JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 11, 357-409 (1965)

Critical Point Theory in Hilbert Space Under General Boundary Conditions*

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Submitted by C. L. Dolph

1. INTRODUCTION

Let E^n be the n-dimensional real Euclidean space with points

$$x = (x_1, x_2, \cdots, x_n).$$

Let R and R_2 be bounded open connected subsets of E^n with $\bar{R} \subset R_2$ where the bar denotes the closure operation. The boundary S of R is a suitable manifold. In the Morse theory of the critical points of a real valued function $f(x)$ defined in R_2 one distinguishes "regular" and "general" boundary conditions with respect to R. In both cases it is required that no critical points of $f(x)$ lie on S. In addition, the *regular* boundary condition requires that the vector field

$$g(x) = \operatorname{grad} f(x) \tag{1.1}$$

is exteriorly directed at all points of S. The *general* case, not subject to this restriction was first treated in a joint paper by Morse and van Schaack [1].

The main object of the present paper is to prove at least some of the results obtained by these authors, for a Hilbert space E in a direct manner, i.e. without assuming their validity in E^n.[1] Thus the paper may be considered as a further step towards developing the Morse theory in Hilbert space. (See [3] and [4].)

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* The main part of the research for this paper was done while the author had a research appointment with the Institute of Science and Technology at the University of Michigan and, as a consequence, had a reduced teaching load in the Department of Mathematics. He takes this opportunity to express to the Institute as well as to the Department his thanks for having provided this opportunity.

[1] An exception is Section 9 where theorems on vectorfields in Hilbert space are used which were proved in [2] by assuming their validity in E^n. No "direct" proofs for these theorems are known to the author.

For the purpose of exposition it seems best to recall the basic steps of the procedure followed by Morse and van Schaack in the finite dimensional case.

Under the assumption that all critical points of $f(x)$ in R are nondegenerate they reduced the general case to the regular one in essentially the following way: let R_1 be an open set such that $\bar{R} \subset R_1 \subset \bar{R}_1 \subset R_2$ and such that moreover to each point x of $\bar{R}_1 - R$ corresponds exactly one point $\bar{x}$ on S such that the normal $\bar{n}$ to S at $\bar{x}$ contains x. Let x_0 be a point on S and let N_0 be a neighborhood of x_0 such that

$$x = \bar{x}(u), \qquad x_0 = \bar{x}(0), \qquad u = (u_1, u_2, \cdots, u_{n-1}) \tag{1.2}$$

is a parametric representation of S valid in the intersection $N_0 \cap S$. Then for $x \in N_0$

$$x = \bar{x}(u) + sn(u), \tag{1.2a}$$

and (s, u) may be regarded as local coordinates at x_0. If R_1 is taken as the union of R with the set of all points (1.2a) for which $0 \leqslant s < s_1$ (s_1 small enough) then the modification $\tilde{f}$ of f is defined in $\bar{R}_1$ by

$$\begin{aligned} \tilde{f}(x) &= f(x) && \text{for} \quad x \in R \\ &= f(\bar{x}(u) + sn(u)) + sM/3s_0^2 && \text{for} \quad 0 \leqslant s \leqslant s_1 \end{aligned} \tag{1.3}$$

where M denotes a number satisfying the inequality

$$\| g((x)) \| < M/2 \qquad \text{for} \qquad x \in \bar{R}_1 \tag{1.4}$$

with g defined by (1.1) and with $\| \; \|$ denoting the Euclidean norm while s_0 is a positive constant $< s_1$.

For a suitable choice of s_0 and s_1 it is then shown in [1]:

(i) $\tilde{f}(x)$ satisfies regular boundary conditions with respect to R_1.

(ii) Let the "boundary function" $\phi(u)$ be defined by

$$\phi(u) = f(\bar{x}(u)), \qquad \bar{x} \in S \cap N_0. \tag{1.5}$$

Then the vector (1.1) is normal to S at $x = x_0$ if and only if $u = 0$ is a critical point of $\phi(u)$. The critical points of the boundary function are supposed to be nondegenerate.

