical line. The δ is the partial 1-factor whose lines are the horizontal lines in the bottom of δ. Again this is a linear combination of $B(\delta, j, \pi_i)$ $1 \leqslant i \leqslant f_\mu$ and $E\Delta$ minus the corresponding linear combination of $EB(\delta, j, \pi_i)$ is in $N_{f-1}\mathscr{A}_{f-1}^{(x)}$.

This shows $\overline{E\mathscr{A}_{f-1}^{(x)}}$ is spanned by $\bar{S}$. If $\mathscr{A}_{f-2}^{(x)}$ is semisimple $N_{f-2}=0$ and ES is linearly independent as S is linearly independent. This completes the proof of Lemma 6.1.9.

At this point we define a matrix $\mathscr{H}_\mu$ which is closely related to the matrix H_μ (in fact equal to H_μ when $\mathscr{A}_{f-2}^{(x)}$ is semisimple). We show that $\mathscr{H}_\mu$ and $Z_\mu(x)$ are equal and H_μ is a quotient of them.

LEMMA 6.1.12. *Let Σ_1 and Σ_2 be elements of S. Then for some λ, $\varepsilon(E\Sigma_1, (E\Sigma_2)') = \lambda E$.*

Proof. We know from Section 2 that $\tilde{\varepsilon}((E\Sigma_1) * (E\Sigma_2)')$ is a multiple of E.

DEFINITION 6.1.13. Define an $|S|$ by $|S|$ matrix $\mathscr{H}_\mu$ by saying that the Σ_1, Σ_2 entry of $\mathscr{H}_\mu$ is the multiple of E given by $\varepsilon(E\Sigma_1, (E\Sigma_2)')$. In other words,

$$\tilde{\varepsilon}(E\Sigma_1, (E\Sigma_2)') = ((\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2})\, E.$$

Before continuing we must recall exactly how the matrices $Z_\mu(x)$ were defined. Recall that a *labelled* $(m, k+1)$ partial 1-factor is a partial 1-factor where the m free points have been labelled with the numbers 1, 2, ..., m. The set of labelled $(m, k+1)$ partial 1-factors is denoted by $B_{m,k+1}$. In this section we usually denote a labelled $(m, k+1)$ partial 1-factor by a pair $(\delta, \sigma) \in \mathscr{B}_{m,k+1} \times S_m$. The matrix $Z_{m,k+1}(x)$ has rows and columns indexed by $B_{m,k+1}$. The (δ_1, σ_1), (δ_2, σ_2) entry is

$$(Z_{m,k+1}(x))_{(\delta_1,\sigma_1),(\delta_2,\sigma_2)} = \begin{cases} x^{\gamma(\delta_1,\delta_2)} & \text{if } \sigma_1 = \sigma_2 \tau(\delta_1, \delta_2)^{-1} \\ 0 & \text{otherwise.} \end{cases} \tag{6.1.14}$$

Let $W_{m,k+1}$ and $\mathscr{W}_{m,k+1}$ denote the $\mathbb{R}$-vector spaces with bases $B_{m,k+1}$ and $\mathscr{B}_{m,k+1}$, respectively. Thus

$$W_{m,k+1} = \mathscr{W}_{m,k+1} \otimes \mathbb{R}S_m$$

and $Z_{m,k+1}(x)$ represents a linear transformation of $W_{m,k+1}$. The symmetric group S_m acts on $W_{m,k+1}$ via the left-regular representation on the $\mathbb{R}S_m$ tensor component. The matrix $Z_{m,k+1}(x)$ commutes with this action thus $Z_{m,k+1}(x)$ preserves the space

$$W_\mu = \mathscr{W}_{m,k+1} \otimes (\pi_1 \mathbb{R}S_m). \tag{6.1.15}$$

The matrix $Z_\mu(x)$ is defined to be the restriction of $Z_{m,k+1}(x)$ to W_μ.

DEFINITION 6.1.16. Let V be the span in $\mathscr{A}_{f-1}^{(x)}$ of ES.

(a) For $\delta \in B_{m-1,k+1}$, let δ_0 be the diagram obtained from δ by adding a new free point at the end (position f). Then

$$\varphi(EA(\delta, \pi_l)) = \delta_0 \otimes \pi_l$$

(b) For $\delta \in B_{m+1,k}$ and $j \in \{1, 2, ..., m+1\}$ let δ_0 be the diagram obtained from δ by adding a new point at the end (position f) and joining this new point by a new edge to the jth free point of δ. Then

$$\varphi(EB(\delta, j, \pi_l)) = \delta_0 \otimes \pi_l.$$

It is straightforward to check that φ is a 1–1 map. By comparing dimensions one finds that φ is an isomorphism from V to W_μ. We now come to a basic result.

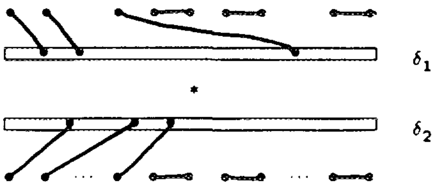
THEOREM 6.1.17. $\varphi \circ \mathscr{H}_\mu = Z_\mu(x) \circ \varphi$.

Proof. For each pair Σ_1, Σ_2 of elements of S we will compare the Σ_1, Σ_2 entry of $\mathscr{H}_\mu$ to the $\varphi(E\Sigma_1)$, $\varphi(E\Sigma_2)$ entry of $Z_\mu(x)$.

Case 1. $\Sigma_1 = A(\delta_1, \pi_r)$ and $\Sigma_2 = A(\delta_2, \pi_s)$ where $\delta_1, \delta_2 \in B_{m-1,k+1}$ and $r, s \in \{1, 2, ..., f_\mu\}$. In the following arguments we only need terms in $F_{f-2}[k]$ for our computations and so we do not need to compute $\tilde{\varepsilon}(E\Sigma_1, (E\Sigma_2)')$ but rather $\tilde{\varepsilon}(X(\pi_1)\Sigma_1, (X(\pi_1)\Sigma_2)') = \tilde{\varepsilon}(\Sigma_1, \Sigma_2)$. Note that

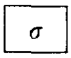
$$\tilde{\varepsilon}(A(\delta_1, \pi_r) * (A(\delta_2, \pi_s))') = X(\pi_r) * \tilde{\varepsilon}(Y(\delta_1) * Y(\delta_2)') * X(\pi_s)'. \qquad (6.1.18)$$

