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SU_3 RECOUPLING AND FRACTIONAL PARENTAGE IN THE 2s-1d SHELL

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Abstract: Explicit algebraic expressions have been calculated for both the SU_3 Wigner coefficients and the SU_3 Racah coefficients which are of particular interest in the recoupling problem involving 2s–1d shell nucleons and basis functions of SU_3 symmetry involving the $SU_3 \supset SU_2$ chain. The Wigner coefficients are those for the Kronecker products $(\lambda_1\mu_1)\times(\lambda_2\mu_2)$, where $(\lambda_1\mu_1)$ is arbitrary and $(\lambda_2\mu_2)$ is the six-dimesional representation (20) of a single 2s–1d shell nucleon or one of the representations (02), (40), (21) for two coupled 2s-1d shell nucleons or the basic eight-dimensional representation (11). The Racah coefficients are those involved in the coupling of two 2s–1d shell nucleons to a function of arbitrary $(\lambda\mu)$. Calculations of SU_3 fractional parentage coefficients are illustrated by a few examples which have been evaluated without recourse to a full chain calculation for representations with large values of λ and μ. The SU_3 fractional parentage coefficients are used to give expressions for single-particle spectroscopic factors for 2s–1d shell nuclei.

1. Introduction

Although shell model calculations using Elliott's projection technique and intrinsic wave functions of SU_3 symmetry [1,2]††† have so far been carried out [2–5,22]) without the use of fractional parentage coefficients, expressions for spectroscopic factors may be given in convenient form in terms of such fractional parentage coefficients, and the energy calculations may be simplified if the needed $\langle n\{|n-2\rangle$ c.f.p. for the intrinsic states can be calculated. The states of interest (the intrinsic states from which the angular momentum eigenfunctions are constructed by the projection technique) are the harmonic oscillator states which form a basis for irreducible representations of SU_3 in which its subgroup SU_2 is explicitly reduced. For the normal parity states of 2s-1d shell nuclei, in particular, the n-particle wave functions are characterized by the group chain $SU_6 \supset SU_3 \supset SU_2$, where the irreducible representations of SU_6, characterized by partitions of n, describe the symmetry of the wave functions under permutations of the n particles, while the irreducible representations of SU_3, characterized by partitions of N, the number of oscillator quanta (which is $2n$ in this case), describe the symmetry of the wave function under permutation of these quanta.

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††† The present paper will follow the notation and general approach of Elliott and Harvey [2]).

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The task of constructing a complete table of c.f.p. for these intrinsic states of s-d shell nuclei is out of the question. The total number of states is very large, and no convenient set of operators which commute with the Casimir operators and the additive infinitesimal operators of the subgroups SU_3 and SU_2 has as yet been constructed to give a full specification of the states. However, a complete table of c.f.p. is not needed since the states of greatest interest in those cases in which SU_3 is a good tool are those which transform according to the leading representations of SU_3 or the next few representations, (those representations with large values of the Casimir operator), for which the quantum numbers of SU_3 itself are in most cases sufficient to completely specify the states. It is the purpose of this note to show that c.f.p. involving such states can be calculated by developing the machinery for the computation of SU_3 reduction coefficients. Explicit algebraic expressions are given for both the SU_3 Wigner coefficients and the SU_3 recoupling (Racah) coefficients which are needed in the computation of the $\langle n\{|n-1\rangle$ and $\langle n\{|n-2\rangle$ c.f.p. Calculations of the fractional parentage coefficients, which can be carried out without recourse to a full chain calculation, are illustrated by a few examples. Finally, the c.f.p. are used to give expressions for single-particle spectroscopic factors for 2s–1d shell nuclei. More detailed tables of c.f.p. and a fuller discussion of the calculation of matrix elements of the Hamiltonian between intrinsic states will be presented in a subsequent paper. In order to express the latter in most convenient form the tables of SU_3 Wigner and Racah coefficients must be extended.

2. Notation. Review of Some Properties of the Infinitesimal Operators of SU_3

The irreducible representations of SU_3 are characterized by $(\lambda\mu)$, where $\lambda = h_1-h_2$ and $\mu = h_2-h_3$. The three-rowed Young tableaux are characterized by the partition $[h_1h_2h_3]$ of the N oscillator quanta. The infinitesimal operators of U_3 are denoted by A_{ij} with $i, j = x, y,$ or z, and commutators $[A_{ij}, A_{kl}] = \delta_{jk}A_{il}-\delta_{il}A_{kj}$. The commuting infinitesimal operators for SU_3 are further chosen as

$$Q_0 = 2A_{zz}-A_{xx}-A_{yy}, \qquad \Lambda_0 = \tfrac{1}{2}(A_{xx}-A_{yy}),$$

where Λ_0 together with $\Lambda_+ = A_{xy}$ and $\Lambda_- = A_{yx}$ form the subgroup SU_2 which is singled out. The basis states of an irreducible representation of SU_3 are characterized by the eigenvalues of Q_0, Λ_0 and Λ^2, the Casimir operator of SU_2. These eigenvalues are specified by the quantum numbers ε, ν and Λ

$$\begin{aligned} Q_0\varphi((\lambda\mu)\varepsilon\Lambda\nu) &= \varepsilon\varphi((\lambda\mu)\varepsilon\Lambda\nu), \\ \Lambda_0\varphi((\lambda\mu)\varepsilon\Lambda\nu) &= \tfrac{1}{2}\nu(\varphi(\lambda\mu)\varepsilon\Lambda\nu), \\ \Lambda^2\varphi((\lambda\mu)\varepsilon\Lambda\nu) &= \Lambda(\Lambda+1)\varphi((\lambda\mu)\varepsilon\Lambda\nu). \end{aligned} \tag{1a}$$

The possible values of ε and Λ can be enumerated through the integers p and q

$$\begin{aligned} \varepsilon &= 2\lambda+\mu-3p-3q = \varepsilon_{\mathrm{H}}-3p-3q, \\ \Lambda &= \tfrac{1}{2}\mu+\tfrac{1}{2}p-\tfrac{1}{2}q = \Lambda_{\mathrm{H}}+\tfrac{1}{2}p-\tfrac{1}{2}q, \end{aligned} \tag{1b}$$

where p and q range over the values $p = 0, 1, 2, \ldots, \lambda$ and $q = 0, 1, \ldots, \mu$, and where the subscript H is used to characterize the state of highest weight [6]), the state with highest possible ε and for this ε highest possible ν.

Irreducible tensor operators [6, 7]) under SU_3 can be denoted by $T^{(\lambda\mu)}_{\varepsilon\Lambda\nu}$. They can be defined through the commutation relations

$$\begin{aligned} &[Q_0, T^{(\lambda\mu)}_{\varepsilon\Lambda\nu}] = \varepsilon T^{(\lambda\mu)}_{\varepsilon\Lambda\nu}, \qquad [\Lambda_0, T^{(\lambda\mu)}_{\varepsilon\Lambda\nu}] = \tfrac{1}{2}\nu T^{(\lambda\mu)}_{\varepsilon\Lambda\nu}, \\ &[A_{ij}, T^{(\lambda\mu)}_{\varepsilon\Lambda\nu}] = \sum_{\Lambda'} \langle(\lambda\mu)\varepsilon'\Lambda'\nu'|A_{ij}|(\lambda\mu)\varepsilon\Lambda\nu\rangle T^{(\lambda\mu)}_{\varepsilon'\Lambda'\nu'}, \quad (i \neq j). \end{aligned} \tag{2}$$

Matrix elements of the infinitesimal operators of SU_3 have been derived by several authors [8]). They also follow at once from the work of Elliott and Harvey, who have shown explicitly how to construct a state with arbitrary ε, Λ and ν from the state of highest weight through step-down operations. (See appendix).

Matrix elements of the infinitesimal operators A_{xy} and A_{yx} are the well-known ones of SU_2, for example,

$$\langle(\lambda\mu)\varepsilon\Lambda(\nu+2)|A_{xy}|(\lambda\mu)\varepsilon\Lambda\nu\rangle = [(\Lambda-\tfrac{1}{2}\nu)(\Lambda+\tfrac{1}{2}\nu+1)]^{\frac{1}{2}}. \tag{3}$$

Matrix elements of A_{xz}, A_{yz}, A_{zx} and A_{zy}, the ε-lowering and raising operators, can be expressed in terms of a single algebraic function f

$$\begin{aligned} A_{xz}|(\lambda\mu)\varepsilon\Lambda\nu\rangle &= f[(\lambda\mu)\varepsilon\Lambda\nu]|(\lambda\mu)(\varepsilon-3)(\Lambda+\tfrac{1}{2})(\nu+1)\rangle \\ &\quad + f[(\lambda\mu)\varepsilon, -(\Lambda+1)\nu]|(\lambda\mu)(\varepsilon-3)(\Lambda-\tfrac{1}{2})(\nu+1)\rangle, \\ A_{yz}|(\lambda\mu)\varepsilon\Lambda\nu\rangle &= f[(\lambda\mu)\varepsilon\Lambda, -\nu]|(\lambda\mu)(\varepsilon-3)(\Lambda+\tfrac{1}{2})(\nu-1)\rangle \\ &\quad - f[(\lambda\mu)\varepsilon, -(\Lambda+1), -\nu]|(\lambda\mu)(\varepsilon-3)(\Lambda-\tfrac{1}{2})(\nu-1)\rangle, \\ A_{zx}|(\lambda\mu)\varepsilon\Lambda\nu\rangle &= f[(\lambda\mu)(\varepsilon+3), -(\Lambda+\tfrac{3}{2})(\nu-1)]|(\lambda\mu)(\varepsilon+3)(\Lambda+\tfrac{1}{2})(\nu-1)\rangle \\ &\quad + f[(\lambda\mu)(\varepsilon+3)(\Lambda-\tfrac{1}{2})(\nu-1)]|(\lambda\mu)(\varepsilon+3)(\Lambda-\tfrac{1}{2})(\nu-1)\rangle, \\ A_{zy}|(\lambda\mu)\varepsilon\Lambda\nu\rangle &= -f[(\lambda\mu)(\varepsilon+3), -(\Lambda+\tfrac{3}{2}), -(\nu+1)]|(\lambda\mu)(\varepsilon+3)(\Lambda+\tfrac{1}{2})(\nu+1)\rangle \\ &\quad + f[(\lambda\mu)(\varepsilon+3)(\Lambda-\tfrac{1}{2}), -(\nu+1)]|(\lambda\mu)(\varepsilon+3)(\Lambda-\tfrac{1}{2})(\nu+1)\rangle, \end{aligned} \tag{4}$$

where

$$f[(\lambda\mu)\varepsilon\Lambda\nu] = \left[\frac{(\Lambda+\frac{1}{2}\nu+1)(\Lambda+1+\frac{1}{3}(\lambda-\mu-\frac{1}{2}\varepsilon))(\Lambda+2+\frac{1}{3}(\lambda+2\mu-\frac{1}{2}\varepsilon))(\frac{1}{3}(2\lambda+\mu+\frac{1}{2}\varepsilon)-\Lambda)}{(2\Lambda+1)(2\Lambda+2)}\right]^{\frac{1}{2}}.$$

It is also convenient to express f in terms of the integers p and q of eq. (1b). In terms of these parameters

$$f[(\lambda\mu)\varepsilon\Lambda\nu] = \left[\frac{(\Lambda+\frac{1}{2}\nu+1)(p+1)(\lambda-p)(\mu+2+p)}{(2\Lambda+1)(2\Lambda+2)}\right]^{\frac{1}{2}},$$

while

$$f[(\lambda\mu)\varepsilon,\ -(\Lambda+1),\nu] = \left[\frac{(\Lambda-\frac{1}{2}\nu)(q+1)(\mu-q)(\lambda+\mu+1-q)}{2\Lambda(2\Lambda+1)}\right]^{\frac{1}{2}}.$$

According to eq. (2), the infinitesimal operators Q_0, Λ_0, A_{ij}, $(i \neq j)$, themselves are SU_3 tensor operators which transform according to the eight-dimensional irreducible representation, $(\lambda\mu) = (11)$. Since the phases and normalization factors are of some importance, the SU_3 tensor character of the infinitesimal operators is exhibited explicitly in table 1. The overall phase is chosen so that the operators

TABLE 1

The SU_3 tensor character of the infinitesimal operators

$T^{(11)}_{3\frac{1}{2}\frac{1}{2}} = -\frac{1}{\sqrt{2}}A_{zy}$	$T^{(11)}_{3\frac{1}{2}-\frac{1}{2}} = \frac{1}{\sqrt{2}}A_{zx}$
$T^{(11)}_{000} = \begin{array}{l} -\frac{1}{2\sqrt{3}}(2A_{zz}-A_{xx}-A_{yy}) \\ -\frac{1}{2\sqrt{3}}Q_0 \end{array}$	$T^{(11)}_{011} = \begin{array}{l} -\frac{1}{\sqrt{2}}A_{xy} \\ -\frac{1}{\sqrt{2}}\Lambda_+ \end{array}$
$T^{(11)}_{010} = \begin{array}{l} \frac{1}{2}(A_{xx}-A_{yy}) \\ \Lambda_0 \end{array}$	$T^{(11)}_{01-1} = \begin{array}{l} \frac{1}{\sqrt{2}}A_{yx} \\ \frac{1}{\sqrt{2}}\Lambda_- \end{array}$
$T^{(11)}_{-3\frac{1}{2}\frac{1}{2}} = -\frac{1}{\sqrt{2}}A_{xz}$	$T^{(11)}_{-3\frac{1}{2}-\frac{1}{2}} = -\frac{1}{\sqrt{2}}A_{yz}$

SU_3 tensor operators are denoted by $T^{(\lambda\mu)}_{\varepsilon\Lambda\frac{1}{2}\nu}$.

$\Lambda_\pm, \Lambda_0$ are related to the tensor operators with $\Lambda = 1, \frac{1}{2}\nu = \pm 1, 0$ according to the standard phases for SU_2. It should be noted that the relation between the operator Q_0 and $T^{(11)}_{000}$ involves an "unnatural" minus sign. In the elementary particle applications of SU_3 (see, e.g. refs. [9, 10]), the operator $-\frac{1}{3}Q_0$ is identified with the hypercharge (Y), while Λ and its third component $\frac{1}{2}\nu$ are the usual isospin quantum numbers.†

† Because of the opposite sign in the definition of Y, Elliott's state with lowest possible ε, $\varepsilon_L = -(\lambda+2\mu)$, and with $\Lambda = \frac{1}{2}\nu = \frac{1}{2}\lambda$ would be identified as the "highest weight" state in the elementary particle applications. We choose to retain Elliott's notation, including $\frac{1}{2}\nu$, for the eigenvalue of Λ_0 and do not introduce some new quantum number such as M_Λ for the "third component" of the "spin" Λ, although this choice introduces unfamiliar factors of $\frac{1}{2}$ in the ordinary Wigner coefficients.

3. The SU_3 Wigner and Racah Coefficients

Moshinsky [11]) has published a general expression (involving several summations) for the SU_3 Wigner coefficients for one class of simply reducible products of SU_3, the products $(\lambda_1\mu_1)\times(\lambda_2 0) = \sum(\lambda\mu)$. Similar expressions for the general case $(\lambda_1\mu_1)\times(\lambda_2\mu_2)$ have recently been given by Resnikoff [12]). Although very elegant, these expressions are not very useful for actual calculations such as those involving the recoupling and fractional parentage coefficients for s-d shell nuclei. It is the aim of the present work to give explicit algebraic expressions somewhat in the format of the familiar tables from Condon and Shortley [13, 14]) for the relatively simple Wigner coefficients needed for s-d shell calculations. For this purpose it is useful to employ the standard technique of generating the Wigner coefficients through recursion formulae derived from the matrix elements of the infinitesimal operators of the group. In s-d shell calculations the Kronecker products which arise are those in which a single s–d shell nucleon, for which $(\lambda\mu) = (20)$, is coupled to an n-particle function of arbitrary $(\lambda\mu)$-symmetry, or those in which two s–d shell nucleons, each with $(\lambda\mu) = (20)$, are coupled to an $(n-2)$-particle function with arbitrary $(\lambda\mu)$. The SU_3 quantum numbers of the two-particle functions follow from the Kronecker product $(20)\times(20) = (40)+(02)+(21)$. The Wigner coefficients which are of particular interest in the recoupling problem involving s–d shell nucleons are therefore the products $(\lambda_1\mu_1)\times(\lambda_2\mu_2) = \sum(\lambda\mu)$ with $(\lambda_2\mu_2)$ equal to (20), (40), (02) or (21). The possible $(\lambda\mu)$ values in these products can be read off from table 2. All but the last of these products are simply reducible. In the product $(\lambda\mu)\times(21)$ only the three irreducible representations, $(\lambda+1, \mu)$, $(\lambda, \mu-1)$ and $(\lambda-1, \mu+1)$, occur more than once. Because of the central role played by the eight-dimensional representation, (11), explicit expressions for the Wigner coefficients of the Kronecker product $(\lambda\mu)\times(11)$ will also be of interest †.

State vectors for the coupled system $\varphi[(\lambda_1\mu_1)(\lambda_2\mu_2); (\lambda\mu)\rho, \varepsilon\Lambda\nu]$ are given in terms of the state vectors of the representations $(\lambda_1\mu_1)$ and $(\lambda_2\mu_2)$ by a unitary transformation whose coefficients are the SU_3 Wigner coefficients

$$\begin{aligned}\varphi[(\lambda_1\mu_1)(\lambda_2\mu_2); (\lambda\mu)\rho, \varepsilon\Lambda\nu] \\ = \sum_{\substack{\varepsilon_1\Lambda_1\nu_1\\ \varepsilon_2\Lambda_2\nu_2}} \varphi((\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1)\varphi((\lambda_2\mu_2)\varepsilon_2\Lambda_2\nu_2)\langle(\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1; (\lambda_2\mu_2)\varepsilon_2\Lambda_2\nu_2| \\ (\lambda\mu)\varepsilon\Lambda\nu\rangle_\rho, \quad (5)\end{aligned}$$

subject to the restrictions $\varepsilon = \varepsilon_1+\varepsilon_2$, $\nu = \nu_1+\nu_2$ and $\Lambda = \Lambda_1+\Lambda_2, \ldots, |\Lambda_1-\Lambda_2|$. The notation for the Wigner coefficients is an obvious extension of that for the rotation group. Since the number of independent coupled functions of given $(\lambda\mu)$ may be greater than one, an additional quantum number or label (denoted by ρ) is needed in those cases (such as $(\lambda_2\mu_2) = (21)$, $(\lambda\mu) = (\lambda_1+1, \mu_1)$, for example) in which a given $(\lambda\mu)$ occurs more than once in the Kronecker product $(\lambda_1\mu_1)\times(\lambda_2\mu_2)$.

† Numerical values for the Wigner coefficients involving the simple irreducible representations of interest in elementary particle physics have recently been given by Edmonds [17]) and by deSwart [10]).

Matrix elements of the components of SU_3 tensor operators can be factored through the generalized Wigner-Eckart theorem [7]) into an SU_3 Wigner coefficient and a reduced matrix element. However, in those cases in which a given $(\lambda\mu)$ occurs more than once in the Kronecker product $(\lambda_1\mu_1)\times(\lambda_2\mu_2)$, matrix elements of SU_3 tensor operators will involve more than one reduced matrix element (one for each independent mode of coupling)

$$\begin{aligned}&\langle(\lambda\mu)\varepsilon\Lambda\nu|T^{(\lambda_2\mu_2)}_{\varepsilon_2\Lambda_2\nu_2}|(\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1\rangle\\ &\quad=\sum_\rho\langle(\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1;(\lambda_2\mu_2)\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)\varepsilon\Lambda\nu\rangle_\rho\langle(\lambda\mu)||T^{(\lambda_2\mu_2)}||(\lambda_1\mu_1)\rangle_\rho.\end{aligned}\tag{6}$$

The quantum number ρ which labels the independent modes of coupling could be defined by a special choice of tensor operators [19]) with non-zero reduced matrix elements for only one state ρ. In the product $(\lambda_1\mu_1)\times(11)$, for example, the two independent coupled states with $(\lambda\mu)=(\lambda_1\mu_1)$ are most naturally chosen such that the reduced matrix elements of the infinitesimal operators A_{ij} are non-zero only for one of the states, the state $\rho=1$. (In particular (see eqs. (4) and tables 1 and 4), $\langle(\lambda\mu)||A_{ij}||(\lambda\mu)\rangle_1=[\frac{1}{3}(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu)]^{\frac{1}{2}}$ while $\langle(\lambda\mu)||A_{ij}||(\lambda\mu)\rangle_2=0$).