(iii) To each point x_0 in S in which the vector (1.1) has the direction of the interior normal there corresponds a unique point x^* in $R_1 - R$ which is a critical point of $\tilde{f}$, and the points x^* thus obtained together with the critical points of f in R are the only critical points of $\tilde{f}$ in R_1.

(iv) Let x_0 and x^* be as in the preceding paragraph. Then x^* as critical point of $\tilde{f}$ is nondegenerate, and its index equals the index of the critical point $u = 0$ of the boundary function ϕ defined in (1.5).

(v) For $j = 0, 1, \cdots, n$ let M^j be the number of critical points of index j of f in R, and let R^j be the jth Betti number of R. Then under the assumption of regular boundary conditions the Morse relations

$$\begin{aligned} R^0 &\leqslant M^0 \\ R^0 - R^1 &\geqslant M^0 - M^1 \\ R^0 - R^1 + R^2 &\leqslant M^0 - M^1 + M^2 \\ &\cdot \end{aligned} \tag{1.6a}$$

$$R^0 - R^1 + R^2 \cdots + (-1)^n R^n = M^0 - M^1 + M^2 \cdots + (-1)^n M^n \tag{1.6b}$$

hold. The results sketched in the above paragraphs (i) to (iv) enabled Morse and van Schaack to prove that the relations (1.6) still hold under general boundary conditions if M^j is replaced by

$$\bar{M}^j = M^j + M^{j-} \tag{1.7}$$

where, as before, M^j is the number of critical points of index j of f in R and where M^{j-} denotes the number of those critical points of index j of the boundary function ϕ at which grad f has the direction of the interior normal.

To carry out in a Hilbert space E the Morse-van Schaack procedure for E^n outlined in paragraphs (i)-(v) above, the first task is to give a suitable definition of a hypermanifold S in E. For the method of this paper it is essential that tangent and parameter spaces for S (as in the case of E^n) are hyperspaces, i.e. closed linear subspaces of E of codimension 1. Unfortunately the notion of hyperspace is not invariant under bounded linear $1 - 1$ maps as the example of the shift operator shows. However, as proved in Lemma 3.2, the notion of hyperspace is invariant under a map of the special form (3.5). The restrictive definition (3.2) of a smooth hypermanifold takes these facts in account and ensures that tangent and parameter spaces of a smooth hypermanifold are hyperspaces. An example of a smooth hypermanifold is the boundary S of the special domain $R \subset E$ defined at the beginning of section 9.

It is shown in Section 4 that under certain differentiability conditions a smooth hypersurface S bounding a domain $R \subset E$ admits at every point a unique exterior and a unique interior normal, and that these normals are twice differentiable (Theorems 4.1 and 4.3).

Using these results and the implicit function theorem ([5]) it is possible to construct at every point x_0 of S a local coordinate system which is the analogue of the "(u, s)" system used in the representation (1.2a). This is done in Section 5.

Such a local coordinate system makes it obviously possible to define the modification $\tilde{f}$ of a given function f as in (1.3). The Hilbert space analogous to the statements (i) and (ii) above are then proved (Lemma 7.6 and Lemmas 6.3, 6.4).

Section 8 establishes statements (iii) and (iv) above for the Hilbert space case. Here the proofs are quite different from the ones given by Morse and van Schaack for the finite dimensional case. For example their proof of (iv) [1; p. 569] is based on a classical theorem of determinant theory according to which the index of a quadratic form Q can be evaluated in terms of the number of changes of sign in a certain sequence of principal subdeterminants of the matrix of Q. No such theorem is available in Hilbert space.

As to the generalization to Hilbert space of the Morse relations (1.6), the present paper is concerned only with (1.6b). In Section 9 we consider the special domain $R = V_0 - \bigcup_{i=1}^{q} \bar{V}_i$ where each V_i is a ball and the bar denotes closure; moreover for $j = 1, 2, \cdots, q$, the $\bar{V}_i$ are disjoint and contained in V_0. Now for such an R in E^n.