The product $Y(\delta_1) * Y(\delta_2)'$ can be formed using the following picture:

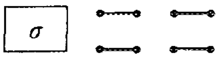


(6.1.19)

Some of the following computations are most easily done by means of pictures. So we need to devise a bit of notation. If σ is a permutation in S_l then



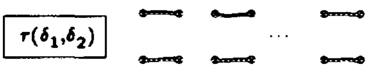
is used to signify the diagram in F_l representing σ. This notation is incorporated into larger pictures. For example, the picture



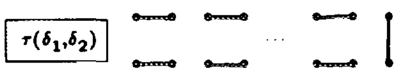
denotes the diagram which has σ in its first l columns followed by horizontal edges from $l+1$ to $l+2$ and from $l+3$ to $l+4$ in each row. We extend this notation linearly. So if $\tau = \sum_\sigma c_\sigma \sigma \in \mathbb{R}S_l$ then

$$\boxed{\tau} \text{ denotes } \sum_\sigma c_\sigma \boxed{\sigma}.$$

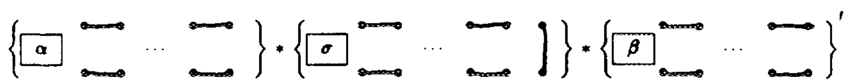
By inspection of (6.1.19) we have

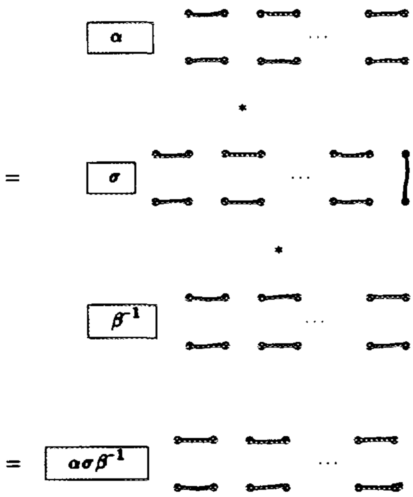
$Y(\delta_1) * Y(\delta_2)' = \left(x^{\gamma(\delta_1,\delta_2)}\right)$  . (6.1.20)

From (6.1.20) it follows that

$\tilde{\varepsilon}\left(Y(\delta_1) * Y(\delta_2)'\right) = \left(x^{\gamma(\delta_1,\delta_2)}\right)$  . (6.1.21)

For $\alpha, \beta \in S_m$ and $\sigma \in S_{m-1}$ we have



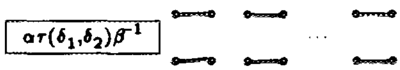


where in the last line $\sigma \in S_{m-1}$ is extended to a permutation in S_m by having it fix m. Combining this with (6.1.21) we have

$X(\pi_r)\,\tilde{\varepsilon}(Y(\delta_1) * Y(\delta_2)')\,X(\pi_s)' =$

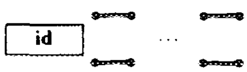
(6.1.22)

$$x^{\gamma(\delta_1,\delta_2)} \sum_{\alpha,\beta} c_\alpha d_\beta$$



where c_α and d_β are the coefficients of α and β in π_r and π_s respectively.

We know that the sum of terms in $F_{f-2}[k]$ in (6.1.22) is a multiple of the sum of terms in $F_{f-2}[k]$ in $X(\pi_1)$. To find out what multiple we can look at a particular diagram, namely,



Recall that the coefficient of the identity in π_1 is 1. From (6.1.22) we have

$(\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2}$ is $(x^{\gamma(\delta_1, \delta_2)})$ times the coefficient of the identity permutation in $\pi_r \tau(\delta_1, \delta_2)\, \pi_s'$. (6.1.23)

An equivalent formulation of (6.1.23) is

$$(\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2} = (x^{\gamma(\delta_1, \delta_2)}) \left(\sum_{\alpha, \beta} c_\alpha d_\beta \right) \tag{6.1.24}$$

where the sum on α, β is over pairs satisfying $\alpha\tau(\delta_1, \delta_2) = \beta$.

Next consider $(Z_\mu(x))_{\varphi(\Sigma_1), \varphi(\Sigma_2)}$. As δ_1 and δ_2 are in $B_{m-1, k+1}$ we have

$$\varphi(\Sigma_1) = (\delta_1)_0 \otimes \pi_r$$
$$\varphi(\Sigma_2) = (\delta_2)_0 \otimes \pi_s,$$

where $(\delta_i)_0$ is obtained from δ_i by adding a new free point at the end. Thus

$$\tau((\delta_1)_0, (\delta_2)_0) = \tau(\delta_1, \delta_2), \tag{6.1.25}$$

where $\tau(\delta_1, \delta_2)$ is considered to be an element of $\mathbb{R}S_m$ via the usual embedding of $\mathbb{R}S_{m-1}$ in $\mathbb{R}S_m$. So

$$(Z_\mu(x))_{\varphi(\Sigma_1), \varphi(\Sigma_2)} = x^{\gamma(\delta_1, \delta_2)} \sum_{\alpha, \beta} c_\alpha d_\beta \tag{6.1.26}$$

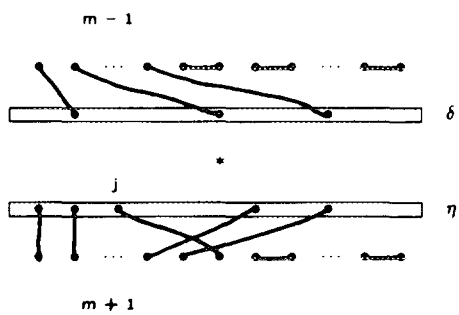
where the sum on the right is over pairs α, β satisfying $\alpha\tau((\delta_1)_0, (\delta_2)_0) = \beta$. Combining (6.1.24), (6.1.25), and (6.1.26), we have

$$(\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2} = (Z_\mu(x))_{\varphi(\Sigma_1), \varphi(\Sigma_2)}.$$

Case 2. $\Sigma_1 = A(\delta, \pi_r)$ and $\Sigma_2 = B(\eta, j, \pi_s)$ where $\delta \in F_{m-1, k+1}$, $\eta \in F_{m+1, k}$, $j \in \{1, 2, ..., m+1\}$ and $r, s \in \{1, 2, ..., f_\mu\}$. In this case we have

$$\tilde{\varepsilon}(A(\delta, \pi_r) * B(\eta, j, \pi_s)') = X(\pi_r) * \tilde{\varepsilon}(Y(\delta) * Z(\eta, j)') * X(\pi_s)'.$$

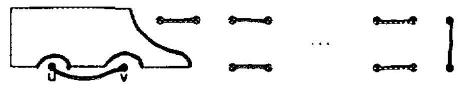
As above we need to examine the structure of $Y(\delta) * Z(\eta, j)'$ to determine its image under $\tilde{\varepsilon}$. To compute that product we draw

$Y(\delta) * Z(\eta, j)' =$ 

(6.1.27)

Note that $Z(\eta, j)$ has k horizontal edges per row and $Y(\delta)$ has $k+1$ horizontal edges per row. So $Y(\delta) * Z(\eta, j)'$ has at least $k+1$ horizontal edges per row.

