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DYNAMIC OUTPUT FEEDBACK CONTROL OF MINIMUM-PHASE MULTIVARIABLE NONLINEAR PROCESSES

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Abstract—This paper concerns the synthesis of dynamic output feedback controllers for minimum-phase multivariable nonlinear processes with a nonsingular characteristic matrix. State-space controller realizations are derived that induce a linear input/output behavior of general form in the closed-loop system. A combination of input/output linearizing state feedback laws and state observers is employed for the derivation of the controllers. For open-loop stable processes, the process model is used as an open-loop state observer. In the more general case of possible open-loop instability, a reduced-order observer is used based on the forced zero dynamics of the process model. The performance and robustness characteristics of the proposed control methodology are illustrated through simulations in a chemical reactor example.

INTRODUCTION

One of the most basic problems in process control is the one of specifying a controller that makes use of measurements of process output variables in order to influence the dynamic behavior of the process in a desirable way. This problem is well-understood and studied within a linear control framework, where both a state-space approach and an input/output approach have led to identical solutions [see e.g. the classical books by Chen (1984), Kailath (1980) and Astrom and Wittenmark (1984)]. In a state-space approach the synthesis of the controllers is based on a combination of state feedback and state observers, while in an input/output approach the controller transfer functions are derived directly.

An obvious limitation of the theory developed in the above framework arises from the fact that physical and chemical phenomena are inherently nonlinear. As a result, real processes can exhibit distinctly peculiar dynamic behavior, which cannot be properly captured and accounted for in a linear control framework. Motivated by such considerations, the control community has lately witnessed an expanding research activity towards the development of nonlinear control methods. Differential geometry has provided powerful mathematical and conceptual tools in this direction, allowing fundamental aspects of nonlinear dynamics to be understood and typical theoretical control problems to be successfully addressed [see e.g. the books by Isidori (1989) and Nijmeijer and van der Schaft (1990)]. The early results in this area have shown that the natural frame for nonlinear control lies within the state-space approach, which allows typical results of linear control theory to be naturally generalized in a nonlinear setting. This is in contrast with the abstract input/output approach for nonlinear systems, which does not have the power and explicitness that transfer functions have in linear systems, although it provides philosophical guidelines and macroscopic perspective.

In a nonlinear state-space approach, the problem of synthesis of dynamic output feedback controllers becomes the problem of deriving state-space realizations of the controllers, viewed as nonlinear dynamic systems. In analogy with the linear case, the most logical and intuitively appealing approach to this problem is the combination of nonlinear state feedback laws and nonlinear state observers. The major difficulties to this end are associated with the observer design problem. Very few results are available on the existence and construction of observers for general nonlinear systems [e.g. Tsinias (1989, 1990) and Grizzle and Moraal (1990)], and moreover there is no general separation principle for nonlinear systems, to guarantee a well-behaved observer–controller combination. One way to cope with this problem is to utilize the natural modes of the process (i.e. the whole process dynamics or the process zero dynamics) for the state observation (Daoutidis and Kravaris, 1992a). In this direction, Daoutidis and Kravaris (1992a) developed a general solution of the dynamic output feedback problem for single-input single-output (SISO) minimum-phase nonlinear processes. In the present work, a general dynamic output feedback control problem is addressed and solved for a large class of multiple-input multiple-output (MIMO) minimum-phase nonlinear processes. In analogy with the SISO treatment, the key features of the approach are:

(1) The globally linearizing control (GLC) methodology (Kravaris and Chung, 1987; Kravaris and Soroush, 1990) provides the conceptual framework for the derivation of the controllers, which is based on the combination of input/output linearizing control laws and open-loop or reduced-order state observers.
(2) The combination of the controller and the observer is treated as a dynamic system itself for analysis and design purposes; thus, the problem of state reconstruction is not studied inde-

pendently, but is incorporated in the controller synthesis.

In addition to the above features, the proposed methodology accounts naturally for the multivariable nature of the control problem, allowing for any desirable degree of coupling to be achieved in the closed-loop system, by an appropriate choice of some adjustable parameters. It provides general and explicit output feedback controller realizations which are directly applicable to a large class of nonlinear multivariable processes of interest.

More specifically, in what follows, we will start with a brief discussion on key differential geometric concepts and alternative state-space realizations of nonlinear multivariable processes. Then, a general output feedback synthesis problem will be formulated for the class of multivariable minimum-phase processes under consideration. In the subsequent sections, the basic results of the paper will be developed: state-space realizations of dynamic output feedback controllers that solve the posed synthesis problem will be derived, and the closed-loop dynamics will be analyzed in terms of the induced input/output behavior and the asymptotic stability characteristics. Finally, the performance and robustness characteristics of the proposed control methodology will be illustrated through simulations in a chemical reactor example.

PRELIMINARIES

We consider MIMO nonlinear processes, with an equal number of inputs and outputs, and a state-space representation of the general form

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^{m} g_j(x) u_j \\ y_i &= h_i(x), \quad i = 1, \ldots, m \end{aligned} \tag{1}$$

where x denotes the vector of state variables, u_j denotes a manipulated input, and y_i denotes an output (to be controlled). For the theoretical development, and without loss of generality, it is assumed that all variables represent deviations from nominal values, and thus the origin is the equilibrium point of interest. It is also assumed that $x \in X \subset \mathbb{R}^n$, where X is open and connected, $u = [u_1 \ldots u_m]^T \in \mathbb{R}^m$, and $y = [y_1 \ldots y_m]^T \in \mathbb{R}^m$. Finally, $f(x), g_j(x), w_\kappa(x)$ are used to denote analytic vector fields on X and $h_i(x)$ is used to denote analytic scalar fields on X. In a more compact vector notation, eq. (1) can take the form

$$\begin{aligned} \dot{x} &= f(x) + g(x) u \\ y_i &= h_i(x), \quad i = 1, \ldots, m \end{aligned} \tag{2}$$

where $g(x)$ is an $(n \times m)$ matrix with columns the vector fields $g_1(x), \ldots, g_m(x)$.

Throughout the paper we will be using the standard Lie derivative notation, where $L_f h_i(x) = \sum_{l=1}^{n} [\partial h_i(x)/\partial x_l] f_l(x)$ and $f_l(x)$ denotes the lth row element of $f(x)$. One can define higher-order Lie derivatives $L_f^k h_i(x) = L_f L_f^{k-1} h_i(x)$ as well as mixed Lie derivatives $L_{g_j} L_f^{k-1} h_i(x)$ in an obvious way.

For the MIMO nonlinear process described by eq. (1), let r_i denote the relative order of the *output* y_i with respect to the *manipulated input vector* u, i.e. the smallest integer for which

$$\begin{aligned} L_g L_f^{r_i-1} h_i(x) = [L_{g_1} L_f^{r_i-1} h_i(x) \;\; L_{g_2} L_f^{r_i-1} h_i(x) \\ \cdots L_{g_m} L_f^{r_i-1} h_i(x)] \not\equiv [0 \;\; 0 \cdots 0]. \end{aligned} \tag{3}$$

If such an integer does not exist, we say that $r_i = \infty$. A graph-theoretic interpretation of the concept of relative order, as well as its interpretation as a measure of how "direct" the effect of the input vector is on an output variable, can be found in Daoutidis and Kravaris (1992b). It is assumed that a finite relative order r_i exists for every i, since this is a necessary condition for output controllability. Then, the matrix

$$C(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix} \tag{4}$$

is called the *characteristic matrix* of the system. It will be assumed that the state-space X does not contain any singular points, i.e. points for which $\det C(x) = 0$. As long as $\det C(0) \neq 0$, one can always redefine X in order to satisfy the above assumption.

In what follows, we selectively review some basic results in alternative state-space realizations and the notion of minimum-phase behavior for the class of processes under consideration. For the nonlinear process described by eq. (1), with finite relative orders r_i, $i = 1, \ldots, m$, and nonsingular characteristic matrix $C(x)$, one can always find scalar fields $t_1(x), \ldots, t_{n-\sum_i r_i}(x)$ such that the scalar fields

$$t_1(x), \ldots, t_{n-\sum_i r_i}(x), h_1(x), L_f h_1(x), \ldots, L_f^{r_1-1} h_1(x), \ldots, h_m(x), L_f h_m(x), \ldots, L_f^{r_m-1} h_m(x)$$

are linearly independent (Isidori, 1989). Then the mapping

$$\zeta = \begin{bmatrix} \zeta^{(0)} \\ \hline \zeta^{(1)} \\ \hline \vdots \\ \hline \zeta^{(m)} \end{bmatrix} = T(x) = \begin{bmatrix} t_1(x) \\ \vdots \\ t_{n-\sum_i r_i}(x) \\ \hline h_1(x) \\ L_f h_1(x) \\ \vdots \\ L_f^{r_1-1} h_1(x) \\ \hline \vdots \\ \hline h_m(x) \\ L_f h_m(x) \\ \vdots \\ L_f^{r_m-1} h_m(x) \end{bmatrix} \tag{5}$$

is invertible and qualifies as a curvilinear coordinate transformation. Assuming also that the vector fields $g_1(x), \dots, g_m(x)$ are involutive (a condition which is usually satisfied in MIMO systems of practical interest, and is trivially satisfied for SISO systems), one can always choose the scalar fields $t_l(x)$ such that $L_{g_j} t_l(x) = 0$ for all l, j. Then, the original system under the coordinate transformation of eq. (5) takes the following normal form (Isidori, 1989):

$$\begin{aligned}
\dot\zeta_1^{(0)} &= F_1(\zeta^{(0)}, \zeta^{(1)}, \dots, \zeta^{(m)})\\
&\vdots\\
\dot\zeta^{(0)}_{n-\sum_i r_i} &= F_{n-\sum_i r_i}(\zeta^{(0)}, \zeta^{(1)}, \dots, \zeta^{(m)})\\
\dot\zeta_1^{(1)} &= \zeta_2^{(1)}\\
&\vdots\\
\dot\zeta^{(1)}_{r_1-1} &= \zeta^{(1)}_{r_1}\\
\dot\zeta^{(1)}_{r_1} &= W_1(\zeta^{(0)}, \zeta^{(1)}, \dots, \zeta^{(m)})\\
&\quad + C_1(\zeta^{(0)}, \zeta^{(1)}, \dots, \zeta^{(m)})u\\
&\vdots\\
\dot\zeta_1^{(m)} &= \zeta_2^{(m)}\\
&\vdots\\
\dot\zeta^{(m)}_{r_m-1} &= \zeta^{(m)}_{r_m}\\
\dot\zeta^{(m)}_{r_m} &= W_m(\zeta^{(0)}, \zeta^{(1)}, \dots, \zeta^{(m)})\\
&\quad + C_m(\zeta^{(0)}, \zeta^{(1)}, \dots, \zeta^{(m)})u\\
y_1 &= \zeta_1^{(1)}\\
&\vdots\\
y_m &= \zeta_1^{(m)}
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
F_l(\zeta^{(0)}, \zeta^{(1)}, \dots, \zeta^{(m)}) &= [L_f t_l(x)]_{x=T^{-1}(\zeta)}\\
&\qquad l = 1, \dots, \left(n - \sum_i r_i\right)\\
C_i(\zeta^{(0)}, \zeta^{(1)}, \dots, \zeta^{(m)}) &= [L_g L_f^{r_i-1} h_i(x)]_{x=T^{-1}(\zeta)}\\
&\qquad i = 1, \dots, m \qquad (7)\\
W_i(\zeta^{(0)}, \zeta^{(1)}, \dots, \zeta^{(m)}) &= [L_f^{r_i} h_i(x)]_{x=T^{-1}(\zeta)}\\
&\qquad i = 1, \dots, m.
\end{aligned}$$