$$\text{left member of (1.6b)} = 1 + q(-1)^{n-1}. \tag{1.8}$$

This quantity does not converge as n goes to infinity. Therefore for the purpose of generalization to Hilbert space we rewrite (1.6b) as follows: denote by M_i^{j-} and M_i^{j+} the number of those critical points of index j of the boundary function ϕ which are situated on the boundary S_i of V_i and at which grad f has the direction of the interior normal and exterior normal respectively; here the terms exterior and interior refer to R. Then from (1.7)

$$\bar{M}^j = M^j + \sum_{i=0}^{q} M_i^{j-}. \tag{1.9}$$

Since the total number of critical points of index j of ϕ situated on V_i is $M_i^{j-} + M_i^{j+}$ application of a Morse relation for the closed Riemannian manifold S_i yields

$$1 + (-1)^{n-1} = \sum_{j=0}^{n} (M_i^{j-} + M_i^{j+})(-1)^j. \tag{1.10}$$

By elementary computation we obtain from (1.6b) (with M^j replaced by $\bar{M}^j$), (1.8), (1.7), (1.9), and (1.10)

$$1 - q = \sum_{j=0}^{n} (-1)^j \left[M^j + M_0^{j-} - \sum_{i=1}^{q} M_i^{j+}\right]. \tag{1.11}$$

It is this formula in which the left member is independent of the dimension n which we will generalize to the Hilbert space case.

2. Preliminaries on Differentials and Gradients

Many of the definitions and lemmas of this section are well known (see e.g. [6], [5], [7]); they are included for reference, and no proofs of such lemmas are given.

E will always denote a real Hilbert space; (x, y) denotes the scalar product of the elements x and y of E, and $\| x \|$ is the nonnegative square root of (x, x).

The differentials occurring in this paper are all Fréchet differentials. We recall the definition: let N be a neighborhood of the point x_0 in E, and let f be a map of N into a Hilbert space $\bar{E}$.[2] If there exists a linear continuous map $l = l(h)$ of N into $\bar{E}$ such that

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - l(h)}{\| h \|} = 0 \tag{2.1}$$

then $l(h) = df(x_0; h)$ is unique and is called the differential of f at x_0 with the increment h. If $df(x; h)$ is defined for all x of some neighborhood of x_0 then it is called continuous at x_0 if to every positive ϵ there exists a positive δ such that for x in the spherical neighborhood $N = N_\delta(x_0)$ of x_0 with radius δ, $\| d(x; h) - d(x_0; h) \| < \epsilon \| h \|$ for $x \in N$. $d(x, h)$ is called locally uniformly continuous at x_0 if there exists a neighborhood N' of x_0 such that the above in quality holds with x_0 replaced by any $x' \in N'$ provided that $\| x - x' \| < \delta$. In this case we also say: $d(x, h)$ is uniformly continuous in N'.

If this is the case and if N' is convex then the limit in (2.1) is uniform in N'. This follows immediately from the mean value theorem [6, Theorem 5]:

$$f(x + h) - f(x) = \int_0^1 df(x + th; h)\, dt.$$

If $d(x; h)$ as function of x has at x_0 a differential with increment k which is bounded in h then this differential is called the second differential of f at x_0 and denoted by $d^2f(x; h, k)$ provided that, in addition,

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{df(x_0 + k; h) - df(x_0; h) - d^2f(x_0; h, k)}{\| h \| \, \| k \|} = 0.$$

The second differential is called continuous at x_0 if to each positive ϵ there exists a neighborhood N of x_0 such that

$$\| d^2f(x; h, k) - d^2f(x_0; h, k) \| < \epsilon \| h \| \, \| k \|;$$

[2] In this paper $\bar{E}$ will always stand for either E or the real line E'.

uniform continuity is defined as for the first differential, and a uniformity statement analogue to the one above concerning the uniformity of the limit (2.1) holds. f is said to be of class C^0 in a subset S of E if it is continuous at every point of S, and of class C' if it has a continuous differential at every point of S. The class C'' is defined correspondingly. The meaning of the term uniformly (locally uniformly) of class C' or C'' will be obvious. The second differential of an $f \in C''$ is symmetric in the increments h and k [6, Theorem 8].