A more general technique for distinguishing the independent coupled functions of the same irreducible representation has been given by Moshinsky [15]). This involves the explicit construction of an operator [15]) (X), which together with the Casimir invariants † for the separate systems 1 and 2, the Casimir invariants of the coupled system, and the operators Q_0, Λ_0 and Λ^2, again for the coupled system, gives a complete specification of the coupled functions $\varphi([\lambda_1\mu_1)(\lambda_2\mu_2);(\lambda\mu)\rho,\varepsilon\Lambda\nu]$. That is, the latter are simultaneously eigenfunctions of the operators $C_2(1)$, $C_3(1)$, $C_2(2)$, $C_3(2)$, $C_2(1,2)$, $C_3(1,2)$, Q_0, Λ_0 and Λ^2, as well as X, where the quantum number ρ can be related to the eigenvalue of X. Except for terms which are functions only of operators of type C_2 and C_3, Moshinsky's operator X can be expressed in terms of the infinitesimal operators as

$$X=\sum_{\alpha,\beta,\mu}(A_{\beta\mu}(1)A_{\mu\alpha}(1)+A_{\mu\alpha}(1)A_{\beta\mu}(1))A_{\alpha\beta}(2),\tag{7}$$

that is, it is an SU_3 invariant of third degree which is unsymmetrical in the coordinates of systems 1 and 2 which make up the coupled state.

Recursion formulae for the Wigner coefficients follow from the matrix elements of the infinitesimal operators. By operating with the operator $A_{xz}=A_{xz}(1)+A_{xz}(2)$ on a wave function of the coupled system, eq. (5), a recursion relation is obtained for the Wigner coefficients in the usual way. Since the operator A_{xz}, the ε-lowering, ν-raising operator, couples the states Λ to both the states $(\Lambda+\frac{1}{2})$ and $(\Lambda-\frac{1}{2})$, the recursion formula for the Wigner coefficients has just double the complexity of the

† Both the Casimir invariants of second and third degree, that is, $C_2=\Sigma_{\alpha\beta}A_{\alpha\beta}A_{\beta\alpha}$ and $C_3=\Sigma_{\alpha\beta\mu}A_{\alpha\beta}(A_{\beta\mu}A_{\mu\alpha}+A_{\mu\alpha}A_{\beta\mu})$ are needed to specify the quantum numbers λ and μ, see Biedenharn [16]).

analogous recursion formula for the rotation group:

$$\begin{aligned}
& f[(\lambda\mu)\varepsilon\Lambda\nu]\langle\varepsilon_1\Lambda_1\nu_1;\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)(\varepsilon-3)(\Lambda+\tfrac{1}{2})(\nu+1)\rangle \\
&\qquad +f[(\lambda\mu),\varepsilon,-(\Lambda+1),\nu]\langle\varepsilon_1\Lambda_1\nu_1;\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)(\varepsilon-3)(\Lambda-\tfrac{1}{2})(\nu+1)\rangle \\
&= f[(\lambda_1\mu_1)(\varepsilon_1+3)(\Lambda_1-\tfrac{1}{2})(\nu_1-1)]\langle(\varepsilon_1+3)(\Lambda_1-\tfrac{1}{2})(\nu_1-1);\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)\varepsilon\Lambda\nu\rangle \\
&\qquad +f[(\lambda_1\mu_1)(\varepsilon_1+3)-(\Lambda_1+\tfrac{3}{2})(\nu_1-1)] \\
&\qquad \times\langle(\varepsilon_1+3)(\Lambda_1+\tfrac{1}{2})(\nu_1-1);\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)\varepsilon\Lambda\nu\rangle \\
&+f[(\lambda_2\mu_2)(\varepsilon_2+3)(\Lambda_2-\tfrac{1}{2})(\nu_2-1)]\langle\varepsilon_1\Lambda_1\nu_1;(\varepsilon_2+3)(\Lambda_2-\tfrac{1}{2})(\nu_2-1)|(\lambda\mu)\varepsilon\Lambda\nu\rangle \\
&\qquad +f[(\lambda_2\mu_2)(\varepsilon_2+3)-(\Lambda_2+\tfrac{3}{2})(\nu_2-1)] \\
&\qquad \times\langle\varepsilon_1\Lambda_1\nu_1;(\varepsilon_2+3)(\Lambda_2+\tfrac{1}{2})(\nu_2-1)|(\lambda\mu)\varepsilon\Lambda\nu\rangle.
\end{aligned} \tag{8a}$$

A similar recursion formula is obtained by application of A_{zx}

$$\begin{aligned}
& f[(\lambda\mu)(\varepsilon+3)(\Lambda-\tfrac{1}{2})(\nu-1)]\langle\varepsilon_1\Lambda_1\nu_1;\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)(\varepsilon+3)(\Lambda-\tfrac{1}{2})(\nu-1)\rangle \\
&\qquad +f[(\lambda\mu)(\varepsilon+3)-(\Lambda+\tfrac{3}{2})(\nu-1)]\langle\varepsilon_1\Lambda_1\nu_1;\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)(\varepsilon+3)(\Lambda+\tfrac{1}{2})(\nu-1)\rangle \\
&= f[(\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1]\langle(\varepsilon_1-3)(\Lambda_1+\tfrac{1}{2})(\nu_1+1);\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)\varepsilon\Lambda\nu\rangle \\
&\qquad +f[(\lambda_1\mu_1)\varepsilon_1-(\Lambda+1)\nu_1]\langle(\varepsilon_1-3)(\Lambda_1-\tfrac{1}{2})(\nu_1+1);\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)\varepsilon\Lambda\nu\rangle \\
&+f[(\lambda_2\mu_2)\varepsilon_2\Lambda_2\nu_2]\langle\varepsilon_1\Lambda_1\nu_1;(\varepsilon_2-3)(\Lambda_2+\tfrac{1}{2})(\nu_2+1)|(\lambda\mu)\varepsilon\Lambda\nu\rangle \\
&\qquad +f[(\lambda_2\mu_2)\varepsilon_2-(\Lambda_2+1)\nu_2]\langle\varepsilon_1\Lambda_1\nu_1;(\varepsilon_2-3)(\Lambda_2-\tfrac{1}{2})(\nu_2+1)|(\lambda\mu)\varepsilon\Lambda\nu\rangle.
\end{aligned} \tag{8b}$$

A more complicated recursion formula follows from application of the Casimir invariant C_2 of the coupled system.

In the actual calculations the recursion formulae may become very simple. The first step in the calculation may involve the coefficients with highest $\varepsilon(=\varepsilon_{\mathrm{H}} = 2\lambda+\mu)$. These are also the ones of greatest physical interest in the present application, since the full set of angular momentum eigenfunctions for a given irreducible representation of SU$_3$ can be constructed by projection from the intrinsic function of highest weight [2]). With $\varepsilon = \varepsilon_{\mathrm{H}}$ the first two Wigner coefficients in eq. (8b) vanish. Since the ν-dependence of the coefficients is known from the properties of SU$_2$, further Wigner coefficients in eq. (8) can be eliminated by proper choice of ν, such as $\nu_1 = 2\Lambda_1$.

The ν-dependence may be factored out by expressing the full SU$_3$ Wigner coefficient as a product of an ordinary SU$_2$ Wigner coefficient which carries the ν-dependence and a ν-independent factor denoted by a double bar, the "isoscalar factor" introduced by Edmonds [17])

$$\begin{aligned}
& \langle(\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1;(\lambda_2\mu_2)\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)\varepsilon\Lambda\nu\rangle_\rho \\
&\qquad = \langle\Lambda_1\tfrac{1}{2}\nu_1\Lambda_2\tfrac{1}{2}\nu_2|\Lambda\tfrac{1}{2}\nu\rangle\langle(\lambda_1\mu_1)\varepsilon_1\Lambda_1;(\lambda_2\mu_2)\varepsilon_2\Lambda_2||(\lambda\mu)\varepsilon\Lambda\rangle_\rho.
\end{aligned} \tag{9}$$

Since both the full SU$_3$ coefficients and the SU$_2$ coefficients form unitary matrices, the double-barred SU$_3$ coefficients also form unitary matrices. In some cases it may be simpler to deal with the full SU_3 coefficients, in others with the double-barred ones.

The factoring of eq. (9) makes it possible to carry out the ν-summations in expressions for the SU_3 recoupling or Racah coefficients [7]). The SU_3 Racah coefficient can thus be expressed in terms of summations involving ordinary Racah coefficients and the double-barred coefficients of eq. (9). (The recoupling coefficients which will actually be used are the generalizations of the unitary U-coefficients rather than the Racah coefficients)

$$\begin{aligned} & U((\lambda_1\mu_1)(\lambda_2\mu_2)(\lambda\mu)(\lambda_3\mu_3); (\lambda_{12}\mu_{12})\rho_{12}\rho_{12,3}(\lambda_{23}\mu_{23})\rho_{23}\rho_{1,23}) \\ & \quad = \sum_{\substack{\varepsilon_1\varepsilon_2(\varepsilon_3) \\ \Lambda_1\Lambda_2\Lambda_3\Lambda_{12}\Lambda_{23}}} \{U(\Lambda_1\Lambda_2\Lambda\Lambda_3; \Lambda_{12}\Lambda_{23})\langle\varepsilon_1\Lambda_1; \varepsilon_2\Lambda_2||(\lambda_{12}\mu_{12})\varepsilon_{12}\Lambda_{12}\rangle_{\rho_{12}} \\ & \quad \times\langle\varepsilon_{12}\Lambda_{12}; \varepsilon_3\Lambda_3||(\lambda\mu)\varepsilon\Lambda\rangle_{\rho_{12,3}}\langle\varepsilon_2\Lambda_2; \varepsilon_3\Lambda_3||(\lambda_{23}\mu_{23})\varepsilon_{23}\Lambda_{23}\rangle_{\rho_{23}} \\ & \quad \times\langle\varepsilon_1\Lambda_1; \varepsilon_{23}\Lambda_{23}||(\lambda\mu)\varepsilon\Lambda\rangle_{\rho_{1,23}}\}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} & \varphi\{[[(\lambda_1\mu_1)(\lambda_2\mu_2)](\lambda_{12}\mu_{12})\rho_{12}, (\lambda_3\mu_3)](\lambda\mu)\rho_{12,3}\varepsilon\Lambda\nu\} \\ & \quad = \sum_{(\lambda_{23}\mu_{23})\rho_{1,23}} U((\lambda_1\mu_1)(\lambda_2\mu_2)(\lambda\mu)(\lambda_3\mu_3); (\lambda_{12}\mu_{12})\rho_{12}\rho_{12,3}(\lambda_{23}\mu_{23})\rho_{23}\rho_{1,23}) \\ & \quad \times\varphi\{[(\lambda_1\mu_1)[(\lambda_2\mu_2)(\lambda_3\mu_3)](\lambda_{23}\mu_{23})\rho_{23}](\lambda\mu)\rho_{1,23}\varepsilon\Lambda\nu\}. \end{aligned} \tag{11}$$

In the s–d shell recoupling problem most of the quantum numbers ρ are redundant labels and therefore not needed. The problem involves the coupling of two particles of (20) symmetry to an $(n-2)$-particle system of arbitrary $(\lambda\mu)$ symmetry. If $(\lambda_1\mu_1)$ and $(\lambda_3\mu_3)$ can be identified with (20), for example, all of the Kronecker products implied by the recoupling coefficient of eq. (11) are simply reducible and *no* labels ρ are needed. If, on the other hand, $(\lambda_2\mu_2)$ and $(\lambda_3\mu_3)$ are identified with (20), the product $(\lambda_1\mu_1)\times(\lambda_{23}\mu_{23})$ may require the quantum number $\rho_{1,23}$ if $(\lambda_{23}\mu_{23}) = (21)$. Whenever the label ρ is not needed it will be omitted from the expressions for the U- and Wigner-coefficients. Relations involving the SU_3 recoupling coefficients and sums over products of two or three Wigner coefficients are again straightforward generalizations of those for the rotation group. In particular, the relation

$$\begin{aligned} & \sum_{\rho_{1,23}} \langle(\lambda_1\mu_1)\varepsilon_1\Lambda_1; (\lambda_{23}\mu_{23})\varepsilon_{23}\Lambda_{23}||(\lambda\mu)\varepsilon\Lambda\rangle_{\rho_{1,23}} U((\lambda_1\mu_1)(\lambda_2\mu_2)(\lambda\mu)(\lambda_3\mu_3); \\ & \qquad\qquad (\lambda_{12}\mu_{12})\rho_{12}\rho_{12,3}(\lambda_{23}\mu_{23})\rho_{23}\rho_{1,23}) \\ & \quad = \sum_{\substack{\varepsilon_2(\varepsilon_3)(\varepsilon_{12}) \\ \Lambda_2\Lambda_3\Lambda_{12}}} \langle\varepsilon_2\Lambda_2; \varepsilon_3\Lambda_3||(\lambda_{23}\mu_{23})\varepsilon_{23}\Lambda_{23}\rangle_{\rho_{23}}\langle\varepsilon_1\Lambda_1; \varepsilon_2\Lambda_2||(\lambda_{12}\mu_{12})\varepsilon_{12}\Lambda_{12}\rangle_{\rho_{12}} \\ & \quad \times\langle\varepsilon_{12}\Lambda_{12}; \varepsilon_3\Lambda_3||(\lambda\mu)\varepsilon\Lambda\rangle_{\rho_{12,3}} U(\Lambda_1\Lambda_2\Lambda\Lambda_3; \Lambda_{12}\Lambda_{23}) \end{aligned} \tag{12}$$

may be useful in generating further recursion formulae for the double-barred Wigner coefficients, although such formulae may be complicated by the summation over $\rho_{1,23}$ in cases such as $(\lambda_{23}\mu_{23}) = (21)$.

3.1. WIGNER COEFFICIENTS FOR THE PRODUCTS $(\lambda_1\mu_1)\times(\lambda_2 0)$.

Wigner coefficients with $\varepsilon = \varepsilon_H = 2\lambda+\mu$ (and $\Lambda_H = \frac{1}{2}\mu$) may be the most useful ones in shell model calculations with functions of SU$_3$ symmetry since the full set of angular momentum eigenfunctions for given $(\lambda\mu)$ can be constructed by projection from the intrinsic function of highest weight [2]). These Wigner coefficients also form a natural starting point for the calculation. Repeated application of the recursion formula (8b) yields the following relation for the double-barred SU$_3$ coefficient:

$$\begin{aligned}
&\langle(\lambda_1\mu_1)(\varepsilon_{1H}-3\alpha-3\beta)(\Lambda_{1H}+\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta);(\lambda_2 0)\varepsilon_2=\varepsilon_{2H}-3\sigma+3\alpha+3\beta,\\
&\qquad\qquad\qquad\qquad\Lambda_2=\tfrac{1}{2}\sigma-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta\|(\lambda\mu)\varepsilon_H\Lambda_H\rangle\\
&=(-1)^{\alpha}\left[\frac{(\lambda_1-\alpha)!(\mu_1-\beta)!(\mu_1+1+\alpha-\beta)(\lambda_1+\mu_1+1-\beta)!(\lambda_2-\sigma+\alpha+\beta)!\,(\lambda_1+\lambda_2-\lambda-\sigma)!(\lambda+\mu-\lambda_1-\lambda_2+\sigma+\alpha)!}{\lambda_1!\alpha!\beta!(\mu_1+1+\alpha)!(\lambda_1+\mu_1+1)!(\lambda_2-\sigma)!(\lambda_1+\lambda_2-\lambda-\sigma-\alpha)!\,(\lambda+\mu-\lambda_1-\lambda_2+\sigma)!}\right]^{\frac{1}{2}}\\
&\times\left[\frac{(\lambda_1+\lambda_2+\mu_1-\lambda+1-\sigma)!(\lambda_1+\lambda_2-\lambda+\mu_1-\mu-\sigma)!}{(\lambda_1+\lambda_2+\mu_1-\lambda+1-\sigma-\beta)!(\lambda_1+\lambda_2-\lambda+\mu_1-\mu-\sigma-\beta)!}\right]^{\frac{1}{2}}\\
&\times\langle(\lambda_1\mu_1)\varepsilon_{1H}\Lambda_{1H};(\lambda_2 0)\varepsilon_{2H}-3\sigma,\Lambda_2=\tfrac{1}{2}\sigma\|(\lambda\mu)\varepsilon_H\Lambda_H\rangle,
\end{aligned}\tag{13}$$

where $\varepsilon_{1H} = 2\lambda_1+\mu_1$, $\Lambda_{1H} = \frac{1}{2}\mu_1$; $\varepsilon_2(\alpha=\beta=0) = \varepsilon_H-\varepsilon_{1H} = (2\lambda+\mu)-(2\lambda_1+\mu_1) = \varepsilon_{2H}-3\sigma = 2\lambda_2-3\sigma$. The last equation defines the fixed integer σ. (Note that $\varepsilon_2 = \varepsilon_{2H}-3\sigma$ implies $\Lambda_2 = \frac{1}{2}\sigma$; Λ_2 is uniquely specified by ε_2 in the representation $(\lambda_2 0)$.) The integers α and β can range over the values $0 \leqq \alpha \leqq \frac{1}{3}(\lambda_1+\lambda_2-\lambda+\mu-\mu_1)$ and $0 \leqq \beta \leqq \frac{1}{3}(\lambda_1+\lambda_2-\lambda+2\mu_1-2\mu)$. The magnitude of $\langle(\lambda_1\mu_1)\varepsilon_{1H}\Lambda_{1H};(\lambda_2 0)\varepsilon_{2H}-3\sigma, \frac{1}{2}\sigma\|(\lambda\mu)\varepsilon_H\Lambda_H\rangle$ can be obtained from the unitary property of the double-barred Wigner coefficients

$$\sum_{\alpha,\beta}|\langle(\lambda_1\mu_1)(\varepsilon_{1H}-3\alpha-3\beta)(\Lambda_{1H}+\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta);(\lambda_2 0)\varepsilon_2\Lambda_2\|(\lambda\mu)\varepsilon_H\Lambda_H\rangle|^2 = 1.$$

No techniques have been discovered for performing the needed sums over α and β in general closed form. For the actual cases needed the sums can easily be carried out explicitly. Expressions for the Wigner coefficients with both $\varepsilon = \varepsilon_H$ and $\varepsilon_1 = \varepsilon_{1H}$ are given in table 2. Algebraic expressions for those with $\varepsilon_1 < \varepsilon_{1H}$ follow from eq. (13). Because of the central role played in s–d shell calculations by the (20) representation the complete table of SU$_3$ double-barred Wigner coefficients for the product $(\lambda\mu)\times(20)$ is given as table 3. The phase of the coefficient $\langle(\lambda_1\mu_1)\varepsilon_{1H}\Lambda_{1H}\nu_{1H};(\lambda_2\mu_2)\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)\varepsilon_H\Lambda_H\nu_H\rangle$ must be chosen. By straightforward generalization of the Condon and Shortley phase convention [13,14]), the coefficients with both $\varepsilon = \varepsilon_H$, $\nu = \nu_H$; $\varepsilon_1 = \varepsilon_{1H}$ and $\nu_1 = \nu_{1H}$ are chosen positive (and real). In the case $(\lambda_2\mu_2) = (\lambda_2 0)$ there is only one coefficient of this type and this convention uniquely specifies the phases of the Wigner coefficients. In the general case with both $\lambda_2 \neq 0$ and $\mu_2 \neq 0$, there will be more than one coefficient with both $\varepsilon = \varepsilon_H$ and $\varepsilon_1 = \varepsilon_{1H}$ in all those cases in which there is more than one independent coupled function of $(\lambda\mu)$ symmetry in the product $(\lambda_1\mu_1)\times(\lambda_2\mu_2)$. In this case there is more than one possible value of Λ_2 in the state with $\varepsilon_2 = \varepsilon_H - \varepsilon_{1H}$. The phases of the Wigner coefficients can then be uniquely specified by the additional phase convention that the Wigner coefficient with the largest value of Λ_2 and ν_2, (and $\varepsilon = \varepsilon_H, \varepsilon_1 = \varepsilon_{1H}, \nu = \nu_H, \nu_1 = \nu_{1H}$) is positive †.