Referring to the above normal form, let

$$\mathscr{D}y_1 = \begin{bmatrix} y_1 \\ \vdots \\ \dfrac{\mathrm{d}^{r_1-1} y_1}{\mathrm{d}t^{r_1-1}} \end{bmatrix}, \dots, \mathscr{D}y_m = \begin{bmatrix} y_m \\ \vdots \\ \dfrac{\mathrm{d}^{r_m-1} y_m}{\mathrm{d}t^{r_m-1}} \end{bmatrix}. \tag{8}$$

Then, according to Daoutidis and Kravaris (1991), the dynamic system

$$\begin{aligned}
\dot\zeta_1^{(0)} &= F_1(\zeta^{(0)}, \mathscr{D}y_1, \dots, \mathscr{D}y_m)\\
&\vdots\\
\dot\zeta^{(0)}_{n-\sum_i r_i} &= F_{n-\sum_i r_i}(\zeta^{(0)}, \mathscr{D}y_1, \dots, \mathscr{D}y_m)
\end{aligned} \tag{9}$$

represents a reduced-order realization of the inverse system (or, equivalently, the forced zero dynamics) of eq. (1).

Furthermore, the unforced reduced-order inverse, i.e. the dynamic system

$$\begin{aligned}
\dot\zeta_1^{(0)} &= F_1(\zeta^{(0)}, 0, \dots, 0)\\
&\vdots\\
\dot\zeta^{(0)}_{n-\sum_i r_i} &= F_{n-\sum_i r_i}(\zeta^{(0)}, 0, \dots, 0)
\end{aligned} \tag{10}$$

represents the (unforced) zero dynamics of the process described by eq. (1), i.e. the nonlinear analogue of the concept of transmission zeros in MIMO linear systems (Daoutidis and Kravaris, 1991).

In analogy with the linear case, the nonlinear process in the form of eq. (6) is said to be *minimum-phase* if its (unforced) zero dynamics [eq. (10)] is asymptotically stable, while it is said to be *nonminimum-phase* if its (unforced) zero dynamics is unstable.

FORMULATION OF THE OUTPUT FEEDBACK SYNTHESIS PROBLEM FOR MINIMUM-PHASE PROCESSES

In this section, we will formulate the output feedback control problem for MIMO minimum-phase nonlinear processes as an explicit synthesis problem. The objective is to calculate state-space realizations of nonlinear controllers which will be using measurements of the output variables and the output set-points in order to enforce certain properties in the closed-loop system (see Fig. 1). The desirable closed-loop properties will, as usual, include

- input/output stability
- tracking of the output set-points
- rejection of disturbances and modeling errors
- asymptotic stability of the unforced closed-loop system (internal stability).

The assumption of minimum-phase behavior allows the formulation of a generic synthesis problem along the above lines. In particular, the assumption of stable zero dynamics allows requesting a closed-loop response with no zeros, resulting by essentially canceling the zero dynamics of the process. Furthermore,

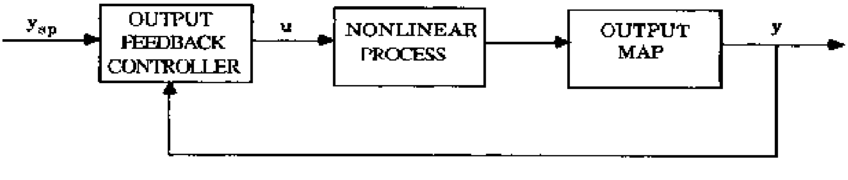

Fig. 1. Generic output feedback control structure.

properness considerations dictate that the relative orders r_i are "preserved" in the closed-loop system, which allows requesting a closed-loop response of order $(r_1 + \cdots + r_m)$. For convenience, a linear input/output behavior will also be requested, allowing for input/output stability and performance characteristics to be transparently incorporated in the design procedure. Taking into account the above considerations, as well as the requirement of a closed-loop static gain matrix equal to the identity matrix, the following synthesis problem is posed:

Given a state-space realization of a MIMO nonlinear process, calculate a state-space realization of a nonlinear controller which induces an input/output behavior of the form

$$
\begin{aligned}
y_1 + \sum_{i=1}^{m} \sum_{k=1}^{r_i} \gamma_{ik}^{1} \frac{d^k y_i}{dt^k} &= y_{sp1} \\
y_2 + \sum_{i=1}^{m} \sum_{k=1}^{r_i} \gamma_{ik}^{2} \frac{d^k y_i}{dt^k} &= y_{sp2} \\
&\vdots \\
y_m + \sum_{i=1}^{m} \sum_{k=0}^{r_i} \gamma_{ik}^{m} \frac{d^k y_i}{dt^k} &= y_{spm}
\end{aligned}
\tag{11}
$$

where γ_{ik}^{j} are adjustable constant parameters, with

$$
\det \begin{bmatrix} \gamma_{1r_1}^{1} & \gamma_{2r_2}^{1} & \cdots & \gamma_{mr_m}^{1} \\ \gamma_{1r_1}^{2} & \gamma_{2r_2}^{2} & \cdots & \gamma_{mr_m}^{2} \\ \vdots & \vdots & & \vdots \\ \gamma_{1r_1}^{m} & \gamma_{2r_2}^{m} & \cdots & \gamma_{mr_m}^{m} \end{bmatrix} \neq 0 \tag{12}
$$

and $y_{sp1}, \ldots, y_{spm}$ are the output set-points.

In a more compact vector form, eq. (11) takes the form

$$
y + \sum_{i=1}^{m} \sum_{k=1}^{r_i} \gamma_{ik} \frac{d^k y_i}{dt^k} = y_{sp} \tag{13}
$$

where $\gamma_{ik} = [\gamma_{ik}^{1} \cdots \gamma_{ik}^{m}]^T$. The condition of eq. (12) guarantees that the closed-loop system will be nonsingular and of order $r_1 + \cdots + r_m$. Furthermore, its bounded-input bounded-output (BIBO) stability characteristics will depend on the roots of the characteristic polynomial:

$$
\det \begin{bmatrix} (1 + \sum_{k=1}^{r_1} \gamma_{1k}^{1} s^k) & (\sum_{k=1}^{r_2} \gamma_{2k}^{1} s^k) & \cdots & (\sum_{k=1}^{r_m} \gamma_{mk}^{1} s^k) \\ (\sum_{k=1}^{r_1} \gamma_{1k}^{2} s^k) & (1 + \sum_{k=1}^{r_2} \gamma_{2k}^{2} s^k) & \cdots & (\sum_{k=1}^{r_m} \gamma_{mk}^{2} s^k) \\ \vdots & \vdots & & \vdots \\ (\sum_{k=1}^{r_1} \gamma_{1k}^{m} s^k) & (\sum_{k=1}^{r_2} \gamma_{2k}^{m} s^k) & \cdots & (1 + \sum_{k=1}^{r_m} \gamma_{mk}^{m} s^k) \end{bmatrix} = 0. \tag{14}
$$

Note that the input/output behavior of eq. (11) is a fully coupled one, capturing the most general form of a linear input/output behavior. However, the role of the adjustable parameters γ_{ik}^{j} is transparent: they determine the input/output stability and performance characteristics as well as the level of input/output coupling in the closed-loop system. A common design objective in practical applications (with the exception of some ill-conditioned processes like high-purity distillation columns) is the requirement of an input/output decoupled closed-loop system. In this case, the postulated input/output behavior in the synthesis problem takes the simplified form

$$
\begin{aligned}
y_1 + \sum_{k=1}^{r_1} \gamma_{1k}^{1} \frac{d^k y_1}{dt^k} &= y_{sp1} \\
y_2 + \sum_{k=1}^{r_2} \gamma_{2k}^{2} \frac{d^k y_2}{dt^k} &= y_{sp2} \\
&\vdots \\
y_m + \sum_{k=1}^{r_m} \gamma_{mk}^{m} \frac{d^k y_m}{dt^k} &= y_{spm}
\end{aligned}
\tag{15}
$$

or in more compact notation

$$
y_i + \sum_{k=1}^{r_i} \gamma_{ik}^{i} \frac{d^k y_i}{dt^k} = y_{spi} \tag{16}
$$

for $i = 1, \ldots, m$. The BIBO stability characteristics of the closed-loop system will then depend on the roots of the characteristic equation:

$$
\det \begin{bmatrix} (1 + \sum_{k=1}^{r_1} \gamma_{1k}^{1} s^k) & 0 & \cdots & 0 \\ 0 & (1 + \sum_{k=1}^{r_2} \gamma_{2k}^{2} s^k) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & (1 + \sum_{k=1}^{r_m} \gamma_{mk}^{m} s^k) \end{bmatrix} = 0. \tag{17}
$$

Finally, one can further simplify the form of the closed-loop input/output behavior by requesting a decoupled, critically damped response, in which case the number of adjustable parameters reduces to m.

