JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 11, 102-130 (1965)

General Theory of Simple Waves in Relaxation Hydrodynamics*

N. COBURN

Department of Mathematics,

University of Michigan, Ann Arbor, Michigan

1. INTRODUCTION

The purposes of this paper are: (1) to determine sufficient conditions for simple waves to exist in one-dimensional, nonsteady, nonmagnetic, relaxation hydrodynamics; (2) to study the properties of these waves; (3) to apply the theory to the flow of a dissociating gas. In a previous paper [1], we obtained the partial differential equation for characteristic manifolds in relaxation hydrodynamics when K, the relaxation scalar (cf. [1, p. 278]), *is constant*. However, such a value for K leads to simple waves which form a family of parallel lines in the one-dimensional nonsteady case (Theorem 12).

In order to determine a more general type of simple wave, we discuss the theory of characteristic manifolds for the following two types of relaxation scalars: *Case I*, K is not constant and is a function of class C^1 of the density, relaxation variable and entropy; *Case II*, K is not constant and is a function of class C^1 of the space and time variables. The following basic result is proved: if a regularity condition is satisfied and if as K approaches the limit value zero, the speed of the characteristic wave approaches the limiting speed c_∞ (as in the case when K is constant), then the characteristic manifolds (for the above two types of K) satisfy the *same partial differential* equation as in the case where K is constant (see Theorems 2-4). Most of Section 2 is concerned with verifying this result. The principal difficulty lies in the fact that the discontinuity theory for Case I is rather involved. Due to this, sufficient conditions (Conditions A and B) for the discontinuity theory problem must be formulated with care. However, when this has been done then the discontinuity theory problem for Case II is easily analyzed. Further, for proper given data (Conditions C and D), the corresponding Cauchy problem is

* This work was done under a grant by the Institute of Science and Technology of the University of Michigan and a grant by the Air Force Office of Scientific Research (20-63) administered by the Office of Research Administration of the University of Michigan.

shown to possess a solution with two branches for *any* unitial manifold of class C^1 (Theorem 6). *This last result has no equivalent in equilibrium hydrodynamics*. Finally, a special case of nonequilibrium hydrodynamics does exist which shows all the properties of discontinuity theory in equilibrium hydrodynamics. That is, a case exists such that: (1) a one-parameter family of discontinuities exist; (2) the characteristic manifolds (or wave fronts) are independent of the discontinuities (Theorem 7).

Three facts should be noted in connection with our discussion of Section 2. First, our basic assumption of Theorems 2-5 is a condition on the *limit partial differential characteristic equation as K approaches zero*. This and another viewpoint have been used in the linearized theory. Stupochenko and Stakhanov [2] have noted that the *limit partial* differential equation for the velocity vector, as K approaches infinity or zero, is the wave equation in the (x, t) plane, with waves propagating at some limit speeds, c_0 or $\bar{c}$. But Broer [3], using another linearized theory, showed that the limit speed, as obtained from the *limit solution*, is *independent* of K. In our work, the *limit partial differential characteristic equation* is basic. This is due to the fact that in the nonlinear theory, very little is known about the solution. Secondly, the discontinuity theory relations for nonequilibrium hydrodynamics differ in two ways from those of equilibrium hydrodynamics: (1) one relation involves an equation in the jumps of *all* partial derivatives of one of the various variables; (2) another relation is a quadratic algebraic equation in these jumps. By contrast, in equilibrium hydrodynamics, the discontinuity relations are linear homogeneous algebraic equations in the jumps. Finally, the fact that *there are no exceptional manifolds* for the Cauchy problem is due to the fact that the discontinuity manifolds are unknown until the second normal derivative of the relaxation variable is specified. Further, the existence of two families of possible solutions of the Cauchy problem for any manifold means that a theory of discontinuities can be developed for any such manifold. Thus, the determination of those manifolds which are wave fronts depends upon the introduction of a *new principle*. In our previous paper, we used the condition: as K approaches zero, the wave speed goes to c_∞. We use the same type of condition in our present work. We shall conclude Section 2 by proving that in *a particular linearized theory, as K approaches zero, the wave speed becomes the speed $\bar{c}$* (see c' of [1, p. 279]) of *Stupochenko and Stakhanov* [2]. Thus c_∞ is the nonlinear equivalent (for K approaching zero) of $\bar{c}$ (Theorems 8, 9).

In Section 3, simple waves are defined as a family of bicharacteristic curves in the (x, t) plane such that along each curve, the density, relaxation variable, and entropy are constant. It should be noted that *simple waves do not exist in nonisentropic equilibrium flows in the (x, t) plane*. This result is obvious if simple waves are defined as bicharacteristics along which entropy is constant, since entropy is also constant along stream lines. Further, even if bicharacteristics are defined by the Courant-Friedrichs method of multipliers [4], no satisfactory scheme exists for defining simple waves in nonisentropic equilibrium flows. As a result, Germain and Gunderson [5] have introduced a method of "nonisentropic perturbations of simple waves." In nonequilibrium hydrodynamics, *some simple waves exist* (Theorem 16). For the cases where the relaxation scalar, K, is of Case I or is a constant [1], the simple waves form a family of parallel lines (Theorem 12). The first case differs from the second in that the entropy and the relaxation variable have different ranges of permissible values in the two cases. Both of these last two cases are characterized by the condition that K *is constant along each curve of the simple wave family*. By contrast, if K is of Case II then K *varies along each curve of the simple wave family. The curves are straight lines of arbitrary slope* (Theorem 15). For K of Case I, the *existence and uniqueness* of a simple wave solution (for entropy, density, etc.) can be verified for proper boundary data and regularity conditions (Theorem 16); for K of Case II, this problem is difficult to discuss since the basic nine equations in ten unknowns are of mixed type, that, is, some are differential and others are algebraic equations. Finally, in Cases I and II, it can be shown that if internal energy is decreasing at a proper rate, then entropy is increasing as one moves from one simple wave to another, and: (1) in the supersonic case, the particle speed is decreasing; (2) in the subsonic case, the particle speed is increasing (Theorems 10, 17).

In Section 4, we shall discuss the application of the theory of Sections 2 and 3 to the theory of a chemically reacting compressible gas. In particular, we shall study a *simple dissociating gas* [6] in the following three states: (1) *nonequilibrium state*; (2) *equilibrium state*; (3) *frozen state*. By use of the theory of Section 2, we can immediately generalize a result of Li [6] for the relation between linearized *nonequilibrium state speeds, c, and the relaxation scalar*, K, to the nonlinear theory of the nonequilibrium state (Theorem 19). Further, we shall obtain the wave speeds (or equations of the characteristic manifolds) in the nonlinear theory of the *equilibrium and frozen states and prove that the wave speeds in these states* (Theorem 20) *are independent of* K, as in the case of the limit solutions in the linear theory [3]. Finally, we prove that *simple waves of Case I and not of Case II* (see Section 3) *exist in the nonequilibrium state but neither case exists in the equilibrium and frozen states* (Theorem 21). The paper of Li [6] was called to the author's attention by Professor S. B. Ong.

2. Characteristic Manifolds

From the linearized partial differential equation for the velocity vector derived by Stupochenko and Stakhanov, it follows that as K approaches infinity, the limit partial differential equation is the wave equation with speed of propagation c_0 (cf. [2, p. 783]). Following, the notation of the above authors, we call, c_0, a limit speed. For the nonlinear theory with constant K, it can be shown from the wave speeds of the characteristics that the limit speed c_0 (or $'''c$ in the notation of [1]) corresponds to K approaching infinity (cf. (3.2b), (4.7) of [1]). Further, it can be shown from the last nonlinear theory that the limit speed c_∞ (or $''c$ of [1]) corresponds to K approaching zero.

In this section of the paper, we shall examine two cases of variable K. First, we study the case where K is a function of ρ, S, q of class C^1 (Case I) in these variables. By definition ρ, S, q denote the density, entropy, relaxation variable, respectively. We shall show that *the limit speed c_∞ characterizes all of the possible wave speeds of characteristic waves* in that if the wave speed approaches c_∞ *when K approaches zero*, then *the characteristic manifolds of the nonlinear theory for K* of Case I *satisfy the same relation as in the case where K is constant*. Finally, the case where K is a specified function of class C^1 of the time and space variables (Case II) will be considered. Here, the basic equations are easily analyzed. The above result remains valid.

In order to verify the above result, we shall need: (1) appropriate forms of the energy relations; (2) a proper formulation of the discontinuity theory problem. Here, we shall discuss only the nonmagnetic case. However, for the magnetic infinitely conductive fluid, it can be easily verified by use of the results of our previous paper [1] that the above result remains valid. In addition to the energy relations, we shall use the conventional Eulerian forms of the equations of continuity and motion. Finally, we show that proper formulation of the discontinuity theory problem leads to a well-posed corresponding Cauchy problem (with a two branch solution).