Chain rules [5, Section 15]. Let x_0, y_0, z_0 be elements of the Hilbert spaces E_0, E_1, E_2 respectively, and let N_0, N_1, N_2 be neighborhoods of x_0, y_0, z_0 respectively. Let f_0, f_1 be maps of class C':

$$f_0 : N_0 \to N_1, \qquad f_1 : N_1 \to N_2$$

with $f_0(x_0) = y_0$, $f_1(y_0) = z_0$. Then the composite map $f = f_1 \circ f_0 : N_0 \to N_2$ is of class C', and

$$df(x_0; h) = df_1(y_0; k) \tag{2.2}$$

where

$$k = df_0(x_0; h). \tag{2.3}$$

If, moreover, f_0 and f_1 are of class C'', then f is also of class C'', and

$$d^2f(x_0; h_1, h_2) = d^2f_1(y_0; k_1, k_2) + df_1(y_0; d^2y) \tag{2.4}$$

where k_i $(i = 1, 2)$ is obtained from h_i replacing h by h_i in (2.3), and where $d^2y = d^2f_0(x_0; h_1, h_2)$.

In order to recall the definition and some properties of a gradient in a Hilbert space E [8, p. 67] we return to the notations of the beginning of this section. If $\bar{E}$ is the real line E' then the differential $l(h) = df(x; h)$ is a linear continuous functional in h. Therefore there exists a unique element $g = g(x)$ in E such that $df(x; h) = (g(x), h)$. This $g(x)$ is called the gradient of f:

$$g(x) = (\operatorname{grad} f)(x). \tag{2.5}$$

LEMMA 2.1. *If the assumptions made for the validity of (2.2) and (2.3) are satiesfied and if E_2 is the real line E', then*

$$g(x) = \operatorname{grad} f(x) = d_0{}^* \operatorname{grad} f_1(y) \qquad \text{at} \qquad x = x_0, \qquad y = y_0 \tag{2.6}$$

where $d_0{}^*(h)$ denotes the map adjoint to the map $d_0(h) = df_0(x_0; h)$.

PROOF. From (2.2) and (2.5) we see that

$$(\operatorname{grad} f(x), h)_{x=x_0} = (\operatorname{grad} f_1(y), k)_{y=y_0} \tag{2.7}$$

where k is given by (2.3). Consequently the right member of (2.7) equals $(d^* \operatorname{grad} f_1(y), h)_{y=y_0}$. This proves (2.6).

LEMMA 2.2. *If $g_1(y) = \operatorname{grad} f_1(y)$ exists in a neighborhood of $y = y_0$, and if the differential $l_1(k) = dg_1(y_0; k)$ exists then f_1 has a second differential at $y = y_0$, and*

$$d^2f_1(y_0; k_1, k_2) = (l_1(k_1), k_2). \tag{2.8}$$

Conversely: if the left member of (2.8) exists then $l_1(k) = dg_1(y_0; k)$ exists and (2.8) holds. Moreover if the left member of (2.8) is continuous then $dg_1(y_0; k)$ as linear operator in k is symmetric.

PROOF. The first part of the lemma is proved in [9, p. 78]. Under the assumption of the second part of the lemma we have

$$df_1(y_0 + k_1; k_2) - df_1(y_0; k_2) = d^2f(y_0; k_1, k_2) + r(y_0, k_1, k_2) \tag{2.9}$$

where

$$\lim_{\substack{k_1 \to 0 \\ k_2 \to 0}} \frac{r(y_0, k_1, k_2)}{\| k_1 \| \, \| k_2 \|} = 0. \tag{2.10}$$

Now the left member of (2.9) is a bounded linear functional in k_2 and so is the first term of the right member. Therefore the same is true of $r(y_0, k_1, k_2)$. Consequently there exists a unique element $\rho = \rho(y_0, k_1)$ such that

$$r(y_0, k_1, k_2) = (\rho(y_0, k_1), k_2). \tag{2.11}$$

By definition of the gradient $g_1(y)$ the left member of (2.9) is the scalar product of $g_1(y_0 + k) - g_1(y_0)$ with k_2. Therefore it follows from (2.9) and (2.11) that (2.8) holds if we define $l_1(k_1)$ by

$$l_1(k_1) = g_1(y_0 + k_1) - g_1(y_0) - \rho(y_0, k_1). \tag{2.12}$$

It remains to prove that this $l_1(k_1)$ is the differential of $g_1(y)$ at $y = y_0$. The linearity of l_1 follows from (2.8) since the left member is linear in k_1. To prove the boundedness of l_1 we note that