In what follows we will address the posed synthesis problem in its most general form, initially for open-loop stable processes and then for general processes that may be open-loop unstable. In analogy with the SISO treatment of the problem (Daoutidis and Kravaris, 1992a), the GLC synthesis methodology will be used in the derivation of the controller realizations. In particular, referring to the GLC structure

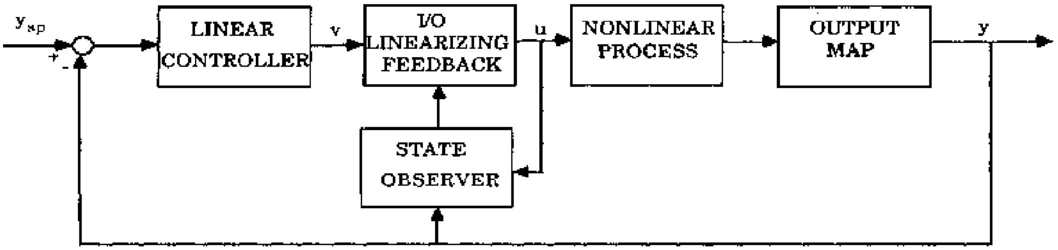

Fig. 2. GLC structure.

of Fig. 2, the following steps will be followed:

- synthesis of a state feedback law that induces an input/output behavior of the form
$$\sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}\frac{\mathrm{d}^k y_i}{\mathrm{d}t^k}=v \tag{18}$$
between y and v, where $\beta_{ik}=[\beta_{ik}^1\cdots\beta_{ik}^m]^T$ are vectors of adjustable parameters;
- reconstruction of the process states through an appropriate state observer;
- combination of the state feedback law (with the states estimated through the observer) with a linear error feedback compensator with integral action imposed on the v–y dynamics, that induces the desired input/output behavior between y_{sp} and y.

OUTPUT FEEDBACK CONTROL OF OPEN-LOOP STABLE MINIMUM-PHASE PROCESSES

Under the assumption of open-loop stability of the process dynamics, the process state variables can be reconstructed by simulating the process dynamics itself. The process model can be used as a (full-order) open-loop state observer for this purpose. Theorem 1 provides a solution to the posed synthesis problem for the class of nonlinear processes under consideration, using the above state observer. The proof can be found in the appendix.

Theorem 1: *Consider the nonlinear process described by eq.* (1), *with finite relative orders* r_i *and* $\det C(x)\neq 0$ *for* $x\in X$. *Then, the dynamic system*

$$\dot\xi_1^{(1)}=\xi_2^{(1)}$$
$$\vdots$$
$$\dot\xi_{r_1-1}^{(1)}=\xi_{r_1}^{(1)}$$
$$\dot\xi_{r_1}^{(1)}=([\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1})_1\times\left[(y_{sp}-y)-\sum_{i=1}^{m}\sum_{k=1}^{r_i-1}\gamma_{ik}\xi_{k+1}^{(i)}\right]$$
$$\vdots$$
$$\dot\xi_1^{(m)}=\xi_2^{(m)}$$
$$\vdots$$
$$\dot\xi_{r_m-1}^{(m)}=\xi_{r_m}^{(m)}$$
$$\dot\xi_{r_m}^{(m)}=([\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1})_m\times\left[(y_{sp}-y)-\sum_{i=1}^{m}\sum_{k=1}^{r_i-1}\gamma_{ik}\xi_{k+1}^{(i)}\right]$$
$$\begin{aligned}\dot w={}&f(w)+g(w)\{[\beta_{1r_1}\cdots\beta_{mr_m}]C(w)\}^{-1}\\&\times\Bigg\{\sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\xi_{k+1}^{(i)}\\&+[\beta_{1r_1}\cdots\beta_{mr_m}][\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1}\\&\times\left[(y_{sp}-y)-\sum_{i=1}^{m}\sum_{k=1}^{r_i-1}\gamma_{ik}\xi_{k+1}^{(i)}\right]\\&-\sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}L_f^k h_i(w)\Bigg\}\end{aligned}$$
$$\begin{aligned}u={}&\{[\beta_{1r_1}\cdots\beta_{mr_m}]C(w)\}^{-1}\\&\times\Bigg\{\sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\xi_{k+1}^{(i)}\\&+[\beta_{1r_1}\cdots\beta_{mr_m}][\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1}\\&\times\left[(y_{sp}-y)-\sum_{i=1}^{m}\sum_{k=1}^{r_i-1}\gamma_{ik}\xi_{k+1}^{(i)}\right]\\&-\sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}L_f^k h_i(w)\Bigg\}\end{aligned} \tag{19}$$

where $\beta_{ik}=[\beta_{ik}^1\cdots\beta_{ik}^m]^T$ *are vectors of adjustable parameters with* $\det[\beta_{1r_1}\cdots\beta_{mr_m}]\neq 0$, *and the symbol* $(\ )_i$ *denotes the* i*th row of a matrix, represents an* $(n+r_1+\cdots+r_m)$*th order state-space realization of a dynamic output feedback controller that induces the closed-loop input/output behavior*:

$$y+\sum_{i=1}^{m}\sum_{k=1}^{r_i}\gamma_{ik}\frac{\mathrm{d}^k y_i}{\mathrm{d}t^k}=y_{sp}.$$

Remark 1: Equation (19) represents a state-space realization of a dynamic *error-feedback* controller. The input to the controller is the error vector $(y_{sp}-y)$, its output is the manipulated input vector for the process u, while it involves $(n+r_1+\cdots+r_m)$ state variables. The state variables denoted by ξ correspond to the state variables of the linear error feedback compensator, while the n state variables denoted by w correspond to the state variables of the full-order

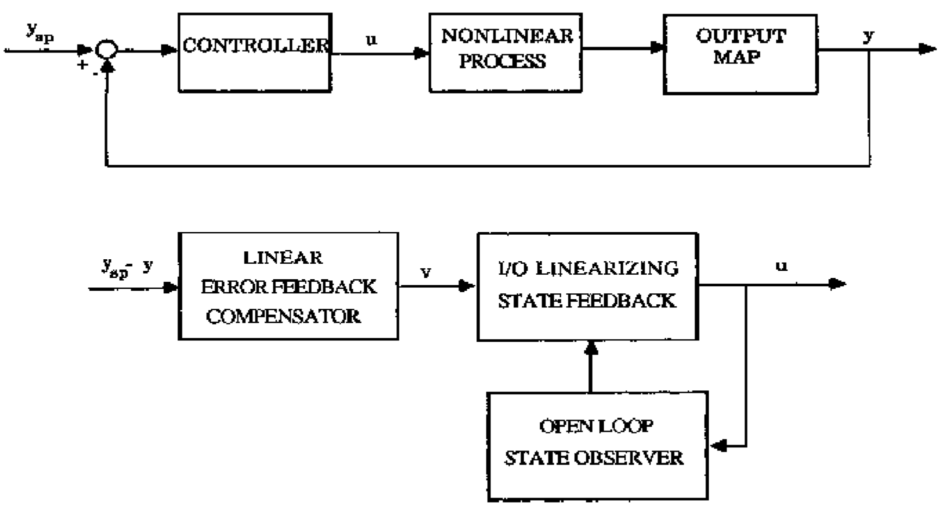

Fig. 3. Error feedback controller and control structure.

open-loop observer. The overall control structure as well as the various components of the controller are shown in Fig. 3.

In complete analogy with the results for SISO systems (Daoutidis and Kravaris, 1992a), appropriate initialization of the states of the controller of eq. (19) can lead to elimination of the states ξ, leading to a reduced-order controller realization. This result is summarized in Corollary 1, whose detailed proof can be found in the appendix.

Corollary 1: *Under the assumptions of Theorem* 1, *the dynamic system*

$$\begin{aligned}\dot{w} &= f(w) + g(w)\{[\gamma_{1r_1}\cdots\gamma_{mr_m}]C(w)\}^{-1}\\ &\quad\times\left[(y_{sp}-y)-\sum_{i=1}^{m}\sum_{k=1}^{r_i}\gamma_{ik}L_f^k h_i(w)\right]\\ u &= \{[\gamma_{1r_1}\cdots\gamma_{mr_m}]C(w)\}^{-1}\\ &\quad\times\left[(y_{sp}-y)-\sum_{i=1}^{m}\sum_{k=1}^{r_i}\gamma_{ik}L_f^k h_i(w)\right]\end{aligned} \tag{20}$$

represents an (n)*th order state-space realization of a dynamic output feedback controller that induces the closed-loop input/output behavior of eq.* (13).

Remark 2. The controller realization of eq. (20) is clearly the most convenient for practical implementation because of its reduced order. It is also interesting to note that eq. (20) can be interpreted as a feedforward controller on the error vector $(y_{sp}-y)$, which enforces the dynamics

$$\sum_{i=1}^{m}\sum_{k=1}^{r_i}\gamma_{ik}\frac{d^k y_i}{dt^k}=(y_{sp}-y).$$

This interpretation suggests an alternative way of derivation of eq. (20), starting from the above postulated dynamics, and using estimates of the output derivatives obtained from the process model. A more detailed development along the above lines is omitted for reasons of brevity.

Remark 3: In order to obtain a decoupled closed-loop input/output behavior of the form of eq. (16), one simply sets $\gamma_{ik}^j = 0$ for $i \neq j$ in the controller realizations of eqs (19) and (20). By also setting $\beta_{ik}^j = 0$ for $i \neq j$ in eq. (19), we essentially request input/output decoupling in the v–y dynamics of eq. (18), in which case the linear compensator used in the proof of Theorem 1 reduces to a cascade of SISO linear compensators. In this case, the resulting error feedback controller assumes the following full-order realization:

$$\begin{aligned}\dot{\xi}_1^{(1)} &= \xi_2^{(1)}\\ &\ \ \vdots\\ \dot{\xi}_{r_1-1}^{(1)} &= \xi_{r_1}^{(1)}\\ \dot{\xi}_{r_1}^{(1)} &= \frac{1}{\gamma_{1r_1}^1}\left[(y_{sp1}-y_1)-\sum_{k=1}^{r_1-1}\gamma_{1k}^1\xi_{k+1}^{(1)}\right]\\ &\ \ \vdots\\ \dot{\xi}_1^{(m)} &= \xi_2^{(m)}\\ &\ \ \vdots\\ \dot{\xi}_{r_m-1}^{(m)} &= \xi_{r_m}^{(m)}\\ \dot{\xi}_{r_m}^{(m)} &= \frac{1}{\gamma_{mr_m}^m}\left[(y_{spm}-y_m)-\sum_{k=1}^{r_m-1}\gamma_{mk}^m\xi_{k+1}^{(m)}\right]\end{aligned} \tag{21}$$