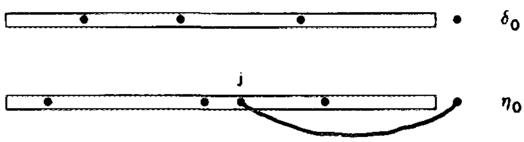
Subcase 1. For some pair (u, v) with $1 \leqslant u < v \leqslant m$ we have a horizontal edge between points u and v in the bottom row of $Y(\delta) * Z(\eta, j)'$. So $\tilde{\varepsilon}(Y(\delta) * Z(\eta, j)') = x^{\gamma(\delta, \eta)}\Delta$, where Δ looks like

$\Delta =$ 

We expand $X(\pi_r) * \varepsilon(Y(\delta) * Z(\eta, j)') * X(\pi_s)'$ as $\sum_{\alpha, \beta} c_\alpha d_\beta x^{\gamma(\delta, \eta)} \alpha \Delta \beta^{-1}$ where again c_α and d_β are the coefficients of α and β in π_r and π_s. Each term has a horizontal edge in the bottom joining $\beta^{-1}(u)$ to $\beta^{-1}(v)$. None of these terms appear with non zero coefficient in $X(\pi_1)$. As $\varepsilon(\Sigma_1 \otimes \Sigma_2)$ is a multiple of $x(\pi_1)$, the multiple must be 0 and so

$$(\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2} = 0. \tag{6.1.28}$$

Next consider δ_0 and η_0:

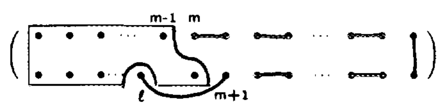


The edge between u and v in Δ comes about because in $\delta \cup \eta$ there is a path between u' and v' where u' and v' are the vertices of η joined to u and v. As this path does not intersect j in η, this is also a path between these vertices in $\delta_0 \cup \eta_0$. This shows that

$$(Z_\mu(x))_{\varphi(\Sigma_1), \varphi(\Sigma_2)} = 0. \tag{6.1.29}$$

This case is now complete by (6.1.28) and (6.1.29).

Subcase 2. There are no horizontal edges between pairs (u, v) with $1 \leqslant u < v \leqslant m$ in the bottom row of $Y(\delta) * Z(\eta, j)'$. As $Y(\delta) * Z(\eta, j)$ has as many horizontal edges on the bottom as on the top, there must be a horizontal edge from $m+1$ to l for some l with $1 \leqslant l \leqslant m$. Now $\tilde{\varepsilon}(Y(\delta) * Z(\eta, j)')$ is of the form

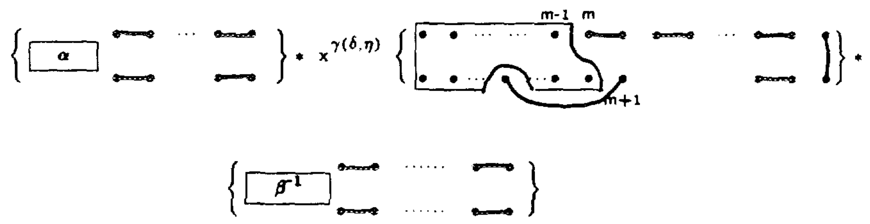
$\tilde{\varepsilon}(Y(\delta) * Z(\eta, j)') = x^{\gamma(\delta, \eta)}$  (6.1.30)

There are $m-1$ vertical lines between $1, 2, \ldots, m-1$ on the top and $\{1, 2, \ldots, l-1, l+1, \ldots, m\}$ which gives a 1-1 map $\hat{\sigma}$ between these sets say from the points on the bottom to $\{1, 2, \ldots, m-1\}$ on the top. Extend this map to a permutation in S_m by defining $\sigma(l) = m$.

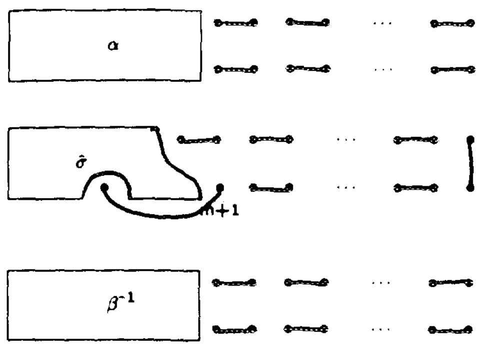
Again let c_α, d_β be the coefficients of α and β in π_r and π_s respectively for α, β in S_m. Expanding $X(\pi_r) * \tilde{\varepsilon}(Y(\delta) * Z(\eta, j)') * X(\pi_s)'$ we obtain

$$X(\pi_r)\, \tilde{\varepsilon}(Y(\delta) * Z(\eta, j)')\, X(\pi_s) = x^{\gamma(\delta,\eta)} \sum_{\alpha,\beta} c_\alpha d_\beta X(\alpha)\, \tilde{\varepsilon}(Y(\delta) * Z(\eta, j)')\, X(\beta^{-1}). \tag{6.1.31}$$

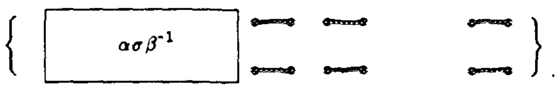
For fixed α, β we have



$= x^{\gamma(\delta,\eta)}$



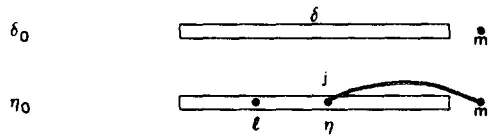
$= x^{\gamma(\delta,\eta)}$



Considering terms in $F_{f-2}[k]$ only we know $X(\pi_r)\, \tilde{\varepsilon}(Y(\delta) * Z(\eta, j)')\, X(\pi_s)$ is $(\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2} X(\pi_1)$. As the coefficient of the identity in $X(\pi_1)$ is 1,

$$(\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2} = x^{\gamma(\delta,\eta)} \sum_{\substack{\alpha, \beta \in S_m \\ \alpha\sigma = \beta}} c_\alpha\, d_\beta. \tag{6.1.32}$$