First, we shall indicate our notation. Consider a Euclidean four-space E_4 with coordinate lines $t = x^0 =$ variable (t is the time) which are parallel E_1. The coordinate variables x^j ($j = 1, 2, 3$) are Cartesian orthogonal coordinates of the ∞^1 Euclidean three-spaces which are orthogonal to the above E_1. In order to discuss *both* $t = x^0$ and x^j, we shall use Greek indices and write x^α ($\alpha = 0, 1, 2, 3$). We note that the components of the metric tensor, g_{00}, g_{0k} become

$$g_{00} = 1, \qquad g_{0k} = 0 \tag{2.1}$$

by proper choice of scale factor along the time axis. Further, if ∂_t denotes the time derivative, then we can write

$$\partial_t \equiv \frac{\partial}{\partial t} \equiv \partial_0 . \tag{2.2}$$

In general, we shall use the notation

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}, \qquad \partial_j \equiv \frac{\partial}{\partial x^j} . \tag{2.3}$$

Although in the Cartesian orthogonal systems x^j, x^α, there is no distinction between covariant and contravariant quantities we shall still use the upper and lower indices in order to indicate summations via the Einstein summation convention. Again, the variables, p, e, T, v^j will denote, pressure, internal energy, temperature, the velocity vector, respectively; the scalars A, B, C, F, G are defined by (cf. [1, p. 271])

$$A \equiv \left.\frac{\partial p}{\partial \rho}\right)_{S,q}, \qquad B \equiv \left.\frac{\partial p}{\partial S}\right)_{\rho,q}, \tag{2.4}$$

$$C \equiv \rho^2 \frac{\partial^2 e}{\partial \rho \partial q} = \left.\frac{\partial p}{\partial q}\right)_{S,\rho},$$

$$F \equiv \frac{\partial^2 e}{\partial q \partial S}, \qquad G \equiv \frac{\partial^2 e}{\partial q^2}. \tag{2.5}$$

We can now write the *two basic energy relations* as

$$KT(\partial_t S + v^k \partial_k S) = \bar{q}^2 \tag{2.6}$$

$$\partial_\alpha \bar{q} = -K\left(G\partial_\alpha q + F\partial_\alpha S + \frac{C}{\rho^2}\partial_\alpha \rho\right) + \bar{q}\partial_\alpha \ln K \tag{2.7}$$

where $\bar{q}$ is defined by

$$\bar{q} \equiv \partial_t q + v^j \partial_j q \tag{2.8}$$

Note that (2.7) follows from one of the Stupochenko and Stakhanov relations (cf. (2.22) of [1]) by differentiation.[1] For the nonmagnetic case, the equations of motion are

$$\rho(\partial_t v_j + v^k \partial_k v_j) = -\partial_j p \tag{2.9}$$

Finally, the equation of continuity is

$$\partial_t \rho + v^j \partial_j \rho + \rho \partial_j v^j = 0 \tag{2.10}$$

Before discussing the application of discontinuity theory and the Cauchy problem to our system (2.6)-(2.10), we shall briefly outline the general basic conceptions of these two theories and stress their similarities and differences.

The simplest approach to discontinuity theory is to imbed in Euclidean four-dimensional space, E_4, a *single* three-dimensional hypersurface S_3, *on each side* of which a dependent variable, Z, is continuous with continuous derivatives of orders up to and including g. However, Z and its derivatives $\partial_\alpha Z$, etc. may be discontinuous along S_3. Then, Hadamard's theorem [7]

[1] A similar scheme is used in reducing a nonlinear partial differential equation of order n to a quasi-linear equation of order $n + 1$ (cf. [11, p. 495]).

states that "any tangential derivative of the jump of Z is equal to the jump of the *same* tangential derivative of Z." The normal derivative of Z may be continuous or discontinuous as one moves from S_3^+ to S_3^-. Note, that the above theorem of Hadamard is concerned with the behavior of two fields of quantities defined *only in an open set containing the given hypersurface* (or manifold) S_3, $Z+$ along S_3^+ and $Z-$ along S_3^-. If one develops the corresponding theory in three-dimensional Euclidean space, E_3, then the above S_3 must be replaced by an S_2. However, the *single* S_2 with its associated fields $Z+$ and $Z-$ is insufficient to develop the theory of wave propagation. For proper development of this last theory, one introduces a *moving* S_2 with its *associated moving fields* $Z+$ and $Z-$ [8]. This last concept is equivalent to considering a one (or more) parameter family of S_2, associated with which are two distinct one parameter families of fields $Z+$ and $Z-$, defined in a *region of a single* E_3. There is only one defect in identifying the *moving* S_2 and the *family of* S_2 in a single E_3; the two families of fields $Z+$ (and $Z-$) may become multivalued. This will be the case when a portion of S_2 is *stationary*. Then, only use of the time coordinate will restore the single-valuedness of $Z+$ (and $Z-$). This problem of multivaluedness of the fields $Z+$ (and $Z-$). This problem of multivaluedness of the fields $Z+$ (and $Z-$) does not exist when the general discontinuity theory for Z is developed on a *one parameter family* of S_3 in E_4. Thus, we shall proceed in the following manner: (1) Z and $\partial_\alpha Z$ will be identified with any of the dependent variables $\bar{q}$, and $\partial_\alpha v_j$, $\partial_\alpha \rho$, $\partial_\alpha S$, $\partial_\alpha q$, $\partial_\alpha \bar{q}$, respectively; (2) ρ, q, S, v_j will be assumed to be functions which are C^2 in $U - S_3$ where U is an open set containing S_3 and are *solutions of (2.6)-(2.10)*. The problem will be to determine the one parameter family of *manifolds* S_3 (characteristics) along which ρ, q, S, and v_j determine two types of C^2 fields $Z+$, $Z-$ (actually, $(Z+) - (Z-)$) and $\partial_\alpha Z)+$, $\partial_\alpha Z)-$. The local *existence* of the above fields will be shown when the Cauchy problem is studied (see (2.59) etc.).

The simplest form of the Cauchy problem is as follows: ρ, q, S, v_j and their tangential derivatives are *known* along some *specified manifold* S_3; $\bar{q}$ and all of its first derivatives, and the first normal derivatives of q, ρ, S, and v_j *are to be determined*. This problem can be connected to the above discontinuity theory by noting that in *conventional hydrodynamics*, the p first *normal derivatives* are determined by a system of *p linear*, *nonhomogeneous*, *algebraic equations*. Hence, by forming the jumps of these equations, we obtain a system of linear, homogeneous algebraic equations for the jumps in the normal derivatives. Obviously, the Cauchy problem does *not possess a unique solution* when the determinant of the system vanishes. However, the manifolds (characteristics) determined by the last condition are those for which the discontinuity theory problem permits a one parameter family of solutions (that is, there exists one independent jump and every jump is a multiple of this jump). This result is not valid for the system (2.6)-(2.10) and is due to several reasons, as will be shown.

Now, we shall discuss the decomposition of $\partial_\alpha \rho$, $\partial_\alpha S$, $\partial_\alpha q$, $\partial_\alpha v_j$ for use in the discontinuity and Cauchy theories of the system (2.6), (2.7), (2.9), (2.10). To do so, we decompose $\partial_\alpha \rho$, $\partial_\alpha S$, $\partial_\alpha q$, $\partial_\alpha v_j$ into components along any three orthogonal unit vectors $\underset{a}{t_\alpha}$ ($a = 1, 2, 3$) lying in the tangent hyperplane at point P^* to any hypersurface S_3 of class C^1, $\phi(x^\alpha) = c$, with normal vector ϕ_α having space magnitude ϕ where c is the parameter and

$$\phi_\alpha \equiv \partial_\alpha \phi, \qquad \phi \equiv \phi^j \phi_j \tag{2.11}$$

by writing

$$\partial_\alpha \rho = \phi_\alpha \bar{P} + \sum_a \underset{a}{t_\alpha} \underset{a}{A} \tag{2.12}$$

$$\partial_\alpha q = \phi_\alpha \bar{Q} + \sum_a \underset{a}{t_\alpha} \underset{a}{B} \tag{2.13}$$

$$\partial_\alpha S = \phi_\alpha \bar{s} + \sum_a \underset{a}{t_\alpha} \underset{a}{C} \tag{2.14}$$

$$\partial_\alpha v_j = \phi_\alpha \bar{V}_j + \sum_a \underset{a}{t_\alpha} \underset{a}{D_j} \tag{2.15}$$

If ρ, q, S, v_j are continuous along S_3 with one-sided derivatives along $S_3{}^+$ and $S_3{}^-$, then by Hadamard's theorem, the tangential derivatives of these quantities are continuous along S_3 but their normal derivatives may be discontinuous [7]. Thus, by forming the jumps of (2.12)-(2.15), we obtain the relations

$$[\partial_\alpha \rho] = \phi_\alpha P, \qquad P \equiv [\bar{P}] \tag{2.16}$$

$$[\partial_\alpha q] = \phi_\alpha Q, \qquad Q \equiv [\bar{Q}] \tag{2.17}$$

$$[\partial_\alpha S] = \phi_\alpha s, \qquad s \equiv [\bar{s}] \tag{2.18}$$

$$[\partial_\alpha v_j] = \phi_\alpha V_j, \qquad V_j \equiv [\bar{V}_j] \tag{2.19}$$

where $[\bar{P}]$, $[\bar{Q}]$, $[\bar{s}]$, $[\bar{V}_j]$ denote the jumps of $\bar{P}$, $\bar{Q}$, $\bar{s}$, $\bar{V}_j$, respectively.

Now, we determine a system of necessary discontinuity theory relations. First, we note that by forming the jump of (2.8) and using (2.17), we obtain

$$[\bar{q}] = LQ \tag{2.20}$$

where ϕ_α are defined by (2.11) and L is defined by

$$L \equiv \phi_0 + v^j \phi_j \tag{2.21}$$

Similarly, we note that (see (2.4), (2.12)-(2.14))

$$\partial_j p = A\partial_j \rho + B\partial_j S + C\partial_j q \tag{2.22}$$

$$\partial_\alpha K = \frac{\partial K}{\partial \rho}\partial_\alpha \rho + \frac{\partial K}{\partial S}\partial_\alpha S + \frac{\partial K}{\partial q}\partial_\alpha q \tag{2.23}$$

where A, B, C and the partial derivatives of K with respect to ρ, S, q are continuous functions of these last variables. By forming the jumps of (2.6), (2.7), (2.9), (2.10), we find by use of (2.16)-(2.23) and the well known rule for forming the jump of a product (cf. [9, p. 280])

$$KTs = LQ^2 + 2\bar{q}_1 Q \tag{2.24}$$

$$[\partial_\alpha \bar{q}] = -\phi_\alpha K\left(\frac{CP}{\rho^2} + GQ + Fs\right) + (LQ + \bar{q}_1)\,[\partial_\alpha \ln K] + LQ(\partial_\alpha \ln K)_1 \tag{2.25}$$

$$\rho L V_j = -\phi_j(AP + Bs + CQ) \tag{2.26}$$

$$LP + \rho\phi_j V^j = 0 \tag{2.27}$$

Here, $\bar{q}_1$, $(\partial_\alpha \ln K)_1$ are the values of $\bar{q}$, $\partial_\alpha \ln K$, respectively, on one side of S_3 (say S_3^-). Note that the subscript "2" indicates the value of a quantity on the other side of S_3 (that is, S_3^+) and the above jumps are determined by the order relation

$$[A] \equiv A_2 - A_1 \tag{2.28}$$

In (2.25) and in our future work, if $\bar{q}$, K, etc. are negative, $\ln \bar{q}$, $\ln K$, etc. will be assumed to represent $\ln|\bar{q}|$, $\ln|K|$, etc. Our final results will be unaltered.