$$\dot{w} = f(w) + g(w)\{\mathrm{diag}[\beta^i_{ir_i}]C(w)\}^{-1}$$
$$\times \begin{bmatrix} \beta^1_{10}\zeta^{(1)}_1 + \sum_{k=1}^{r_1-1}\left(\beta^1_{1k} - \frac{\beta^1_{1r_1}}{\gamma^1_{1r_1}}\gamma^1_{1k}\right)\xi^{(1)}_{k+1} + \frac{\beta^1_{1r_1}}{\gamma^1_{1r_1}}(y_{sp1} - y_1) - \sum_{k=0}^{r_1}\beta^1_{1k}L^k_f h_1(w) \\ \vdots \\ \beta^m_{m0}\zeta^{(m)}_1 + \sum_{k=1}^{r_m-1}\left(\beta^m_{mk} - \frac{\beta^m_{mr_m}}{\gamma^m_{mr_m}}\gamma^m_{mk}\right)\xi^{(m)}_{k+1} + \frac{\beta^m_{mr_m}}{\gamma^m_{mr_m}}(y_{spm} - y_m) - \sum_{k=0}^{r_m}\beta^m_{mk}L^k_f h_m(w) \end{bmatrix}$$

$$u = \{\mathrm{diag}[\beta^i_{ir_i}]C(w)\}^{-1}$$
$$\times \begin{bmatrix} \beta^1_{10}\xi^{(1)}_1 + \sum_{k=1}^{r_1-1}\left(\beta^1_{1k} - \frac{\beta^1_{1r_1}}{\gamma^1_{1r_1}}\gamma^1_{1k}\right)\xi^{(1)}_{k+1} + \frac{\beta^1_{1r_1}}{\gamma^1_{1r_1}}(y_{sp1} - y_1) - \sum_{k=0}^{r_1}\beta^1_{1k}L^k_f h_1(w) \\ \vdots \\ \beta^m_{m0}\xi^{(m)}_1 + \sum_{k=1}^{r_m-1}\left(\beta^m_{mk} - \frac{\beta^m_{mr_m}}{\gamma^m_{mr_m}}\gamma^m_{mk}\right)\xi^{(m)}_{k+1} + \frac{\beta^m_{mr_m}}{\gamma^m_{mr_m}}(y_{spm} - y_m) - \sum_{k=0}^{r_m}\beta^m_{mk}L^k_f h_m(w) \end{bmatrix}$$

and the following reduced-order realization:

$$\dot{w} = f(w) + g(w)\{\mathrm{diag}[\gamma^i_{ir_i}]C(w)\}^{-1}\begin{bmatrix} (y_{sp1} - y_1) - \sum_{k=1}^{r_1}\gamma^1_{1k}L^k_f h_1(w) \\ \vdots \\ (y_{spm} - y_m) - \sum_{k=1}^{r_m}\gamma^m_{mk}L^k_f h_m(w) \end{bmatrix}$$
$$u = \{\mathrm{diag}[\gamma^i_{ir_i}]C(w)\}^{-1}\begin{bmatrix} (y_{sp1} - y_1) - \sum_{k=1}^{r_1}\gamma^1_{1k}L^k_f h_1(w) \\ \vdots \\ (y_{spm} - y_m) - \sum_{k=1}^{r_m}\gamma^m_{mk}L^k_f h_m(w) \end{bmatrix}. \tag{22}$$

Remark 4: In the case of a SISO nonlinear process, i.e. for $m = 1$, the controller realizations of eqs (21) and (22) reduce exactly to the realizations derived in Daoutidis and Kravaris (1992a), as expected.

OUTPUT FEEDBACK CONTROL OF MINIMUM-PHASE PROCESSES

The error feedback controllers developed in the previous section can be applied only to open-loop stable minimum-phase processes. In the case of open-loop instability, any error in the observer initialization would grow indefinitely, leading to obvious internal stability problems. However, the normal-form representation of eq. (6) for minimum-phase nonlinear processes suggests an alternative way of state reconstruction, valid even in the presence of possible open-loop instability. In particular, it is clear from eq. (6) that $\zeta^{(1)} = \mathscr{D}y_1, \ldots, \zeta^{(m)} = \mathscr{D}y_m$, i.e. $(r_1 + \cdots + r_m)$ state variables are exactly the outputs and their derivatives up to $(r_i - 1)$th order, which are assumed to be available. The remaining $(n - \sum_i r_i)$ state variables can be obtained by simulating the forced zero dynamics of eq. (9) under the assumption of minimum-phase behavior. The forced zero dynamics act, then, as a reduced-order observer, forced by the outputs and their derivatives. Theorem 2 provides a solution to the posed synthesis problem for general minimum-phase nonlinear processes based on the above reduced-order observer. In particular, the observer is combined with an input/output linearizing state feedback which makes explicit use of the outputs and their derivatives, while the overall controller is completed with a state-space realization of a linear multivariable error feedback compensator with integral action.

Theorem 2: *Consider the nonlinear process described by eq. (6). Then, the dynamic system*

$$\begin{aligned}
\dot{\xi}^{(1)}_1 &= \xi^{(1)}_2 \\
&\vdots \\
\dot{\xi}^{(1)}_{r_1-1} &= \xi^{(1)}_{r_1} \\
\dot{\xi}^{(1)}_{r_1} &= ([\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1})_1 \\
&\quad\times\left[(y_{sp} - y) - \sum_{l=1}^{m}\sum_{k=1}^{r_l-1}\gamma_{lk}\xi^{(l)}_{k+1}\right] \\
&\vdots \\
\dot{\xi}^{(m)}_1 &= \xi^{(m)}_2 \\
&\vdots \\
\dot{\xi}^{(m)}_{r_m-1} &= \xi^{(m)}_{r_m} \\
\dot{\xi}^{(m)}_{r_m} &= ([\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1})_m \\
&\quad\times\left[(y_{sp} - y) - \sum_{l=1}^{m}\sum_{k=1}^{r_l-1}\gamma_{lk}\xi^{(l)}_{k+1}\right]
\end{aligned} \tag{23}$$

$$
\begin{aligned}
\dot{\eta}_1 &= F_1(\eta, \mathscr{D}y_1, \ldots, \mathscr{D}y_m)\\
&\ \vdots\\
\dot{\eta}_{n-\sum_i r_i} &= F_{n-\sum_i r_i}(\eta, \mathscr{D}y_1, \ldots, \mathscr{D}y_m)\\
u &= \{[\beta_{1r_1} \cdots \beta_{mr_m}] C(\eta, \mathscr{D}y_1, \cdots, \mathscr{D}y_m)\}^{-1}\\
&\quad \times \Bigg\{ \sum_{i=1}^{m} \sum_{k=0}^{r_i-1} \beta_{ik} \xi_{k+1}^{(i)}\\
&\quad + [\beta_{1r_1} \cdots \beta_{mr_m}][\gamma_{1r_1} \cdots \gamma_{mr_m}]^{-1}\\
&\quad \times \left[(y_{sp} - y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i-1} \gamma_{ik} \xi_{k+1}^{(i)} \right]\\
&\quad - \sum_{i=1}^{m} \sum_{k=0}^{r_i-1} \beta_{ik} \frac{\mathrm{d}^k y_i}{\mathrm{d}t^k}\\
&\quad - \sum_{i=1}^{m} \beta_{ir_i} W_i(\eta, \mathscr{D}y_1, \cdots, \mathscr{D}y_m) \Bigg\}
\end{aligned}
$$

where $\beta_{ik} = [\beta_{ik}^1 \cdots \beta_{ik}^m]^T$ are vectors of adjustable parameters with $\det[\beta_{1r_1} \cdots \beta_{mr_m}] \neq 0$, *represents an* (n)*th order state-space realization of a dynamic output feedback controller that induces the closed-loop input/output behavior*:

$$
y + \sum_{i=1}^{m} \sum_{k=1}^{r_i} \gamma_{ik} \frac{\mathrm{d}^k y_i}{\mathrm{d}t^k} = y_{sp}.
$$

Remark 5: The controller of Theorem 2, similarly to the SISO case, is a nonlinear analogue of a two-degree-of-freedom controller, i.e. a *mixed error and output feedback* controller. This is consistent with the intuition from linear systems theory where a two-degree-of-freedom controller is usually employed for open-loop unstable systems, with the output feedback having a stabilizing effect on the overall control action. The overall control structure and the various components of the controller are shown in Fig. 4.

Remark 6: The implementation of the controller of Theorem 2 in the case that $r_i \geqslant 3$ for some i may require filtering of the output signal or approximation of the output derivatives in order to suppress noise effects.

Remark 7: The controller of eq. (23) that induces a decoupled closed-loop input/output behavior of the form of eq. (16) takes the simplified form

$$
\begin{aligned}
\dot{\xi}_1^{(1)} &= \xi_2^{(1)}\\
&\ \vdots\\
\dot{\xi}_{r_1-1}^{(1)} &= \xi_{r_1}^{(1)}\\
\dot{\xi}_{r_1}^{(1)} &= \frac{1}{\gamma_{1r_1}^1}\left[(y_{sp1} - y_1) - \sum_{k=1}^{r_1-1} \gamma_{1k}^1 \xi_{k+1}^{(1)} \right]\\
&\ \vdots\\
\dot{\xi}_1^{(m)} &= \xi_2^{(m)}\\
&\ \vdots\\
\dot{\xi}_{r_m-1}^{(m)} &= \xi_{r_m}^{(m)}\\
\dot{\xi}_{r_m}^{(m)} &= \frac{1}{\gamma_{mr_m}^m}\left[(y_{spm} - y_m) - \sum_{k=1}^{r_m-1} \gamma_{mk}^m \xi_{k+1}^{(m)} \right]\\
\dot{\eta}_1 &= F_1(\eta, \mathscr{D}y_1, \ldots, \mathscr{D}y_m)\\
&\ \vdots\\
\dot{\eta}_{n-\sum_i r_i} &= F_{n-\sum_i r_i}(\eta, \mathscr{D}y_1, \ldots, \mathscr{D}y_m) \qquad (24)\\
u &= \{\mathrm{diag}[\beta_{ir_i}^i] C(\eta, \mathscr{D}y_1, \ldots, \mathscr{D}y_m)\}^{-1}
\end{aligned}
$$

$$
\times \begin{bmatrix}
\beta_{10}^1 \xi_1^{(1)} + \sum_{k=1}^{r_1-1} \left(\beta_{1k}^1 - \frac{\beta_{1r_1}^1}{\gamma_{1r_1}^1} \gamma_{1k}^1 \right) \xi_{k+1}^{(1)} + \frac{\beta_{1r_1}^1}{\gamma_{1r_1}^1} (y_{sp1} - y_1) - \sum_{k=0}^{r_1-1} \beta_{1k}^1 \frac{\mathrm{d}^k y_1}{\mathrm{d}t^k} - \beta_{1r_1}^1 W_1(\eta, \mathscr{D}y_1, \ldots, \mathscr{D}y_m)\\
\vdots\\
\beta_{m0}^m \xi_1^{(m)} + \sum_{k=1}^{r_m-1} \left(\beta_{mk}^m - \frac{\beta_{mr_m}^m}{\gamma_{mr_m}^m} \gamma_{mk}^m \right) \xi_{k+1}^{(m)} + \frac{\beta_{mr_m}^m}{\gamma_{mr_m}^m} (y_{spm} - y_m) - \sum_{k=0}^{r_m-1} \beta_{mk}^m \frac{\mathrm{d}^k y_m}{\mathrm{d}t^k} - \beta_{mr_m}^m W_m(\eta, \mathscr{D}y_1, \ldots, \mathscr{D}y_m)
\end{bmatrix}
$$

in the case that input/output decoupling in the v–y dynamics is also requested.