From eqs. (13) and (9) it can be seen that the coefficient $\langle(\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1; (\lambda_2 0)\varepsilon_{2H} = 2\lambda_2, \Lambda_{2H} = \nu_{2H} = 0|(\lambda\mu)\varepsilon_H\Lambda_H\nu_H\rangle$ differs in sign from the coefficient $\langle(\lambda_1\mu_1)\varepsilon_{1H}\Lambda_{1H}\nu_{1H}; (\lambda_2 0)\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)\varepsilon_H\Lambda_H\nu_H\rangle$ by the factor $(-1)^\alpha$ $(\equiv(-1)^{3\alpha})$ with $3\alpha = \lambda_1+\lambda_2-\lambda+\mu-\mu_1$ which leads to the symmetry property ††

$$\begin{aligned}\langle(\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1; (\lambda_2 0)\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)\varepsilon\Lambda\nu\rangle \\ = (-1)^{\lambda_1+\lambda_2-\lambda+\mu-\mu_1}\langle(\lambda_2 0)\varepsilon_2\Lambda_2\nu_2; (\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1|(\lambda\mu)\varepsilon\Lambda\nu\rangle. \qquad (14a)\end{aligned}$$

Using the analogous symmetry property of the ordinary Wigner coefficient, the symmetry property for the double-barred SU_3 coefficient becomes

$$\begin{aligned}\langle(\lambda_1\mu_1)\varepsilon_1\Lambda_1; (\lambda_2 0)\varepsilon_2\Lambda_2||(\lambda\mu)\varepsilon\Lambda\rangle \\ = (-1)^{\lambda_1+\lambda_2-\lambda+\mu-\mu_1+\Lambda_1+\Lambda_2-\Lambda}\langle(\lambda_2 0)\varepsilon_2\Lambda_2; (\lambda_1\mu_1)\varepsilon_1\Lambda_1||(\lambda\mu)\varepsilon\Lambda\rangle. \qquad (14b)\end{aligned}$$

3.2. WIGNER COEFFICIENTS FOR THE PRODUCTS $(\lambda_1\mu_1)\times(0\mu_2)$.

Wigner coefficients of this type can be obtained from those for the product $(\lambda_1\mu_1)\times(\lambda_2 0)$ through a further symmetry property of the Wigner coefficients (see ref. [10]) and appendix)

$$\begin{aligned}\langle(\lambda_1\mu_1)\varepsilon_1\Lambda_1; (\lambda_2\mu_2)\varepsilon_2\Lambda_2||(\lambda_3\mu_3)\varepsilon_3\Lambda_3\rangle \\ = (-1)^{\frac{1}{3}(\mu_1-\mu_3-\lambda_1+\lambda_3-\frac{1}{2}\varepsilon_2)+\Lambda_3-\Lambda_1}\left[\frac{(\dim(\lambda_3\mu_3))(2\Lambda_1+1)}{(\dim(\lambda_1\mu_1))(2\Lambda_3+1)}\right]^{\frac{1}{2}} \\ \times\langle(\lambda_3\mu_3)\varepsilon_3\Lambda_3; (\mu_2\lambda_2)-\varepsilon_2\Lambda_2||(\lambda_1\mu_1)\varepsilon_1\Lambda_1\rangle, \qquad (15)\end{aligned}$$

where $\dim(\lambda\mu) = \frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2)$.

3.3. WIGNER COEFFICIENTS FOR THE PRODUCTS $(\lambda_1\mu_1)\times(\lambda_2 1)$.

The coefficients of greatest interest in the present work, that is those with $\varepsilon = \varepsilon_H = 2\lambda+\mu$ and $\Lambda_H = \frac{1}{2}\mu$ but arbitrary ε_1 and Λ_1 can again be expressed in

† Unfortunately this phase convention differs from that employed by other authors [10–12, 17]). Because of the "unnatural" minus sign in the relation between the operator Q_0 and the SU_3 tensor operator $T^{(11)}_{000}$ it might have been preferable to fix the phases in terms of the coefficients involving the lowest rather than the highest values of ε. This change would not affect the sign of coefficients with $(\lambda_2\mu_2) = (20)$ and would not bring our phases into agerement with those of others. Since there is as yet no unanimity as to the choice of phases, and since the state with $\varepsilon = \varepsilon_H$ plays the preferred role in Elliott's approach, the above phase convention has been retained in this work. The choice made by deSwart, for example, involves the "highest" weight state for $(\lambda\mu)$ $(\varepsilon = -(\lambda+2\mu)$ in Elliott's notation), but the state with largest Λ_1 to fix the phases [10]).

†† Symmetry properties of the Wigner coefficients have been discussed by deSwart [10]) and Resnikoff [12]). The phase factor in eq. (14a) is an explicit evaluation of deSwart's phase factor of type ξ_1; however, subject to the above rather than deSwart's phase convention.

terms of those with both $\varepsilon = \varepsilon_{\mathrm{H}}$ and $\varepsilon_1 = \varepsilon_{1\mathrm{H}}$ by repeated application of the recursion formulae. There are now two coefficients of this type corresponding to the two possibilities $\Lambda_2 = \frac{1}{2}\sigma' \pm \frac{1}{2}$ for $\varepsilon_2 = \varepsilon_{2\mathrm{H}} - 3\sigma' = 2\lambda_2 + 1 - 3\sigma'$, but explicit algebraic expressions (involving no summations) can again be given to relate both types of coefficients to those with $\varepsilon_1 = \varepsilon_{1\mathrm{H}}$, and $\varepsilon = \varepsilon_{\mathrm{H}}$:

$$\begin{aligned}
&\langle(\lambda_1\mu_1)(\varepsilon_{1\mathrm{H}}-3\alpha-3\beta)(\Lambda_{1\mathrm{H}}+\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta);(\lambda_2 1)(\varepsilon_{2\mathrm{H}}-3\sigma+3\alpha+3\beta)\Lambda_2\\
&\quad=\tfrac{1}{2}\sigma-\tfrac{1}{2}-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta\|(\lambda\mu)\varepsilon_{\mathrm{H}}\Lambda_{\mathrm{H}}\rangle\\
&\quad=(-1)^{\alpha}\left[\frac{(\lambda_1-\alpha)!(\mu_1-\beta)!(\mu_1+1+\alpha-\beta)(\lambda_1+\mu_1+1-\beta)!(\lambda_2+1-\sigma+\alpha+\beta)!\times(\lambda_1+\lambda_2-\lambda-\sigma)!(\lambda+\mu-\lambda_1-\lambda_2+\sigma+\alpha)!}{\alpha!\beta!\lambda_1!(\mu_1+1+\alpha)!(\lambda_1+\mu_1+1)!(\lambda_2+1-\sigma)!(\lambda_1+\lambda_2-\lambda-\sigma-\alpha)!\times(\lambda+\mu-\lambda_1-\lambda_2+\sigma)!}\right]^{\frac{1}{2}}\\
&\quad\times\left[\frac{(\lambda_1+\lambda_2+\mu_1-\lambda+1-\sigma)!(\lambda_1+\lambda_2-\lambda+\mu_1-\mu-\sigma)!(\sigma+1)}{(\lambda_1+\lambda_2+\mu_1-\lambda+1-\sigma-\beta)!(\lambda_1+\lambda_2-\lambda+\mu_1-\mu-\sigma-\beta)!(\sigma+1-\alpha-\beta)}\right]^{\frac{1}{2}}\\
&\quad\times\langle(\lambda_1\mu_1)\varepsilon_{1\mathrm{H}}\Lambda_{1\mathrm{H}};(\lambda_2 1)(\varepsilon_{2\mathrm{H}}-3\sigma)\Lambda_2=\tfrac{1}{2}\sigma-\tfrac{1}{2}\|(\lambda\mu)\varepsilon_{\mathrm{H}}\Lambda_{\mathrm{H}}\rangle,
\end{aligned}\tag{16a}$$

$$\begin{aligned}
&\langle(\lambda_1\mu_1)(\varepsilon_{1\mathrm{H}}-3\alpha-3\beta)(\Lambda_{1\mathrm{H}}+\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta);(\lambda_2 1)(\varepsilon_{2\mathrm{H}}-3\sigma+3\alpha+3\beta)\Lambda_2\\
&\quad=\tfrac{1}{2}\sigma+\tfrac{1}{2}-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta\|(\lambda\mu)\varepsilon_{\mathrm{H}}\Lambda_{\mathrm{H}}\rangle\\
&\quad=(-1)^{\alpha}\left[\frac{(\lambda_1-\alpha)!(\mu_1-\beta)!(\mu_1+1+\alpha-\beta)(\lambda_1+\mu_1+1-\beta)!(\lambda_2-\sigma+\alpha+\beta)!\times(\lambda+\mu-\lambda_1-\lambda_2-1+\sigma+\alpha)!(\sigma+1-\alpha-\beta)}{\alpha!\beta!\lambda_1!(\mu_1+1+\alpha)!(\lambda_1+\mu_1+1)!(\lambda_1+\lambda_2-\lambda+1-\sigma-\alpha)!\times(\lambda_1+\lambda_2-\lambda+\mu_1-\mu+1-\sigma-\beta)!(\lambda_1+\lambda_2-\lambda+\mu_1+2-\sigma-\beta)!}\right]^{\frac{1}{2}}\\
&\quad\times\left\{\left[\frac{(\lambda_1+\lambda_2-\lambda+1-\sigma)!(\lambda_1+\lambda_2-\lambda+\mu_1+2-\sigma)!(\lambda_1+\lambda_2-\lambda+\mu_1-\mu+1-\sigma)!}{(\lambda_2-\sigma)!(\lambda+\mu-\lambda_1-\lambda_2-1+\sigma)!(\sigma+1)}\right]^{\frac{1}{2}}\right.\\
&\quad\times\langle(\lambda_1\mu_1)\varepsilon_{1\mathrm{H}}\Lambda_{1\mathrm{H}};(\lambda_2 1)(\varepsilon_{2\mathrm{H}}-3\sigma)(\tfrac{1}{2}\sigma+\tfrac{1}{2})\|(\lambda\mu)\varepsilon_{\mathrm{H}}\Lambda_{\mathrm{H}}\rangle\\
&\quad+\frac{[(\lambda_1+\lambda_2-\lambda-\sigma)!(\lambda_1+\lambda_2-\lambda+\mu_1+1-\sigma)!(\lambda_1+\lambda_2-\lambda+\mu_1-\mu-\sigma)!(\lambda_2+2)]^{\frac{1}{2}}}{(\sigma+1-\alpha)(\sigma+1-\alpha-\beta)[(\sigma+1)(\lambda_2+1-\sigma)!(\lambda+\mu-\lambda_1-\lambda_2+\sigma)!]^{\frac{1}{2}}}\\
&\quad\left.\times F_{\alpha\beta}\langle(\lambda_1\mu_1)\varepsilon_{1\mathrm{H}}\Lambda_{1\mathrm{H}};(\lambda_2 1)(\varepsilon_{2\mathrm{H}}-3\sigma)(\tfrac{1}{2}\sigma-\tfrac{1}{2})\|(\lambda\mu)\varepsilon_{\mathrm{H}}\Lambda_{\mathrm{H}}\rangle\right\},
\end{aligned}\tag{16b}$$

where, in (16b),

$$\begin{aligned}
F_{\alpha\beta}=\alpha(\sigma+1-\alpha-\beta)(\lambda_1+\lambda_2-\lambda+\mu_1+2-\sigma)(\lambda_1+\lambda_2-\lambda+\mu_1-\mu+1-\sigma)&\\
+\beta(\sigma+1)(\lambda_1+\lambda_2-\lambda+1-\sigma-\alpha)(\lambda_1+\lambda_2-\lambda-\mu-\sigma-\alpha).&
\end{aligned}$$

Coefficients with both $\varepsilon_1 = \varepsilon_{1\mathrm{H}}$ and $\varepsilon = \varepsilon_{\mathrm{H}}$ can be evaluated through the unitary property of the Wigner coefficients. They are shown in table 2 for the case $(\lambda_2\mu_2)$

$=(21)$. Note that there is only one coefficient of this type in all those cases in which the representation $(\lambda'\mu')$ occurs only once in the Kronecker product. In these cases the unitary property is sufficient to determine the one coefficient. In those cases in which the quantum number ρ is needed, that is for $(\lambda'\mu') = (\lambda+1, \mu)$, $(\lambda-1, \mu+1)$ and $(\lambda, \mu-1)$, two independent linear combinations of $\langle\varepsilon_{1\mathrm{H}}\Lambda_{1\mathrm{H}};\, \varepsilon_{2\mathrm{H}}-3\sigma, \Lambda_2 = \frac{1}{2}\sigma-\frac{1}{2}\|(\lambda\mu)\varepsilon_{\mathrm{H}}\Lambda_{\mathrm{H}}\rangle$ and $\langle\varepsilon_{1\mathrm{H}}\Lambda_{1\mathrm{H}};\, \varepsilon_{2\mathrm{H}}-3\sigma, \Lambda_2 = \frac{1}{2}\sigma+\frac{1}{2}\|(\lambda\mu)\varepsilon_{\mathrm{H}}\Lambda_{\mathrm{H}}\rangle$ can be found such that the coupled functions are eigenfunctions of Moshinsky's operator X (see eq. (7)). In practice, the resultant algebraic expressions for these coefficients have been found to be rather complicated and an alternate approach has been used. For one of the coupled states, the Wigner coefficients with both $\varepsilon_1 = \varepsilon_{1\mathrm{H}}$ and $\varepsilon = \varepsilon_{\mathrm{H}}$ have been chosen arbitrarily. The choice $\langle\varepsilon_{1\mathrm{H}}\Lambda_{1\mathrm{H}};\, \varepsilon_{2\mathrm{H}}-3\sigma, \Lambda_2 = \frac{1}{2}\sigma-\frac{1}{2}\|(\lambda\mu)\varepsilon_{\mathrm{H}}\Lambda_{\mathrm{H}}\rangle_1 = 0$ has proved convenient. By making the coefficients with $\rho = 2$ orthogonal to those with $\rho = 1$, all of the unitary (orthogonal) properties of the transformation coefficients have been preserved so that this technique for making the distinction between the two independent coupled functions does not suffer the worst faults of an arbitrary labelling.

Because of the central role played by the eight-dimensional irreducible representation $(\lambda_2\mu_2) = (11)$, the complete table of SU_3 Wigner coefficients for the product $(\lambda\mu)\times(11)$ is given as table 4. Coefficients in adjacent columns of the table are related through the symmetry relation (15).

3.4. THE U-COEFFICIENTS.

With the Wigner coefficients given in tables 2 and 3, and through eqs. (13) and (16), the U-coefficients needed for s–d shell calculations can now be computed. These are the coefficients

$$U((\lambda\mu)(20)(\lambda'\mu')(20);\, (\lambda_{12}\mu_{12})(\lambda_{23}\mu_{23})\rho_{1,23}),$$

which are needed in the recoupling transformation

$$\begin{aligned}
&\varphi([[1, 2, \ldots, (n-2)(\lambda\mu), (n-1)(20)](\lambda_{12}\mu_{12}), n(20)](\lambda'\mu')\varepsilon'\Lambda'\nu') \\
&\quad = \sum_{(\lambda_{23}\mu_{23})\rho_{1,23}} U((\lambda\mu)(20)(\lambda'\mu')(20);\, (\lambda_{12}\mu_{12})(\lambda_{23}\mu_{23})\rho_{1,23}) \\
&\quad \times\varphi([1, 2, \ldots (n-2)(\lambda\mu), [(n-1)(20), n(20)](\lambda_{23}\mu_{23})](\lambda'\mu')\rho_{1,23}\varepsilon'\Lambda'\nu'). \qquad (17)
\end{aligned}$$

The summation over the states in the coupled scheme, involving $(\lambda_{23}\mu_{23})$, includes a sum over the two independent functions $\rho_{1,23} = 1$ and 2, which can be constructed with the representation (21) for three of the final states $(\lambda'\mu')$. The full U-matrix divides into three unit matrices, six (2×2), three (3×3) and three (4×4) real, unitary (orthogonal) matrices. These are tabulated as table 5. The U-coefficients of the form

$$U((20)(\lambda\mu)(\lambda'\mu')(20);\, (\lambda_{12}\mu_{12})(\lambda_{23}\mu_{23}))$$

may also be needed. They can be obtained through a composition of recoupling transformations which gives

$$U((20)(\lambda\mu)(\lambda'\mu')(20);(\lambda_{12}\mu_{12})(\lambda_{13}\mu_{13})) = (-1)^{\lambda+\lambda'-\lambda_{12}-\lambda_{13}-\mu-\mu'+\mu_{12}+\mu_{13}}$$
$$\times \sum_{(\lambda_{23}\mu_{23})\rho_{1,23}} (-1)^{\mu_{23}} U((\lambda\mu)(20)(\lambda'\mu')(20);(\lambda_{12}\mu_{12})(\lambda_{23}\mu_{23})\rho_{1,23})$$
$$\times U((\lambda\mu)(20)(\lambda'\mu')(20);(\lambda_{13}\mu_{13})(\lambda_{23}\mu_{23})\rho_{1,23}), \tag{18}$$

a straightforward generalization of the analogous formula for ordinary Racah coefficients in which repeated use has been made of the symmetry relation (14a). The behaviour of the coupled functions of eq. (17) under permutation of the $(n-1)$th and nth particles is given by these U-coefficients

$$P_{(n-1)n}\varphi([[1,2,\dots(n-2)(\lambda\mu),(n-1)(20)](\lambda_{12}\mu_{12}),n(20)](\lambda'\mu')\varepsilon'\Lambda'\nu')$$
$$= (-1)^{\lambda+\lambda'-\lambda_{12}-\mu-\mu'+\mu_{12}} \sum_{(\lambda_{13}\mu_{13})} (-1)^{\mu_{13}-\lambda_{13}} U((20)(\lambda\mu)(\lambda'\mu')(20);(\lambda_{12}\mu_{12})(\lambda_{13}\mu_{13}))$$
$$\times \varphi([[1,2,\dots(n-2)(\lambda\mu),(n-1)(20)](\lambda_{13}\mu_{13}),n(20)](\lambda'\mu')\varepsilon'\Lambda'\nu').$$

4. The SU_3 Fractional Parentage Coefficients

The $\langle n\{|n-1\rangle$ fractional parentage coefficients are factored into orbital and charge-spin c.f.p., using the technique of Jahn and van Wieringen [18]). For the states of high orbital symmetry which are the most interesting for the s–d shell calculations the Jahn and van Wieringen tables of charge-spin c.f.p. will suffice for nuclei with $n \leqq 10$. The orbital c.f.p. of interest here are not those involving the angular momentum eigenfunctions but the intrinsic oscillator functions of SU_3 symmetry in the $SU_3 \supset SU_2$ chain. These are also characterized by the partitions $[f]$ which specify the irreducible representations of SU_6 and the symmetry properties of the wave functions under permutation of the n particles. The latter can further be made explicit by the Yamanouchi symbols [18]), but these are omitted here for short-hand purposes so that the parentage expansion for the intrinsic orbital states can be written

$$\varphi(n[f](\lambda\mu)\varepsilon\Lambda\nu) = \sum_{(\lambda'\mu')} \langle n[f](\lambda\mu)\{|(n-1)(\lambda'\mu')\rangle$$
$$\times \varphi([1,\dots,(n-1)[f'](\lambda'\mu'),n(20)](\lambda\mu)\varepsilon\Lambda\nu), \tag{19}$$

where the coupling of the $(n-1)$-particle function of $(\lambda'\mu')$ symmetry to the nth particle of (20) symmetry to give an n-particle function of $(\lambda\mu)$ symmetry is effected through the SU_3 Wigner coefficients of table 3:

$$\varphi([1,\dots,(n-1)[f'](\lambda'\mu'),n(20)](\lambda\mu)\varepsilon\Lambda\nu)$$
$$= \sum_{\varepsilon'\Lambda'\nu'} \langle(\lambda'\mu')\varepsilon'\Lambda'\nu';(20)\varepsilon_2\Lambda_2\nu_2|(\lambda\mu)\varepsilon\Lambda\nu\rangle \varphi(1,\dots,(n-1)[f'](\lambda'\mu')\varepsilon'\Lambda'\nu')$$
$$\times \varphi(n(20)\varepsilon_2\Lambda_2\nu_2). \tag{20}$$