To complete our study of necessary conditions, we analyze (2.25). First, we prove

THEOREM 1. *For all nonvanishing K of class C^1 in ρ, p, S, q*

$$t^\alpha[\partial_\alpha K] = 0, \qquad kK = LQ$$

where k is a function of ϕ of class C^1 (the characteristic manifolds are determined by $\phi(x^\alpha) = constant$).

To verify Theorem 1, we note that the general form of Hadamard's theorem leads to

$$t^\alpha[\partial_\alpha \bar{q}] = t^\alpha \partial_\alpha[\bar{q}] \tag{2.29}$$

where t^α is any tangent vector to S_3. Since K is of class C^1 in ρ, S, q, it follows by forming the jump of (2.23), multiplying the resulting equation by t^α, and using (2.16)-(2.18) that $t^\alpha\partial_\alpha K$ *is continuous* as one crosses S_3 or

$$t^\alpha[\partial_\alpha K] = 0 \tag{2.30a}$$

Forming the scalar product of (2.25) with t^α and using (2.20), (2.29), (2.30a), we obtain

$$t^\alpha\partial_\alpha \ln LQ = t^\alpha(\partial_\alpha \ln K)_1 \tag{2.30b}$$

From (2.30a), we see that the subscript "1" may be dropped from the right-hand side of (2.30b). If we integrate the resulting equation, we obtain for K not constant

$$LQ = kK \tag{2.31a}$$

where k is constant on S_3. Thus, if $\phi =$ constant is a characteristic manifold then

$$k = k(\phi) \tag{2.31b}$$

where $k(\phi)$ is an unknown function of class C^1 in ϕ. For the case, $K =$ constant, we showed previously (cf. (3.11b) of [1]) that

$$LQ = k \tag{2.31c}$$

Further, we note that (2.31c) can be written in the form (2.31a) for all finite constant K, except $K = 0$. Again, as K is given the limit value zero, Theorem 1 may not be valid.

Finally, we form the scalar product of (2.25) with ϕ^α and obtain

$$\phi^\alpha[\partial_\alpha\bar{q}] = -\phi^\alpha\phi_\alpha K\left(\frac{CP}{\rho^2} + GQ + Fs\right) + LQ(\phi^\alpha\partial_\alpha \ln K)_1$$
$$+ (LQ + \bar{q}_1)[\phi^\alpha\partial_\alpha \ln K] \tag{2.32}$$

Either the relations (2.20), (2.24)-(2.27) or (2.20), (2.24), (2.26), (2.27), (2.31a), (2.32) form the desired *systems of necessary equations for the discontinuity theory problem*. It should be noted that the decompositions (2.12)-(2.14), which did not enter in any provious discontinuity theory, are now in the basic necessary equations.

The necessary conditions for the Cauchy problem are formed in a similar manner by substituting (2.12)-(2.14) into (2.6)-(2.10). However, here ϕ_α as well as v_j, ρ, S, q, $\underset{a}{t^\alpha}$ and the tangential derivatives of v_j, ρ, S, q are known. Thus, from (2.8) we find by use of (2.13)

$$\bar{q} = L\bar{Q} + K_1 \tag{2.33}$$

where K_1 and all future K_j are known on the given S_3 and depend on the known Cauchy data $\overset{a}{t^\alpha}$, $\overset{a}{B}$, etc. By use of (2.12)-(2.14), we find that (2.6), (2.7), and (2.33) furnish for a *properly chosen known tangent vector* t_α of the given S_3 and LK_2 defined by $(K_1)^2$

$$KT\bar{s} = -L\bar{Q}^2 + 2\bar{q}\bar{Q} + K_2 \tag{2.34}$$

$$\partial_\alpha \bar{q} = -\phi_\alpha K\left(G\bar{Q} + F\bar{s} + \frac{C\bar{P}}{\rho^2}\right) + t_\alpha K_3 + \bar{q}\partial_\alpha \ln K. \tag{2.35}$$

Multiplying (2.35) by any known tangent vector T^α of S_3, we obtain the linear differential equation

$$\frac{d\bar{q}}{ds} - \bar{q}\frac{d}{ds}\ln K = K_4\,, \qquad \frac{d}{ds} \equiv T^\alpha \partial_\alpha\,. \tag{2.36}$$

Integrating (2.36), and using (2.33) we obtain a relation similar to (2.31a)

$$L\bar{Q} = kK - K_1 \tag{2.37}$$

where k is known when S_3 and a family of curves on S_3 with tangent vector T^α are known. Forming the scalar product of (2.35) with ϕ^α, we obtain the Cauchy equation which corresponds to (2.32)

$$\phi^\alpha \partial_\alpha \bar{q} = -\phi^\alpha \phi_\alpha K\left(G\bar{Q} + F\bar{s} + \frac{C\bar{P}}{\rho^2}\right) + \bar{q}\phi^\alpha \partial_\alpha \ln K. \tag{2.38}$$

Finally, from (2.9), (2.10), we find the Cauchy analogs of (2.26), (2.27)—where $'K_j$ is a known vector

$$\rho L\bar{V}_j = -\phi_j(A\bar{P} + B\bar{s} + C\bar{Q}) + {}'K_j \tag{2.39}$$

$$L\bar{P} + \rho\phi_j \bar{V}^j = K_6 \tag{2.40}$$

The relations (2.33), (2.34), (2.37)-(2.40) constitute a system of necessary Cauchy relations which are the nonhomogeneous equations corresponding to the homogeneous necessary discontinuity relations (2.20), (2.24), (2.31a), (2.32), (2.26), (2.27), respectively, with the following three modifications: (1) the first term in the right-hand side of (2.34) is the negative of the corresponding term in (2.24); (2) the quantity k in the first term of the right-hand side of (2.37) depends upon the family of curves used to span S_3 but k of (2.31a) is constant over S_3; (3) the right-hand sides of (2.38), (2.32) are similar but differ in complexity. Except for (2.34), (2.38), the Cauchy relations are linear in $\bar{V}_j$, $\bar{P}$, $\bar{s}$, $\bar{Q}$. Similar remarks are applicable to the equivalent necessary system (2.33)-(2.35), (2.39), (2.40) when compared to the necessary system (2.20), (2.24)-(2.27).

Now, we shall formulate two problems involving *sufficiency conditions*.

Problem I (discontinuity theory). What conditions are sufficient to determine: (1) a one parameter family of S_3 (the characteristic manifolds or wave fronts) along which the jump relations either of (2.20), (2.24)-(2.27) or of (2.20), (2.24), (2.26), (2.27), (2.31a), (2.32) are valid; (2) the jumps P, s, Q, V_j (and hence the jumps in the normal derivatives of ρ, S, q, $\bar{q}$, v_j) except possibly for a nonvanishing proportionality factor?

Problem II (the Cauchy problem). For a specified manifold S_3 with equation $\phi(x^\alpha) = c$, what conditions are sufficient to determine $\bar{P}$, $\bar{s}$, $\bar{Q}$, $\bar{q}$, $\bar{V}_j$ the family of fields $Z+$, $Z-$, $\partial_\alpha Z)+$, $\partial_\alpha Z)-$) of (2.12)-(2.15) (and hence the normal derivatives of ρ, S, q, $\bar{q}$, v_j) so that either (2.33)-(2.35), (2.39), (2.40) or (2.33), (2.34), (2.37)-(2.40) are satisfied? Will the solution be unique, or have at most a finite number of branches? Are the manifolds which satisfy Problem I not admissible (as is the case in conventional or equilibrium hydrodynamics)?

Before turning to the study of the solutions of these two problems, we note the following points. First, we shall restrict most of our future work to one of the above two systems, that is, the discontinuity relations (2.20), (2.24), (2.31a), (2.32), (2.26), (2.27) and the corresponding Cauchy relations (2.33), (2.34), (2.37)-(2.40). Both of these systems of necessary conditions are *complete* in the sense that they are equivalent to *all* of the jump conditions on $\partial_\alpha \rho$, $\partial_\alpha q$, $\partial_\alpha S$, $\partial_\alpha v_j$, $\partial_\alpha \bar{q}$ that can be obtained from the basic equations (the energy relations (2.6), (2.7); the motion equations (2.9); the continuity equation (2.10); the definition of $\bar{q}$ (2.8)) by forming jumps and using the tensor decomposition of (2.16)-(2.19) for Problem I or directly from the tensor decompositions of (2.12)-(21.5) for Problem II. Secondly, we note that: for Problem I, ρ, S, q, v_j are *continuous* and differentiable on S_3 and hence their tangential derivatives are *continuous* on S_3 but the behaviors of $\bar{q}$ and its normal and tangential derivatives along S_3 are to be determined; in Problem II, ρ, S, q, v_j and their tangential derivatives are *known* along S_3. Thirdly, we note that two new mathematical phenomena occur in the necessary equations of Problems I and II: (1) one of the equations is quadratic (see (2.24) and (2.34)); (2) another one of these equations is a partial differential equation (see (2.30b) and (2.35)). This is in sharp contrast to the case of equilibrium hydrodynamics where the corresponding relations are all algebraic linear equations.