We next consider $(Z_\mu(x))_{\varphi(\Sigma_1), \varphi(\Sigma_2)}$. We have



Note that $\gamma(\delta, \eta) = \gamma(\delta_0, \eta_0)$ and that there is a path in $\delta \cup \eta$ between the jth and the lth free points in η. This path yields a path in $\delta_0 \cup \eta_0$ from the mth free point of δ_0 to the lth free point in η_0. With this observation it is easy to see

$$\tau(\delta_0, \eta_0) = \sigma. \tag{6.1.33}$$

Let α, β be in S_m. The $(\delta_0 \otimes \alpha, \eta_0 \otimes \beta)$ entry of $Z_{m,k}(x)$ is

$$(Z_{m,k}(x))_{\delta_0 \otimes \alpha, \eta_0 \otimes \beta} = \begin{cases} x^{\gamma(\delta_0, \eta_0)} & \text{if } \alpha\sigma(i) = \beta(i) \quad \text{for all } i \\ 0 & \text{otherwise.} \end{cases}$$

This means

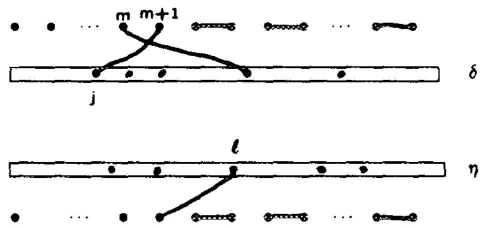
$$(Z_\mu(x))_{\varphi(\Sigma_1), \varphi(\Sigma_2)} = x^{\gamma(\delta_0, \eta_0)} \sum_{\substack{\alpha, \beta \in S_m \\ \alpha\sigma = \beta}} c_\alpha d_\beta \tag{6.1.34}$$

Case 2 is now complete by (6.1.32) and (6.1.34).

Case 3. $\Sigma_1 = B(\delta, j, \pi_r)$ and $\Sigma_2 = B(\eta, l, \pi_s)$. As above we have

$$\tilde{\varepsilon}(B(\delta, j, \pi_r) * B(\eta, l, \pi_s)) = X(\pi_r)\, \tilde{\varepsilon}(Z(\delta, j) * Z(\eta, l))\, X(\pi_s)$$

The diagrams here are

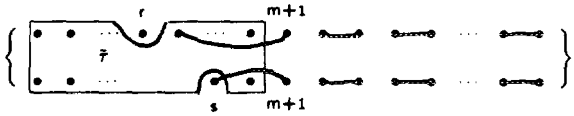
$Z(\delta, j) * Z(\eta, l)' =$  (6.1.35)

This is also divided into subcases.

Subcase 1. For some u, v with $1 \leqslant u < v \leqslant m$ there is a horizontal edge joining u to v in either the top or the bottom row of $Z(\delta, j) * Z(\eta, l)'$. In this case $(\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2} = 0 = (Z_\mu(x))_{\varphi(\Sigma_1), \varphi(\Sigma_2)}$ using the same argument as in Subcase 1 of Case 2.

Subcase 2. The vertices $m+1$ on the top and bottom of $Z(\delta, j) * Z(\eta, l)'$ are joined by *horizontal* lines to vertices which must be less than $m+1$ (see (6.1.35)). In particular suppose $m+1$ on the top is joined to r and $m+1$ on the bottom is joined to s where $1 \leqslant r,\ s \leqslant m$. The diagram is now

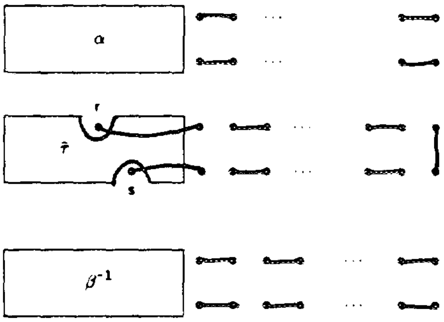
$Z(\delta, j) * Z(\eta, \ell)' = x^{\gamma(\delta,\eta)}$



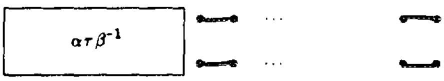
where $\tilde{\tau}$ is a 1–1 map from $\{1, 2, ..., m\}\backslash\{s\}$ to $\{1, 2, ..., m\}\backslash\{r\}$. Extend $\tilde{\tau}$ to a permutation τ by defining $\tau(s) = r$.

If α and β are in S_m we have

$X(\alpha)\tilde{\varepsilon}\big(Z(\delta, j) * Z(\eta, \ell)'\big)X(\beta)' = x^{\gamma(\delta,\eta)}$



$= x^{\gamma(\delta,\eta)}$

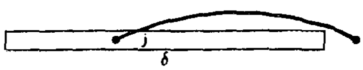


Again let c_α, d_β be the coefficients of α and β in π_r and π_s. As in Case 2, $(\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2}$ is the coefficient of the identity in $X(\pi_r)\, \tilde{\varepsilon}(Z(\delta, j) * Z(\eta, l)')\, X(\pi_s)$. This is

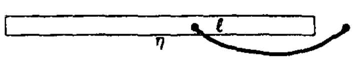
$$(\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2} = x^{\gamma(\delta,\eta)} \sum_{\substack{\alpha,\, \beta \in S_m \\ \alpha\tau = \beta}} c_\alpha d_\beta. \tag{6.1.36}$$

Now consider δ_0 and η_0:

δ_0



η_0



Note that $\gamma(\delta_0, \eta_0) = \gamma(\delta, \eta)$. Because we have horizontal edges from $m+1$ to r in the top row and from $m+1$ to s in the bottom row of $Z(\delta, j) * Z(\eta, l)'$, the following paths are present in $\delta \cup \eta$:

1. a path from the jth free point of δ to the rth free point of δ if $r < j$ and the $(r+1)$st free point of δ if $r > j$.

2. a path from the lth free point of η to the sth free point of η if $s < l$ and to the $(s+1)$st free point of η if $s > l$.