$$\| d^2f(y_0; k_1, k_2) \| \leqslant \mu \| k_1 \| \, \| k_2 \| \tag{2.13}$$

for some positive μ. It therefore follows from (2.8) with $k_2 = l_1(k_1)$ that

$$\| l_1(k_1) \|^2 \leqslant \mu \| k_1 \| \, \| l_1(k_1) \|$$

which proves the boundedness. Finally, we have to estimate the remainder term $\rho(y_0, k_1)$ in (2.12). It follows from (2.10) and (2.11) that to every positive $\epsilon > 0$ there exists a δ such that

$$\| \rho(y_0; k_1), k_2) \| < \epsilon \| k_1 \| \, \| k_2 \| \tag{2.14}$$

if $\| k_1 \|$ and $\| k_2 \|$ are less than δ. Obviously (2.14) holds for $\| k_1 \| < \delta$ and arbitrary k_2. Therefore we may set $k_2 = \rho(y_0, k_1)$, and we obtain from (2.14) the estimate $\| \rho(y_0, k_1) \| < \epsilon \| k_1 \|$ for $\| k_1 \| < \delta$. This completes the proof that the l_1 defined by (2.12) is the differential of $g_1(y)$.

Finally, the symmetry of $dg_1(y_0; k)$ follows from (2.8) since the left member of that equality is symmetric in k_1 and k_2.

LEMMA 2.3. *With the notations and assumptions used in the statement of the chain rules, with $\bar{E} = E^1$, the real line, with $g(x)$ defined by (2.5), and with $g_1(y) = (\operatorname{grad} f_1)(y)$ we have*

$$(dg(x; h_2), h_1) = (dg_1(y; k_2), k_1)_{y=f_0(x)} + (g_1(y), d^2f_0(x; h_2, h_1))_{y=f_0(x)} \tag{2.15}$$

where k_1, k_2 are defined as in (2.4).

PROOF. We see from (2.7) that for any $x \in N_0$

$$(g(x), h_1) = (g_1(y), df_0(x; h_1)) \tag{2.16}$$

and (2.15) follows easily by differentiating (2.16) with respect to x and observing the chain rule (2.2), (2.3) in differentiating the right member.

3. PRELIMINARIES ON HYPERSURFACES IN A HILBERT SPACE

DEFINITION 3.1. A subspace E_1 of the real Hilbert space E is a closed linear subset of E with the scalar product induced by that of E. The subspace E_1 is called maximal if it is the nullspace of some nonzero linear continuous functional. A hyperplane is the translate of a maximal subspace.

The following lemma states some well known facts.

LEMMA 3.1. (a) *The subspace E_1 of Definition 3.1 is a Hilbert space;* (b) *the subspace E_1 is maximal if and only if there exists an element p of E with $\| p \| = 1$ such that $x \in E_1$ if and only if $(x, p) = 0$, and if there exists such a p then $-p$ is the only other element satisfying the conditions stated;* (c) *if E_1 is a proper subspace of E and if p is a nonzero element of E orthogonal to E_1 then E_1 is maximal if and only if every element x in E allows the unique decomposition*

$$x = \lambda p + x_1, \qquad x_1 \in E_1, \qquad \lambda \text{ real.} \tag{3.1}$$

In what follows $N_\rho(x_0)$ will always denote the open ball with center x_0 and radius ρ, i.e.

$$N_\rho(x_0) = \{x \in E \mid \| x - x_0 \| < \rho\}; \tag{3.2}$$

for $N_\rho(0)$ we will shortly write N_ρ.