Additional labels may be needed to fully specify the states if a function of definite $[f]$ contains a representation $(\lambda\mu)$ more than once. With the U-functions of table 5, the full set of $\langle n\{|n-1\rangle$ c.f.p. could, in principle, be computed through a chain calculation starting with $n = 2$, similar to that used by Jahn and van Wieringen. Such a calculation, however, would involve not only a large number of physically uninteresting states but also a large number of states for which the quantum numbers of SU_3 do not give a full labelling of the wave functions. Although partial chains of this type may be an aid in the calculation they are not needed. For the states of greatest physical interest, states of high SU_3 symmetry (that is large values of λ and μ, therefor large values of the Casimir operator), the $\langle n\{|n-1\rangle$ c.f.p. for particular values of n can be computed without recourse to a chain calculation. The method involves a direct comparison of explicit expressions for both the n-particle and the $(n-1)$-particle functions, the coupling of the latter to the nth particle being effected through the SU_3 Wigner coefficients. The technique is illustrated through a simple example, the calculation of the $\langle 9\{|8\rangle$ c.f.p. of the type $\langle[441](66)\{|[431](\lambda'\mu')\rangle$ for which there are two possible representations $(\lambda'\mu')$, namely (46) and (65), see ref. [1]). The parentage expansion eq. (19) is then specifically

$$\begin{aligned}\varphi([441](66);\varepsilon_{\mathrm{H}}\Lambda_{\mathrm{H}}\nu_{\mathrm{H}}) = {} & \langle[441](66)\{|[431](46)\rangle\varphi((46);\varepsilon'_{\mathrm{H}}\Lambda'_{\mathrm{H}}\nu'_{\mathrm{H}})\varphi((20);400)\\ & +\langle[441](66)\{|[431](65)\rangle[\tfrac{1}{2}\sqrt{3}\varphi((65);\varepsilon''_{\mathrm{H}}\Lambda''_{\mathrm{H}}\nu''_{\mathrm{H}})\varphi((20);1\tfrac{1}{2}1)\\ & -\tfrac{1}{2}\varphi((65);(\varepsilon''_{\mathrm{H}}-3)(\Lambda''_{\mathrm{H}}+\tfrac{1}{2})(\nu''_{\mathrm{H}}+1))\varphi((20);400)],\end{aligned} \tag{21}$$

in which the 9th particle of (20) symmetry is coupled to the 8-particle functions of (46) and (65) symmetry through the SU_3 Wigner coefficients whose numerical values have been obtained from table 3. A (66)-function of highest weight can be chosen since the c.f.p. are independent of ε, Λ and ν. The parent state is thus the intrinsic state used by Elliott and Harvey. Explicit expressions for the various functions of eq. (21) can be given in the following highly abbreviated notation:

$$\begin{aligned}&\varphi([441](66);\varepsilon_{\mathrm{H}}\Lambda_{\mathrm{H}}\nu_{\mathrm{H}}) = \varphi\{440100\},\\ &\varphi([431](65);\varepsilon''_{\mathrm{H}}\Lambda''_{\mathrm{H}}\nu''_{\mathrm{H}}) = \varphi\{430100\},\\ &\varphi([431](65);(\varepsilon''_{\mathrm{H}}-3)(\Lambda''_{\mathrm{H}}+\tfrac{1}{2})(\nu''_{\mathrm{H}}+1)) = -\sqrt{\tfrac{2}{3}}\varphi\{420200\}-\frac{1}{\sqrt{3}}\varphi\{340100\},\\ &\varphi([431](46);\varepsilon'_{\mathrm{H}}\Lambda'_{\mathrm{H}}\nu'_{\mathrm{H}}) = \frac{1}{\sqrt{3}}\varphi\{420200\}-\sqrt{\tfrac{2}{3}}\varphi\{340100\}.\end{aligned} \tag{22}$$

The numbers in the curly brackets are the occupation numbers of the six single-particle states of (20) symmetry which are listed in the following specific order. If the six single-particle states are described by the harmonic oscillator quantum numbers n_z, n_x and n_y, then the order of the single-particle states which has been chosen is $\varphi(n_z, n_x, n_y) = \varphi(200)$, $\varphi(110)$, $\varphi(101)$, $\varphi(020)$, $\varphi(011)$, $\varphi(002)$. In terms of the quantum numbers ε Λ and ν, these same functions are $\varphi(400)$, $\varphi(1\tfrac{1}{2}1)$, $\varphi(1\tfrac{1}{2}-1)$,

$\varphi(-2\ 1\ 2)$, $\varphi(-2\ 1\ 0)$ and $\varphi(-2\ 1\ -2)$, in the above order. The occupation numbers in the 9-particle function of (66) SU_3 symmetry are $\{440100\}$. By uncoupling the 9th particle from such a wave function it is possible only to obtain wave functions with occupation numbers $\{440000\}$, $\{430100\}$ and $\{340100\}$ but not $\{420200\}$. If the explicit expressions for the various wave functions are substituted into eq. (21) the coefficient of $\varphi\{420200\}$ must therefore vanish, giving a relation between the two c.f.p.:

$$\langle[441](66)\{|431](46)\rangle\tfrac{1}{3}\sqrt{3}+\langle[441](66)\{|[431](65)\rangle\tfrac{1}{6}\sqrt{6} = 0.$$

This together with the normalization determines the coefficients

$$\langle[441](66)\{|[431](46)\rangle = -\tfrac{1}{3}\sqrt{3}, \quad \langle[441](66)\{|[431](65)\rangle = \sqrt{\tfrac{2}{3}}.$$

States with occupation number $\{420200\}$ can of course also occur in the 8-particle states with $[f'] = [44]$, the other possible daughter of $[f] = [441]$. Since $[f'] = [44]$ implies $T = 0$, $S = 0$, while $T = 1$, $S = 1$ are possible isospin-spin values of [431], there can be no cancellation of the $\{420200\}$ states of [431] with those of [44]. In particular, the *total* wave function $\varphi\{420200\}$ with $T = 1$, $S = 1$, $M_T = 1$, $M_S = 1$ can be written in terms of Slater determinants as

$$\begin{aligned}&\tfrac{1}{2}\sqrt{2}|(200n^+)(200n^-)(200p^+)(200p^-)(110n^+)(110n^-)(020n^+)(020p^+)|\\ &-\tfrac{1}{2}\sqrt{2}|(200n^+)(200n^-)(200p^+)(200p^-)(110n^+)(110p^+)(020n^+)(020n^-)|,\end{aligned}$$

in which the first single-particle quantum numbers are, for example, ($n_z = 2$, $n_x = 0$, $n_y = 0$, neutron, spin up). This cannot be a daughter of the parent state with $T = S = \frac{1}{2}$, $M_T = M_S = \frac{1}{2}$:

$$\begin{aligned}&\varphi([441]T = S = \tfrac{1}{2}, M_T = M_S = \tfrac{1}{2}, (66)\varepsilon_H \Lambda_H \nu_H)\\ &\quad = |(200n^+)(200n^-)(200p^+)(200p^-)(110n^+)(110n^-)(110p^+)(110p^-)(020n^+)|.\end{aligned}$$

By uncoupling the 9th particle in the p^- or $m_t = m_s = -\frac{1}{2}$ state from this wave function, we get the only possible $T = 1$, $S = 1$, $M_T = 1$, $M_S = 1$ *total* wave functions, which are of type $\{340100\}$ and $\{430100\}$ each with a coefficient $\frac{1}{9}\sqrt{9}$. (The remaining 8×8 Slater determinants have a normalization constant $1/\sqrt{8!}$ rather than the $1/\sqrt{9!}$ of the parent wave function). By writing the analogue of eq. (21) for the total wave function and again substituting the explicit expressions of eq. (22) into this equation, we could now use the coefficients of $\varphi\{340100\}$ and $\varphi\{430100\}$ directly to evaluate the c.f.p. The coefficient of $\varphi\{430100\}$, for example, becomes

$$\langle[441](66)T = S = \tfrac{1}{2}\{|[431](65)T = S = 1\rangle\tfrac{1}{2}\sqrt{3}\langle 11; \tfrac{1}{2}-\tfrac{1}{2}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\langle 11; \tfrac{1}{2}-\tfrac{1}{2}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

and this must be equal to $\frac{1}{9}\sqrt{9}$. (The ordinary Wigner coefficients describe the coupling of the isospin and spin functions of the 9th particle to the $T = 1$ and $S = 1$ 8-paricle functions to form resultant $T = \frac{1}{2}$ and $S = \frac{1}{2}$ wave functions). The full c.f.p. can be written in terms of orbital and spin-charge coefficients (Jahn and van Wieringen)

$$\langle [f](\lambda\mu)TS\{|[f'](\lambda'\mu')T'S'\rangle = \sqrt{\frac{n_{f'}}{n_f}}\,\langle [f](\lambda\mu)\{|[f'](\lambda'\mu')\rangle\langle[\tilde{f}']TS\{|[\tilde{f}']T'S'\rangle. \tag{23}$$

The spin-charge c.f.p. can be taken from the tables of Jahn and van Wieringen. The ratio $n_{f'}/n_f$ of the dimensions of the irreducible representations $[f']$ and $[f]$ of the symmetric group are also tabulated by Jahn and van Wieringen. The above relation can therefore be used to evaluate the orbital c.f.p.

Although these methods may be very inelegant, they are relatively simple for states of large λ and μ. Explicit expressions are needed only for wave functions which are at *most* two steps removed from the function of highest weight ($p+q+r \leqq 2$ in the notation of eq. (A.1) of the appendix). These are easy to calculate by the methods of Elliott and Harvey. A few examples of SU_3 c.f.p. are illustrated in table 6. It is seen that the quantum numbers of SU_3 are not sufficient to completely specify all of the states, so that some additional arbitrary labels are needed. (Thus the two types of (73) and (53) functions of [431] and [43] symmetry, respectively, are chosen arbitrarily but orthogonal to each other.) It should be noted that these c.f.p. have orthogonality properties identical with those of the c.f.p. for orbital angular momentum functions [18]). The $\langle n\{|n-2\rangle$ c.f.p. can be calculated by standard techniques using the U-coefficients of table 5.

5. Single-Particle Spectroscopic Factors for 2s-1d Shell Nuclei

Expressions for single-particle spectroscopic factors may be given in convenient form in terms of the $\langle n\{|n-1\rangle$ c.f.p. for the intrinsic states. The total n-particle wave function is assumed to be a known linear combination of projected angular momentum eigenfunctions of the Elliott form

$$\psi = N_0 \int \mathrm{d}\Omega\, D^{J_0}_{M_{J_0}K_{J_0}}(\Omega)\varphi_\Omega(nT_0S_0[f_0](\lambda_0\mu_0)_{\mathrm{H}}M_{T_0}\Sigma_0). \tag{24}$$

The subscript zero is used to denote the n-particle parent state, and J_0 is the total angular momentum, K_{J_0} and Σ_0 are the z-components of the total and spin angular momenta in the rotated system (denoted by the subscript Ω), while M_{J_0} gives the projection of J_0 along the space-fixed z direction. The subscript H on $(\lambda_0\mu_0)$ indicates the intrinsic SU_3 state of highest weight. (The symbol φ is used for intrinsic functions, ψ for angular momentum eigenfunctions.) The normalization constant N_0 involves a sum over the possible orbital angular momentum quantum numbers L_0

$$N_0 = \frac{2J_0+1}{8\pi^2}\,\frac{1}{[\sum_{L_0} |(a(L_0K_0)\langle L_0K_0,\, S_0\Sigma_0|J_0K_{J_0}\rangle|^2]^{\frac{1}{2}}}, \tag{25}$$

in which $K_0 = K_{J_0}-\Sigma_0$ and the $a(L_0K_0)$ are the expansion coefficients which give the intrinsic function in terms of orbital angular momentum eigenfunctions

$$\varphi(\lambda_0\mu_0)_{\mathrm{H}} = \sum_{K_0L_0} a(L_0K_0)\psi((\lambda_0\mu_0)K_0, L_0K_0). \tag{26}$$

Using the parentage expansion for the n-particle intrinsic state, φ,

$$\begin{aligned}\psi = &\sum_{TS[f](\lambda\mu)} \langle n[f_0](\lambda_0\mu_0)T_0S_0\{|(n-1)[f](\lambda\mu)TS\rangle \\ &\times \sum_{M_T\Sigma'} \langle TM_T, \tfrac{1}{2}(M_{T_0}-M_T)|T_0M_{T_0}\rangle\langle S\Sigma', \tfrac{1}{2}(\Sigma_0-\Sigma')|S_0\Sigma_0\rangle \\ &\times \sum_{\varepsilon\Lambda\nu} \langle(\lambda\mu)\varepsilon\Lambda\nu; (20)\varepsilon_2\Lambda_2\nu_2|(\lambda_0\mu_0)\varepsilon_{0\mathrm{H}}\Lambda_{0\mathrm{H}}\nu_{0\mathrm{H}}\rangle \\ &\times N_0\int \mathrm{d}\Omega\, D^{J_0}_{M_{J_0}K_{J_0}}(\Omega)\varphi_\Omega((n-1)TS[f](\lambda\mu)\varepsilon\Lambda\nu, M_T\Sigma')\varphi_\Omega((20)\varepsilon_2\Lambda_2\nu_2, \\ &\qquad (M_{T_0}-M_T)(\Sigma_0-\Sigma')).\end{aligned} \tag{27}$$

The intrinsic space function for the nth particle is written in terms of angular momentum eigenfunctions

$$\varphi_\Omega((20), \varepsilon\Lambda\nu) = \sum_{lk} \alpha((l, k)\varepsilon\Lambda\nu)\psi_\Omega(l, k), \tag{28a}$$

in which k is the z-component of l in the rotated system. Explicitly, using the Elliott and Harvey phase conventions,

$$\begin{aligned}&\varphi((20), 400) = \frac{1}{\sqrt{3}}\psi(0, 0)+\sqrt{\tfrac{2}{3}}\psi(2, 0), \\ &\varphi((20), 1\tfrac{1}{2}1) = \frac{1}{\sqrt{2}}\{\psi(2, -1)-\psi(2, 1)\}, \quad \varphi((20), 1\tfrac{1}{2}-1) = \frac{i}{\sqrt{2}}\{\psi(2, -1)+\psi(2, 1)\}, \\ &\varphi((20), -21\pm 2) = \frac{1}{2\sqrt{3}}\{2\psi(0, 0)-\sqrt{2}\psi(2, 0)\pm\sqrt{3}[\psi(2, -2)+\psi(2, 2)]\}, \\ &\varphi((20), -210) = \frac{i}{\sqrt{2}}\{\psi(2, -2)-\psi(2, 2)\}.\end{aligned} \tag{28b}$$

The intrinsic space functions $\varphi((\lambda\mu)\varepsilon\Lambda\nu)$ with $\varepsilon < \varepsilon_{\mathrm{H}}$ which arise through the parentage expansion of eq. (27) can be expressed in terms of the function of highest weight by the technique of Elliott and Harvey [2]). A function with $\varepsilon < \varepsilon_{\mathrm{H}}$ is first expressed in terms of SU$_3$ step-down operators acting on the function of highest weight (eq. (A.1) of the appendix). The step-down operators when acting on a function of highest weight can then be replaced by functions of the angular momentum operators L_+, L_- and L_0. Thus [2]),

$$\varphi((\lambda\mu), \varepsilon\Lambda\nu) = F((\lambda\mu), \varepsilon\Lambda\nu, L_+, L_-, L_0)\varphi((\lambda\mu))_{\mathrm{H}}. \tag{29}$$

Expanding $\varphi((\lambda\mu))_{\mathrm{H}}$ in terms of angular momentum functions, as in eq. (26), and using the matrix elements of L_+, L_- and L_0 in the angular momentum scheme, the intrinsic states of the $(n-1)$-particle daughter can be expressed as

$$\varphi_\Omega((\lambda\mu), \varepsilon\Lambda\nu) = \sum_{KL\gamma} a(LK) f(\lambda\mu, \varepsilon\Lambda\nu, LK\gamma)\psi_\Omega((\lambda\mu)K; L(K+\gamma)), \tag{30}$$

where the label K in the angular momentum eigenfunction ψ_Ω plays the role of the band quantum number, while $(K+\gamma)$ gives the z-component of L in the rotated coordinate system. As a specific simple example, consider $\varphi((\lambda\mu), \varepsilon_{\mathrm{H}}-3, \Lambda_{\mathrm{H}}+\frac{1}{2}, \nu_{\mathrm{H}}+1)$ which is equal to

$$\frac{1}{\sqrt{\lambda}} A_{xz}\varphi(\lambda\mu)_{\mathrm{H}} = \frac{-i}{\sqrt{\lambda}} i(A_{xz}-A_{zx})\varphi(\lambda\mu)_{\mathrm{H}} = \frac{1}{2\sqrt{\lambda}}(L_+ - L_-)$$

so that in this case

$$F((\lambda\mu), \varepsilon_{\mathrm{H}}-3, \Lambda_{\mathrm{H}}+\tfrac{1}{2}, \nu_{\mathrm{H}}+1; L_+, L_-, L_0) = \frac{1}{2\sqrt{\lambda}}(L_+ - L_-), \tag{31a}$$

$$f(\lambda\mu, \varepsilon_{\mathrm{H}}-3, \Lambda_{\mathrm{H}}+\tfrac{1}{2}, \nu_{\mathrm{H}}+1, KL, \gamma = \pm 1) = \pm\frac{1}{2\sqrt{\lambda}}[(L\mp K)(L\pm K+1)]^{\frac{1}{2}}. \tag{31b}$$

By transforming the orbital and spin angular momentum functions ψ_Ω back to the space-fixed reference frame (e.g. $\psi_\Omega(L, K+\gamma) = \sum_M D^L_{M(K+\gamma)}(\Omega)^*\psi(L, M)$), the integral of eq. (27) becomes

$$\begin{aligned}
&\int \mathrm{d}\Omega\, D^{J_0}_{M_{J_0}K_{J_0}}(\Omega)\varphi_\Omega((n-1)TS[f](\lambda\mu)\varepsilon\Lambda\nu, M_T\Sigma')\varphi_\Omega((20)\varepsilon_2\Lambda_2\nu_2(M_{T_0}-M_T)(\Sigma_0-\Sigma'))\\
&\quad= \frac{8\pi^2}{2J_0+1}\sum_{LK\gamma}\sum_{lk}\sum_{Jj} a(LK)\alpha((lk), \varepsilon_2\Lambda_2\nu_2) f(\lambda\mu, \varepsilon\Lambda\nu, KL\gamma)\\
&\quad\times\langle L(K+\gamma)S\Sigma'|JK_J\rangle\langle lk\tfrac{1}{2}(\Sigma_0-\Sigma')|j(K_{J_0}-K_J)\rangle\langle JK_J\, j(K_{J_0}-K_J)|J_0K_{J_0}\rangle\\
&\quad\times\sum_{M_J}\langle JM_J\, jm|J_0M_{J_0}\rangle\Psi([f](\lambda\mu)K\Sigma'; LSJM_J; TM_T)\psi(l\tfrac{1}{2}jm; tm_t),
\end{aligned}\tag{32}$$

in which Ψ is an L–S coupled wave function with band quantum number K, and ψ is the single-particle angular momentum function for the nth particle, in a state of definite l and j. With this value of the integral the wave function of eq. (27) is now in a form from which an overlap integral can be calculated. For this purpose the $(n-1)$-particle daughter wave function, which is also assumed to be of the form of eq. (24), is expanded out in L–S coupled wave functions

$$\begin{aligned}
&N\int \mathrm{d}\Omega\, D^J_{M_JK_J}(\Omega)\varphi_\Omega((n-1)TS[f](\lambda\mu)_{\mathrm{H}}, M_T\Sigma)\\
&\quad= N\int \mathrm{d}\Omega\, D^J_{M_JK_J}(\Omega)\sum_{LK} a(LK)D^L_{MK}(\Omega)^*D^S_{M_S\Sigma}(\Omega)^*\psi([f](\lambda\mu)K\Sigma; LM, SM_S, TM_T),\\
&\quad= \sum_L \frac{a(LK)\langle LKS\Sigma|JK_J\rangle}{[\sum_{L'}|a(L'K)\langle L'KS\Sigma|JK_J\rangle|^2]^{\frac{1}{2}}}\psi([f](\lambda\mu)K\Sigma; LSJM_J, TM_T).
\end{aligned}\tag{33}$$