Now in $\delta_0 \cup \eta_0$ there is a path from the sth free point of η_0 to the rth free point of δ_0. It follows that

$$\tau(\delta_0, \eta_0) = \tau. \tag{6.1.37}$$

The condition for the $(\delta_0 \otimes \alpha, \eta_0 \otimes \beta)$ entry of $Z_\mu(x)$ to be nonzero is that $\beta = \alpha\tau$. This gives

$$(Z_\mu(x))_{\varphi(\Sigma_1), \varphi(\Sigma_2)} = x^{\gamma(\delta, \eta)} \sum_{\substack{\delta, \beta \in S_m \\ \alpha\tau = \beta}} c_\alpha d_\beta. \tag{6.1.38}$$

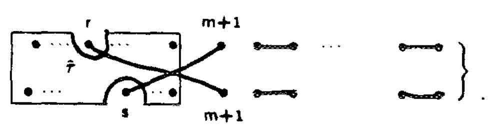
Now

$$(\mathscr{H}_\mu)_{\Sigma_1, \Sigma_2} = (Z_\mu(x))_{\varphi(\Sigma_1), \varphi(\Sigma_2)} \tag{6.1.39}$$

follows from (6.1.34) and (6.1.38). This completes Subcase 2.

Subcase 3. The vertices $m+1$ on the top and bottom are joined by *vertical* lines to vertices which again must be less than $m+1$ (see (6.1.35)). In particular suppose $m+1$ on the top is joined to s on the bottom and $m+1$ on the bottom is joined to r on the top. The diagram is now

$$\left(Z(\delta, j) * Z(\eta, \ell)'\right) = X^{\gamma(\delta, \eta)} \Bigg\{$$



This subcase can now be completed in exactly the same way as Subcase 2.

This completes all parts of Theorem 6.1.17.

6.2. *More about* φ

In this subsection we are going to prove another important property of the isomorhism $\varphi: V \to \pi_1 W_{m,k+1}$. Recall from Section 2.3 that $K_{m,k}$ denotes the kernel of the map $Z_{m,k}(x)$ and that $K^{(1)}_{m,k}$ denotes the subspace of $W_{m,k+1}$ obtained from $K_{m,k}$ via the inheritance construction. In Section 3.3 we showed that $K^{(1)}_{m,k}$ is contained in the kernel of $Z_{m,k+1}(x)$. The main result of this section is

THEOREM 6.2.1. $\varphi^{-1}(\pi_1 K_{m,k}^{(1)}) \subseteq EN_{f-2}\mathscr{A}_{f-1}^{(x)}$.

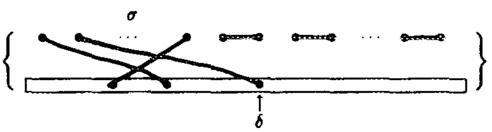
Proof. Let n be an element of $K_{m,k}$ and let (i,j) be a pair with $1 \leqslant i < j \leqslant f$. We write $\varphi^{-1}\pi_1 n^{ij}$ in the form $E * u * v$ where $u \in N_{f-2}$ and $v \in \mathscr{A}_{f-1}^{(x)}$. Write

$$n = \sum_{(\delta,\sigma)} c_{\delta,\sigma}(\delta \otimes \sigma),$$

where the sum is over pairs (δ, σ) consisting of an unlabelled (m, k) partial 1-factor δ and a permutation σ in S_m. Define $r(n) \in \mathscr{A}_{f-2}^{(x)}$ by

$$r(n) = \sum_{(\delta,\sigma)} c_{\delta,\sigma}(\delta_0 \otimes \delta \otimes \sigma),$$

where δ_0 is the (unlabelled) (m, k) partial 1-factor with free points $1, 2, ..., m$ and edges from $m + (2l-1)$ to $m + (2l)$ for $l = 1, 2, ..., k$. Pictorially we have

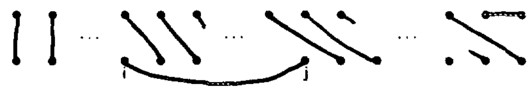
$$r(n) = \sum_{(\delta,\sigma)} c_{\delta,\sigma}$$ 

Since $n \in K_{m,k}$, $r(n)$ is in N_{f-2}.

It is convenient here to define γ on the linearly independent set $A(\delta, \sigma) \cup B(\delta, \sigma)$ $\sigma \in S_m$ by $\gamma(A(\delta, \sigma)) = \delta_0 \otimes \sigma$, $\gamma(B(\delta, j, \sigma)) = \delta_0 \times \sigma$ (see (6.1.16)) and extend γ linearly. Clearly γ and φ are connected by $\varphi(E(A(\delta, \pi_i))) = \gamma(A(\delta, \pi_i))$ and $\varphi(E(B(\delta, j, \pi_i))) = \gamma(B(\delta, j, \pi_i))$.

Case 1. $j < f$. In this case the following Claim proves our result.

CLAIM. *$\varphi^{-1}(\pi_1 n^{ij}) = x^{-k}E * r(n) * \Gamma_{ij}$ which will hold if $\gamma^{-1}(\pi_1 n^{ij}) = X(\pi_1) * r(n) * \Gamma_{ij}$ where*

Γ_{ij} = 

Proof of the Claim. By linearity of γ it is enough to show that

$$\gamma^{-1}(\eta \cdot (\delta \otimes \sigma)^{ij}) = x^{-k}X(\eta) * (\delta_0 \otimes \delta \otimes \sigma) * \Gamma_{ij}$$

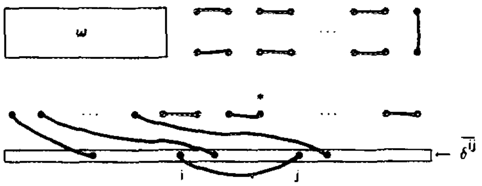
for all $\eta, \sigma \in S_m$ and all $\delta \in \mathscr{B}_{m,k}$. Equivalently we need to show that

$$\gamma^{-1}(\omega \cdot (\delta \otimes \mathrm{id})^{ij}) = x^{-k}X(\omega) * (\delta_0 \otimes \delta \otimes \mathrm{id}) * \Gamma_{ij} \qquad (6.2.2)$$

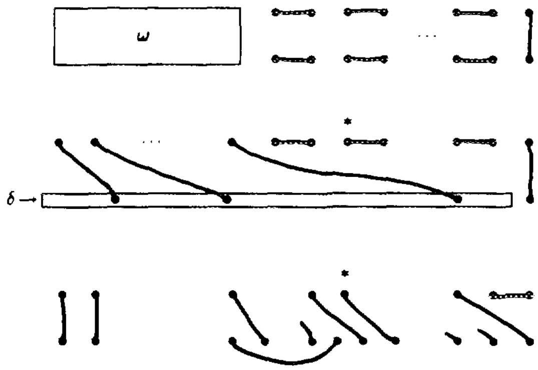
for all $\omega \in S_m$ and all $\delta \in \mathscr{B}_{m,k}$ where id denotes the identity permutation in S_m. To see this we split into two subcases.