DEFINITION 3.2. A subset S of E is called a hypermanifold if to every point x_0 of S there exists a maximal subspace U of E, a neighborhood U_0 of 0 relative to U, and a neighborhood $N(x_0)$ of x_0 such that there is a one-to-one correspondence

$$x = x(u), \qquad x_0 = x(0) \tag{3.3}$$

between the points u of U_0 and the points of $S(x_0) = S \cap N(x_0)$. U is called a parameter space for x_0, and the set U_0 is called a system (u) of local parameters. A transformation

$$u = f(v) \tag{3.4}$$

is called a parameter transformation if it is a one-to-one map between the points u of a neighborhood of 0 relative to U and a neighborhood of 0 with respect to a maximal subspace V. Moreover:

(α) S is said to be of class C^0 if for each $x_0 \in S$ there exists a local parameter system (u) such that the map (3.3) is bicontinuous. For an S of class C^0 the parameter transformation (3.4) is called admissible if it is bicontinuous.

(β) S is said to be of class C' if to every $x_0 \in S$ there exists a system (u) of local parameters such that the map (3.3) is of class C'. For an S of class C' the parameter transformation (3.4) is called admissible if together with its inverse it is of class C'.

(γ) S is said to be smooth if it is of class C' and if to each $x_0 \in S$ there exists a local parameter system (u) such that for $u_0 \in U_0$ the linear map U into E given by the differential $dx(u_0; u)$ of (3.3) is nonsingular, i.e. has a bounded inverse, and if moreover $dx(u_0; u) - u$ is completely continuous.[3] For a smooth S the parameter transformation (3.4) is admissible if it is admissible in the sense of the preceding paragraph, and if in addition $df(0; v)$ is nonsingular and $df(0; v) - v$ is completely continuous.

(δ) S is of class C'' if it is smooth and if for each $x_0 \in S$ and a suitable parameter system (u) the map (3.3) is of class C''. For an S of class C'' the parameter transformation (3.4) is said to be admissible if it is admissible in the sense of the preceding paragraph and if, in addition, the map (3.4) and its inverse are of class C''. If S is of class C'' and if the map (3.3) is of class C''' then S si said to be of slass C'''.

Remark. The definitions of a hypermanifold of class C^0, C', C'' and of a smooth hypermanifold are invariant under admissible parameter transformations. This follows from the chain rules (2.2), (2.4) together with the fact that the composition of a continuous map with continuous or completely continuous map is continuous or completely continuous respectively.

Definition 3.3. Let S be a hypermanifold of class C' and let $x_0 \in S$. Let $dx(0; u)$ be the differential of the map (3.3) at the zero point of U and

[3] The significance of this restriction will be clear from Theorem 3.1.

denote by $T(x_0)$ the image of U under this linear map, and by $T_0(x_0)$ the translate of $T(x_0)$ by x_0. Then $T(x_0)$ and $T_0(x_0)$ are called the tangent space and tangent plane to S at x_0 respectively.

U is by definition a hyperspace but $T(x_0)$ is not necessarily a hyperspace. However we have

THEOREM 3.1. *If S is a smooth hypersurface then the tangent space $T(x_0)$ (Definition 3.3) is a hyperspace.*

Theorem 3.1 is obviously a consequence of the following

LEMMA 3.2. *Let U be a hyperspace in the Hilbert space E. Let $d : U \to E$ be linear and of the form*

$$t = d(u) = u + D(u) \tag{3.5}$$

where D is completely continuous. Let T be the image of U under d. We suppose that d is nonsingular, i.e. that $\delta = d^{-1} : T \to U$ exists as a bounded map. Then T is a hyperspace.

PROOF. If we define $\Delta(t)$ by

$$u = \delta(t) = t + \Delta(t). \tag{3.6}$$

then Δ is completely continuous: indeed comparison of (3.5) with (3.6) shows that $\Delta(t) = -D(u)$, and our assertion follows from the complete continuity of D together with the boundedness of $d^{-1} = \delta$. We claim next that T is a proper subset of E: otherwise δ would be a map $E \to E$ which moreover by our assumptions is one-to-one. But this latter property together with the complete continuity of Δ implies by a well-known theorem that the map (3.6) is a map onto E. Thus $U = E$, a contradiction since U is a hyperspace in E. Thus $T \neq E$. Since T is obviously linear and is easily seen to be closed, T is a proper subspace of E. Consequently there exists a $q \in E$ of norm 1 which is orthogonal to T. Our lemma will be proved if we can show that T and q together span E (see Lemma 3.1c).