The computation of the overlap integrals is somewhat complicated by the fact that wave functions of the form $\psi([f](\lambda\mu)K\Sigma; LSJM_J)$ with different band quantum numbers K are not orthogonal to each other. Their overlap is most conveniently expressed in terms of the integrals $A(KLK')$ evaluated by Elliott and Harvey [2])

$$A(KLK') = a(LK)a(LK')\langle\psi(K'; LM)|\psi(K; LM)\rangle. \tag{34}$$

We are finally in a position to evaluate the overlap integrals $\mathscr{I}(lj)$ from which the single-particle spectroscopic factors [20]) can be calculated. In particular

$$\mathscr{I}(lj) = \langle nT_0S_0[f_0](\lambda_0\mu_0)K_0\Sigma_0, J_0M_{J_0}M_{T_0} \\ \times|[(n-1)TS[f](\lambda\mu)K\Sigma;\ J;\ tslj]J_0M_{J_0}, T_0M_{T_0}\rangle \tag{35}$$

becomes

$$\begin{aligned}\mathscr{I}(lj) = {} & \langle n[f_0](\lambda_0\mu_0)T_0S_0\{|(n-1)[f](\lambda\mu)TS\rangle \sum_{\varepsilon\Lambda\nu'\Sigma'}\langle S\Sigma'\tfrac{1}{2}(\Sigma_0-\Sigma')|S_0\Sigma_0\rangle \\ & \times\langle(\lambda\mu)\varepsilon\Lambda\nu; (20)\varepsilon_2\Lambda_2\nu_2|(\lambda_0\mu_0)\varepsilon_{0\mathrm{H}}\Lambda_{0\mathrm{H}}\nu_{0\mathrm{H}}\rangle \sum_{k\gamma L}\alpha((lk); \varepsilon_2\Lambda_2\nu_2)A(K_0-k-\gamma, LK) \\ & \times f(\lambda\mu, \varepsilon\Lambda\nu, (K_0-k-\gamma)L, \gamma)\langle lk\tfrac{1}{2}(\Sigma_0-\Sigma'|)|j(\Sigma_0-\Sigma'+k)\rangle \\ & \times\langle J(K_0-k+\Sigma')j(\Sigma_0-\Sigma'+k)|J_0(K_0+\Sigma_0)\rangle \\ & \times\frac{\langle L(K_0-k)S\Sigma'|J(K_0-k+\Sigma')\rangle}{[\sum_{L'}|a(L'K)\langle L'KS\Sigma|JK_J\rangle|^2]^{\frac{1}{2}}}\frac{\langle LKS\Sigma|J(K+\Sigma)\rangle}{[\sum_{L_0}|a(L_0K_0)\langle L_0K_0S_0\Sigma_0|J_0K_{J_0}\rangle|^2]^{\frac{1}{2}}}.\end{aligned} \tag{36}$$

In terms of coefficients $\beta(lk; \varepsilon_2\Lambda_2\nu_2; L)$ defined as

$$\begin{aligned}\beta(lk; \varepsilon_2\Lambda_2\nu_2; L) \equiv {} & \sum_{\Sigma'}\alpha((lk), \varepsilon_2\Lambda_2\nu_2) \\ & \times\langle lk\tfrac{1}{2}(\Sigma_0-\Sigma')|j(\Sigma_0-\Sigma'+k)\rangle\langle J(K_0-k+\Sigma')_j(\Sigma_0-\Sigma'+k)|J_0(K_0+\Sigma_0)\rangle \\ & \times\langle n[f_0](\lambda_0\mu_0)T_0S_0\{|(n-1[f](\lambda\mu)TS\rangle\langle S\Sigma'\tfrac{1}{2}(\Sigma_0-\Sigma')|S_0\Sigma_0\rangle \\ & \times\frac{\langle L(K_0-k)S\Sigma'|J(K_0-k+\Sigma')\rangle}{[\sum_{L_0}|a(L_0K_0)\langle L_0K_0S_0\Sigma_0|J_0K_{J_0}\rangle|^2]^{\frac{1}{2}}}\frac{\langle LKS\Sigma|J(K+\Sigma)\rangle}{[\sum_{L'}|a(L'K)\langle L'KS\Sigma|JK_J\rangle|^2]^{\frac{1}{2}}},\end{aligned} \tag{37}$$

the overlap integrals can be written

$$\begin{aligned}\mathscr{I}(lj) = {} & \sum_{k\gamma L}\sum_{\substack{\varepsilon\Lambda\nu \\ (\varepsilon_2\Lambda_2\nu_2)}}\beta(lk; \varepsilon_2\Lambda_2\nu_2; L)\langle(\lambda\mu)\varepsilon\Lambda\nu; (20)\varepsilon_2\Lambda_2\nu_2|(\lambda_0\mu_0)\varepsilon_{0\mathrm{H}}\Lambda_{0\mathrm{H}}\nu_{0\mathrm{H}}\rangle \\ & \times f(\lambda\mu, \varepsilon\Lambda\nu, L(K_0-k-\gamma), \gamma)A(K_0-k-\gamma, LK).\end{aligned} \tag{38}$$

The summations have been carried out explicitly for the six possible representations $(\lambda_0\mu_0)$ of the s–d shell. The results are presented in table 7. Since the n-particle parent and $(n-1)$-particle daughter wave functions are in general linear combinations of wave functions of the form of eq. (24), the overlap integrals involving these states and an nth particle of definite l and j will in general involve linear combinations of the $\mathscr{I}(lj)$ given in table 7.

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Appendix

A1. MATRIX ELEMENTS OF THE INFINITESIMAL OPERATORS

The matrix elements of the infinitesimal operators follow at once from the explicit construction of the SU_3 basis functions given by Elliott and Harvey. A function with arbitrary $\varepsilon\Lambda\nu$ is given in terms of stepdown operators acting on the function of highest weight by (see refs. [2, 21]),

$$\varphi((\lambda\mu)\varepsilon\Lambda\nu) = N(\lambda\mu, pqr)A^r_{yx}O^q_{yz}A^p_{xz}\varphi((\lambda\mu)\varepsilon_{\rm H}\Lambda_{\rm H}\nu_{\rm H}), \tag{A.1}$$

with

$$\varepsilon = \varepsilon_{\rm H}-3p-3q, \quad \Lambda = \Lambda_{\rm H}+\tfrac{1}{2}p-\tfrac{1}{2}q, \quad \tfrac{1}{2}\nu = \Lambda-r, \tag{A.2}$$

where $O_{yz} = A_{yx}A_{xz}-A_{yz}(A_{xx}-A_{yy}+1)$ and where the normalization constant has the value [2])

$$N(\lambda\mu, p, q, r) = \left[\frac{(\lambda-p)!(\mu-q)!(\lambda+\mu+1-q)!(\mu+p-q-r)!(\mu+p-q+1)}{\lambda!\mu!(\lambda+\mu+1)!(\mu+p+1)!p!q!r!}\right]^{\frac{1}{2}}. \tag{A.3}$$

To determine the matrix element of A_{xz}, for example, it is necessary only to act with A_{xz} on a function of arbitrary $\varepsilon\Lambda\nu$ and commute the operator A_{xz} through to the right in eq. (A.1). Using the commutator properties

$$[A_{xz}, O_{yz}] = 0, \qquad [A_{xz}, A^r_{yx}] = -rA^{r-1}_{yx}A_{yz},$$

and recplacing $A_{yz}\varphi(p, q, r = 0)$ by

$$\frac{(A_{yx}A_{xz}-O_{yz})}{(\nu(p, q, r = 0)+1)}\varphi(p, q, r = 0),$$

we get

$$A_{xz}\varphi(p, q, r) = \frac{(\Lambda+\frac{1}{2}\nu+1)}{(2\Lambda+1)}\frac{N(p, q, r)}{N(p+1, q, r)}\varphi(p+1, q, r)+\frac{(\Lambda-\frac{1}{2}\nu)}{(2\Lambda+1)}\frac{N(p, q, r)}{N(p, q+1, r-1)} \times\varphi(p, q+1, r-1), \tag{A.4}$$

which is another form of the first of eqs. (4) of the text.

A2. THE ADJOINT IRREDUCIBLE REPRESENTATION

By expanding the operator O_{yz} in eq. (A.1), a basis function of $(\lambda\mu)$ symmetry and arbitrary $\varepsilon\Lambda\nu$ can also be expressed in the following form:

$$\varphi((\lambda\mu)\varepsilon\Lambda\nu) = N(\lambda\mu pqr)\sum_{a=0}^{q} \frac{q!(\mu+p+1)!}{a!(q-a)!(\mu+p+1-q+a)!}(-1)^{q-a}A_{yx}^{r+a}A_{yz}^{q-a}A_{xz}^{p+a} \times \varphi((\lambda\mu)\varepsilon_{\rm H}\Lambda_{\rm H}\nu_{\rm H}). \quad \text{(A.5)}$$

By using step-up operators acting on the function of lowest weight, to be denoted by the subscript L, a similar expression can be derived. In particular, for functions of $(\mu\lambda)$ symmetry,

$$\varphi((\mu\lambda)-\varepsilon, \Lambda, -\nu) = N(\lambda\mu pqr)\sum_{a=0}^{q} \frac{q!(\mu+p+1)!}{a!(q-a)!(\mu+p+1-q+a)!}A_{xy}^{r+a}A_{zy}^{q-a}A_{zx}^{p+a} \times \varphi((\mu\lambda)\varepsilon_{\rm L}\Lambda_{\rm L}\nu_{\rm L}), \quad \text{(A.6)}$$

where the magnitudes of ε, Λ and ν are the same in eqs. (A.5) and (A.6) for identical p, q, and r. By expressing the infinitesimal operators of the group in terms of harmonic oscillator creation and annihilation operators † and by expressing the function of highest weight in terms of creation operators of type a_z^+ and a_x^+ acting on a "closed shell" state of $3N$ oscillator quanta coupled to $(\lambda\mu) = (00)$ (Moshinsky [11]), the relation (A.5) can be expressed as a polynomial in the creation operators [11]) acting on the closed shell state $|c\rangle$

$$\varphi((\lambda\mu)\varepsilon\Lambda\nu) = \mathscr{P}(a_z^+, a_x^+, a_y^+)|c\rangle. \quad \text{(A.7)}$$

Similarly, the function of lowest weight can be expressed in terms of annihilation operators of type a_z and a_x acting on a closed shell state of $3N$ oscillator quanta coupled to $(\lambda\mu) = (00)$, so that the relation (A.6) can be expressed as a polynomial in the oscillator quanta annihilation operators acting on a closed shell state $|c'\rangle$

$$\varphi((\mu\lambda)-\varepsilon, \Lambda, -\nu) = (-1)^{p+r}\mathscr{P}(a_z, a_x, a_y)|c'\rangle, \quad \text{(A.8)}$$

where the polynomial functions $\mathscr{P}$ of eqs. (A.7) and (A.8) are identical. Comparing the two relations we see that the basis functions of the adjoint representation $(\mu\lambda)$ are related to those of the irreducible representation $(\lambda\mu)$ by the relation

$$\varphi((\lambda\mu)\varepsilon\Lambda\nu)^* = (-1)^{p+r}\varphi((\mu\lambda)-\varepsilon, \Lambda, -\nu), \quad \text{(A.9a)}$$

where the phase factor can be expressed also in terms of the quantum numbers ε and ν through eq. (A.2)

$$\varphi((\lambda\mu)\varepsilon\Lambda\nu)^* = (-1)^{\frac{1}{3}(\lambda-\mu)+\frac{1}{2}\nu-\frac{1}{6}\varepsilon}\varphi((\mu\lambda)-\varepsilon, \Lambda, -\nu). \quad \text{(A.9b)}$$

† See refs. [1,2]) or ref. [11]): $A_{ij} = \Sigma_k a_i^+(k)a_j(k)$, $i, j = x, y, z$, k = particle index = $1, \ldots, n$.

A3. A SYMMETRY PROPERTY OF THE SU_3 WIGNER COEFFICIENTS, Eq. (15)

Using eq. (A.9) and the techniques employed by de Swart [10]) to derive the symmetry properties of the SU_3 Wigner coefficients, we obtain the relation

$$\begin{aligned}&\langle(\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1;(\lambda_2\mu_2)\varepsilon_2\Lambda_2\nu_2|(\lambda_3\mu_3)\varepsilon_3\Lambda_3\nu_3\rangle\\ &= c(\lambda_i\mu_i)(-1)^{\frac{1}{3}(\lambda_2-\mu_2)+\frac{1}{2}\nu_2-\frac{1}{6}\varepsilon_2}\left[\frac{\dim(\lambda_3\mu_3)}{\dim(\lambda_1\mu_1)}\right]^{\frac{1}{2}}\\ &\times\langle(\lambda_3\mu_3)\varepsilon_3\Lambda_3\nu_3;(\mu_2\lambda_2)-\varepsilon_2,\Lambda_2-\nu_2|(\lambda_1\mu_1)\varepsilon_1\Lambda_1\nu_1\rangle,\end{aligned} \tag{A.10}$$

where $c(\lambda_i\mu_i)$ is a phase factor which is independent of the quantum numbers ε_i, Λ_i, ν_i; $|c| = 1$. The phase factor $c(\lambda_i\,\mu_i)$ can be determined by setting both $\varepsilon_1 = \varepsilon_{1\mathrm{H}}$, $\nu_1 = \nu_{1\mathrm{H}}$, and $\varepsilon_3 = \varepsilon_{3\mathrm{H}}$, $\nu_3 = \nu_{3\mathrm{H}}$, and letting Λ_2 have its largest possible value. For these values of the quantum numbers our phase convention for the SU_3 Wigner coefficients implies that *both* SU_3 Wigner coefficients in eq. (A.10) are positive. Hence

$$c(\lambda_i\mu_i) = (-1)^{\frac{1}{3}(\mu_1+\mu_2-\mu_3-\lambda_1-\lambda_2+\lambda_3)}. \tag{A.11}$$

The symmetry relation for the double-barred SU_3 Wigner coefficient, eq. (15) of the text, is obtained by combining (A.10) with the analogous symmetry relation for the ordinary (SU_3) Wigner coefficient.

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TABLE 2

SU_3-Wigner coefficients $\langle(\lambda\mu)\varepsilon_H\Lambda_H;\ (\lambda_2\mu_2)\varepsilon_2\Lambda_2||(\lambda'\mu')\varepsilon'_H\Lambda'_H\rangle$
with $\varepsilon_H = 2\lambda+\mu$, $\Lambda_H = \frac{1}{2}\mu$ and $\varepsilon'_H = 2\lambda'+\mu'$, $\Lambda'_H = \frac{1}{2}\mu'$

$(\lambda_2\mu_2)$	$(\lambda'\mu')$	$\langle(\lambda\mu)\varepsilon_H\Lambda_H;\ (\lambda_2\mu_2)\varepsilon_2\Lambda_2\| \times \|(\lambda'\mu')\varepsilon'_H\Lambda'_H\rangle$	$(\lambda_2\mu_2)$	$(\lambda'\mu')$	$\langle\lambda\mu)\varepsilon_H\Lambda_H;\ (\lambda_2\mu_2)\varepsilon_2\Lambda_2\| \times \|(\lambda'\mu')\varepsilon'_H\Lambda'_H\rangle$
(20)	$(\lambda+2, \mu)$	1	(02)	$(\lambda-2, \mu)$	$\left[\frac{(\lambda-1)(\lambda+\mu)}{(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
	$(\lambda, \mu+1)$	$\left[\frac{\lambda}{\lambda+2}\right]^{\frac{1}{2}}$		$(\lambda, \mu-1)$	$\left[\frac{\lambda(\lambda+\mu+1)}{(\lambda+2)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
	$(\lambda+1, \mu-1)$	$\left[\frac{\lambda+\mu+1}{\lambda+\mu+3}\right]^{\frac{1}{2}}$		$(\lambda-1, \mu+1)$	$\left[\frac{\lambda(\lambda+\mu+1)}{(\lambda+1)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$
	$(\lambda-2, \mu+2)$	$\left[\frac{\lambda-1}{\lambda+1}\right]^{\frac{1}{2}}$		$(\lambda+2, \mu-2)$	1
	$(\lambda-1, \mu)$	$\left[\frac{\lambda(\lambda+\mu+1)}{(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$		$(\lambda+1, \mu)$	1
	$(\lambda, \mu-2)$	$\left[\frac{\lambda+\mu}{\lambda+\mu+2}\right]^{\frac{1}{2}}$		$(\lambda, \mu+2)$	1
(40)	$(\lambda+4, \mu)$	1	(40)	$(\lambda, \mu-1)$	$\left[\frac{\lambda(\lambda+\mu)(\lambda+\mu+1)}{(\lambda+2)(\lambda+\mu+2)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$
	$(\lambda+2, \mu+1)$	$\left[\frac{\lambda}{\lambda+4}\right]^{\frac{1}{2}}$		$(\lambda+1, \mu-3)$	$\left[\frac{(\lambda+\mu)(\lambda+\mu-1)}{(\lambda+\mu+2)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$
	$(\lambda+3, \mu-1)$	$\left[\frac{\lambda+\mu+1}{\lambda+\mu+5}\right]^{\frac{1}{2}}$		$(\lambda-4, \mu+4)$	$\left[\frac{\lambda-3}{\lambda+1}\right]^{\frac{1}{2}}$
	$(\lambda, \mu+2)$	$\left[\frac{\lambda(\lambda-1)}{(\lambda+2)(\lambda+3)}\right]^{\frac{1}{2}}$		$(\lambda-3, \mu+2)$	$\left[\frac{(\lambda-2)(\lambda+\mu+1)}{(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
	$(\lambda+1, \mu)$	$\left[\frac{\lambda(\lambda+\mu+1)}{(\lambda+3)(\lambda+\mu+4)}\right]^{\frac{1}{2}}$		$(\lambda-2, \mu)$	$\left[\frac{(\lambda-1)(\lambda+\mu)}{(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
	$(\lambda+2, \mu-2)$	$\left[\frac{(\lambda+\mu)(\lambda+\mu+1)}{(\lambda+\mu+3)(\lambda+\mu+4)}\right]^{\frac{1}{2}}$		$(\lambda-1, \mu-2)$	$\left[\frac{\lambda(\lambda+\mu-1)}{(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
	$(\lambda-2, \mu+3)$	$\left[\frac{(\lambda-1)(\lambda-2)}{(\lambda+1)(\lambda+2)}\right]^{\frac{1}{2}}$		$(\lambda, \mu-4)$	$\left[\frac{(\lambda+\mu-2)}{(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
	$(\lambda-1, \mu+1)$	$\left[\frac{\lambda(\lambda-1)(\lambda+\mu+1)}{(\lambda+2)(\lambda+1)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$			

where $\varepsilon_2 = \varepsilon'_H - \varepsilon_H = (2\lambda_2+\mu_2-3\sigma)$
$\Lambda_2 = \frac{1}{2}\sigma$ if $\mu_2 = 0$, $\Lambda_2 = \frac{1}{2}\mu_2 - \frac{1}{2}\sigma$ if $\lambda_2 = 0$.