Subcase 1. Suppose that the last point of δ (i.e., the $(f-2)$nd point) is a free point. Let $\overline{\delta^{ij}}$ denote the unlabelled $(m-1, k+1)$ partial 1-factor obtained from δ^{ij} by removing the fth point.

By the definition of γ we have

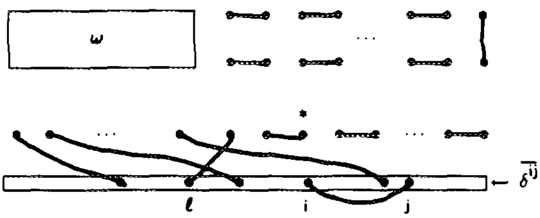
$\gamma^{-1}(\omega \cdot (\delta \otimes \mathrm{id})^{ij}) =$  (6.2.3)

Observe that the diagram on the right-hand side of (6.2.3) can be factored as

$\gamma^{-1}(\omega \cdot (\delta \otimes \mathrm{id})^{ij}) = x^{-k}$ 

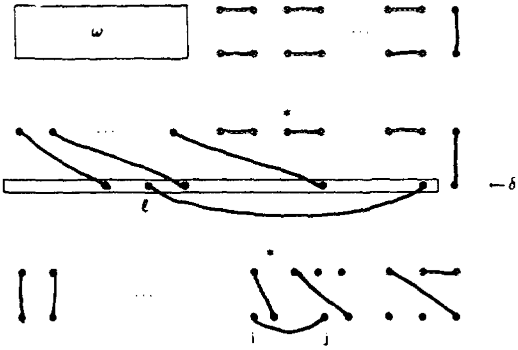
This proves the claim in Subcase 1.

Subcase 2. Suppose the last point of δ is not a free point. Say the last point of δ is adjacent to the lth point. Let $\overline{\delta^{ij}}$ be the $(m+1, k)$ (unlabelled) partial 1-factor obtained from δ^{ij} by removing the point f and the edge from f to l. From the definition of φ we have

$\gamma^{-1}(\omega \cdot (\delta \otimes \mathrm{id})^{ij}) =$ 

The diagram on the right-hand side of the above equation can be factored as

$\gamma^{-1}(\omega\cdot(\delta\otimes \mathrm{id})^{ij}) = x^{-k}$



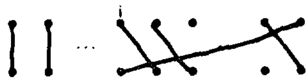
$$= x^{-k}X(\omega) * (\delta_0\otimes\delta\otimes \mathrm{id}) * \Gamma_{ij}$$

This proves the claim in Subcase 2 and completes the proof of the Theorem in Case 1.

Case 2. $j=f$. In this case the following Claim proves our result.

CLAIM. *$\gamma^{-1}(\pi_1 n^{ij}) = X(\pi_1)\, r(n)\, \Omega_i$ where*

$\Omega_i =$

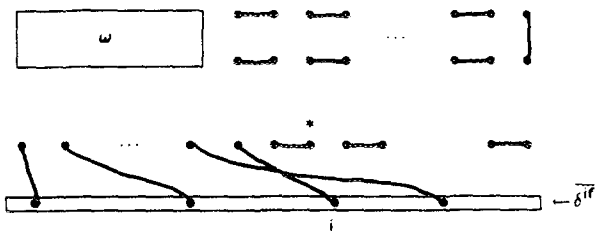


Proof of the Claim. As in Case 1 it is enough to show that

$$\gamma^{-1}(\omega\cdot(\delta\otimes \mathrm{id})^{if}) = x^{-k}X(\omega) * (\delta_0\otimes\delta\otimes \mathrm{id}) * \Omega_i.$$

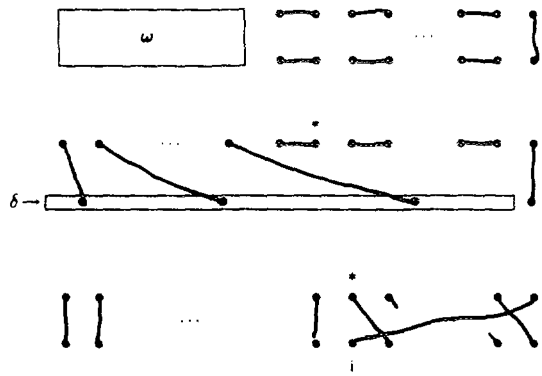
for all $\omega\in S_m$, $\delta\in\mathscr{B}_{m,k}$. Let $\overline{\delta^{if}}$ be obtained from δ^{if} by removing the point f and the line from i to f. Then

$\gamma^{-1}(\omega\cdot(\delta\otimes \mathrm{id})^{if}) =$



It is straightforward to check that the diagram on the righ-hand side has the factorization

$\varphi^{-1}\left(\omega \cdot (\delta \otimes \mathrm{id})^{if}\right) = x^{-k}$



$= x^{-k} X(\omega) * (\delta_0 \otimes \delta \otimes \mathrm{id}) * \Omega_i$.

This completes the proof of Theorem 6.2.1. ▮

7. The Structure of the Radical of $\mathscr{A}_f^{(x)}$

We can now determine fairly precise information about the structure of the radical N_f in terms of the matrices $Z_\mu(x)$.

7.1. *A Filtration on* N_f

Let μ be a partition of m and let f be a positive integer with $f-m=2(k+1)$ even. Recall that $K_f^{(1)}[\mu]$ denotes the part of the nullspace of $Z_\mu(x)$ which is inherited from $K_{f-2}[\mu]$. Let $\hat{Z}_\mu(x)$ denote the induced action of $Z_\mu(x)$ on $\pi_1 W_{m,k+1}/K_f^{(1)}[\mu]$. According to the results in Section 6, $\hat{Z}_\mu(x)$ is equal to the matrix H_μ in the characteristic sequence for ε. Let $P_f[\mu]$ and $Q_f[\mu]$ denote the nullspace and range of $\hat{Z}_\mu(x)=H_\mu$. Denote the radical N_f of $\mathscr{A}_f^{(x)}$ by $N_f^{(0)}$. Denote by $N_f^{(0)}[\mu]$ the μ piece of the radical of $\mathscr{A}_f^{(x)}[k]$.