To do this we note that by Lemma 3.1 there exists a p of norm 1 orthogonal to U such that every point $x \in E$ has the unique representation

$$x = \lambda p + u, \qquad \lambda \text{ real}, \qquad u \in U. \tag{3.7}$$

We now extend the map d to a map $\bar{d}$ with domain E by assigning to the point x given by (3.7) the point

$$\bar{d}(x) = \lambda q + d(u). \tag{3.8}$$

Using (3.7) this may be rewritten as

$$\bar{d}(x) = x + \bar{D}(x), \qquad \bar{D}(x) = \lambda(q - p) + D(u). \tag{3.9}$$

Now since p has norm 1 it is seen from the orthogonal decomposition (3.7) that $|\lambda|$ and $\|u\|$ are both bounded by $\|x\|$. From this together with the complete continuity of $D(u)$ it follows routinely that $\bar{D}(x)$ is also completely continuous. Moreover the obviously linear map $\bar{d}$ is one-to-one, for $\bar{d}(x) = 0$ implies $\lambda = 0$ and $d(u) = 0$ by (3.8). But since $d(u)$ is one-to-one we also have $u = 0$. Thus $x = 0$ by (3.7). $\bar{d}$ is then a one-to-one linear map $E \to E$ of the form (3.9) with completely continuous $\bar{D}$. It follows that the map is onto, i.e., the range of $\bar{d}$ is E. On the other hand it is clear from (3.8) that this range is the set $\{T, q\}$ spanned by the range T of d and q. Thus $\{T, q\} = E$ which we wanted to prove.

THEOREM 3.2. *Let S be a smooth hypersurface (Definition 3.2(γ)) given by (3.3) in a neighborhood of $x_0 \in S$. Let*

$$v = dx(0; u), \qquad u \in U, \tag{3.10}$$

and let the map f in (3.4) be the inverse δ of (3.10). Then:

(i) *(3.4) is an admissible parameter transformation (Definition 3.2(γ)) (such that the tangent space $T(x_0)$ is a parameter space), and*

$$dx_1(0; v) = v \tag{3.11}$$

where

$$x_1(v) = x(u(v)). \tag{3.12}$$

(ii) *If $t = t(v)$ is the projection of $x_1(v) - x_0$ on $T(x_0)$ then $t(v)$ is invertible and $v = v(t)$ is an admissible parameter transformation; moreover*

$$dx_2(0; t) = t \tag{3.13}$$

where $x_2(t) = x_1(v(t))$.

PROOF. (i) Follows immediately from the definitions involved, the chain rule (2.2) and Theorem 3.1.

(ii) if p_0 is orthogonal to $T(x_0)$ and of norm 1 then the projection of $x_1(v) - x_0$ on $T(x_0)$ is

$$t = x_1(v) - x_0 + p_0(x_1(v) - x_0\,, p_0). \tag{3.14}$$

Taking into account that $(p_0\,, v) = 0$ one sees that (3.14) may be rewritten as

$$t = v + F(v) \tag{3.15}$$

where

$$F(v) = r(v) + p_0(r(v), p_0), \qquad r(v) = x_1(v) - x_0 - v. \tag{3.16}$$

Now (3.15) is satisfied with $t = v = 0$. It is well known that then (3.15) for small enough $\|v\|$ and $\|t\|$ has a unique solution $v = v(t)$ provided that there exist positive numbers η and k with $k < 1$ such that

$$\| F(v'') - F(v') \| < k \| v'' - v' \| \qquad \text{for} \qquad \| v' \|, \qquad \| v'' \| < \eta. \tag{3.17}$$

(See e.g. [5, Section II]). Now since $\| p_0 \| = 1$ one sees easily from (3.16) that

$$\| F(v'') - F(v') \| \leqslant 2 \| r(v'') - r(v') \|. \tag{3.18}$$

But using the mean value theorem [6, Theorem 5] and (3.11) we see from (3.16) that

$$r(v'') - r(v') = \int_0^1 [dx_1(v' + \alpha(v'' - v'; v'' - v') - dx_1(0; v'' - v')] \, d\alpha. \tag{3.19}$$

Now by the definition of the continuity of a differential (Section 2) there exists to given positive $k < 1$ a number η such that

$$\| dx_1(v; h) - dx_1(0; h) \| < \| h \| k/2 \qquad \text{for} \qquad \| v \| < \eta. \tag{3.20}$$

(3.17) follows now from (3.18), (3.19), and (3.20). Thus the existence of a unique solution $v = v(t)$ of (3.14) is assured. That this solution is of class C' follows from the fact that the right member of our equation (3.14) for $v(t)$ is of class C'.