Remark 8: In the case of a SISO nonlinear process, i.e. for $m = 1$, the controller realization of eq. (24) reduces exactly to the realization derived in Daoutidis and Kravaris (1992a), as expected.

Remark 9: The controller realizations derived in Theorem 1 and 2 can find a transparent input/output interpretation from an input/output operator perspective, in complete analogy with the SISO results (Daoutidis and Kravaris, 1992a). In particular, the controller realizations can be decomposed in realizations of the postulated operator between the error and the output, and internally stable realizations of the process inverse, illustrating thus the importance of alternative realizations of the inverse for the controller synthesis and implementation.

STABILITY CONSIDERATIONS IN THE CLOSED-LOOP SYSTEM

Under the controllers of Theorems 1 and 2, the input/output characteristics of the closed-loop system

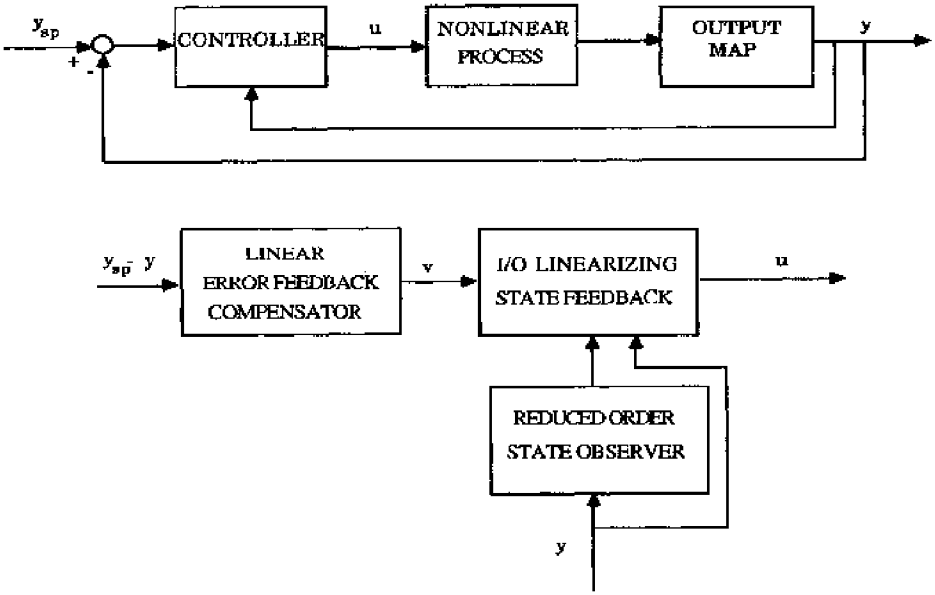

Fig. 4. Error and output feedback controller and control structure.

clearly depend on the roots of the closed-loop characteristic polynomial [eq. (14)]. The designer has the flexibility to choose the adjustable parameters in order to achieve a desirable speed of the response and level of input/output coupling, as well as other closed-loop design objectives, and such that they do not violate the constraints in the manipulated inputs. In addition to input/output stability, the internal stability of the closed-loop system must also be guaranteed, i.e. the asymptotic stability of the states in the unforced closed-loop system, for perturbations in the initial conditions. To this end, assume that

(1) the process dynamics is locally exponentially stable,
(2) the zero dynamics of the process is locally exponentially stable,
(3) the roots of eq. (14) lie in the open left-half of the complex plane, and
(4) The roots of the characteristic equation

$$\det \begin{bmatrix} (\sum_{k=0}^{r_1} \beta_{1k}^1 s^k) & (\sum_{k=0}^{r_2} \beta_{2k}^1 s^k) & \cdots & (\sum_{k=0}^{r_m} \beta_{mk}^1 s^k) \\ (\sum_{k=0}^{r_1} \beta_{1k}^2 s^k) & (\sum_{k=0}^{r_2} \beta_{2k}^2 s^k) & \cdots & (\sum_{k=0}^{r_m} \beta_{mk}^2 s^k) \\ \vdots & \vdots & & \vdots \\ (\sum_{k=0}^{r_1} \beta_{1k}^m s^k) & (\sum_{k=0}^{r_2} \beta_{2k}^m s^k) & \cdots & (\sum_{k=0}^{r_m} \beta_{mk}^m s^k) \end{bmatrix} = 0 \quad (25)$$

lie in the open left-half of the complex plane.

A stability analysis of the unforced closed-loop system ($y_{spi} = 0$) based on Lyapunov's first theorem can then be used to show that the above conditions guarantee the local internal stability of the closed-loop system under the controller of eq. (19). Under conditions 2, 3 and 4 from above, a similar procedure can be used to guarantee the local internal stability of the closed-loop system under the controller of eq. (23). The detailed proofs of the above results are omitted for reasons of brevity.

ILLUSTRATIVE EXAMPLE

Consider the ideal continuous stirred tank reactor (CSTR) shown in Fig. 5. A solution stream at concentration C_{A0} and temperature T_0 enters the reactor, where the following chemical reactions take place:

$$A \leftrightarrows U_1$$
$$A \rightarrow U_2$$
$$A \rightarrow P$$
$$A \rightarrow U_3$$
$$A \rightarrow U_4.$$

U_1, U_2, U_3, U_4 represent undesirable products, while P represents a desirable one. The effluent stream leaves the reactor at concentrations C_A, C_{U_1}, C_{U_2}, C_{U_3}, C_{U_4}, C_P, and temperature T. The values of the various process parameters are shown in Table 1. Figure 6 provides a plot of the selectivity with respect to P, denoted by S_P, as a function of C_A and T. S_P is defined as

$$S_P = \frac{r_P}{r_{U_1} + r_{U_2} + r_{U_3} + r_{U_4}}$$

where

$$r_{U_1} = k_1 C_A - k_{-1} C_{U_1}$$
$$r_{U_2} = k_2 C_A^3$$
$$r_P = k_P C_A^2$$
$$r_{U_3} = k_3 C_A^3$$
$$r_{U_4} = k_4 C_A$$

evaluated at steady-state, and $k_i = Z_i \exp(-E_i/RT)$. As can be seen, there is a well-defined maximum for the selectivity at $S_P = 1.0$, corresponding to $C_A = 1.0\ \text{kmol m}^{-3}$ and $T = 400$ K. Based on the above, the control problem is formulated as the one of operating the reactor at the above reactant concentration and temperature. Under standard assumptions, the dynamic behavior of the process is then described by the following material and energy balances:

$$\begin{aligned}
\frac{dC_A}{dt} &= \frac{1}{\tau}(C_{A0} - C_A) - (r_{U_1} + r_{U_2} + r_P + r_{U_3} + r_{U_4}) \\
\frac{dC_{U_1}}{dt} &= -\frac{1}{\tau}C_{U_1} + r_{U_1} \\
\frac{dT}{dt} &= \frac{1}{\tau}(T_0 - T) + \frac{1}{V\rho c}Q \\
&\quad + \frac{1}{\rho c}[(-\Delta H_1)r_{U_1} + (-\Delta H_2)r_{U_2} \\
&\quad + (-\Delta H_P)r_P + (-\Delta H_3)r_{U_3} \\
&\quad + (-\Delta H_4)r_{U_4}].
\end{aligned} \tag{26}$$

It is assumed that measurements of the controlled outputs C_A and T are available, while C_{U_1} cannot be measured. The inlet reactant concentration and the heat input to the reactor are used as the two manipulated input variables. Setting $x_1 = C_A$, $x_2 = C_{U_1}$, $x_3 = T$, $u_1 = C_{A0}$, $u_2 = Q$, $y_1 = C_A$ and $y_2 = T$, the dynamic model of the process can easily be put in the form of eq. (1), with

$$g_1(x) = \begin{bmatrix} 1/\tau \\ 0 \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1/V\rho c \end{bmatrix}$$

$$h_1(x) = x_1, \quad h_2(x) = x_3.$$

A straightforward calculation of the relative orders gives

$$r_1 = 1, \quad r_2 = 1$$

while the characteristic matrix of the above system is found to be equal to

$$C(x) = \begin{bmatrix} L_{g_1}h_1(x) & 0 \\ 0 & L_{g_2}h_2(x) \end{bmatrix} = \begin{bmatrix} F/V & 0 \\ 0 & 1/V\rho c \end{bmatrix}$$

and is nonsingular in the entire state-space. Moreover, the process model is already in normal form, and for this reason the controller of Theorem 2 was employed in the simulations. More specifically, for the above process and for an input/output decoupled closed-loop response of the form

$$\begin{aligned}
y_1 + \gamma_{11}^1 \frac{dy_1}{dt} &= y_{sp1} \\
y_2 + \gamma_{21}^2 \frac{dy_2}{dt} &= y_{sp2}
\end{aligned} \tag{27}$$

the controller takes the form

$$\begin{aligned}
\xi_1^{(1)} &= \frac{1}{\gamma_{11}^1}(y_{sp1} - y_1) \\
\xi_1^{(2)} &= \frac{1}{\gamma_{21}^2}(y_{sp2} - y_2) \\
\dot{\eta} &= -\frac{1}{\tau}\eta + r_{U_1}(\eta, y_1, y_2) \\
u_1 &= \frac{V}{F\beta_{11}^1}\Bigg[\beta_{10}^1 \xi_1^{(1)} + \frac{\beta_{11}^1}{\gamma_{11}^1}(y_{sp1} - y_1) \\
&\quad - \beta_{10}^1 y_1 - \beta_{11}^1 f_1(\eta, y_1, y_2)\Bigg] \\
u_2 &= \frac{V\rho c}{\beta_{21}^2}\Bigg[\beta_{20}^2 \xi_1^{(2)} + \frac{\beta_{21}^2}{\gamma_{21}^2}(y_{sp2} - y_2) \\
&\quad - \beta_{20}^2 y_2 - \beta_{21}^2 f_3(\eta, y_1, y_2)\Bigg].
\end{aligned} \tag{28}$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\tau}x_1 - [r_{U_1}(x_1, x_2, x_3) + r_{U_2}(x_1, x_3) + r_P(x_1, x_3) + r_{U_3}(x_1, x_3) + r_{U_4}(x_1, x_3)] \\ -\frac{1}{\tau}x_2 + r_{U_1}(x_1, x_2, x_3) \\ \left\{ \frac{1}{\rho c}[(-\Delta H_1)r_{U_1}(x_1, x_2, x_3) + (-\Delta H_2)r_{U_2}(x_1, x_3) + (-\Delta H_P)r_P(x_1, x_3) \right. \\ \left. + (-\Delta H_3)r_{U_3}(x_1, x_3) + (-\Delta H_4)r_{U_4}(x_1, x_3)] + \frac{1}{\tau}(T_0 - x_3) \right\} \end{bmatrix}$$