TABLE 2 (continued)

$\langle(\lambda\mu)\varepsilon_{H}\Lambda_{H};\ (\lambda_2\mu_2)\varepsilon_2\Lambda_2\|(\lambda'\mu')\varepsilon'_{H}\Lambda'_{H}\rangle$

$(\lambda_2\mu_2) = (21)$

$\varepsilon_2\Lambda_2$	$(\lambda'\mu')_\rho$	$\langle(\lambda\mu)\varepsilon_{H}\Lambda_{H};\ (21)\varepsilon_2\Lambda_2\|(\lambda'\mu')\varepsilon'_{H}\Lambda'_{H}\rangle$	$\varepsilon_2\Lambda_2$	$(\lambda'\mu')_\rho$	$\langle(\lambda\mu)\varepsilon_{H}\Lambda_{H};\ (21)\varepsilon_2\Lambda_2\|(\lambda'\mu')\varepsilon'_{H}\Lambda'_{H}\rangle$
$5\ \frac{1}{2}$	$(\lambda+3,\mu-1)$	1	$5\ \frac{1}{2}$	$(\lambda+2,\mu+1)$	1
$2\ 1$	$(\lambda,\mu+2)$	$\left[\frac{\lambda}{\lambda+2}\right]^{\frac{1}{2}}$	$2\ 1$	$(\lambda+2,\mu-2)$	$\left[\frac{\lambda+\mu+1}{\lambda+\mu+3}\right]^{\frac{1}{2}}$
$-1\ \frac{3}{2}$	$(\lambda-2,\mu+3)$	$\left[\frac{\lambda-1}{\lambda+1}\right]^{\frac{1}{2}}$	$-1\ \frac{3}{2}$	$(\lambda+1,\mu-3)$	$\left[\frac{\lambda+\mu}{\lambda+\mu+2}\right]^{\frac{1}{2}}$
$2\ 0$	$(\lambda+1,\mu)_1$	0	$2\ 0$	$(\lambda+1,\mu)_2$	$-\left[\frac{\lambda(\lambda+3)+\mu(\lambda+1)}{(\lambda+3)(\lambda+\mu+4)}\right]^{\frac{1}{2}}$
$2\ 1$		$\left[\frac{\lambda(\lambda+\mu+1)}{\lambda(\lambda+3)+\mu(\lambda+1)}\right]^{\frac{1}{2}}$	$2\ 1$		$\left[\frac{2\mu(\mu+2)}{(\lambda+3)(\lambda+\mu+4)[\lambda(\lambda+3)+\mu(\lambda+1)]}\right]^{\frac{1}{2}}$
$-1\ \frac{3}{2}$	$(\lambda,\mu-1)_1$	$\left[\frac{3\lambda(\lambda+\mu)(\lambda+\mu+1)}{(\lambda+\mu+2)[3\lambda(\lambda+\mu+1)+2(\mu-1)]}\right]^{\frac{1}{2}}$	$-1\ \frac{3}{2}$	$(\lambda,\mu-1)_2$	$\left[\frac{8(\mu-1)(\mu+2)(\lambda+\mu+1)}{3(\lambda+2)(\lambda+\mu+2)(\lambda+\mu+3)[3\lambda(\lambda+\mu+1)+2(\mu-1)]}\right]^{\frac{1}{2}}$
$-1\ \frac{1}{2}$		0	$-1\ \frac{1}{2}$		$-\left[\frac{(\lambda+\mu+1)[3\lambda(\lambda+\mu+1)+2(\mu-1)]}{3(\lambda+2)(\lambda+\mu+2)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$
$-1\ \frac{3}{2}$	$(\lambda-1,\mu+1)_1$	$\left[\frac{3\lambda(\lambda-1)(\lambda+\mu+1)}{(\lambda+1)[3\lambda(\lambda+\mu+1)-2(\mu+3)]}\right]^{\frac{1}{2}}$	$-1\ \frac{3}{2}$	$(\lambda-1,\mu+1)_2$	$\left[\frac{8\lambda\mu(\mu+3)}{3(\lambda+1)(\lambda+2)(\lambda+\mu+3)[3\lambda(\lambda+\mu+1)-2(\mu+3)]}\right]^{\frac{1}{2}}$
$-1\ \frac{1}{2}$		0	$-1\ \frac{1}{2}$		$-\left[\frac{\lambda[3\lambda(\lambda+\mu+1)-2(\mu+3)]}{3(\lambda+1)(\lambda+2)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$
$-4\ 1$	$(\lambda-3,\mu+2)$	$\left[\frac{(\lambda-2)(\lambda+\mu+1)}{(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$-4\ 1$	$(\lambda-1,\mu-2)$	$\left[\frac{\lambda(\lambda+\mu-1)}{(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
$-4\ 1$	$(\lambda-2,\mu)$	$\left[\frac{(\lambda-1)(\lambda+\mu)}{(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$			

Expressions for SU_3 Wigner coefficients with $\varepsilon' = \varepsilon'_{H}$ but $\varepsilon < \varepsilon_{H}$ follow from eqs. (13) and (16).

TABLE 3

$\langle(\lambda\mu)\varepsilon_1\Lambda_1;\ (20)\varepsilon_2\Lambda_2 \| (\lambda'\mu')\varepsilon\Lambda\rangle$

with $\varepsilon = 2\lambda'+\mu'-3p-3q,\quad \Lambda = \frac{1}{2}\mu'+\frac{1}{2}p-\frac{1}{2}q$

$\varepsilon_2\Lambda_2$ Λ_1	$(\lambda'\mu') = (\lambda+2, \mu)$	$(\lambda'\mu') = \lambda, \mu+1)$
$\varepsilon_2\Lambda_2 = 40$ $\Lambda_1 = \Lambda$	$\left[\frac{(\lambda+1-p)(\lambda+2-p)(\lambda+\mu+2-q)(\lambda+\mu+3-q)}{(\lambda+1)(\lambda+2)(\lambda+\mu+2)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{2(p+1)(\lambda-p)(\mu+1-q)(\lambda+\mu+2-q)}{\lambda(\lambda+2)(\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 1\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$\left[\frac{2p(\lambda+2-p)(\mu+1+p)(\lambda+\mu+2-q)(\lambda+\mu+3-q)}{(\lambda+1)(\lambda+2)(\lambda+\mu+2)(\lambda+\mu+3)(2\Lambda+1)}\right]^{\frac{1}{2}}$	$(\lambda-2p)\left[\frac{(\mu+2+p)(\mu+1-q)(\lambda+\mu+2-q)}{\lambda(\lambda+2)(\mu+1)(\lambda+\mu+2)(2\Lambda+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 1\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$-\left[\frac{2(\lambda+1-p)(\lambda+2-p)q(\mu+1-q)(\lambda+\mu+3-q)}{(\lambda+1)(\lambda+2)(\lambda+\mu+2)(\lambda+\mu+3)(2\Lambda+1)}\right]^{\frac{1}{2}}$	$(\lambda+2\mu+4-2q)\left[\frac{(p+1)(\lambda-p)q}{\lambda(\lambda+2)(\mu+1)(\lambda+\mu+2)(2\Lambda+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -21$ $\Lambda_1 = \Lambda-1$	$\left[\frac{p(p-1)(\mu+p)(\mu+1+p)(\lambda+\mu+2-q)(\lambda+\mu+3-q)}{(\lambda+1)(\lambda+2)(\lambda+\mu+2)(\lambda+\mu+3)2\Lambda(2\Lambda+1)}\right]^{\frac{1}{2}}$	$\left[\frac{2p(\lambda+1-p)(\mu+1+p)(\mu+2+p)(\mu+1-q)(\lambda+\mu+2-q)}{\lambda(\lambda+2)(\mu+1)(\lambda+\mu+2)2\Lambda(2\Lambda+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -21$ $\Lambda_1 = \Lambda+1$	$\left[\frac{(\lambda+1-p)(\lambda+2-p)q(q-1)(\mu+1-q)(\mu+2-q)}{(\lambda+1)(\lambda+2)(\lambda+\mu+2)(\lambda+\mu+3)(2\Lambda+1)(2\Lambda+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{2(p+1)(\lambda-p)q(q-1)(\mu+2-q)(\lambda+\mu+3-q)}{\lambda(\lambda+2)(\mu+1)(\lambda+\mu+2)(2\Lambda+1)(2\Lambda+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -21$ $\Lambda_1 = \Lambda$	$-\left[\frac{p(\lambda+2-p)(\mu+1+p)q(\mu+1-q)(\lambda+\mu+3-q)}{(\lambda+1)(\lambda+2)(\lambda+\mu+2)(\lambda+\mu+3)2\Lambda(\Lambda+1)}\right]^{\frac{1}{2}}$	$-\frac{\{\lambda(\mu+1)-(\lambda+2\mu+4)p-\lambda q+2pq\}[q(\mu+2+p)]^{\frac{1}{2}}}{[\lambda(\lambda+2)(\mu+1)(\lambda+\mu+2)2\Lambda(2\Lambda+2])^{\frac{1}{2}}}$

TABLE 3 (continued)

$\langle(\lambda\mu)\varepsilon_1\Lambda_1;\ (20)\varepsilon_2\Lambda_2\|(\lambda'\mu')\varepsilon\Lambda\rangle$

with $\varepsilon = 2\lambda'+\mu'-3p-3q,\quad \Lambda = \frac{1}{2}\mu'+\frac{1}{2}p-\frac{1}{2}q$

$\varepsilon_2\Lambda_2$ Λ_1	$(\lambda'\mu') = (\lambda+1,\mu-1)$	$(\lambda'\mu') = (\lambda-2,\mu+2)$
$\varepsilon_2\Lambda_2 = 40$ $\Lambda_1 = \Lambda$	$\left[\frac{2(\lambda+1-p)(\mu+1+p)(q+1)(\lambda+\mu+1-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(p+1)(p+2)(\mu+1-q)(\mu+2-q)}{\lambda(\lambda+1)(\mu+1)(\mu+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 1\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$-(\lambda-\mu+1-2p)\left[\frac{p(q+1)(\lambda+\mu+1-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+3)(2\Lambda+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{2(p+1)(\lambda-1-p)(\mu+3+p)(\mu+1-q)(\mu+2-q)}{\lambda(\lambda+1)(\mu+1)(\mu+2)(2\Lambda+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 1\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$(\lambda+\mu+1-2q)\left[\frac{(\lambda+1-p)(\mu+1+p)(\mu-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+3)(2\Lambda+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{2(p+1)(p+2)q(\mu+2-q)(\lambda+\mu+2-q)}{\lambda(\lambda+1)(\mu+1)(\mu+2)(2\Lambda+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -21$ $\Lambda_1 = \Lambda-1$	$-\left[\frac{2p(p-1)(\mu+p)(\lambda+2-p)(q+1)(\lambda+\mu+1-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+3)2\Lambda(2\Lambda+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda-p)(\lambda-1-p)(\mu+2+p)(\mu+3+p)(\mu+1-q)(\mu+2-q)}{\lambda(\lambda+1)(\mu+1)(\mu+2)2\Lambda(2\Lambda+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -21$ $\Lambda_1 = \Lambda+1$	$-\left[\frac{2(\lambda+1-p)(\mu+1+p)q(\mu-q)(\mu+1-q)(\lambda+\mu+2-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+3)(2\Lambda+1)(2\Lambda+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(p+1)(p+2)q(q-1)(\lambda+\mu+2-q)(\lambda+\mu+3-q)}{\lambda(\lambda+1)(\mu+1)(\mu+2)(2\Lambda+1)(2\Lambda+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -21$ $\Lambda_1 = \Lambda$	$\frac{\{(\lambda+\mu+1)(\mu+1+p)+q(\lambda-\mu+1)-2pq\}[p(\mu-q)]^{\frac{1}{2}}}{[(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+3)2\Lambda(2\Lambda+2)]^{\frac{1}{2}}}$	$\left[\frac{(p+1)(\lambda-1-p)(\mu+3+p)q(\mu+2-q)(\lambda+\mu+2-q)}{\lambda(\lambda+1)(\mu+1)(\mu+2)2\Lambda(\Lambda+1)}\right]^{\frac{1}{2}}$

TABLE 3 (continued)

$\langle(\lambda\mu)\varepsilon_1\Lambda_1;\ (20)\varepsilon_2\Lambda_2\|(\lambda'\mu')\varepsilon\Lambda\rangle$

with $\varepsilon = 2\lambda'+\mu'-3p-3q,\quad \Lambda = \frac{1}{2}\mu'+\frac{1}{2}p-\frac{1}{2}q$

$\varepsilon_2\Lambda_2$ Λ_1	$(\lambda'\mu') = (\lambda-1,\mu)$	$(\lambda'\mu') = (\lambda,\mu-2)$
$\varepsilon_2\Lambda_2 = 40$ $\Lambda_1 = \Lambda$	$-\left[\frac{2(p+1)(\mu+2+p)(q+1)(\mu-q)}{(\lambda+1)\mu(\mu+2)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(\mu+p)(\mu+1+p)(q+1)(q+2)}{\mu(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 1\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$(\mu+2+2p)\left[\frac{(\lambda-p)(q+1)(\mu-q)}{(\lambda+1)\mu(\mu+2)(\lambda+\mu+2)(2\Lambda+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{2p(\lambda+1-p)(\mu+p)(q+1)(q+2)}{\mu(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)(2\Lambda+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 1\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$(2q-\mu)\left[\frac{(p+1)(\mu+2+p)(\lambda+\mu+1-q)}{(\lambda+1)\mu(\mu+2)(\lambda+\mu+2)(2\Lambda+1)}\right]^{\frac{1}{2}}$	$\left[\frac{2(\mu+p)(\mu+1+p)(q+1)(\mu-1-q)(\lambda+\mu-q)}{\mu(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)(2\Lambda+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -21$ $\Lambda_1 = \Lambda-1$	$-\left[\frac{2p(\lambda-p)(\lambda+1-p)(\mu+1+p)(q+1)(\mu-q)}{(\lambda+1)\mu(\mu+2)(\lambda+\mu+2)2\Lambda(2\Lambda+1)}\right]^{\frac{1}{2}}$	$\left[\frac{p(p-1)(\lambda+1-p)(\lambda+2-p)(q+1)(q+2)}{\mu(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)2\Lambda(2\Lambda+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -21$ $\Lambda_1 = \Lambda+1$	$\left[\frac{2(p+1)(\mu+2+p)q(\mu+1-q)(\lambda+\mu+1-q)(\lambda+\mu+2-q)}{(\lambda+1)\mu(\mu+2)(\lambda+\mu+2)(2\Lambda+1)(2\Lambda+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(\mu+p)(\mu+1+p)(\mu-q)(\mu-1-q)(\lambda+\mu-q)(\lambda+\mu+1-q)}{\mu(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)(2\Lambda+1)(2\Lambda+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -21$ $\Lambda_1 = \Lambda$	$\frac{\{(\mu+2)(\mu-q)+\mu p-2pq\}[(\lambda-p)(\lambda+\mu+1-q)]^{\frac{1}{2}}}{[(\lambda+1)\mu(\mu+2)(\lambda+\mu+2)2\Lambda(2\Lambda+2)]^{\frac{1}{2}}}$	$-\left[\frac{p(\mu+p)(\lambda+1-p)(q+1)(\mu-1-q)(\lambda+\mu-q)}{\mu(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)2\Lambda(\Lambda+1)}\right]^{\frac{1}{2}}$

TABLE 4

$\langle(\lambda\mu)\varepsilon_1\Lambda_1;\quad(11)\varepsilon_2\Lambda_2\|(\lambda'\mu')\varepsilon'\Lambda'\rangle$

with $\varepsilon' = 2\lambda'+\mu'-3p-3q,\qquad \Lambda' = \frac{1}{2}\mu'+\frac{1}{2}p-\frac{1}{2}q$

$\varepsilon_2\Lambda_2$ Λ_1	$(\lambda'\mu') = (\lambda+2,\mu-1)$	$(\lambda'\mu') = (\lambda-2,\mu+1)$
$\varepsilon_2\Lambda_2 = 3\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$\left[\frac{(\lambda+1-p)(\lambda+2-p)(\mu+1+p)(\mu-q)(\lambda+\mu+2-q)}{(\lambda+1)(\lambda+2)(\mu+1)(\lambda+\mu+2)(\mu+p-q)}\right]^{\frac{1}{2}}$	$\left[\frac{(p+1)(p+2)(\mu+p+3)(\lambda+\mu+1-q)(\mu+1-q)}{\lambda(\lambda+1)(\mu+1)(\lambda+\mu+2)(\mu+p-q+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 3\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$-\left[\frac{p(\lambda+2-p)(q+1)(\lambda+\mu+1-q)(\lambda+\mu+2-q)}{(\lambda+1)(\lambda+2)(\mu+1)(\lambda+\mu+2)(\mu+p-q)}\right]^{\frac{1}{2}}$	$\left[\frac{(p+1)(\lambda-1-p)(q+1)(\mu-q)(\mu+1-q)}{\lambda(\lambda+1)(\mu+1)(\lambda+\mu+2)(\mu+p-q+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 00$ $\Lambda_1 = \Lambda$	$\left[\frac{3p(\lambda+2-p)(\mu-q)(\lambda+\mu+2-q)}{2(\lambda+1)(\lambda+2)(\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{3(p+1)(\lambda-1-p)(\mu+1-q)(\lambda+\mu+1-q)}{2\lambda(\lambda+1)(\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda+1$	$-\left[\frac{(\lambda+1-p)(\lambda+2-p)(\mu+1+p)q(\mu-q)(\mu+1-q)}{(\lambda+1)(\lambda+2)(\mu+1)(\lambda+\mu+2)(\mu+p-q)(\mu+p-q+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(p+1)(p+2)(\mu+p+3)q(\lambda+\mu+1-q)(\lambda+\mu+2-q)}{\lambda(\lambda+1)(\mu+1)(\lambda+\mu+2)(\mu+p-q+2)(\mu+p-q+3)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda-1$	$-\left[\frac{p(p-1)(\mu+p)(q+1)(\lambda+\mu+1-q)(\lambda+\mu+2-q)}{(\lambda+1)(\lambda+2)(\mu+1)(\lambda+\mu+2)(\mu+p-q)(\mu+p-q-1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\lambda-p)(\lambda-1-p)(\mu+2+p)(q+1)(\mu-q)(\mu+1-q)}{\lambda(\lambda+1)(\mu+1)(\lambda+\mu+2)(\mu+p-q+2)(\mu+p-q+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda$	$\frac{(\mu+1+p+q)[p(\lambda+2-p)(\mu-q)(\lambda+\mu+2-q)]^{\frac{1}{2}}}{[2(\lambda+1)(\lambda+2)(\mu+1)(\lambda+\mu+2)(\mu+p-q-1)(\mu+p-q+1)]^{\frac{1}{2}}}$	$-\frac{(\mu+3+p+q)[(p+1)(\lambda-1-p)(\mu+1-q)(\lambda+1+\mu-q)]^{\frac{1}{2}}}{[2\lambda(\lambda+1)(\mu+1)(\lambda+\mu+2)(\mu+p-q+1)(\mu+p-q+3)]^{\frac{1}{2}}}$
$\varepsilon_2\Lambda_2 = -3\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$-\left[\frac{p(\lambda+2-p)q(\mu-q)(\mu+1-q)}{(\lambda+1)(\lambda+2)(\mu+1)(\lambda+\mu+2)(\mu+p-q)}\right]^{\frac{1}{2}}$	$\left[\frac{(p+1)(\lambda-1-p)q(\lambda+\mu+1-q)(\lambda+\mu+2-q)}{\lambda(\lambda+1)(\mu+1)(\lambda+\mu+2)(\mu+p-q+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -3\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$\left[\frac{p(p-1)(\mu+p)(\mu-q)(\lambda+\mu+2-q)}{(\lambda+1)(\lambda+2)(\mu+1)(\lambda+\mu+2)(\mu+p-q)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda-p)(\lambda-1-p)(\mu+2+p)(\mu+1-q)(\lambda+\mu+1-q)}{\lambda(\lambda+1)(\mu+1)(\lambda+\mu+2)(\mu+p-q+2)}\right]^{\frac{1}{2}}$

TABLE 4 (continued)

$\langle(\lambda\mu)\varepsilon_1\Lambda_1;\ (11)\varepsilon_2\Lambda_2 \| (\lambda'\mu')\varepsilon\Lambda\rangle$

with $\varepsilon = 2\lambda'+\mu'-3p-3q,\quad \Lambda = \frac{1}{2}\mu'+\frac{1}{2}p-\frac{1}{2}q$

$\varepsilon_2\Lambda_2$ Λ_1	$(\lambda'\mu') = (\lambda+1,\mu+1)$	$(\lambda'\mu') = (\lambda-1,\mu-1)$
$\varepsilon_2\Lambda_2 = 3\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$\left[\frac{(p+1)(\lambda-p)(\lambda+1-p)q(\lambda+\mu+3-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+2)(\lambda+\mu+3)(\mu+p-q+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(p+1)(\mu+p+1)(\mu+p+2)(q+1)(\lambda+\mu-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)(\mu+p-q)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 3\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$\left[\frac{(\lambda+1-p)(\mu+2+p)(\mu+1-q)(\lambda+\mu+2-q)(\lambda+\mu+3-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+2)(\lambda+\mu+3)(\mu+p-q+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\lambda-p)(\mu+1+p)(q+1)(q+2)(\mu-1-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)(\mu+p-q)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 00$ $\Lambda_1=\Lambda$	$\left[\frac{3(\lambda+1-p)(\mu+2+p)q(\lambda+\mu+3-q)}{2(\lambda+1)(\mu+1)(\lambda+\mu+2)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$\left[\frac{3(\lambda-p)(\mu+1+p)(q+1)(\lambda+\mu-q)}{2(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda+1$	$-\left[\frac{q(q-1)(\mu+2-q)(p+1)(\lambda-p)(\lambda+1-p)}{(\lambda+1)(\mu+1)(\lambda+\mu+2)(\lambda+\mu+3)(\mu+p-q+2)(\mu+p-q+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{(p+1)(\mu+1+p)(\mu+2+p)(\mu-q)(\lambda+\mu-q)(\lambda+\mu+1-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)(\mu+p-q)(\mu+p-q+1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda-1$	$\left[\frac{p(\mu+1+p)(\mu+2+p)(\mu+1-q)(\lambda+\mu+2-q)(\lambda+\mu+3-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+2)(\lambda+\mu+3)(\mu+p-q+2)(\mu+p-q+1)}\right]^{\frac{1}{2}}$	$\left[\frac{p(\lambda-p)(\lambda+1-p)(q+1)(q+2)(\mu-q-1)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)(\mu+p-q)(\mu+p-q-1)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda$	$\frac{-(\mu+1-p-q)[(\lambda+1-p)(\mu+2+p)q(\lambda+\mu+3-q)]^{\frac{1}{2}}}{[2(\lambda+1)(\mu+1)(\lambda+\mu+2)(\lambda+\mu+3)(\mu+p-q+1)(\mu+p-q+3)]^{\frac{1}{2}}}$	$\frac{-(\mu-1-p-q)[(\lambda-p)(\mu+1+p)(q+1)(\lambda+\mu-q)]^{\frac{1}{2}}}{[2(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)(\mu+p-q+1)(\mu+p-q-1)]^{\frac{1}{2}}}$
$\varepsilon_2\Lambda_2 = -3\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$-\left[\frac{(\lambda+1-p)(\mu+2+p)q(q-1)(\mu+2-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+2)(\lambda+\mu+3)(\mu+p-q+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda-p)(\mu+1+p)(\mu-q)(\lambda+\mu-q)(\lambda+\mu+1-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)(\mu+p-q)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -3\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$\left[\frac{p(\mu+1+p)(\mu+2+p)q(\lambda+\mu+3-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+2)(\lambda+\mu+3)(\mu+p-q+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{p(\lambda-p)(\lambda+1-p)(q+1)(\lambda+\mu-q)}{(\lambda+1)(\mu+1)(\lambda+\mu+1)(\lambda+\mu+2)(\mu+p-q)}\right]^{\frac{1}{2}}$