By the construction in this paper we have

$$N_f^{(0)}[\mu]=(P_f[\mu]\otimes Q_f[\mu])\oplus(Q_f[\mu]\otimes P_f[\mu])\oplus(P_f[\mu]\otimes P_f[\mu])\oplus N_f^{(1)}[\mu]. \tag{7.1.1}$$

Definition 7.1.2. Define ideals $N_f^{(i,L)}[\mu]$, $N_f^{(i,R)}[\mu]$ and $M^{(i)}[\mu]$ as follows:

(A) $(i=0)$ Referring to the decomposition in (7.1.1) we have

$$N_f^{(0,L)}[\mu]=(P_f[\mu]\otimes Q_f[\mu])\oplus(P_f[\mu]\otimes P_f[\mu])\oplus N_f^{(1)}[\mu]$$

$$N_f^{(0,R)}[\mu]=(Q_f[\mu]\otimes P_f[\mu])\oplus(P_f[\mu]\otimes P_f[\mu])\oplus N_f^{(1)}[\mu]$$

$$M_f^{(0)}[\mu]=(P_f[\mu]\otimes P_f[\mu])\oplus N_f^{(1)}[\mu]$$

(B) $(i>0)$

$$N_f^{(i,L)}[\mu]=(N_{f-2}^{(i-1,L)}[\mu])^{(1)}$$
$$N_f^{(i,r)}[\mu]=(N_{f-2}^{(i-1,r)}[\mu])^{(1)}$$
$$M_f^{(i)}[\mu]=(M_{f-2}^{(i-1)}[\mu])^{(1)}.$$

(C) Also we define $N_f^{(i)}$ for $i\geqslant 1$ by $N_f^{(i)}=(N_{f-2}^{(i-1)})^{(1)}$.

The inclusions amongst the ideals $N_f^{(i,L)}[\mu]$, $N_f^{(i,R)}[\mu]$, $M_f^{(i)}[\mu]$ are shown in Fig. 1.

The next theorem explains multiplication in N_f in terms of Fig. 1.

THEOREM 7.1.3. *For each i we have*

(a) $N_f^{(i,iL)}[\mu]\,N_f^{(i,R)}[\mu]=M^{(i)}[\mu]$

(b) *The other eight products,*

$$\begin{array}{lll} N_f^{(i,L)}[\mu]\,N_f^{(i,L)}[\mu] & & N_f^{(i,L)}[\mu]\,M_f^{(i)}[\mu] \\ N_f^{(i,R)}[\mu]\,N_f^{(i,L)}[\mu] & N_f^{(i,R)}[\mu]\,N_f^{(i,R)}[\mu] & N_f^{(i,R)}[\mu]\,M_f^{(i)}[\mu] \\ M_f^{(i)}[\mu]\,N_f^{(i,L)}[\mu] & M_f^{(i)}[\mu]\,N_f^{(i,R)}[\mu] & M_f^{(i)}[\mu]\,M_f^{(i)}[\mu] \end{array}$$

are all contained in $N_f^{(i+1)}[\mu]$.

Proof. For $i=0$ these results follow from the tower construction. For $i>0$ the results follow (by induction on i) from Lemma 3.2.2. ∎

This theorem has some interesting corollaries.

COROLLARY 7.1.4. *For each i we have* $N_f^{(i)}[\mu]^3\subseteq N_f^{(i+1)}[\mu]$.

It is not known whether equality occurs in Corollary 7.1.4 for all i and μ. In every case the authors have tried there has been equality.

COROLLARY 7.1.5. *Let l be the number of integers i such that* $P_{f-2i}[\mu]$ *is nonempty. Then*

$$(N_f[\mu])^{3^{l+1}}=0.$$

In particular, if s denotes the integer part of $3f/2$ *then*

$$N_f^{3^{s-1}}=0.$$

7.2. *The Case* $x=-2$

To end this paper we look at the case $x=-2$. We will work out the dimensions of the $N_f^{(i,L)}[\mu]$, $N_f^{(i,R)}[\mu]$ and $M_f^{(i)}[\mu]$ for small values of f.

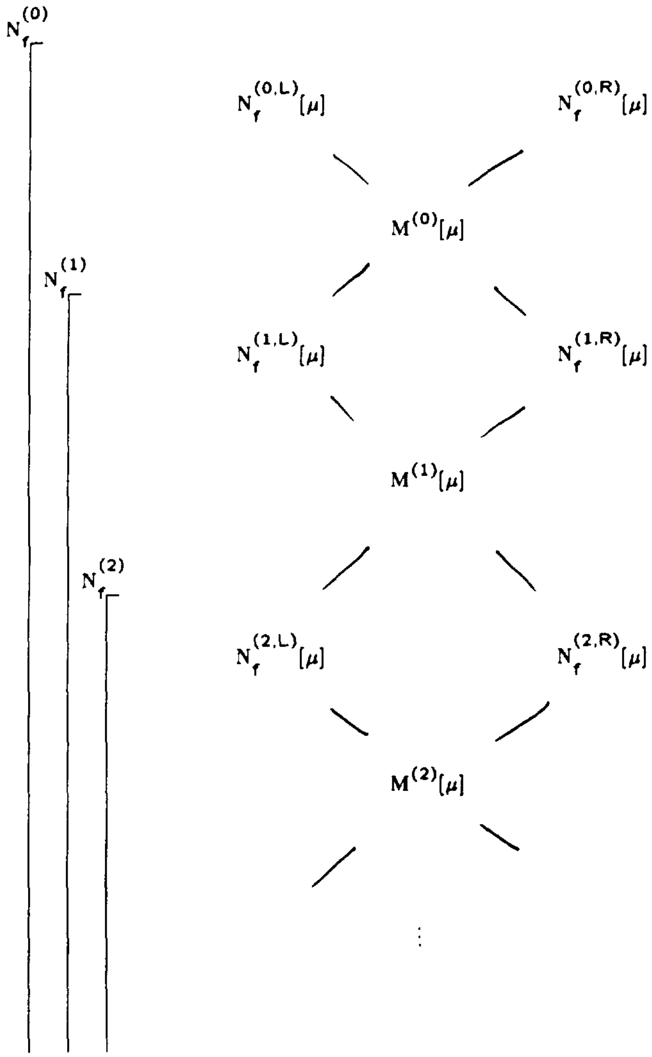


FIGURE 1

In addition we will compute the dimensions of the irreducible modules V_μ of the semisimple quotients $\mathscr{A}_f^{(-2)}/N_f$. The information is presented in the following way. We tabulate the information first according to the value of f. For each value of f we then consider each partition μ with $f-|\mu|$ even. For each such μ we represent the filtration of $N_f[\mu]$ with an inclusion diagram. Beside each component of the filtration we given the dimension of the corresponding piece in the associated graded module. For example, the diagram