First, we show that (2.20), (2.24), (2.26), (2.27), (2.31a), (2.32) are insufficient for solving Problem I. From (2.26), we see that V_j lies along ϕ_j. Further, by forming the scalar product of (2.26) with ϕ_j, using the resulting equation to eliminate $\phi_j V^j$ in (2.27), and then replacing s by the right hand side of (2.24) in the resulting equation, we obtain (for ϕ defined by (2.11))

$$(L^2 - A\phi) P = \frac{\phi Q}{KT}(LQB + 2\bar{q}_1 B + KTC). \tag{2.41}$$

Thus, (2.20), (2.24), (2.41), and hence (2.26) enable us to determine $[\bar{q}]$, s, P, and V_j in terms of Q, when $L^2 - A\phi$ does not vanish. However, because of the fact that (2.24), (2.41) are quadratic in Q, the ratios of the jumps P/Q, s/Q, V^j/Q cannot be uniquely determined. In fact, from (2.20), we see that $[\bar{q}]/Q$ is known; but from (2.24), (2.41), it follows that unless Q is known s/Q, P/Q cannot be determined. However, Q can be determined from (2.31a) when $k(\phi)$ is known. This leads to a new problem. For what functions $k(\phi)$ will the relations (see (2.20), (2.31a))

$$[\bar{q}] = LQ = kK$$

be consistent with (2.32)? Thus, we must analyze (2.32).

To study the two sides of (2.32), we must determine the structure of $\phi^\alpha[\partial_\alpha \bar{q}]$. If δ^α_β is the Kronecker delta tensor, $\underset{a}{t}^\alpha$ $(a = 1, 2, 3)$ are three orthogonal unit vectors which span S_3 (see (2.12)-(2.15)), $\bar{n}^\alpha$ is *space-time* unitized normal to S_3, then (cf. p. 96 of [10])

$$\delta^\alpha_\beta = \sum_a \underset{a}{t}^\alpha \underset{a}{t}_\beta + \bar{n}^\alpha \bar{n}_\beta, \qquad \bar{n}^\alpha = \frac{\phi^\alpha}{((\phi^0)^2 + \phi)^{1/2}}. \tag{2.42}$$

Further, the time derivative $\partial_0 q$ may be written as

$$\partial_0 q = \delta^\alpha_0 \partial_\alpha q \tag{2.43}$$

From (2.42), (2.43) we see that $\bar{q}$, defined by (2.8), is a sum of the *first normal* and the first tangential derivatives of q. Thus, the scalar $\phi^\alpha[\partial_\alpha \bar{q}]$ in the left hand side of (2.32) is a sum of the jumps of the first and the second derivatives of q and one term of this sum is the jump in the *second normal* derivative of q (cf. (6.13) of [1]). So far, we have introduced only the *jumps in the first normal derivatives* of ρ, S, q. Hence, it is evident that *(2.32) cannot be analyzed unless a new assumption is made.* It is evident by inspection that the *right-hand side of (2.32) contains only the jumps* P, s, Q in the first normal derivatives of ρ, S, q (as well as the unspecified $\bar{P}_1$, $\bar{Q}_1$, $\bar{s}_1$, $\bar{q}_1$).

In order to treat Problem I, we shall consider two conditions.

Condition A. $\bar{q}_1$, $\bar{P}_1$, $\bar{s}_1$, $\bar{Q}_1$ of S_3- are to be specified in terms of P, Q, s so that the normal derivative of K (see (2.12)-(2.14), (2.23)) satisfies

$$LQ(\phi^\alpha \partial_\alpha \ln K)_2 + \bar{q}_1[\phi^\alpha \partial_\alpha \ln K] = 0 \tag{2.44}$$

Condition B. The jump in the second normal derivative of q is to be specified so that

$$\phi^\alpha[\partial_\alpha \bar{q}] = 0 \tag{2.45}$$

When the jumps P, Q, s have been determined in terms of ρ, S, q, v_j, ϕ_j then Condition A furnishes a linear relation, for a given K and nonvanishing $\bar{q}_1$, by means of which $\bar{q}_1$ can be determined. Since $\bar{q}$ is defined by (2.8), we find by substituting (2.13) into (2.8) that $\bar{q}_1$ depends on the tangential derivatives. $\underset{a}{B}$. Thus we see that (2.44) of Condition A is a condition on $\underset{a}{B}$. When $\bar{q}_1$ is zero but Q is not zero, the relation (2,44) is a condition on the first normal derivative of K or on $\bar{P}_1$, $\bar{s}_1$, $\bar{Q}_1$. As we have noted, *Condition B is a condition on the jump of the second normal derivative of* q. This jump is independent of the jumps P, Q, s. If we substitute Condition A into (2.32), we obtain

$$\phi^\alpha(\partial_\alpha \bar{q}] = -\phi^\alpha \phi_\alpha K\left(\frac{PC}{\rho^2} + QG + sF\right). \tag{2.46}$$

Now, we use (2.24) and (2.41) to eliminate s and P, respectively, in (2.46). The resulting equation can be expressed in a useful form by introducing c, c_∞, c_0, $\bar{c}$ the speed of propagation of a characteristic wave, and the three limit speeds of such waves, respectively (cf. p. 279 of [1], noting that $''c \equiv c_\infty$, $'''c \equiv c_0$, $'c \equiv \bar{c}$)

$$c^2 \equiv \frac{L^2}{\phi}, \qquad c_\infty{}^2 \equiv A - \frac{BC}{\rho^2 F}, \qquad c_0{}^2 \equiv A - \frac{C^2}{\rho^2 G}, \qquad \bar{c}^2 \equiv A \tag{2.47}$$

and using the definitions

$$n_j \equiv \frac{\phi_j}{\phi^{1/2}}, \qquad n_0 \equiv V \equiv \frac{\phi_0}{\phi^{1/2}}. \tag{2.48}$$

We obtain after a lengthy but direct computation

$$\begin{aligned}\frac{(\bar{c}^2 - c^2)\, c}{1 + V} n^\alpha[\partial_\alpha \bar{q}] = F(LQ)^2 (c^2 - c_\infty{}^2) + KGTLQ(c^2 - c_0{}^2) \\ + 2\bar{q}_1 LQF(c^2 - c_\infty{}^2).\end{aligned} \tag{2.49}$$

We shall consider (2.49) for the case when K *approaches zero as a limiting value* on some S_3. As we noted in our proof of Theorem 1, for such a case, we can not prove that $LQ = kK$. In fact, from the condition that the rate of change of internal energy, e, with respect to the relaxation variable, q, is, $-\bar{q}/K$ (cf. p. 783 of [2]), we see that if both K and $\bar{q}$ approach zero in such a manner that $\bar{q}/K$ is finite and nonvanishing then nonequilibrium flows with $Q \neq 0$ may exist. We assume that *such a flow exists and propagates with limit speed* $\pm c_\infty$. Further, we note that the right hand side of (2.49) is independent of the jump of the second normal derivative of q but the left hand side of (2.49) depends on this jump (cf. (6.13) of [1]). Hence, we assume that one more condition is satisfied: *the jump in the second normal derivative of q across any characteristic S_3 is chosen so that the right-hand side of (2.49) has the same value for all K and the corresponding speeds $\pm c$.*

Now, we verify

THEOREM 2. *If*: (1) *Condition A is satisfied for all permissible $K, \phi^\alpha \partial_\alpha K$, $\bar{q}_1$;* (2) *a nonequilibrium flow exists with limit speed $\pm c_\infty$ corresponding to K, $\phi^\alpha \partial_\alpha \ln K$ approaching zero on some characteristic S_3;* (3) *the jump in the second normal derivative of q has the same value for all such K, $\phi^\alpha \partial_\alpha \ln K$ and their corresponding c, then Condition B is satisfied.*

To prove this result, we allow K, $\phi^\alpha \partial_\alpha \ln K$ to approach zero on some characteristic S_3. By Condition A, we find that $\bar{q}_1$ approaches zero on S_3. Further, if c for this S_3 approaches $\pm c_\infty$, then the right-hand side of (2.49) is zero. Thus, by condition (3) of our theorem $[\phi^\alpha \partial_\alpha \bar{q}]$ vanishes for all K and their corresponding c. Hence, Condition B is satisfied for all such K, c.

The above theorem shows what conditions on the jumps lead to Condition B. However, the next result follows directly from (2.49) and Condition A.

THEOREM 3. *If Conditions A and B are satisfied, then the speed of a characteristic wave, c, is given by*

$$F(LQ)^2(c^2 - c_\infty^2) + KGTLQ(c^2 - c_0^2) + 2\bar{q}_1 FLQ(c^2 - c_\infty^2) = 0. \quad (2.50)$$

Finally, we can rewrite (2.50) by replacing c by the first relation of (2.47) substituting $\phi_0 + v^j \phi_j$ for L (see 2.21), and kK for LQ. We find

THEOREM 4. *If Conditions A and B are satisfied, then the characteristic manifolds are the solutions of the first-order partial differential equation (when $K \neq 0$, $k = k(\phi)$),*

$$(2\bar{q}_1 + kK)F\{c_\infty^2 \phi - (\phi_0 + v^j \phi_j)^2\} + GTK\{c_0^2 \phi - (\phi_0 + v^j \phi_j)^2\} = 0. \quad (2.51)$$

Our next result is

THEOREM 5. *If Conditions A and B are satisfied then for specified $k(\phi)$, $\bar{q}_1$, Problem I has a unique solution.*

With the aid of (2.51), (2.24), (2.26), (2.31a), (2.41), we can determine the solution of Problem I (the discontinuity theory problem) when Conditions A and B are satisfied. This solution is given by the following steps: (1) choose an arbitrary function $k(\phi)$; (2) choose functions $\bar{q}_1$, $\bar{P}_1$, $\bar{s}_1$ which satisfy (2.44) for properly determined ρ, S, q, v^j (that is, solutions of (2.6)-(2.10)), then the equation (2.51) determines the ∞^1 characteristic S_3, and Q, s, P, V_j are determined by (2.31a), (2.24), (2.41), (2.26), respectively.