TABLE 4 (continued)

$\langle(\lambda\mu)\varepsilon_1\Lambda_1;(11)\varepsilon_2\Lambda_2||(\lambda'\mu')\varepsilon\Lambda\rangle$

with $\varepsilon = 2\lambda'+\mu'-3p-3q,\quad \Lambda = \frac{1}{2}\mu'+\frac{1}{2}p-\frac{1}{2}q$

$\varepsilon_2\Lambda_2$ Λ_1	$(\lambda'\mu') = (\lambda+1, \mu-2)$	$(\lambda'\mu') = (\lambda-1, \mu+2)$
$\varepsilon_2\Lambda_2 = 3\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$\left[\frac{(\mu+p)(\mu+1+p)(\lambda+1-p)(q+1)(\mu-q-1)}{(\lambda+1)\mu(\mu+1)(\lambda+\mu+2)(\mu+p-q-1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(p+1)(p+2)(\lambda-1-p)q(\mu+2-q)}{(\lambda+1)(\mu+1)(\mu+2)(\lambda+\mu+2)(\mu+p-q+3)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 3\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$-\left[\frac{p(\mu+p)(q+2)(q+1)(\lambda+\mu-q)}{(\lambda+1)\mu(\mu+1)(\lambda+\mu+2)(\mu+p-q-1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(p+1)(\mu+p+3)(\mu+1-q)(\mu+2-q)(\lambda+\mu+2-q)}{(\lambda+1)(\mu+1)(\mu+2)(\lambda+\mu+2)(\mu+p-q+3)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 00$ $\Lambda_1 = \Lambda$	$\left[\frac{3p(\mu+p)(q+1)(\mu-1-q)}{2(\lambda+1)\mu(\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{3(p+1)(\mu+3+p)q(\mu+2-q)}{2(\lambda+1)(\mu+1)(\mu+2)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda+1$	$\left[\frac{(\mu+p)(\mu+p+1)(\lambda+1-p)(\mu-q)(\mu-1-q)(\lambda+\mu+1-q)}{(\lambda+1)\mu(\mu+1)(\lambda+\mu+2)(\mu+p-q)(\mu+p-q-1)}\right]^{\frac{1}{2}}$	$\left[\frac{(p+1)(p+2)(\lambda-1-p)q(q-1)(\lambda+\mu+3-q)}{(\lambda+1)(\mu+1)(\mu+2)(\lambda+\mu+2)(\mu+p-q+3)(\mu+p-q+4)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda-1$	$\left[\frac{p(p-1)(\lambda+2-p)(q+1)(q+2)(\lambda+\mu-q)}{(\lambda+1)\mu(\mu+1)(\lambda+\mu+2)(\mu+p-q-2)(\mu+p-q-1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda-p)(\mu+2+p)(\mu+3+p)(\mu+1-q)(\mu+2-q)(\lambda+\mu+2-q)}{(\lambda+1)(\mu+1)(\mu+2)(\lambda+\mu+2)(\mu+p-q+3)(\mu+p-q+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda$	$\frac{-(2\lambda+\mu+2-p-q)[p(\mu+p)(q+1)(\mu-1-q)]^{\frac{1}{2}}}{[2(\lambda+1)\mu(\mu+1)(\lambda+\mu+2)(\mu+p-q)(\mu+p-q-2)]^{\frac{1}{2}}}$	$\frac{(2\lambda+\mu+2-p-q)[(p+1)(\mu+p+3)q(\mu+2-q)]^{\frac{1}{2}}}{[2(\lambda+1)(\mu+1)(\mu+2)(\lambda+\mu+2)(\mu+p-q+2)(\mu+p-q+4)]^{\frac{1}{2}}}$
$\varepsilon_2\Lambda_2 = -3\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$\left[\frac{p(\mu+p)(\mu-1-q)(\mu-q)(\lambda+\mu+1-q)}{(\lambda+1)\mu(\mu+1)(\lambda+\mu+2)(\mu+p-q-1)}\right]^{\frac{1}{2}}$	$\left[\frac{(p+1)(\mu+3+p)q(q-1)(\lambda+\mu+3-q)}{(\lambda+1)(\mu+1)(\mu+2)(\lambda+\mu+2)(\mu+p-q+3)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = -3\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$-\left[\frac{p(p-1)(\lambda+2-p)(q+1)(\mu-1-q)}{(\lambda+1)\mu(\mu+1)(\lambda+\mu+2)(\mu+p-q-1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda-p)(\mu+2+p)q(\mu+3+p)(\mu+2-q)}{(\lambda+1)(\mu+1)(\mu+2)(\lambda+\mu+2)(\mu+p-q+3)}\right]^{\frac{1}{2}}$

TABLE 4 (continued)

$\langle(\lambda\mu)\varepsilon_1\Lambda_1;\ (11)\varepsilon_2\Lambda_2\|(\lambda\mu)\varepsilon\Lambda\rangle_\rho$

with $\varepsilon = 2\lambda+\mu-3p-3q,\quad \Lambda = \frac{1}{2}\mu+\frac{1}{2}p-\frac{1}{2}q$

$\varepsilon_2\Lambda_2$ Λ	$(\lambda'\mu') = (\lambda\mu)$ $\rho = 1$	$(\lambda'\mu') = (\lambda\mu)$ $\rho = 2$
$\varepsilon_2\Lambda_2 = 3\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$\left[\dfrac{3(p+1)(\lambda-p)(\mu+2+p)}{2g_{\lambda\mu}(\mu+p-q+1)}\right]^{\frac{1}{2}}$	$\dfrac{\{2g_{\lambda\mu}q-\mu(\lambda+\mu+1)(\lambda+2\mu+6)\}[(p+1)(\lambda-p)(\mu+2+p)]^{\frac{1}{2}}}{[\lambda(\lambda+2)\mu(\mu+2)(\lambda+\mu+1)(\lambda+\mu+3)2g_{\lambda\mu}(\mu+p-q+1)]^{\frac{1}{2}}}$
$\varepsilon_2\Lambda_2 = 3\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$\left[\dfrac{3(q+1)(\mu-q)(\lambda+\mu+1-q)}{2g_{\lambda\mu}(\mu+p-q+1)}\right]^{\frac{1}{2}}$	$\dfrac{\{2g_{\lambda\mu}p+\lambda(\mu+2)(\lambda-\mu+3)\}[(q+1)(\mu-q)(\lambda+\mu+1-q)]^{\frac{1}{2}}}{[\lambda(\lambda+2)\mu(\mu+2)(\lambda+\mu+1)(\lambda+\mu+3)2g_{\lambda\mu}(\mu+p-q+1)]^{\frac{1}{2}}}$
$\varepsilon_2\Lambda_2 = 00$ $\Lambda_1 = \Lambda$	$\dfrac{-(2\lambda+\mu-3p-3q)}{2\sqrt{g_{\lambda\mu}}}$	$\dfrac{\sqrt{3}\{\lambda\mu(\mu+2)(\lambda+\mu+1)-\mu(\lambda+\mu+1)(\lambda+2\mu+6)p+\lambda(\mu+2)(\lambda-\mu+3)q+2g_{\lambda\mu}pq\}}{2[\lambda(\lambda+2)\mu(\mu+2)(\lambda+\mu+1)(\lambda+\mu+3)g_{\lambda\mu}]^{\frac{1}{2}}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda+1$	0	$\left[\dfrac{2(p+1)(\lambda-p)(\mu+2+p)q(\mu+1-q)(\lambda+\mu+2-q)g_{\lambda\mu}}{\lambda(\lambda+2)\mu(\mu+2)(\lambda+\mu+1)(\lambda+\mu+3)(\mu+p-q+1)(\mu+p-q+2)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda-1$	0	$-\left[\dfrac{2p(\lambda+1-p)(\mu+1+p)(q+1)(\mu-q)(\lambda+\mu+1-q)g_{\lambda\mu}}{\lambda(\lambda+2)\mu(\mu+2)(\lambda+\mu+1)(\lambda+\mu+3)(\mu+p-q+1)(\mu+p-q)}\right]^{\frac{1}{2}}$
$\varepsilon_2\Lambda_2 = 01$ $\Lambda_1 = \Lambda$	$\dfrac{[3(\mu+p-q)(\mu+p-q+2)]^{\frac{1}{2}}}{2\sqrt{g_{\lambda\mu}}}$	$\dfrac{\left\{\begin{array}{l}\lambda(\lambda+\mu+1)\mu(\mu+2)(2\lambda+\mu+6)+2(\lambda+\mu+1)\mu[\lambda(\lambda+2)-(\mu+2)(\mu+3)]p-\mu(\lambda+\mu+1)(\lambda+2\mu+6)p^2\\ \quad -2\lambda[(\mu+1)(\lambda+\mu+1)(2\lambda+\mu+6)-\mu g_{\lambda\mu}]q+\lambda(\mu+2)(\lambda-\mu+3)q^2\\ \quad -2[\lambda(\lambda+\mu+1)(2\lambda+\mu+6)-g_{\lambda\mu}]pq+2g_{\lambda\mu}(p^2q+pq^2)\end{array}\right\}}{2[\lambda(\lambda+2)\mu(\mu+2)(\lambda+\mu+1)(\lambda+\mu+3)g_{\lambda\mu}(\mu+p-q)(\mu+p-q+2)]^{\frac{1}{2}}}$
$\varepsilon_2\Lambda_2 = -3\frac{1}{2}$ $\Lambda_1 = \Lambda+\frac{1}{2}$	$\left[\dfrac{3q(\mu+1-q)(\lambda+\mu+2-q)}{2g_{\lambda\mu}(\mu+p-q+1)}\right]^{\frac{1}{2}}$	$\dfrac{\{2g_{\lambda\mu}p+\lambda(\mu+2)(\lambda-\mu+3)\}[q(\mu+1-q)(\lambda+\mu+2-q+1)]^{\frac{1}{2}}}{[\lambda(\lambda+2)\mu(\mu+2)(\lambda+\mu+1)(\lambda+\mu+3)2g_{\lambda\mu}(\mu+p-q)]^{\frac{1}{2}}}$
$\varepsilon_2\Lambda_2 = -3\frac{1}{2}$ $\Lambda_1 = \Lambda-\frac{1}{2}$	$-\left[\dfrac{3p(\lambda+1-p)(\mu+1+p)}{2g_{\lambda\mu}(\mu+p-q+1)}\right]^{\frac{1}{2}}$	$\dfrac{-\{2g_{\lambda\mu}q-\mu(\lambda+\mu+1)(\lambda+2\mu+6)\}[p(\lambda+1-p)(\mu+1+p)]^{\frac{1}{2}}}{[\lambda(\lambda+2)\mu(\mu+2)(\lambda+\mu+1)(\lambda+\mu+3)2g_{\lambda\mu}(\mu+p-q+1)]^{\frac{1}{2}}}$

where $g_{\lambda\mu} = \lambda^2+\lambda\mu+\mu^2+3\lambda+3\mu$

SU$_3$ Wigner coefficients with $\rho = 1$ are defined by the matrix elements of the infinitesimal operators A_{ij}, eqs. (3) and (4). Coefficients with $\rho = 2$ have been made orthogonal to these.

$$\langle(\lambda\mu)\|A_{ij}\|(\lambda\mu)\rangle_1 = \sqrt{\tfrac{1}{3}g_{\lambda\mu}},\qquad \langle(\lambda\mu)\|A_{ij}\|(\lambda\mu)\rangle_2 = 0.$$

TABLE 5

The recoupling coefficients

$U(\lambda\mu)(20)(\lambda'\mu')(20);\ \ (\lambda_{12}\mu_{12})(\lambda_{23}\mu_{23})\rho_{1,23})$

$U((\lambda\mu)(20)(\lambda+4,\mu)(20);\ (\lambda+2,\mu)(40)) = 1, \qquad U((\lambda\mu)(20)(\lambda,\mu-4)(20);\ (\lambda,\mu-2)(40)) = 1$

$U((\lambda\mu)(20)(\lambda-4,\mu+4)(20);\ (\lambda-2,\mu+2)(40)) = 1$

$(\lambda'\mu')=(\lambda+2,\mu+1)$	$(\lambda_{23}\mu_{23})=$ (40)	(21)
$(\lambda_{12}\mu_{12})=(\lambda+2,\mu)$	$\left[\frac{\lambda}{2(\lambda+2)}\right]^{\frac{1}{2}}$	$\left[\frac{\lambda+4}{2(\lambda+2)}\right]^{\frac{1}{2}}$
$(\lambda,\mu+1)$	$\left[\frac{\lambda+4}{2(\lambda+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{\lambda}{2(\lambda+2)}\right]^{\frac{1}{2}}$

$(\lambda'\mu')=(\lambda-3,\mu+2)$	$(\lambda_{23}\mu_{23})=$ (40)	(21)
$(\lambda_{12}\mu_{12})=(\lambda-2,\mu+2)$	$\left[\frac{\mu}{2(\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{\mu+4}{2(\mu+2)}\right]^{\frac{1}{2}}$
$(\lambda-1,\mu)$	$\left[\frac{\mu+4}{2(\mu+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{\mu}{2(\mu+2)}\right]^{\frac{1}{2}}$

$(\lambda'\mu')=(\lambda-2,\mu+3)$	$(\lambda_{23}\mu_{23})=$ (40)	(21)
$(\lambda_{12}\mu_{12})=(\lambda,\mu+1)$	$\left[\frac{\lambda-2}{2\lambda}\right]^{\frac{1}{2}}$	$\left[\frac{\lambda+2}{2\lambda}\right]^{\frac{1}{2}}$
$(\lambda-2,\mu+2)$	$\left[\frac{\lambda+2}{2\lambda}\right]^{\frac{1}{2}}$	$-\left[\frac{\lambda-2}{2\lambda}\right]^{\frac{1}{2}}$

$(\lambda'\mu')=(\lambda-1,\mu-2)$	$(\lambda_{23}\mu_{23}=$ (40)	(21)
$(\lambda_{12}\mu_{12})=(\lambda-1,\mu)$	$\left[\frac{\mu-2}{2\mu}\right]^{\frac{1}{2}}$	$\left[\frac{\mu+2}{2\mu}\right]^{\frac{1}{2}}$
$(\lambda,\mu-2)$	$\left[\frac{\mu+2}{2\mu}\right]^{\frac{1}{2}}$	$-\left[\frac{\mu-2}{2\mu}\right]^{\frac{1}{2}}$

$(\lambda'\mu')=(\lambda+3,\mu-1)$	$(\lambda_{23}\mu_{23})=$ (40)	(21)
$(\lambda_{12}\mu_{12})=(\lambda+2,\mu)$	$\left[\frac{(\lambda+\mu+1)}{2(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda+\mu+5)}{2(\lambda+\mu+3)}\right]^{\frac{1}{2}}$
$(\lambda+1,\mu-1)$	$\left[\frac{(\lambda+\mu+5)}{2(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\lambda+\mu+1)}{2)\lambda+\mu+3)}\right]^{\frac{1}{2}}$

$(\lambda'\mu')=(\lambda+1,\mu-3)$	$(\lambda_{23}\mu_{23})=$ (40)	(21)
$(\lambda_{12}\mu_{12})=(\lambda+1,\mu-1)$	$\left[\frac{(\lambda+\mu-1)}{2(\lambda+\mu+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda+\mu+3)}{2(\lambda+\mu+1)}\right]^{\frac{1}{2}}$
$(\lambda,\mu-2)$	$\left[\frac{(\lambda+\mu+3)}{2(\lambda+\mu+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\lambda+\mu-1)}{2(\lambda+\mu+1)}\right]^{\frac{1}{2}}$

$(\lambda'\mu')=(\lambda+2,\mu-2)$	$(\lambda_{23}\mu_{23})=$ (40)	(02)	(21)
$(\lambda_{12}\mu_{12})=(\lambda+2,\mu)$	$\left[\frac{(\lambda+\mu)(\lambda+\mu+1)}{6(\lambda+\mu+2)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda+\mu+4)}{3(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda+\mu+1)(\lambda+\mu+4)}{2(\lambda+\mu+2)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$
$(\lambda+1,\mu-1)$	$\left[\frac{2(\lambda+\mu)(\lambda+\mu+4)}{3(\lambda+\mu+1)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$\frac{-1}{\sqrt{3}}$	$\left[\frac{2}{(\lambda+\mu+1)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$
$(\lambda,\mu-2)$	$\left[\frac{(\lambda+\mu+3)(\lambda+\mu+4)}{6(\lambda+\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda+\mu)}{3(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\lambda+\mu)(\lambda+\mu+3)}{2(\lambda+\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$

$(\lambda'\mu')=(\lambda,\mu+2)$	$(\lambda_{23}\mu_{23})=$ (40)	(02)	(21)
$(\lambda_{12}\mu_{12})=(\lambda+2,\mu)$	$\left[\frac{\lambda(\lambda-1)}{6(\lambda+1)(\lambda+2)}\right]^{\frac{1}{2}}$	$\left[\frac{\lambda+3}{3(\lambda+1)}\right]^{\frac{1}{2}}$	$\left[\frac{\lambda(\lambda+3)}{2(\lambda+1)(\lambda+2)}\right]^{\frac{1}{2}}$
$(\lambda,\mu+1)$	$\left[\frac{2(\lambda-1)(\lambda+3)}{3\lambda(\lambda+2)}\right]^{\frac{1}{2}}$	$\frac{-1}{\sqrt{3}}$	$\left[\frac{2}{\lambda(\lambda+2)}\right]^{\frac{1}{2}}$
$(\lambda-2,\mu+2)$	$\left[\frac{(\lambda+2)(\lambda+3)}{6\lambda(\lambda+1)}\right]^{\frac{1}{2}}$	$\left[\frac{\lambda-1}{3(\lambda+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\lambda-1)(\lambda+2)}{2\lambda(\lambda+1)}\right]^{\frac{1}{2}}$

$(\lambda'\mu')=(\lambda-2,\mu)$	$(\lambda_{23}\mu_{23})=$ (40)	(02)	(21)
$(\lambda_{12}\mu_{12})=(\lambda-2,\mu+2)$	$\left[\frac{\mu(\mu-1)}{6(\mu+1)(\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{\mu+3}{3(\mu+1)}\right]^{\frac{1}{2}}$	$\left[\frac{\mu(\mu+3)}{2(\mu+1)(\mu+2)}\right]^{\frac{1}{2}}$
$(\lambda-1,\mu)$	$\left[\frac{2(\mu-1)(\mu+3)}{3\mu(\mu+2)}\right]^{\frac{1}{2}}$	$\frac{-1}{\sqrt{3}}$	$\left[\frac{2}{\mu(\mu+2)}\right]^{\frac{1}{2}}$
$(\lambda,\mu-2)$	$\left[\frac{(\mu+2)(\mu+3)}{6\mu(\mu+1)}\right]^{\frac{1}{2}}$	$\left[\frac{\mu-1}{3(\mu+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\mu-1)(\mu+2)}{2\mu(\mu+1)}\right]^{\frac{1}{2}}$

TABLE 5 (Continued)

$U((\lambda\mu)(20)(\lambda'\mu')(20);\ (\lambda_{12}\mu_{12})(\lambda_{23}\mu_{23})\rho_{1,23})$