$$N_4^{(0,L)}[\phi] \qquad\qquad N_4^{(0,R)}[\phi]$$
$$\dim = 2 \qquad\qquad \dim = 2$$
$$M_4^{(0)}[\phi]$$
$$\dim = 1$$
$$N_4^{(1)}[\phi]$$
$$\dim = 0$$

means that the dimension of $N_4^{(0,L)}[\phi]/M_4^{(0)}[\phi]$ is 2, the dimension of $N_4^{(0,R)}[\phi]/M_4^{(0)}[\phi]$ is 2 and the dimension of $M_4^{(0)}[\phi]/N_4^{(1)}[\phi]$ is 1.

Finally, we give the dimension of the irreducible $(\mathscr{A}_f^{(x)}/N_f)$-module V_μ indexed by μ:

$f=1$ $\quad N_1[1]=0$, $\dim V_1 = 1$

$f=2$
- $\mu=2$ $\quad N_2[2]=0$, $\dim V_2=1$
- $\mu=1^2$ $\quad N_2[1^2]=0$, $\dim V_{1^2}=1$
- $\mu=\phi$ $\quad N_2[\phi]=0$, $\dim V_\phi=1$

$f=3$
- $\mu=3$ $\quad N_3[3]=0$, $\dim V_3=1$
- $\mu=21$ $\quad N_3[21]=0$, $\dim V_{21}=2$
- $\mu=1^3$ $\quad N_3[1^3]=0$, $\dim V_{1^3}=1$
- $\mu=1$ $\quad \dim(N_3^{(0)}[1])=5$, $\dim(N_3^{(1)}[1])=0$, $\dim V_1=2$

$N_3^{(0,L)}[1]$ dim $= 2$ $\qquad$ $N_3^{(0,R)}[1]$ dim $= 2$

$M_3^{(0)}[1]$ dim $= 1$

$N_3^{(1)}[1] = 0$

$f = 4$

$\mu = 4$ $\quad$ $N_4[4] = 0$, dim $V_4 = 1$

$\mu = 31$ $\quad$ $N_4[31] = 0$, dim $V_{31} = 3$

$\mu = 2^2$ $\quad$ $N_4[2^2] = 0$, dim $V_{2^2} = 2$

$\mu = 21^2$ $\quad$ $N_4[21^2] = 0$, dim $V_{21^2} = 3$

$\mu = 1^4$ $\quad$ $N_4[1^4] = 0$, dim $V_{1^4} = 1$

$\mu = 2$ $\quad$ $N_4[2] = 0$, dim $V_2 = 6$

$\mu = 1^2$ $\quad$ $\dim(N_4^{(0)}[1^2]) = 27$, $N_4^{(1)}[1^2] = 0$, dim $V_{1^2} = 3$

$N_4^{(0,L)}[1^2]$ dim $= 9$ $\qquad$ $N_4^{(0,R)}[1^2]$ dim $= 9$

$M_4^{(0)}[1^2]$ dim $= 9$

$N_4^{(1)}[1^2] = 0$

$\mu = \phi$ $\quad$ $N_4^{(0)}[\phi] = N_4^{(1)}[\phi]$, $\dim(N_4^{(1)}[\phi]) = 5$, dim $V_\phi = 2$

$N_4^{(1,L)}[\phi]$ dim $= 2$ $\qquad$ $N_4^{(1,R)}[\phi]$ dim $= 2$

$M_4^{(1)}[\phi]$ dim $= 1$

$N_4^{(2)}[\phi] = 0.$

We end with the Bratelli diagram for the tower of algebras $\mathscr{A}_f^{(-2)}$ (see Fig. 2). Each irreducible is denoted by a partition. Beside each partition is the degree of the corresponding irreducible.

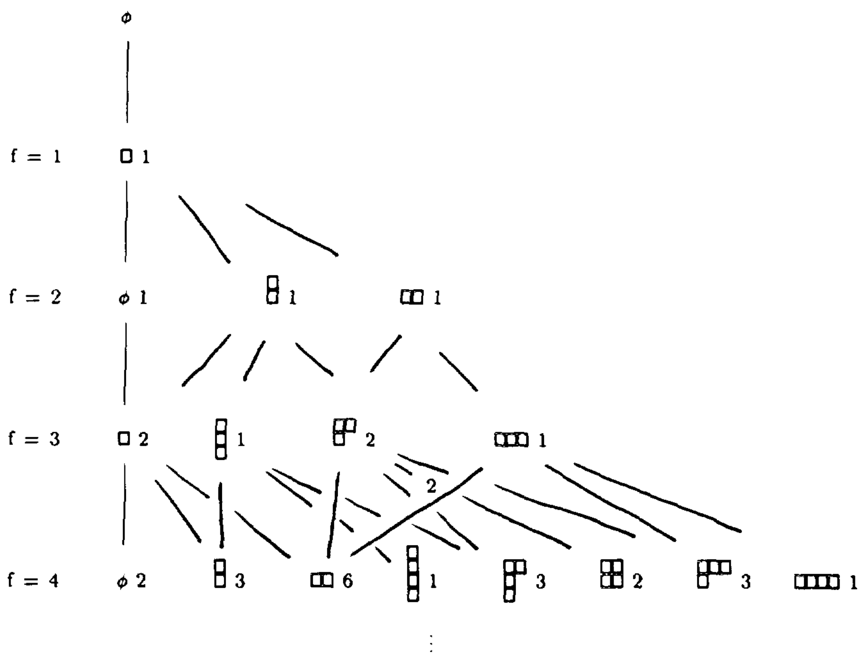


FIG. 2. The Bratelli diagram for the tower $\mathscr{A}_f^{(-2)}$.

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