Now, we seek the analogs of Conditions A and B which are sufficient to solve Problem II (the Cauchy problem). First, we show that the necessary conditions (2.33), (2.34), (2.37)-(2.40) are insufficient to solve this problem. Let us choose some family of curves which span a given S_3 (that is, one curve passes thru each point of S_3), then k of (2.37) is known. Hence $\bar{Q}$, $\bar{s}$, $\bar{P}$, $\bar{V}^j$ can be determined by use of (2.37), (2.34), (2.40), (2.39), respectively. But, the question arises: "do *all* families of curves on *all* S_3 lead to the *same* k?" If we eliminate (2.37) from our system then (2.33), (2.34), (2.39), (2.40) form a system of six equations, of which five are linear and one is quadratic, in the seven unknowns $\bar{q}$, $\bar{Q}$, $\bar{s}$, $\bar{P}$, $\bar{V}^j$. Thus, *we must make some new assumption in order to be able to solve the Cauchy problem.* One method for doing this is to specify $\bar{q}$, then the above equations furnish a unique solution for the six unknowns $\bar{Q}$, $\bar{s}$, $\bar{P}$, $\bar{V}^j$. The question now is the following one: "are the differential equations (2.35), or the equivalent equation (2.37) and the differential equation (2.38), satisfied for this choice of $\bar{q}$?" Hence, this direct method leads to difficulties.

In order to solve Problem II, we shall introduce two conditions whose role in the Cauchy problem is similar to that of Conditions A and B in the discontinuity theory problem.

Condition C. K_1, K_3 of (2.33), (2.35) are specified in terms of the Cauchy data $\underset{a}{A}$, $\underset{a}{B}$, $\underset{a}{C}$ of (2.12)-(2.14) so that

$$K_3 = 0, \qquad K_1 = kK - L\bar{Q}, \tag{2.52}$$

where k is any constant but $\bar{Q}$ will be determined by (2.55), (2.58) or (2.55), (2.59).

Condition D. The second normal derivative of q is specified so that for $\bar{q}$ defined by (2.8)

$$\phi^\alpha \partial_\alpha \frac{\bar{q}}{K} = 0. \tag{2.53}$$

Now, we shall prove

Theorem 6. *If Conditions C and D are satisfied then Problem II (the Cauchy problem) has two solutions for* any S_3 *of class C^1. Hence the Cauchy problem is well-posed when the range of $\bar{Q}$ is specified. Further, the family of fields $Z+$, $Z-$, satisfy the discontinuity theory relations of Theorem 5.*

To show that Conditions C and D suffice to determine at most two distinct values of $\bar{P}$, $\bar{s}$, $\bar{Q}$, $\bar{q}$, $\bar{V}^j$ which satisfy the Cauchy equations of Problem II (2.33)-(2.35), (2.39), (2.40), we form the scalar product of (2.35) with any tangent vector t^α to the given S_3 at any point, and obtain by use of the first equation of Condition C.

$$t^\alpha \partial_\alpha \frac{\bar{q}}{K} = 0. \tag{2.54}$$

From (2.54) and (2.53), we find that for an arbitrary constant k

$$\bar{q} = Kk. \tag{2.55}$$

Note that k will have different values on S_3^+ and S_3^-. The equation (2.55) and the second relation of Condition C imply that (2.33) is valid. By substituting (2.55) and the first equation of Condition C into (2.35), we find the Cauchy equation which corresponds to the vanishing of the right hand side of (2.46), that is,

$$\frac{C\bar{P}}{\rho^2} + F\bar{s} + G\bar{Q} = 0. \tag{2.56}$$

Thus, (2.34), (2.39), (2.40), (2.56) consist of five linear and one quadratic equation in the six unknowns $\bar{P}$, $\bar{Q}$, $\bar{s}$, $\bar{V}_j$. It is easily verified that *the jumps of these relations coincide with the discontinuity equations*. To determine when this system can be solved, we multiply (2.39) by ϕ^j, solve for $\phi^j\bar{V}_j$, use this last expression to eliminate $\phi^j\bar{V}_j$ in (2.40), and then eliminate $\bar{s}$ in (2.40) by use of (2.34). By this procedure, we obtain the equation for the Cauchy problem which corresponds to (2.41), where L, ϕ, A are related to c, $\bar{c}$ by (2.47),

$$-(L^2 - A\phi)\,\bar{P} = \frac{\phi\bar{Q}}{KT}(-L\bar{Q}B + 2\bar{q}B + KTC) + K_7. \tag{2.57}$$

Assuming that $L^2 - A\phi$ does not vanish, we can use (2.57) to eliminate $\bar{P}$ in (2.56). In addition, we can eliminate $\bar{s}$ in (2.56) by use of (2.34) and then find the following expression for $\bar{Q}$ in terms of the speeds c_0 of (2.47) and $'c_\infty$

$$\begin{aligned}(-2\bar{q} + L\bar{Q})F\bar{Q}\{'c_\infty{}^2\phi - (\phi_0 + v^j\phi_j)^2\}\\ + \bar{Q}KTG\{c_0{}^2\phi - (\phi_0 + v^j\phi_j)^2\} + K_8 = 0\end{aligned} \tag{2.58}$$

where $'c_\infty$ is obtained by replacing A by $-A$ in c_∞ of (2.47). This last relation is similar to equation (2.51) of discontinuity theory. *If $L^2 - A\phi$ vanishes*, then from (2.57), the equation for $\bar{Q}$ is

$$LB\bar{Q}^2 - (2\bar{q}B + KTC)\bar{Q} + K_9 = 0. \tag{2.59}$$

Since two solutions exist for $\bar{Q}$, $\bar{s}$, etc., the fields $Z+$, $Z-$, etc. needed for the determination of (2.16)-(2.20) exist.

The theory for the case when K is a specified function of class C^1 in the variables t, x^j is *similar* to the theory for the case where K is a constant [1]. In order to verify this result, we shall outline the discontinuity theory problem. Here (2.20), (2.24), (2.26), (2.27) remain valid but (2.25) is replaced by the simpler relation (note by (2.20), $[\bar{q}] = LQ$)

$$[\partial_\alpha\bar{q}] = -\phi_\alpha K\left(GQ + Fs + \frac{CP}{\rho^2}\right) + LQ\partial_\alpha \ln K. \tag{2.60}$$

From (2.60), it can be easily shown that Theorems 1-5 remain valid but that Condition A is superfluous. Note, since (2.31a) is still valid, the characteristic manifolds are determined by (2.51).

Now, we study a special case of relaxation hydrodynamics where the theory of discontinuities is similar to that of conventional hydrodynamics. We shall prove

THEOREM 7. *If a one parameter family of k, $\bar{q}$ defined by (2.31a), (2.8), respectively, exist such that*

$$2\bar{q}_1 = -kK \tag{2.61}$$

and the Conditions A and B of (2.44), (2.45), respectively, are satisfied then: (1) *the characteristic manifolds (2.51) are independent of k, K;* (2) *the jumps Q, s, P. V^j form a one parameter family. Further, the Condition A of (2.44) and the relation (2.61) can be satisfied by proper choice of $\bar{q}_1$ and the normal derivative of K (that is, $\bar{P}_1$, $\bar{Q}_1$, $\bar{s}_1$).*

If we substitute (2.61) into (2.51), we find that the characteristic equation is

$$c^2 = c_0^2. \tag{2.62}$$

Since (2.62) is independent of k, K, this establishes the first part of our result. Further, by substituting (2.61), into (2.24), we obtain

$$s = 0. \tag{2.63a}$$

The relations (2.20), (2.31a), (2.26), (2.27) furnish six linear algebraic equations for the six unknown jumps Q, $[\bar{q}]$, P, V^j in terms of kK, which posses unique solutions since $c^2 \neq \bar{c}^2$. Hence, *the jumps form a one parameter family*. Finally, if we replace LQ by kK (see (2.31a)) and express the normal derivatives of K in terms of $\bar{P}_1$, $\bar{Q}_1$, $\bar{s}_1$, P, Q, s by use of (2.12)-(2.14), (2.23), we see that (2.44), (2.61) are two linear equations for the unknown $\bar{q}_1$, $\bar{P}_1$, $\bar{Q}_1$, $\bar{s}_1$ in terms of kK. Hence, (2.44), (2.61) can be satisfied by properly choosing the above unknowns.

Finally, we note that the Cauchy problem corresponding to (2.61) will possess a *unique* solution. The basic conditions for this problem are (2.52), (2.55), (2.34), (2.35), (2.37), (2.39), (2.40) and

$$-2\bar{q} + L\bar{Q} = -K_1 . \tag{2.63b}$$

The solution of this system is easily determined. Replacing $\bar{q}$ of (2.63b) by kK (see (2.55)) and comparing with (2.37), we see that *k (and hence $\bar{q}$) must vanish*. Further, by use of this last result and (2.63b), we see that *$\bar{s}$ vanishes* (see (2.34)). Since $\bar{q}$ vanishes, (2.35) reduces to (2.56), with $\bar{s} = 0$. Again, forming the scalar product of (2.39) with ϕ^j and eliminating $\phi^j V_j$ from the resulting equation by use of (2.40), we obtain a linear equation in $\bar{P}$, $\bar{Q}$.

The determinant of $\bar{P}$, $\bar{Q}$ in the last relation and (2.56) is $c^2 - c_0{}^2$. Hence, *if c is not $\pm c_0$, a unique solution exists for $\bar{P}$, $\bar{Q}$ such that $s = \bar{q} = k = 0$.* From (2.63b), we see that K_1 is $-L\bar{Q}$. Note, the right hand sides of (2.39), (2.40) depend on the Cauchy data $'K_j$, K_6.