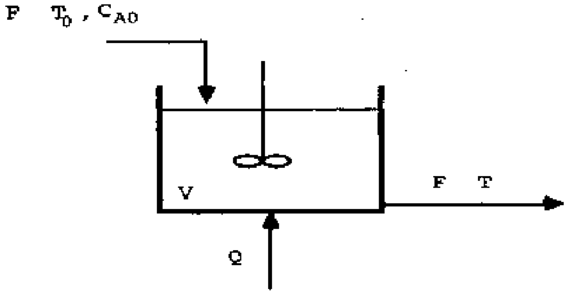

Fig. 5. A continuous stirred tank reactor (CSTR).

The adjustable parameters were chosen as $\beta^1_{10} = 1$, $\beta^2_{20} = 1$, $\beta^1_{11} = \gamma^1_{11} = 25$ and $\beta^2_{21} = \gamma^2_{21} = 60$, while a sampling period of 2.5 s was used in the simulations. The performance of the above output feedback controller was tested in terms of the tracking and regulatory characteristics of the closed-loop system, giving excellent results. All the simulations were carried out under both

(1) the assumption of a perfect model, and
(2) a 10% error in the reaction rates

in order to also test the robustness characteristics

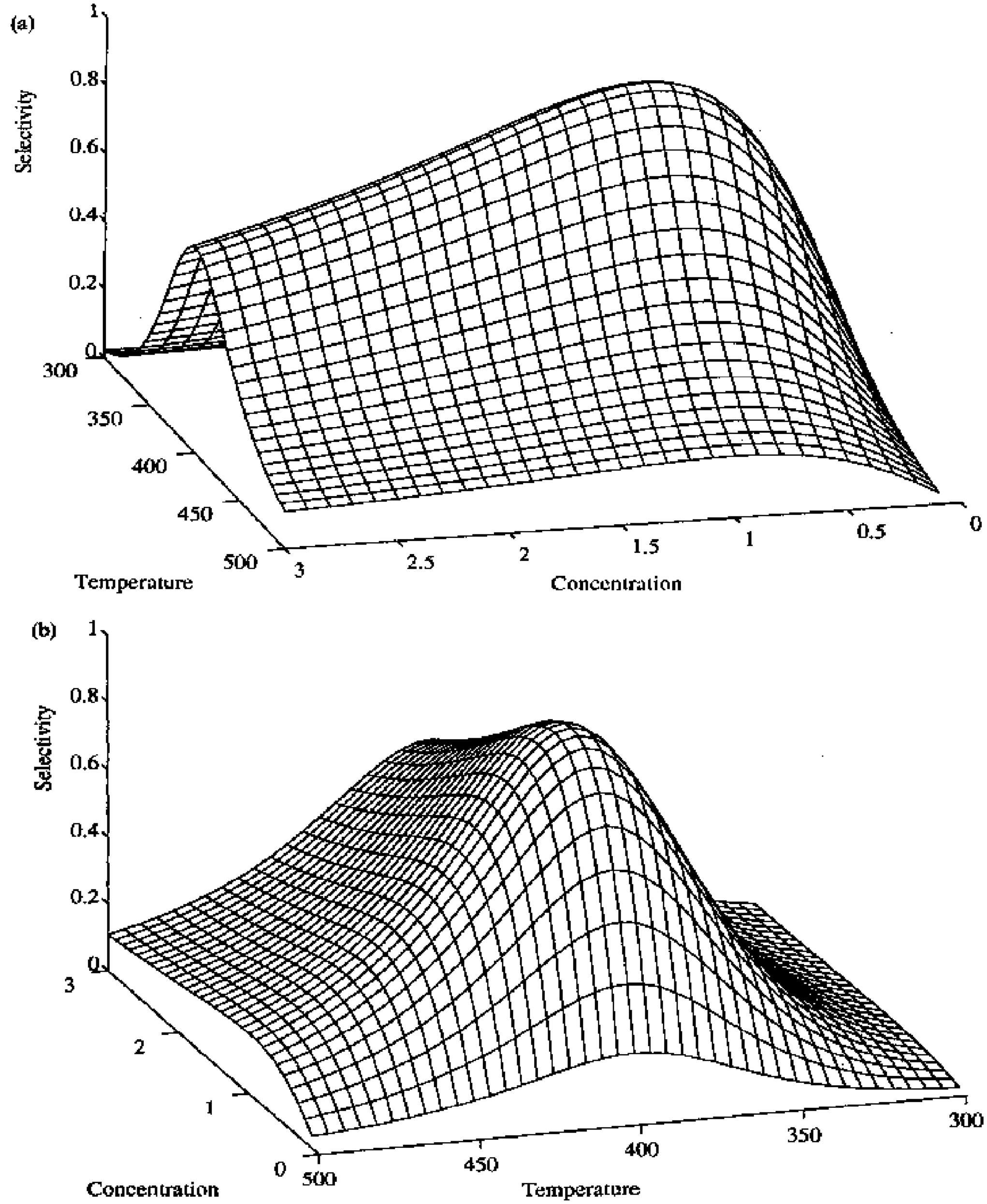

Fig. 6. Three-dimensional plot of sensitivity vs reactant concentration and reactor temperature.

of the controller. The first representative run corresponds to the start-up of the reactor. Starting from the initial conditions: $C_A = 0.0\ \mathrm{kmol\,m^{-3}}$, $C_{U_1} = 0.0\ \mathrm{kmol\,m^{-3}}$ and $T = 300$ K, the control objective is to bring the reactor to the desired operating conditions. Figures 7–10 illustrate the profiles for the two controlled outputs and the two manipulated inputs. The output responses under the perfect model clearly verify the theoretically predicted ones, while under the modeling error the responses are also very satisfactory. Figure 11 provides a comparison between the actual process value and the observer estimate for C_{U_1} in the case that the above modeling error is assumed. As expected, there is a bias in the estimate of C_{U_1} due to the process–model mismatch. Note, however, that although the state observer does not perform perfectly, the controller compensates for this, with the overall performance being excellent. In the second representative run, the reactor is initially operating at the nominal steady-state: $C_A = 1.0\ \mathrm{kmol\,m^{-3}}$, $C_{U_1} = 0.585\ \mathrm{kmol\,m^{-3}}$ and $T = 400$ K, and at time $t = 100$ s an increase of 20 K is imposed in

Table 1. Process parameters

$\tau = 60$ s	$R = 8.345\ \mathrm{kJ\,kmol^{-1}\,K^{-1}}$
$\rho = 1 \times 10^3\ \mathrm{kg\,m^{-3}}$	$c = 4.2\ \mathrm{kJ\,kg^{-1}\,K^{-1}}$
$V = 1 \times 10^{-2}\ \mathrm{m^3}$	$T_0 = 355$ K
$Z_1 = 3.906\ \mathrm{s^{-1}}$	$Z_{-1} = 900\ \mathrm{s^{-1}}$
$Z_2 = 3.906\ \mathrm{m^6\,kmol^{-2}\,s^{-1}}$	$Z_P = 2.4993 \times 10^6\ \mathrm{m^3\,kmol^{-1}\,s^{-1}}$
$Z_3 = 9.99 \times 10^{10}\ \mathrm{m^6\,kmol^{-2}\,s^{-1}}$	$Z_4 = 9.99 \times 10^{10}\ \mathrm{s^{-1}}$
$E_1 = 2 \times 10^4\ \mathrm{kJ\,kmol^{-1}}$	$E_{-1} = 6 \times 10^4\ \mathrm{kJ\,kmol^{-1}}$
$E_2 = 2 \times 10^4\ \mathrm{kJ\,kmol^{-1}}$	$E_P = 6 \times 10^4\ \mathrm{kJ\,kmol^{-1}}$
$E_3 = 1 \times 10^5\ \mathrm{kJ\,kmol^{-1}}$	$E_4 = 1 \times 10^5\ \mathrm{kJ\,kmol^{-1}}$
$-\Delta H_1 = 2 \times 10^3\ \mathrm{kJ\,kmol^{-1}}$	$-\Delta H_2 = 2 \times 10^3\ \mathrm{kJ\,kmol^{-1}}$
$-\Delta H_P = 6 \times 10^4\ \mathrm{kJ\,kmol^{-1}}$	$-\Delta H_3 = 2 \times 10^3\ \mathrm{kJ\,kmol^{-1}}$
$-\Delta H_4 = 2 \times 10^3\ \mathrm{kJ\,kmol^{-1}}$	
$n_1 = 1$	$n_{-1} = 1$
$n_2 = 3$	$n_P = 2$
$n_3 = 3$	$n_4 = 1$

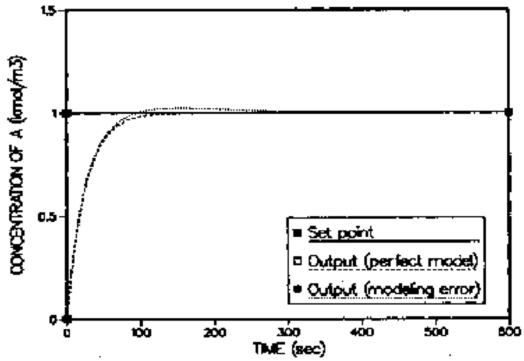

Fig. 7. Reactant concentration profile during reactor start-up.