$(\lambda'\mu') = (\lambda+1, \mu)$	$(\lambda_{23}\mu_{23})\rho_{1,23} =$ (40)	(02)	$(21)_1$	$(21)_2$
$(\lambda_{12}\mu_{12}) = (\lambda, \mu+1)$	$\left[\frac{\mu(\lambda+3)(\lambda+\mu+1)}{3(\mu+1)(\lambda+2)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{\mu\lambda(\lambda+\mu+4)}{6(\mu+1)(\lambda+2)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{\mu(\lambda+\mu+1)(\lambda+\mu+4)}{(\mu+1)(\lambda+2)(\lambda+\mu+2)\varphi}\right]^{\frac{1}{2}}$	$-\left[\frac{(\mu+2)(\lambda+3)(\lambda+\mu+2)}{2(\mu+1)(\lambda+2)\varphi}\right]^{\frac{1}{2}}$
$(\lambda-1, \mu)$	$\left[\frac{(\lambda+3)(\lambda+\mu+4)}{6(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{\lambda(\lambda+\mu+1)}{3(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{\varphi}{2(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	0
$(\lambda+1, \mu-1)$	$\left[\frac{(\mu+2)\lambda(\lambda+\mu+4)}{3(\mu+1)(\lambda+1)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\mu+2)(\lambda+3)(\lambda+\mu+1)}{6(\mu+1)(\lambda+1)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$\left[\frac{\lambda(\mu+2)(\lambda+3)}{(\lambda+1)(\mu+1)(\lambda+\mu+3)\varphi}\right]^{\frac{1}{2}}$	$\left[\frac{\mu(\lambda+1)(\lambda+\mu+1)(\lambda+\mu+4)}{2(\mu+1)(\lambda+\mu+3)\varphi}\right]^{\frac{1}{2}}$
$(\lambda+2, \mu)$	$\left[\frac{\lambda(\lambda+\mu+1)}{6(\lambda+2)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(\lambda+3)(\lambda+\mu+4)}{3(\lambda+2)(\lambda+\mu+3)}\right]^{\frac{1}{2}}$	$\left[\frac{\lambda(\lambda+3)(\lambda+\mu+1)(\lambda+\mu+4)}{2(\lambda+2)(\lambda+\mu+3)\varphi}\right]^{\frac{1}{2}}$	$\left[\frac{\mu(\mu+2)}{(\lambda+2)(\lambda+\mu+3)\varphi}\right]^{\frac{1}{2}}$

where $\varphi = \lambda(\lambda+3)+\mu(\lambda+1)$

$(\lambda'\mu') = (\lambda-1, \mu+1)$	$(\lambda_{23}\mu_{23})\rho_{1,23} =$ (40)	(02)	$(21)_1$	$(21)_2$
$(\lambda_{12}\mu_{12}) = (\lambda, \mu+1)$	$\left[\frac{\mu(\lambda-1)(\lambda+\mu+1)}{3(\mu+1)\lambda(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(\mu+3)(\lambda+2)(\lambda+\mu+1)}{6(\mu+1)\lambda(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{\mu(\lambda-1)(\lambda+2)(\lambda+\mu+1)(\lambda+\mu+3)}{(\mu+1)\lambda(\lambda+\mu+2)\theta}\right]^{\frac{1}{2}}$	$\frac{-\{\lambda^2+(\lambda-2)(\mu+1)\}\sqrt{(\mu+3)}}{[2(\mu+1)\lambda(\lambda+\mu+2)\theta]^{\frac{1}{2}}}$
$(\lambda-1, \mu)$	$\left[\frac{(\mu+3)(\lambda+2)(\lambda+\mu+3)}{3(\mu+2)(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{\mu(\lambda-1)(\lambda+\mu+3)}{6(\mu+2)(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\frac{-\{\lambda(\lambda+\mu+1)-2\}\sqrt{(\mu+3)}}{[(\mu+2)(\lambda+1)(\lambda+\mu+2)\theta]^{\frac{1}{2}}}$	$\left[\frac{\mu(\lambda-1)(\lambda+2)(\lambda+\mu+1)(\lambda+\mu+3)}{2(\mu+2)(\lambda+1)(\lambda+\mu+2)\theta}\right]^{\frac{1}{2}}$
$(\lambda+1, \mu-1)$	$\left[\frac{(\mu+3)(\lambda-1)}{6(\mu+1)(\lambda+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{\mu(\lambda+2)}{3(\mu+1)(\lambda+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\mu+3)(\lambda-1)(\lambda+2)(\lambda+\mu+3)}{2(\mu+1)(\lambda+1)\theta}\right]^{\frac{1}{2}}$	$\frac{\lambda[\mu(\lambda+\mu+1)]^{\frac{1}{2}}}{[(\mu+1)(\lambda+1)\theta]^{\frac{1}{2}}}$
$(\lambda-2, \mu+2)$	$\left[\frac{\mu(\lambda+2)}{6\lambda(\mu+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\mu+3)(\lambda-1)}{3\lambda(\mu+2)}\right]^{\frac{1}{2}}$	$\frac{-(\lambda-2)[\mu(\lambda+\mu+3)]^{\frac{1}{2}}}{[2\lambda(\mu+2)\theta]^{\frac{1}{2}}}$	$-\left[\frac{(\mu+3)(\lambda-1)(\lambda+2)(\lambda+\mu+1)}{\lambda(\mu+2)\theta}\right]^{\frac{1}{2}}$

where $\theta = 3\lambda(\lambda+\mu+1)-2(\mu+3)$

$(\lambda'\mu') = (\lambda, \mu-1)$	$(\lambda_{23}\mu_{23})\rho_{1,23} =$ (40)	(02)	$(21)_1$	$(21)_2$
$(\lambda_{12}\mu_{12}) = (\lambda, \mu+1)$	$\left[\frac{(\mu-1)(\lambda+\mu)}{6(\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(\mu+2)(\lambda+\mu+3)}{3(\mu+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(\mu-1)(\lambda+2)(\lambda+\mu)(\lambda+\mu+3)}{2(\mu+1)(\lambda+\mu+2)\chi}\right]^{\frac{1}{2}}$	$\frac{-(\lambda+\mu+1)[\lambda(\mu+2)]^{\frac{1}{2}}}{[(\mu+1)(\lambda+\mu+2)\chi]^{\frac{1}{2}}}$
$(\lambda-1, \mu)$	$\left[\frac{(\mu-1)(\lambda+2)(\lambda+\mu+3)}{3\mu(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\mu+2)(\lambda+2)(\lambda+\mu)}{6\mu(\lambda+1)(\lambda+\mu+2)}\right]^{\frac{1}{2}}$	$\frac{-\{\lambda(\lambda+\mu+1)-2\}\sqrt{(\mu-1)}}{[(\mu(\lambda+1)(\lambda+\mu+2)\chi]^{\frac{1}{2}}}$	$-\left[\frac{(\mu+2)\lambda(\lambda+2)(\lambda+\mu)(\lambda+\mu+3)}{2\mu(\lambda+1)(\lambda+\mu+2)\chi}\right]^{\frac{1}{2}}$
$(\lambda+1, \mu-1)$	$\left[\frac{(\mu+2)\lambda(\lambda+\mu)}{3(\mu+1)(\lambda+1)(\lambda+\mu+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\mu-1)\lambda(\lambda+\mu+3)}{6(\mu+1)(\lambda+1)(\lambda+\mu+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\mu+2)\lambda(\lambda+2)(\lambda+\mu)(\lambda+\mu+3)}{(\mu+1)(\lambda+1)(\lambda+\mu+1)\chi}\right]^{\frac{1}{2}}$	$\frac{\{\lambda(\lambda+\mu+1)+2(\mu+1)\}\sqrt{(\mu-1)}}{[2(\mu+1)(\lambda+1)(\lambda+\mu+1)\chi]^{\frac{1}{2}}}$
$(\lambda, \mu-2)$	$\left[\frac{(\mu+2)(\lambda+\mu+3)}{6\mu(\lambda+\mu+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\mu-1)(\lambda+\mu)}{3\mu(\lambda+\mu+1)}\right]^{\frac{1}{2}}$	$\frac{-(\lambda+\mu-1)[(\lambda+2)(\mu+2)]^{\frac{1}{2}}}{[2\mu(\lambda+\mu+1)\chi]^{\frac{1}{2}}}$	$\left[\frac{(\mu-1)\lambda(\lambda+\mu)(\lambda+\mu+3)}{\mu(\lambda+\mu+1)\chi}\right]^{\frac{1}{2}}$

where $\chi = 3\lambda(\lambda+\mu+1)+2(\mu-1)$

TABLE 6

Examples of fractional parentage coefficients $\langle n[f](\lambda\mu)\{|n-1)[f'](\lambda'\mu')\rangle$

		[44] (84)	(73)	(46)		[431] (92)	(65)	$(73)_1$	$(73)_2$	(46)
[441]	(93)	$\frac{5}{\sqrt{26}}$	$\frac{-1}{\sqrt{26}}$		...	$-\frac{1}{3}\frac{\sqrt{11}}{\sqrt{2}}$		$-\frac{1}{3}\frac{\sqrt{7}}{\sqrt{2}}$	0	
	(66)	$\frac{\sqrt{6}}{\sqrt{7}}$		$\frac{-1}{\sqrt{7}}$	...		$\frac{\sqrt{2}}{\sqrt{3}}$			$-\frac{1}{\sqrt{3}}$
	⋮				...					

		[43] (83)	(64)	(72)	(45)	(80)	$(53)_1$	$(53)_2$	(26)
[44]	(84)	$\frac{\sqrt{5}}{\sqrt{8}}$	$\frac{\sqrt{3}}{\sqrt{8}}$						
	(73)	$\frac{-\sqrt{15}}{8\sqrt{2}}$	$\frac{5\sqrt{5}}{8\sqrt{22}}$	$\frac{\sqrt{195}}{8\sqrt{7}}$			$\frac{-\sqrt{13}}{2\sqrt{10}}$	$\frac{-\sqrt{13}}{\sqrt{385}}$	
	(46)		$\frac{1}{\sqrt{5}}$		$\frac{\sqrt{5}}{2\sqrt{2}}$				$\frac{\sqrt{7}}{2\sqrt{10}}$
	⋮		... ...						
[431]	(92)	$\frac{\sqrt{35}}{6}$		$\frac{-1}{6}$					
	(65)	$\frac{3\sqrt{5}}{2\sqrt{14}}$	$\frac{-\sqrt{5}}{6\sqrt{2}}$		$\frac{-2\sqrt{2}}{3\sqrt{7}}$				
	$(73)_1$	$\frac{5\sqrt{13}}{24\sqrt{14}}$	$\frac{15\sqrt{39}}{8\sqrt{154}}$	$\frac{-17}{168}$			$\frac{\sqrt{3}}{2\sqrt{14}}$	$\frac{\sqrt{3}}{7\sqrt{11}}$	
	$(73)_2$	$\frac{\sqrt{65}}{4\sqrt{7}}$	$\frac{-\sqrt{65}}{4\sqrt{231}}$	$\frac{5\sqrt{10}}{28}$			$\frac{2}{\sqrt{105}}$	$\frac{-9\sqrt{3}}{7\sqrt{110}}$	
	(46)		$\frac{-\sqrt{7}}{3}$		$\frac{\sqrt{7}}{6\sqrt{2}}$				$\frac{-1}{2\sqrt{2}}$
	⋮	...							

TABLE 7

The overlap integrals, $\mathscr{I}(lj)$, eq. (38)

1. $(\lambda_0\mu_0) = (\lambda+2, \mu)$

$$\mathscr{I}(lj) = \sum_L A(K_0, LK)\beta(l0; 400; L)$$

2. $(\lambda_0\mu_0) = (\lambda, \mu+1)$

$$\mathscr{I}(lj) = \frac{1}{[2\lambda(\lambda+2)]^{\frac{1}{2}}} \sum_L \{A(K_0-1, LK)[\beta(l0; 400; L)[(L+K_0)(L-K_0+1)]^{\frac{1}{2}}+\lambda\sqrt{2}\beta(l1; 1\tfrac{1}{2}1; L)]$$
$$+A(K_0+1, LK)[[-\beta(l0; 400; L)[(L-K_0)(L+K_0+1)]^{\frac{1}{2}}+\lambda\sqrt{2}\beta(l, -1; 1\tfrac{1}{2}1; L)]\}$$

3. $(\lambda_0\mu_0) = (\lambda+1, \mu-1)$

$$\mathscr{I}(lj) = i[(\lambda+\mu+3)(\lambda+\mu+1)\mu(\mu+1)]^{-\frac{1}{2}}$$
$$\sum_L \{A(K_0-1, LK)(K_0-\mu-1)[\beta(l0; 400; L)[\tfrac{1}{2}(L+K_0)(L-K_0+1)]^{\frac{1}{2}}+(\lambda+\mu+1)\beta(l1; 1\tfrac{1}{2}1; L)]$$
$$+A(K_0+1, LK)(K_0+\mu+1)[-\beta(l0; 400; L)[\tfrac{1}{2}(L-K_0)(L+K_0+1)]^{\frac{1}{2}}+(\lambda+\mu+1)\beta(l, -1; 1\tfrac{1}{2}1; L)]\}$$

4. $(\lambda_0\mu_0) = (\lambda-2, \mu+2)$

$$\mathscr{I}(lj) = \frac{1}{[(\lambda-1)(\lambda+1)]^{\frac{1}{2}}} \sum_L \Bigg[A(K_0LK)\Bigg\{\left(1-\frac{1}{2\lambda}L(L+1)+\frac{1}{2\lambda}K_0^2\right)\beta(l0; 400; L)$$
$$+(\lambda-1)\beta(l0; -212; L)+\frac{(\lambda-1)}{\lambda\sqrt{2}}[\beta(l, -1; 1\tfrac{1}{2}1; L)[(L-K_0)(L+K_0+1)]^{\frac{1}{2}}$$
$$-\beta(l1; 1\tfrac{1}{2}1; L)[(L+K_0)(L-K_0+1)]^{\frac{1}{2}}]\Bigg\} +A(K_0-2, LK)\Bigg\{[(L-K_0+2)(L+K_0-1)]^{\frac{1}{2}}$$
$$\times\left(\frac{1}{4\lambda}\beta(l0; 400; L)[(L+K_0)(L-K_0+1)]^{\frac{1}{2}}+\frac{(\lambda-1)}{\lambda\sqrt{2}}\beta(l1; 1\tfrac{1}{2}1; L)\right)+(\lambda-1)\beta(l2; -212; L)\Bigg\}$$
$$+A(K_0+2, LK)\Bigg\{[(L+K_0+2)(L-K_0-1)]^{\frac{1}{2}}\Bigg(\frac{1}{4\lambda}\beta(l0; 400; L)[(L-K_0)(L+K_0+1)]^{\frac{1}{2}}$$
$$-\frac{(\lambda-1)}{\lambda\sqrt{2}}\beta(l, -1; 1\tfrac{1}{2}1; L)\Bigg)+(\lambda-1)\beta(l, -2; -212; L)\Bigg\}\Bigg]$$

5. $(\lambda_0\mu_0) = (\lambda-1, \mu)$

$$\mathscr{I}(lj) = i\sqrt{2}[\mu(\mu+2)\lambda(\lambda+1)(\lambda+\mu+1)(\lambda+\mu+2)]^{-\frac{1}{2}}$$
$$\times\sum_L\Bigg[[A(K_0\, LK)\Bigg\{\beta(l0; 400; L)\tfrac{1}{2}K_0[2\lambda+\mu-L(L+1)+K_0^2]+\lambda(\lambda+\mu+1)K_0\beta(l0; -212; L)$$
$$-\beta(l1; 1\tfrac{1}{2}1; L)\left[\frac{K_0(2\lambda+\mu)}{2\sqrt{2}}+\frac{\mu[\lambda-(\mu+1)^2]}{2\sqrt{2}(\mu+1)}\right][(L+K_0)(L-K_0+1)]^{\frac{1}{2}}$$
$$+\beta(l, -1; 1\tfrac{1}{2}1; L)\left[\frac{K_0(2\lambda+\mu)}{2\sqrt{2}}-\frac{\mu[\lambda-(\mu+1)^2]}{2\sqrt{2}(\mu+1)}\right][(L-K_0)(L+K_0+1)]^{\frac{1}{2}}\Bigg\}$$
$$+A(K_0-2, LK)\Bigg\{\beta(l2; -212; L)\lambda(\lambda+\mu+1)(K_0-\mu-2)+\tfrac{1}{4}\beta(l0; 400; L)(K_0-2\mu-6)$$
$$\times[(L+K_0)(L-K_0+1)(L+K_0-1)(L-K_0+2)]^{\frac{1}{2}}+[(L-K_0+2)(L+K_0-1)]^{\frac{1}{2}}$$

TABLE 7
(continued)

$$\times \left[\beta(l1;1\tfrac{1}{2}1;L)\left[\frac{K_0(2\lambda+\mu)}{2\sqrt{2}}-\frac{[\lambda(2\mu^2+7\mu+4)+\mu(\mu+1)(\mu+3)]}{2\sqrt{2}(\mu+1)}\right]\right]\Bigg\}$$

$$+A(K_0+2,LK)\Bigg\{\beta(l,-2;-212;L)\lambda(\lambda+\mu+1)(K_0+\mu+2)+\tfrac{1}{4}\beta(l0;400;L)(K_0+2\mu+6)$$

$$\times[(L-K_0)(L+K_0+1)(L-K_0-1)(L+K_0+2)]^{\frac{1}{2}}-[(L+K_0+2)(L-K_0-1)]^{\frac{1}{2}}$$

$$\times\left[\beta(l,-1;1\tfrac{1}{2}1;L)\left[\frac{K_0(2\lambda+\mu)}{2\sqrt{2}}+\frac{[\lambda(2\mu^2+7\mu+4)+\mu(\mu+1)(\mu+3)]}{2\sqrt{2}(\mu+1)}\right]\right]\Bigg\}\Bigg]$$

6. $(\lambda_0\mu_0)=(\lambda,\mu-2)$

$$\mathscr{I}(lj)=[\mu^2(\lambda+\mu+1)^2(\mu-1)(\mu+1)(\lambda+\mu)(\lambda+\mu+2)^{-\frac{1}{2}}$$

$$\times\sum_L\Big[[A(K_0,LK)\{\beta(l0;400;L)[\mu^2(\lambda+\mu+1)-K_0^2(\lambda+\mu+1-\tfrac{1}{2}\mu^2)$$

$$-\tfrac{1}{4}K_0^4+\tfrac{1}{2}L(L+1)(K_0^2-\mu^2)]+\beta(l0;-212;L)(\lambda+\mu)(\lambda+\mu+1)(\mu^2-K_0^2)$$

$$+\sqrt{2}(\lambda+\mu)\beta(l1;1\tfrac{1}{2}1;L)[\tfrac{1}{2}K_0^2+K_0-\tfrac{1}{2}(\mu^2-3\mu-2)][(L+K_0)(L-K_0+1)]^{\frac{1}{2}}$$

$$+\sqrt{2}(\lambda+\mu)\beta(l,-1;1\tfrac{1}{2}1;L)[-\tfrac{1}{2}K_0^2+K_0+\tfrac{1}{2}(\mu^2-3\mu-2)][(L-K_0)(L+K_0+1)]^{\frac{1}{2}}\}$$

$$+A(K_0-2,LK)\{\beta(l0;400;L)[\tfrac{1}{2}(\mu+1)(K_0-2)-\tfrac{1}{4}[\mu(\mu+2)+(K_0-2)^2]]$$

$$\times[(L+K_0)(L-K_0+1)(L+K_0-1)(L-K_0+2)]^{\frac{1}{2}}$$

$$-\sqrt{2}\beta(l1;1\tfrac{1}{2}1;L)(\lambda+\mu)[\tfrac{1}{2}(K_0-2)^2-\mu(K_0-2)+\tfrac{1}{2}(\mu^2+\mu+2)][(L-K_0+2)(L+K_0-1)]^{\frac{1}{2}}$$

$$-\beta(l2;-212;L)[(K_0-2)^2-2(\mu-1)(K_0-2)+\mu(\mu-2)]\}$$

$$+A(K_0+2,LK)\{\beta(l0;400;L)[-\tfrac{1}{2}(\mu+1)(K_0+2)+\tfrac{1}{4}[\mu(\mu+2)+(K_0+2)^2]]$$

$$\times[(L-K_0)(L+K_0+1)(L-K_0-1)(L+K_0+2)]^{\frac{1}{2}}$$

$$+\sqrt{2}\beta(l,-1;1\tfrac{1}{2}1;L)(\lambda+\mu)[\tfrac{1}{2}(K_0+2)^2+\mu(K_0+2)+\tfrac{1}{2}(\mu^2+\mu+2)][(L+K_0+2)(L-K_0-1)]^{\frac{1}{2}}$$

$$-\beta(l,-2;-212;L)[(K_0+2)^2+2(\mu-1)(K_0+2)+\mu(\mu-2)]\}\Big]$$