Now, we shall show how the *limit values of* K, c and Condition B are related in the *nonlinear theory*. We assume that Condition A of (2.44) is satisfied on the S_3 under discussion and prove

THEOREM 8. (a) *If* $\lim K = \lim \bar{q}_1 = 0$ *and* $\lim Q = b$, *where b is finite and does not vanish, then* $\lim c = c_\infty$ *implies that Condition B of (2.45) is valid and conversely;* (b) *if* $\lim \bar{q}_1 = a$, $\lim Q = b$, *where a, b are finite and b does not vanish, then* $\lim K = \pm \infty$ *implies* $\lim c = c_0$*;* (c) *if* $\lim \bar{q}_1 = a$, $\lim Q = b$, *where a, b are finite and b does not vanish, and Condition B is satisfied, then* $\lim K = 0$ *implies* $\lim c = c_\infty$, *and conversely.*

The proof follows directly from (2.49). We note that in deriving (2.49), we assumed that none of the jumps P, s vanish.

Now, we shall prove that for a *properly linearized theory*, results similar to those of Theorem 8 are valid. In addition to Condition A of (2.44), $\bar{q}_1$, K must satisfy (2.61). Then, we show

THEOREM 9. (a) *If* $\lim K = 0$ *and* $\lim Q = 0$, *then* $\lim c = \bar{c}$ *and Condition B of (2.45) are valid and if Condition B is valid and* $\lim Q = 0$ *then* $\lim K = 0$, $\lim c = \bar{c}$*;* (b) *if* $\lim K$ *is nonvanishing and finite and* $\lim Q$ *is finite and nonvanishing then* $\lim c = c_0$ *implies Condition B is valid and conversely.*

In case (a), since $\lim Q = 0$, (2.24) becomes

$$s = 0. \tag{2.64}$$

Further, in case (b), Theorem 1 is valid. From theorem 1 (see (2.31a)) and the conditions (2.61), (2.24), we obtain (2.64). Thus, the quadratic equation (2.24) in Q is replaced in both cases (a) and (b) by the linear equation (2.64). The proof of Theorem 9 can be obtained in two steps. First, if we require that s vanish in (2.26) and then eliminate V^j in (2.26) by use of (2.27). We obtain by use of (2.47)

$$P(A - c^2) + CQ = 0, \qquad c \neq 0. \tag{2.65}$$

Secondly, we note that (2.25) is equivalent to the two equations (2.31a) and (2.46). From (2.31a), we find a linear relation for Q and the unknown k. However, from (2.46), we obtain

$$-K\left(P\frac{C}{\rho^2} + GQ\right) = \phi^\alpha[\partial_\alpha \bar{q}]\,(\phi^\gamma\phi_\gamma)^{-1}. \tag{2.66}$$

Theorem 9 follows from Eq. (2.65), (2.66), the fact that the determinant of the left hand sides is $G(c_0^2 - c^2)$, and the relation $\bar{c}^2 \equiv A$ of (2.47).

From Theorems 8, 9 we see that the speed $\bar{c}$ of the linearized theory plays the same role as speed c_∞ of the nonlinear theory. This result leads to Theorem 19 in a chemically reacting fluid.

3. Simple Waves in One-Dimensional Nonsteady Motion

The bicharacteristics can be determined by differentiating the equation (2.50), which is homogeneous of degree zero in ϕ_α when LQ is replaced by kK[11]. In order to express the results of this differentiation in a simple form, we define two invariants I, J by

$$I \equiv (Fk + GT)\, K + 2\bar{q}_1 F$$

$$J \equiv (Fkc_\infty^2 + GTc_0^2)\, K + 2\bar{q}_1 F c_\infty^2. \tag{3.1}$$

By use of (3.1), the relation (2.50) becomes

$$X \equiv Ic^2 - J = 0, \tag{3.2}$$

where by definitions (2.11), (2.47)

$$c^2 \equiv \frac{(\phi_0 + v^j\phi_j)^2}{\phi}, \qquad L \equiv \phi_0 + v^j\phi_j\,. \tag{3.3}$$

Differentiating (3.2) with respect to ϕ_α, and using (3.3), (3.2), we find that the bicharacteristics are determined by

$$\frac{dx^j}{d\sigma} \equiv \frac{1}{2}\frac{\partial X}{\partial \phi_j} = ILv^j - J\phi^j \tag{3.4}$$

$$\frac{dx^0}{d\sigma} \equiv \frac{1}{2}\frac{\partial X}{\partial \phi_0} = IL = \pm (IJ\phi)^{1/2}. \tag{3.5}$$

The case of equilibrium or conventional hydrodynamics corresponds to $I = 1$, $J = A$. Dividing the left hand side of (3.4) by the left hand side of (3.5) and then dividing the corresponding right hand sides, we obtain

$$\frac{dx^j}{dx^0} = v^j \pm cn^j \tag{3.6}$$

where c^2 is J/I (see (3.2)) and n^j is the space-unitized normal to the characteristic space-time manifolds S_3. We note that

$$n^j \equiv \phi^j/\phi^{1/2} \tag{3.7}$$

and (3.6) coincides with the corresponding equation in equilibrium hydrodynamics when $(J/I)^{1/2}$ is replace by the sound speed. In particular, for the important nonsteady one-dimensional case, x^j, v^j, n^j have the components $(x, 0, 0)$, $(u, 0, 0)$, $(1, 0, 0)$, respectively, and (3.6) may be replaced by (cf. [9, p. 290])

$$\frac{\partial x}{\partial \alpha} = (u + c)\frac{\partial t}{\partial \alpha}, \qquad \frac{\partial x}{\partial \beta} = (u - c)\frac{\partial t}{\partial \beta} \tag{3.8}$$

where α, β are parameters along the two families of bicharacteristics. The following two inverse relations for transforming from the (α, β) to the (x, t) variables will be used in our future work

$$\frac{\partial \alpha}{\partial t} = -\bar{J}\frac{\partial x}{\partial \beta}, \qquad \frac{\partial \alpha}{\partial x} = \bar{J}\frac{\partial t}{\partial \beta} \tag{3.9}$$

where by (3.8)

$$\bar{J}^{-1} \equiv \frac{\partial x}{\partial \alpha}\frac{\partial t}{\partial \beta} - \frac{\partial x}{\partial \beta}\frac{\partial t}{\partial \alpha} = 2c\frac{\partial t}{\partial \alpha}\frac{\partial t}{\partial \beta}. \tag{3.10}$$

By eliminating $\partial x/\partial \alpha$, $\partial x/\partial \beta$ in (3.9) by use of (3.8), we obtain the relations

$$\frac{\partial \alpha}{\partial t} = \frac{c - u}{2c}\left(\frac{\partial t}{\partial \alpha}\right)^{-1}, \qquad \frac{\partial \alpha}{\partial x} = \frac{1}{2c}\left(\frac{\partial t}{\partial \alpha}\right)^{-1}. \tag{3.11}$$

The formulas (3.11) can be used only when $\bar{J}^{-1} \neq 0$.

In order to develop the theory of simple waves in the x, t plane we introduce.

Definition. A family of simple waves consists of a family of bicharacteristics (with parameter, β = variable or α = constant, along each curve of the family) such that

$$\frac{\partial q}{\partial \beta} = \frac{\partial S}{\partial \beta} = \frac{\partial p}{\partial \beta} = 0 \tag{3.12a}$$

$$\frac{\partial v_j}{\partial \beta} = 0. \tag{3.12b}$$

We shall consider Cases I and II separately. However, first we shall determine those consequences of the energy relations (2.6), (2.7), the definition of $\bar{q}$ (2.8), and the geometry of the bicharacteristics (3.2), (3.8), (3.11) which are valid in *both* Cases I and II. In order to express (2.6), (2.8) in terms of the derivatives of S, q with respect to α, we use the chain rule for differentiation and simplify by use of (3.11) and the definition of simple waves (3.12). We find

$$KT\frac{dS}{d\alpha} = 2\bar{q}^2\frac{\partial t}{\partial \alpha} \tag{3.13}$$

$$\frac{dq}{d\alpha} = 2\bar{q}\frac{\partial t}{\partial \alpha}. \tag{3.14}$$

Further, from (2.7), we obtain

$$\frac{\partial \bar{q}}{\partial \alpha} = -K\left(G\frac{dq}{d\alpha} + F\frac{dS}{d\alpha} + \frac{C}{\rho^2}\frac{d\rho}{d\alpha}\right) + \bar{q}\frac{\partial}{\partial \alpha}\ln K \tag{3.15a}$$

$$\frac{\partial \bar{q}}{\partial \beta} = \bar{q}\frac{\partial}{\partial \beta}\ln K. \tag{3.15b}$$

Again, eliminating x by differentiating the two equations of (3.8), we find

$$\frac{\partial^2 t}{\partial \alpha \partial \beta} - \frac{1}{2c}\frac{\partial t}{\partial \beta}\frac{\partial}{\partial \alpha}(u - c) + \frac{1}{2c}\frac{\partial t}{\partial \alpha}\frac{\partial}{\partial \beta}(u + c) = 0. \tag{3.16}$$

Finally, we note that any bicharacteristic curve, $\alpha =$ constant, is a characteristic manifold, $\phi =$ constant, in the (x, t) plane. Hence, we may write (see (2.31b))

$$k = k(\alpha) \tag{3.17}$$

First, we prove

THEOREM 10. *In both Cases I and II, if the rate of change of internal energy with respect to the relaxation variable is positive (negative) then, as one moves from one simple wave to another, entropy decreases (increases) as the relaxation variable increases.*

By eliminating $\partial t/\partial \alpha$ in (3.13) by using (3.14), we find when $\bar{q}$ does not vanish (that is, the rate of change of internal energy with respect to the relaxation variable is not zero)

$$KT\frac{dS}{d\alpha} = \bar{q}\frac{dq}{d\alpha}. \tag{3.18a}$$

The energy relation, from which (2.7) was derived, can be written as (where e is the internal energy, cf. p. 782 of [2])

$$\frac{de}{dq} = -\frac{\bar{q}}{K}. \tag{3.18b}$$

Eliminating $\bar{q}/K$ in (3.18a) by use of (3.18b), we obtain

$$\frac{de}{dq} = - T \frac{dS}{dq}. \tag{3.18c}$$

The relation (3.18c) leads to the desired result.