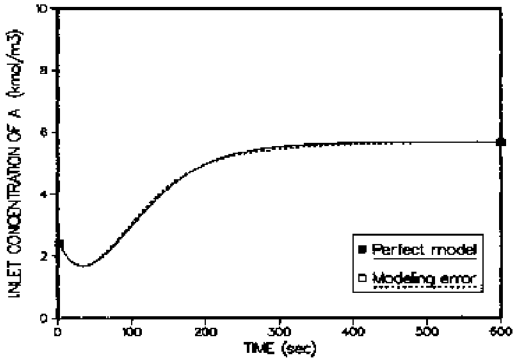

Fig. 9. Inlet reactant concentration profile during reactor start-up.

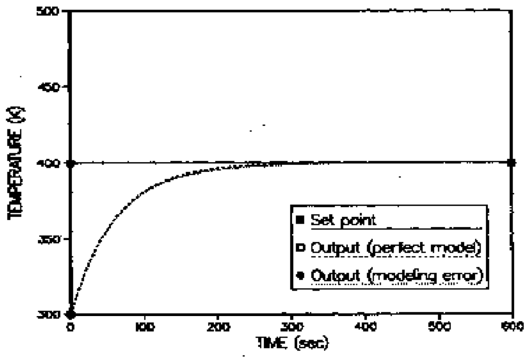

Fig. 8. Reactor temperature profile during reactor start-up.

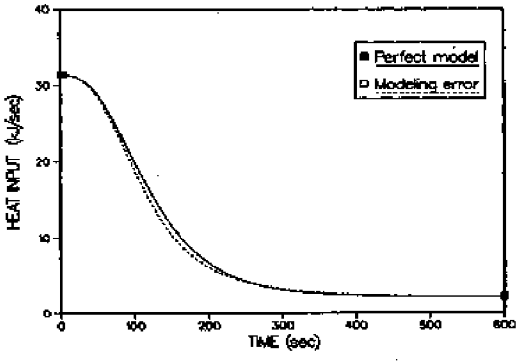

Fig. 10. Heat input profile during reactor start-up.

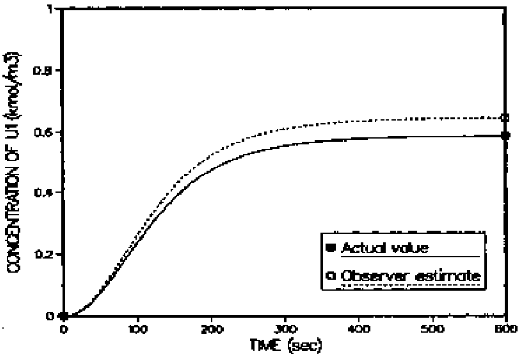

Fig. 11. Actual and estimated state variable under modeling error.

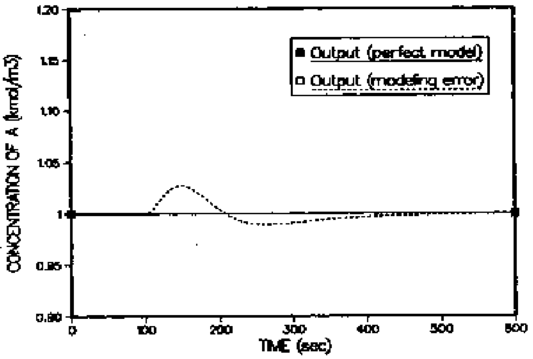

Fig. 12. Reactant concentration profile for disturbance rejection.

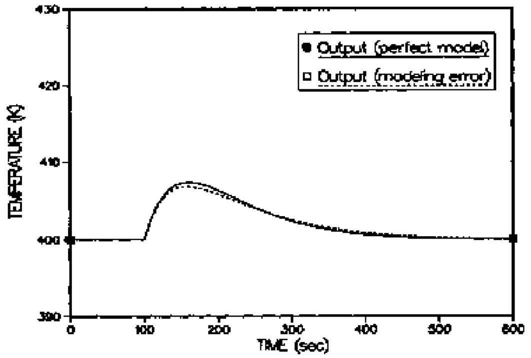

Fig. 13. Reactor temperature profile for disturbance rejection.

the inlet temperature T_0. Figures 12–15 illustrate the profiles for the two controlled outputs and the two manipulated inputs as the reactor returns safely to the desired steady-state. A comparison between the actual process value and the observer estimate for C_U, does not show any difference in the case of the perfect model (Fig. 16), which is expected since the disturbance does not appear in the reduced observer dynamics. In the case of modeling error, a constant difference between the estimated and the actual value

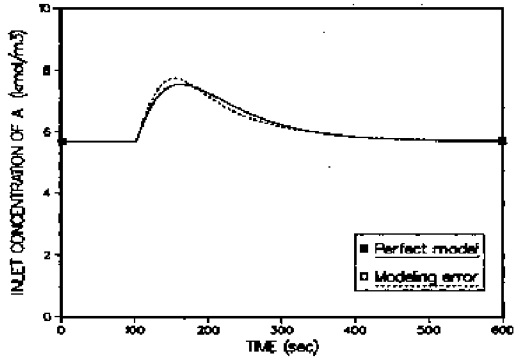

Fig. 14. Inlet reactant concentration profile for disturbance rejection.

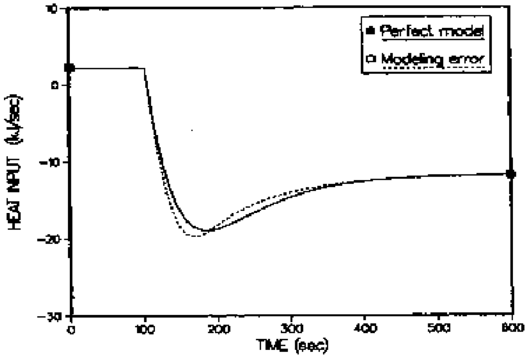

Fig. 15. Heat input profile for disturbance rejection.

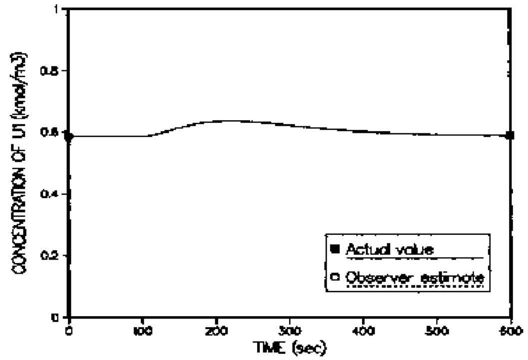

Fig. 16. Actual and estimated state variable in the presence of disturbance.

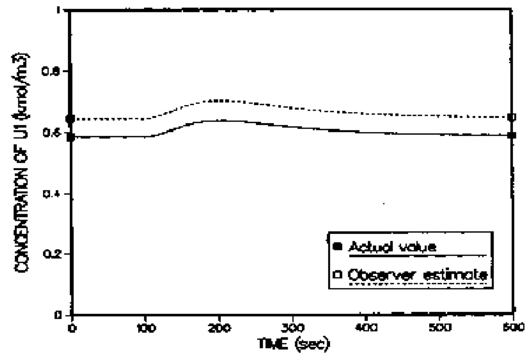

Fig. 17. Actual and estimated state variable in the presence of disturbance, under modeling error.

of C_v, can be seen in Fig. 17; despite this difference, the overall control action is very satisfactory.

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NOTATION

c	heat capacity of the reacting mixture, $\mathrm{kJ\,kg^{-1}\,K^{-1}}$
C	characteristic matrix
C_i	molar concentration of species i, $\mathrm{kmol\,m^{-3}}$
C_{A0}	inlet molar concentration of species A, $\mathrm{kmol\,m^{-3}}$
E_i	activation energy, $\mathrm{kJ\,kmol^{-1}}$
f	vector field
g_j	vector field
h_i	scalar field
n_i	order of reaction
Q	heat input to the reactor, $\mathrm{kJ\,s^{-1}}$
r_i	relative order
R	ideal gas constant, $\mathrm{kJ\,kmol^{-1}\,K^{-1}}$
s	the Laplace domain variable
t	time
T	reactor temperature, K
T_0	inlet temperature, K
u	manipulated input
v	auxiliary variable
V	reactor volume, $\mathrm{m^3}$
w	state vector of process model
x	state vector of process
y_i	process output
y_{spi}	output set-point
Z_i	frequency factor

Greek letters

β^j_{ik}	adjustable parameters
γ^j_{ik}	adjustable parameters
$-\Delta H_i$	heat of reaction, $\mathrm{kJ\,kmol^{-1}}$
ζ	state vector of process in normal-form coordinates
η	controller state variables
ξ	state vector of linear compensator
ρ	density of the reacting mixture, $\mathrm{kg\,m^{-3}}$
τ	reactor residence time, s

Math symbols

det	determinant of a matrix
diag	diagonal matrix
$\not\equiv$	not equivalently equal to
$\mathbb{R}^n$	n-dimensional Euclidean space
T	transpose

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APPENDIX

Proof of Theorem 1

Define the auxiliary vector variable

$$v = [v_1 \cdots v_m]^T = \sum_{i=1}^{m} \sum_{k=0}^{r_i-1} \beta_{ik} \xi^{(i)}_{k+1} + [\beta_{1r_1} \cdots \beta_{mr_m}][\gamma_{1r_1} \cdots \gamma_{mr_m}]^{-1} \left[(y_{sp} - y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i-1} \gamma_{ik} \xi^{(i)}_{k+1} \right]. \tag{29}$$