Case I: $K(\rho, q, S)$ of class C^1 in ρ, q, S

From (3.12a), (3.18a), it follows that in Case I, $\bar{q}$ *is a function of* only α. Thus, from (3.2), we find by use of (3.17), (3.1) that *c is a function of only α*. We summarize Case I when $K \neq 0$, $\bar{q} \neq 0$ by

THEOREM 11. *For simple waves of Case I, the dependent variables $\bar{q}$, c, u (the particle velocity), ρ, S, q, K and all of the thermodynamical variables are functions of* only α. *The case $K =$ constant of* [1] *is a subcase of Case I (see Theorem 14).*

Now, we shall determine the geometry of the simple waves of Case I. By use of Theorem 11, the equation (3.16) reduces to

$$\frac{\partial^2 t}{\partial\alpha\partial\beta} - \frac{1}{2c}\frac{\partial t}{\partial\beta}\frac{d}{d\alpha}(u - c) = 0. \tag{3.19}$$

The differential equation (3.19) is linear in $\partial t/\partial\beta$. If we define the function $F^*(\alpha)$ by

$$\ln F^*(\alpha) \equiv \int^{\alpha} \frac{1}{2c}\left(\frac{du}{d\alpha} - \frac{dc}{d\alpha}\right) d\alpha \tag{3.20}$$

then two integrations of (3.19) furnish the possible solutions, where $\bar{a}$ is an arbitrary constant,

$$t = A^*(\alpha) + B^*(\beta), \qquad u - c = \bar{a} \tag{3.21a}$$

$$t = A^*(\alpha) + B^*(\beta) F^*(\alpha), \qquad u - c \neq \bar{a} \tag{3.21b}$$

and $A^*(\alpha)$, $B^*(\beta)$ are of class C^1 in α, β respectively.

By use of Theorem 11, we see from the relations (3.13), (3.14) that $\partial t/\partial\alpha$ is a function of only α. Hence, *only (3.21a) is possible*. The relations (3.13), (3.14), (3.15a) furnish

$$KT\frac{dS}{d\alpha} = 2\bar{q}^2 \frac{dA^*}{d\alpha} \tag{3.22}$$

$$\frac{dq}{d\alpha} = 2\bar{q}\frac{dA^*}{d\alpha} \tag{3.23}$$

$$\frac{d\bar{q}}{d\alpha} = - K\left(G\frac{dq}{d\alpha} + F\frac{dS}{d\alpha} + \frac{C}{\rho^2}\frac{d\rho}{d\alpha}\right) + \bar{q}\frac{d}{d\alpha}\ln K. \tag{3.24}$$

Finally, by noting that, $u - c = \bar{a}$, in (3.21a) we can integrate the second equation of (3.8) and obtain

THEOREM 12. *The family of simple waves for Case I consists of* ∞^1 parallel *lines in the* (x, t) *plane with constant slope,* $\bar{a} = u - c$. *The scalars* q, $\bar{q}$, S *are determined by the unknown density* ρ, *the arbitrary function* A^*, *and the known function* K; *the differential equations (3.22)-(3.24) determine* S, q, $\bar{q}$ *in terms of* A^*, K, ρ. *The case,* $K =$ *constant* [1], *is of this type.*

Case II: $K(x^j, t)$ *of class* C^1 *in* x^j, t.

From (3.18b) and (3.1), (3.2), (3.17), we obtain the following "weak" form of Theorem 11 (for $K \neq 0$, $\bar{q} \neq 0$):

THEOREM 13. *A necessary condition for simple waves of Case II to exist is that* $K/\bar{q}$ *be a function of* only α.

THEOREM 14. *In simple waves of Case II, the dependent variables* $K/\bar{q}$ *(but neither* K *nor* $\bar{q}$ *separately),* c, u, ρ, S, q, *and all of the thermodynamical variables are functions of* only α.

Again, the geometry of simple waves is determined by the equations (3.19)-(3.21b) but (3.22)-(3.24) are no longer valid. This last result is due to the fact that Theorem 14 for Case II implies both $\bar{q}$ and K are product functions of the type

$$\bar{q} = p(\alpha)\, s(\alpha, \beta), \qquad K = w(\alpha)\, s(\alpha, \beta) \tag{3.25}$$

where, p, w are functions of α and s is a function of α and β. If we substitute (3.25) into (3.13) or (3.14), we see that (3.21a) *is not possible* except for the case $s = s(\alpha)$ which is in Case I. Let us write

$$\frac{dA^*}{d\alpha} = C^* \frac{dF^*}{d\alpha} \tag{3.26}$$

where C^* is an arbitrary function of α. If (3.21b) is used to evaluate $\partial t/\partial\alpha$, and K, $\bar{q}$ are replaced by (3.25), we find that (3.13), (3.14) lead to

$$wT \frac{dS}{d\alpha} = 2fp^2 \frac{dF^*}{d\alpha} \tag{3.27}$$

$$\frac{dq}{d\alpha} = 2fp \frac{dF^*}{d\alpha} \tag{3.28}$$

$$(C^* + B^*)\, s = f \tag{3.29}$$

where f is an arbitrary constant. Further, we find by use of (3.25) that (3.15b) is an identity and (3.15a) reduces to

$$\frac{d}{d\alpha}\ln p = \frac{-p}{w}\left(G\frac{dq}{d\alpha} + F\frac{dS}{d\alpha} + \frac{C}{\rho^2}\frac{d\rho}{d\alpha}\right) + \frac{d}{d\alpha}\ln w. \tag{3.30}$$

In particular, if $s = s(\beta)$ then C^* is an arbitrary constant. Note, by (3.21b), and Theorem 14, $u - c$ is a function of α and hence the simple waves are *not parallel* lines. We can summarize our results by (compare with Theorem 12 for Case I).

THEOREM 15. *The family of simple waves for Case II consists of* ∞^1 non-parallel *lines in the* (x, t) *plane. The scalars* q, S, p, w *are determined by the unknown density* ρ, *the function* F^*, *related to* u, c, *by (3.20), and the arbitrary constant* f; *the differential equations (3.27), (3.28), (3.30) are three relations for* q, p, w, S *in terms of* ρ, F^*, f. *The equation (3.29) determines* s *in terms of the arbitrary functions* $C^*(\alpha)$, $B^*(\beta)$. *Then,* $\bar{q}$, K *are determined by (3.25) as functions of* both α and β.

Now, we shall consider the equations of continuity and motion, (2.10) and (2.9), respectively, for simple waves defined by (3.12). By use of the chain rule for differentiation, the definition (3.12), the expression for $\partial\alpha/\partial t$, $\partial\alpha/\partial x$ of (3.11), the definitions of A, B, C in (2.4), and the relation (3.18a) for $dS/d\alpha$ in terms of $dq/d\alpha$, we find that (2.10), (2.9) become, respectively,

$$c\frac{d\rho}{d\alpha} + \rho\frac{du}{d\alpha} = 0 \tag{3.31}$$

$$\rho c\frac{du}{d\alpha} + A\frac{d\rho}{d\alpha}\rho + \left(\frac{B\bar{q}}{KT} + C\right)\frac{dq}{d\alpha} = 0. \tag{3.32}$$

The relations (3.31), (3.32) are valid for both Cases I and II. However, for Case I, we have the additional relation

$$u - c = \bar{a}, \tag{3.33}$$

where $\bar{a}$ is a constant. We note that c^2 is equal to J/I (see (3.2)), which is a known function of ρ, q, S, $K/\bar{q}_1$. Hence, as we noted in Theorems 9 and 12, c as well as u, ρ, q, S are functions of α for Cases I, II. However, the question of *existence* and *uniqueness* of a solution for c, u, ρ, q, S differs in Cases I and II. In Case I, if K, k are specified then c can be determined by (3.2) in terms of $\bar{q}$, which coincides with $\bar{q}_1$ when $\bar{q}$ is continuous. Hence, u is determined in terms of $\bar{q}$ by (3.33). In Case II, the algebraic and differential equations are linked. We shall prove

THEOREM 16. *The sufficient conditions to determine simple waves in Case I for specified* $K(\alpha)$, *and properly chosen*, $k(\alpha)$ *are that:*
q, $\bar{q}$, ρ, S, A^* *satisfy the five differential equations (3.22)-(3.24), (3.31), (3.32) when* c, u *are determined in terms of* $\bar{q}$, q, ρ, S *by (3.2), (3.33), respectively.*

For Case I, the differential equations (3.22)-(3.24), (3.31), (3.32) are five equations in the first derivatives of the five unknowns, q, $\bar{q}$, ρ, S, A^*; these equations consist of a system of quasi-linear, nonhomogeneous, ordinary differential equations. Thus, the *existence* and *uniqueness* of a solution will depend on choosing $k(\alpha)$ so that the determinant of the system does not vanish (for $K =$ constant, ∞^1 (i.e., a one parameter family, solutions exist).