Then, based on standard linear systems realization theory [e.g. Chen (1984)], it can be verified that the dynamic system

$$\begin{aligned}
\dot{\xi}^{(1)}_1 &= \xi^{(1)}_2 \\
&\;\vdots \\
\dot{\xi}^{(1)}_{r_1-1} &= \xi^{(1)}_{r_1} \\
\dot{\xi}^{(1)}_{r_1} &= ([\gamma_{1r_1} \cdots \gamma_{mr_m}]^{-1})_1 \times \left[(y_{sp} - y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i-1} \gamma_{ik} \xi^{(i)}_{k+1} \right] \\
&\;\vdots \\
\dot{\xi}^{(m)}_1 &= \xi^{(m)}_2 \\
&\;\vdots \\
\dot{\xi}^{(m)}_{r_m-1} &= \xi^{(m)}_{r_m} \\
\dot{\xi}^{(m)}_{r_m} &= ([\gamma_{1r_1} \cdots \gamma_{mr_m}]^{-1})_m \times \left[(y_{sp} - y) - \sum_{i=1}^{m} \sum_{k=1}^{r_i-1} \gamma_{ik} \xi^{(i)}_{k+1} \right]
\end{aligned} \tag{30}$$

and the output map defined by eq. (29) represent an irreducible state-space realization of the linear input/output dynamics between $(y_{sp} - y)$ and v, with the following representation in differential operator form:

$$\begin{aligned}
\mathscr{P}(D)\xi^{(i)}_1 &= \mathscr{Q}(D)(y_{sp} - y) \\
v &= \mathscr{R}(D)\xi^{(i)}_1 + \mathscr{W}(D)(y_{sp} - y)
\end{aligned} \tag{31}$$

where

$$\mathscr{P}(D) = \mathrm{diag}\left[\frac{\mathrm{d}^{r_i}}{\mathrm{d}t^{r_i}}\right] + [\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1}\left[\sum_{i=1}^{m}\sum_{k=1}^{r_i-1}\gamma_{ik}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right]$$

$$\mathscr{Q}(D) = [\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1} \tag{32}$$

$$\mathscr{R}(D) = \left[\sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right] - [\beta_{1r_1}\cdots\beta_{mr_m}] \times[\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1}\left[\sum_{i=1}^{m}\sum_{k=1}^{r_i-1}\gamma_{ik}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right]$$

$$\mathscr{W}(D) = [\beta_{1r_1}\cdots\beta_{mr_m}][\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1}$$

and $\xi_1^{(i)} = [\xi_1^{(1)}\cdots\xi_1^{(m)}]$. Equation (31) can be interpreted as a linear compensator with integral action imposed on the v–y dynamics. The other component of eq. (19) then becomes

$$\dot{w} = f(w) + g(w)\{[\beta_{1r_1}\cdots\beta_{mr_m}]C(w)\}^{-1} \times\left[v - \sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}L_f^k h_i(w)\right] \tag{33}$$

$$u = \{[\beta_{1r_1}\cdots\beta_{mr_m}]C(w)\}^{-1}\left[v - \sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}L_f^k h_i(w)\right]$$

which is an input/output linearizing state feedback law (Kravaris and Soroush, 1990), with the states reconstructed through an open-loop state observer, the process model itself. Under consistent initialization of w and x [i.e. $w(0) = x(0)$], it easily follows that $w(t) = x(t)$ and eq. (33) induces the dynamics

$$\sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}\frac{\mathrm{d}^k y_i}{\mathrm{d}t^k} = v \tag{34}$$

or, equivalently,

$$[\beta_{1r_1}\cdots\beta_{mr_m}]\,\mathrm{diag}\left[\frac{\mathrm{d}^{r_i}y_i}{\mathrm{d}t^{r_i}}\right] + \sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\frac{\mathrm{d}^k y_i}{\mathrm{d}t^k} = v. \tag{35}$$

Combining eq. (35) with the second equation from eq. (31), we obtain the following equality:

$$\sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\frac{\mathrm{d}^k\xi_1^{(i)}}{\mathrm{d}t^k} + [\beta_{1r_1}\cdots\beta_{mr_m}][\gamma_{1r_1}\cdots\gamma_{mr_m}]^{-1} \times\left[(y_{sp}-y) - \sum_{i=1}^{m}\sum_{k=1}^{r_i-1}\gamma_{ik}\frac{\mathrm{d}^k\xi_1^{(i)}}{\mathrm{d}t^k}\right] = [\beta_{1r_1}\cdots\beta_{mr_m}]\,\mathrm{diag}\left[\frac{\mathrm{d}^{r_i}y_i}{\mathrm{d}t^{r_i}}\right] + \sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\frac{\mathrm{d}^k y_i}{\mathrm{d}t^k}. \tag{36}$$

The first equation from eq. (31) also yields

$$[\gamma_{1r_1}\cdots\gamma_{mr_m}]\,\mathrm{diag}\left[\frac{\mathrm{d}^{r_i}\xi_1^{(i)}}{\mathrm{d}t^{r_i}}\right] + \sum_{i=1}^{m}\sum_{k=1}^{r_i-1}\gamma_{ik}\frac{\mathrm{d}^k\xi_1^{(i)}}{\mathrm{d}t^k} = (y_{sp}-y). \tag{37}$$

Eliminating $(y_{sp}-y)$ from eqs (36) and (37), we easily obtain

$$[\beta_{1r_1}\cdots\beta_{mr_m}]\,\mathrm{diag}\left[\frac{\mathrm{d}^{r_i}y_i}{\mathrm{d}t^{r_i}}\right] + \sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\frac{\mathrm{d}^k y_i}{\mathrm{d}t^k} = [\beta_{1r_1}\cdots\beta_{mr_m}]\,\mathrm{diag}\left[\frac{\mathrm{d}^{r_i}\xi_1^{(i)}}{\mathrm{d}t^{r_i}}\right] + \sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\frac{\mathrm{d}^k\xi_1^{(i)}}{\mathrm{d}t^k}. \tag{38}$$

Under consistent initialization of $\xi_1^{(i)}$ and y_i, i.e.

$$\frac{\mathrm{d}^k\xi_1^{(i)}(0)}{\mathrm{d}t^k} = \frac{\mathrm{d}^k y_i(0)}{\mathrm{d}t^k}, \quad k = 0,\ldots,r_i,\ i = 1,\ldots,m \tag{39}$$

we obtain $\xi_1^{(i)} = y_i$, $i = 1,\ldots,m$, and eq. (37) takes the form

$$[\gamma_{1r_1}\cdots\gamma_{mr_m}]\,\mathrm{diag}\left[\frac{\mathrm{d}^{r_i}y_i}{\mathrm{d}t^{r_i}}\right] + \sum_{i=1}^{m}\sum_{k=1}^{r_i-1}\gamma_{ik}\frac{\mathrm{d}^k y_i}{\mathrm{d}t^k} = (y_{sp}-y) \tag{40}$$

which is equivalent to the desired closed-loop input/output dynamics:

$$y + \sum_{i=1}^{m}\sum_{k=1}^{r_i}\gamma_{ik}\frac{\mathrm{d}^k y_i}{\mathrm{d}t^k} = y_{sp}.$$

Proof of Corollary 1

From eq. (33), it easily follows that

$$\sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}\frac{\mathrm{d}^k(h_i(w))}{\mathrm{d}t^k} = v \tag{41}$$

which, combined with eq. (31), yields

$$[\beta_{1r_1}\cdots\beta_{mr_m}]\,\mathrm{diag}\left[\frac{\mathrm{d}^{r_i}(h_i(w))}{\mathrm{d}t^{r_i}}\right] + \sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\frac{\mathrm{d}^k(h_i(w))}{\mathrm{d}t^k} = [\beta_{1r_1}\cdots\beta_{mr_m}]\,\mathrm{diag}\left[\frac{\mathrm{d}^{r_i}\xi_1^{(i)}}{\mathrm{d}t^{r_i}}\right] + \sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\frac{\mathrm{d}^k\xi_1^{(i)}}{\mathrm{d}t^k} \tag{42}$$

or, equivalently,

$$\sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}\frac{\mathrm{d}^k(h_i(w))}{\mathrm{d}t^k} = \sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}\frac{\mathrm{d}^k\xi_1^{(i)}}{\mathrm{d}t^k}. \tag{43}$$

Under consistent initialization of $h_i(w)$ and $\xi_1^{(i)}$, i.e.

$$\frac{\mathrm{d}^k\xi_1^{(i)}(0)}{\mathrm{d}t^k} = \frac{\mathrm{d}^k(h_i(w)(0))}{\mathrm{d}t^k}, \quad k = 0,\ldots,r_i, \quad i = 1,\ldots,m \tag{44}$$

or, equivalently,

$$\xi_l^{(i)}(0) = L_f^{l-1}h_i(w)(0), \quad l = 1,\ldots,r_i, \quad i = 1,\ldots,m$$

it easily follows that $\xi_1^{(i)} = h_i$ or, equivalently, $\xi_l^{(i)} = L_f^{l-1}h_i(w)$, $l = 1,\ldots,r_i$, $i = 1,\ldots,m$. Substituting the above relations into eq. (19), we easily obtain eq. (20).

Proof of Theorem 2

Similarly to the proof of Theorem 1, the linear error feedback compensator defined by eqs (29) and (30) induces the input/output dynamics described by eq. (31), while the output feedback compensator

$$\dot{\eta}_1 = F_1(\eta, \mathscr{D}y_1,\ldots,\mathscr{D}y_m)$$
$$\vdots$$
$$\dot{\eta}_{n-\sum_i r_i} = F_{n-\sum_i r_i}(\eta, \mathscr{D}y_1,\ldots,\mathscr{D}y_m) \tag{45}$$
$$u = \{[\beta_{1r_1}\cdots\beta_{mr_m}]C(\eta, \mathscr{D}y_1,\ldots,\mathscr{D}y_m)\}^{-1} \times\left[v - \sum_{i=1}^{m}\sum_{k=0}^{r_i-1}\beta_{ik}\frac{\mathrm{d}^k y_i}{\mathrm{d}t^k} - \sum_{i=1}^{m}\beta_{ir_i}W_i(\eta, \mathscr{D}y_1,\ldots,\mathscr{D}y_m)\right]$$

is an input/output linearizing state feedback law, with the states reconstructed through the reduced-order observer. Under consistent initialization of η in eq. (45) and $\zeta^{(0)}$ in the process normal form of eq. (6), i.e. $\zeta_l^{(0)}(0) = \eta_l(0)$, $l = 1,\ldots,(n-\sum_i r_i)$ it easily follows that $\zeta_l^{(0)} = \eta_l$, $l = 1,\ldots,(n-\sum_i r_i)$. Then, given that $\zeta_k^{(i)} = \mathrm{d}^{k-1}y_i/\mathrm{d}t^{k-1}$, $k = 1,\ldots,r_i$, $i = 1,\ldots,m$, eq. (45) induces exactly the dynamics of eq. (34); combining eq. (31) with eq. (34) through a similar procedure as in the proof of Theorem 1 results in the desired closed-loop input/output behavior.