In Case II, the theory is different. Again, k must be specified. But, the six differential equations ((3.27, (3.28), (3.30)-(3.32), and (3.20) when differentiated) and the three algebraic equations (3.2), (3.29) are linked. Note, we assume K, $\bar{q}$ will be expressed in terms of ρ, w, s by (3.25). Thus, our system of differential equations will contain ρ, u, c, q, S, p, w, F^* and the algebraic system will contain the additional variables s, $C^* + B^*$. Since this system contains nine equations of mixed type (differential and algebraic) in ten dependent variables, there will exist ∞^1 solutions (in general) for given boundary data of the ten dependent variables. Hence, unless one of the above variables is specified, the solution *will not be unique*. Since K should possess some degree of arbitrariness, one would expect to specify *one* of the following variables, w, s, or $C^* + B^*$. If this is done, the resulting system of differential and algebraic equations consists of nine equations in nine dependent variables. However, the question of *existence* and *uniqueness* of solution can be discussed only for particular cases.

Now, we return to (3.31), (3.32). If we eliminate $d\rho/d\alpha$ in these last equations, and use (3.18b), (3.18c) to simplify $C + (B\bar{q}/KT)$, we obtain

$$\frac{du}{dq} = - \frac{c}{\rho(c^2 - A)} \left(B \frac{dS}{dq} + C \right). \tag{3.34}$$

From Theorem 10 and the relation (3.34), we find

THEOREM 17. *If the rate of increase of internal energy or of entropy with respect to the relaxation variable is such that* $C + B(dS/dq)$ *is positive then:* (1) *in the supersonic case*, $c^2 > A$, *the particle speed*, u, *is decreasing as the relaxation variable increases;* (2) *in the subsonic case*, $c^2 < A$, *the particle speed*, u, *is increasing as the relaxation variable increases.*

Finally, we obtain two results by use of: (1) the equation (3.31) for both Cases I and II; (2) the equations (3.31), (3.33) for Case I. The first result is similar to that of equilibrium hydrodynamics.

THEOREM 18. *In both Cases I and II, as one moves from one simple wave to another such wave, the density increases (decreases) as particle speed decreases (increases). For the parallel simple waves of Cases I, we obtain*

$$(u + \bar{a}) = \rho c = \bar{b} \tag{3.35}$$

where $\bar{b}$ is a constant.

4. NONEQUILIBRIUM, EQUILIBRIUM, AND FROZEN STATES OF A CHEMICALLY REACTING FLUID

In this section, we shall consider a special class of relaxation flows, namely, those which are associated with a chemically reacting fluid. To simplify the discussion, we consider the case of a *simple dissociating gas* and determine how the results of Sections 2 and 3 must be modified. The hydrodynamical theory of such a gas has been discussed by Ting Y. Li [6]. In this last paper, Li has given a survey of the theory for a general reacting fluid.

A related theory has been recently developed by C. Yuan [12] using the constituitive coefficients of Onsager and Curie [13]. By use of this theory, Yuan has shown that the relaxation scalar K, of Section 2 and 3 is a particular function of the Curie coefficient, the density, and the temperature. Hence, if the Curie coefficient is constant (or more generally, a function of ρ, S, q), *then K is of Case I.*

The usual assumptions are that nonequilibrium, equilibrium, and frozen states exist and are defined by the following conditions [6]:

DEFINITIONS:

(a) *Nonequilibrium state*

$$\bar{q} \equiv \frac{dq}{dt} = f(q, \rho, T) \tag{4.1}$$

where f is a differentiable function of q, ρ, T.

(b) *Equilibrium state*

$$\bar{q} \equiv 0 \tag{4.2}$$

(c) *Frozen state*

$$q \equiv \text{constant} \tag{4.3}$$

For the *simple dissociating gas* of the *nonequilibrium* state flow, f of (4.1) is given by [6]

$$f(q, \rho) \equiv \frac{1}{\tau} [{}'K(1 - q) - q^2] \tag{4.4}$$

where $'K$, a new relaxation scalar, and τ can be expressed in terms of K_f, K_r, the forward and reverse specific reaction rate constants, M, the molecular weight of the gas, ρ, the density, and, q, the relaxation variable (the degree of dissociation), respectively, by

$$'K \equiv \frac{MK_f}{4\rho K_r}, \qquad \tau \equiv \frac{M^2}{4\rho^2 K_r(1+q)}. \tag{4.5}$$

Broer [3] has used a linear function of q for f in his theory and Yuan [12] has shown how to obtain Broer's relation from the constituitive equations. The relation (4.4) can also be obtained from the constituitive equations when *additional assumptions are made.*

To relate the theory of Sections 2 and 3 to the theory of flows in the nonequilibrium state (4.1) of a simple dissociating gas, we need only to determine the relaxation scalar, K, of Section 2. Following the work of Stupochenko and Stakhanov [2], we write

$$-K\left.\frac{\partial e}{\partial q}\right)_{S,\rho} = f(\rho, q) = \bar{q}. \tag{4.6}$$

Thus, when $\partial e/\partial q$ is known, *K can be determined in terms of the known f* of (4.4). Since $\partial e/\partial q$ is a function of ρ, S, q, it follows from (4.6) that K is of Case I for a simple dissociating gas. The specific function can be determined when T is known as a function of ρ, S, q.

On the basis of linearized theory, Li (cf. p. 174 of [6]) has concluded that the gas speed in *nonequilibrium state* flow lies between c_0, the equilibrium speed, and $\bar{c}$, the frozen speed. The variable which determines the gas speed is the relaxation variable, K (cf. (4), (6) of [2]). In Theorem 9, we noted that $K = 0$ is associated with $c = \bar{c}$, and $K \neq 0$ (in particular, the limiting case of infinite K) is associated with $c = c_0$. Further, the basic equations (2.65), (2.66) lead to no other gas speeds. However, from Theorems 8 and 3, and the fact that for a simple dissociative gas, K is a function of ρ, S, q we are led to the following result:

THEOREM 19. *In the nonlinear theory of the nonequilibrium state of a simple dissociative gas, the gas speeds vary between c_0 and c_∞, depending upon the relaxation scalar K, $LQ = kK$, and $\bar{q}_1$ of (2.50).*

Now, we shall consider the equilibrium state, defined by (4.2), and the frozen state, defined by (4.3), and prove

THEOREM 20. *The characteristic equations for the equilibrium state and frozen states are independent of K and are given by*

$$Fk(c^2 - c_\infty^2) + GT(c^2 - c_0^2) = 0 \tag{4.7}$$

$$c = \bar{c}, \tag{4.8}$$

respectively.

These results follow from (2.50) and the fact that (2.6)-(2.10) reduce to the relations of conventional gas dynamics for the frozen state.

Finally, we shall prove

THEOREM 21. *Simple waves* do not exist *in the equilibrium and frozen states. Simple waves* do exist *in nonequilibrium flows and are of Case I.*

The proof follows immediately from the fact that (4.2), (4.3), (2.8) imply that

$$\frac{dS}{dt} = 0. \tag{4.9}$$

Since S is constant along stream lines, it cannot be constant along the bicharacteristics. Hence, simple waves do not exist for the equilibrium and frozen states. On the other hand all the conditions for simple waves of Case I (see Section 3) are satisfied in the nonequilibrium state.

NOTE ADDED IN PROOF

For further discussions of the theory of discontinuities, see [14-16]. Also, the simple waves of Section 3 are generalized simple waves in the sense of P. D. Lax, although the basic system of this paper, (2.6)-(2.10), is not of conservation type (as in Lax's theory) and the discontinuity theory for the characteristics of this basic system differs from that of the quasi-linear conservation system ([16], pp. 37-38). This difference is due to the fact that our system is quadratic in the dependent variable, $\bar{q}$, which is discontinuous along a characteristic manifold. For the conservation theory of generalized simple waves, see [16], p. 86 and [17].

REFERENCES

1. N. COBURN, The limiting speeds of characteristics in relaxation hydrodynamics. *J. Math. Anal. Appl.* **5** (1962), 269-286.
2. E. V. STUPOCHENKO AND I. P. STAKHANOV, The equations of relaxation hydrodynamics. *Soviet Phys.-Doklady* **4** (1960), 782-785.
3. L. J. F. BROER, Characteristics of the equations of motion of a reacting gas. *J. Fluid Mech.* **4** (1958), 276-282.
4. R. COURANT AND K. O. FRIEDRICHS, "Supersonic Flow and Shock Waves." Interscience, New York, 1948.
5. P. GERMAIN AND R. GUNDERSEN, Sur les écoulements unidemsionnels d'un fluide parfait à entropie faiblement variable. *C. R. Acad. Sci., Paris* **241** (1955), 925.
6. TING Y. LI, Recent advances on non-equilibrium dissociating gas dynamics. *AIAA J.* **31** (1961), 170-178.
7. J. HADAMARD, "Leçons sur la Propagation des Ondes." Hermann, Paris, 1903.
8. T. Y. THOMAS, "Plastic Flow and Fracture in Solids." Academic Press, New York, 1961.
9. N. COBURN, "Vector and Tensor Analysis." Macmillan, New York, 1955.

10. L. P. EISENHART, "Riemannian Geometry." Princeton Univ. Press, 1926.
11. R. COURANT AND D. HILBERT, "Methods of Mathematical Physics, vol. II, Partial Differential Equations." (by R. COURANT) Interscience, New York, 1962.
12. C. YUAN, "Non-Equilibrium Hydrodynamics of a Chemically Reacting Fluid" (Univ. of Mich. Report, O.R.A., 05424, 1-P, 1963.
13. S. R. DEGROOT AND P. MAZUR, "Non-Equilibrium Thermodynamics." North Holland, Amsterdam, 1962.
14. C. FRUESDELL AND R. A. TOUPIN, "Classical Field Theories," Handbuch der Physik, Vol. III/1, p. 526. Springer, Berlin, 1960.
15. J. B. KELLER, Geometrical acoustics, I. The theory of weak shock waves. *J. Appl. Phys.* **25** (1954), 938-947.
16. A. JEFFREY AND T. TANIUTI, "Non-linear Wave Propagation with Applications to Physics and Magnetohydrodynamics," p. 103. Academic Press, New York and London, 1964.
17. P. D. LAX, Hyperbolic systems of conservation laws, II. *Commun. Pure Appl. Math.* **10** (1957), 561.