Finally to prove (3.13) we note that by the chain rule

$$dx_2(0; dt) = dx_1(0; dv) \qquad \text{where} \qquad dt = dt(0; dv), \tag{3.20a}$$

the differential at $v = 0$ of the right member of (3.14). But the differential of the third term of this right member equals 0 since by (3.11), $d(x_1(0; dv) = dv$ and dv is orthogonal to p_0. Thus we obtain from (3.14) $dt = dx_1(0; dv) = dv$ and therefore from (3.20a) and (3.11), $dx_2(0; dv) = dv$ which is (3.13) since dv is an arbitrary element of $T(x_0)$.

DEFINITION 3.4. A system of parameters $(v) \in T(x_0)$ for which (3.11) holds is called tangential. The system of parameters (t) defined in Theorem (3.2ii) (which by (3.13) is tangential) is called normal.

DEFINITION 3.5. Let S be a hypersurface of class C', and let $\phi(x)$ be a real valued function defined on S. We then say ϕ is of class C' if the following is true for every $x_0 \in S$: if (u) is an admissible parameter system at x_0 and if the positive number ρ is so small that the representation (3.3) is valid in $S \cap N_\rho(x_0)$, then the function $\phi_0(u)$ defined by $\phi_0(u) = \phi(x(u))$ is of class C' in some neighborhood of $u = 0$. $\phi(x)$ is of class C'' if $\phi_0(u)$ is of class C''.

LEMMA 3.3. *Let S, x_0, and ρ be as in the preceding definition. Let $f(x)$ be a real valued function of class C' defined in $V_\rho(x_0)$, and let*

$$\phi_0(u) = f(x(u)) \tag{3.21}$$

where $x(u)$ is the map (3.3.) Then

$$df(x_0; h) = d\phi_0(0; u), \qquad h = dx(0; u). \tag{3.22}$$

If S is smooth (Definition 3.2(γ)) and if (u) is a tangential parameter system we have

$$df(x_0; u) = d\phi_0(0; u) \qquad u \in T(x_0). \tag{3.23}$$

If S and f are of class C'' then

$$d^2\phi_0(0; u_1, u_2) = d^2f(x_0; h_1, h_2) + df(x_0; d^2x(0; u_1, u_2) \tag{3.24}$$

where for $i = 1, 2$

$$h_i = \begin{matrix} u_i \text{ for a tangential parameter system } (u) \\ dx(0; u_i) \qquad \text{otherwise.} \end{matrix} \tag{3.25}$$

The lemma is an immediate consequence of the chain rules (2.2), (2.4), Theorem 3.2, and Definition (3.4).

DEFINITION 3.6. With the notations of the preceding lemma let S be smooth and f of class C'. Then by Theorem 3.1 and Lemma 3.1 we have with $g = \operatorname{grad} f$ the unique decomposition

$$g(x_0) = g_t(x_0) + g_n(x_0), \qquad g_t(x_0) \in T(x_0)$$

$$g_n(x_0) \qquad \text{orthogonal to} \qquad T(x_0). \tag{3.26}$$

$g_t(x_0)$ and $g_n(x_0)$ are called the tangential and the normal part of the gradient respectively.

LEMMA 3.4. *With the assumptions and notations of the preceding definition let*

$$\gamma(u) = \operatorname{grad} \phi_0(u). \tag{3.27}$$

Then for a tangential parameter system (u)

$$g_t(x_0) = \gamma(0). \tag{3.28}$$

Moreover if S and f are of class C''

$$(d\gamma(u; h_2), h_1) = (dg(x; k_2), k_1) + (g(x), d^2x(u; k_2, k_1) \tag{3.29}$$

where $x = x(u)$ and for $i = 1, 2$

$$k_i = \begin{matrix} h_i \text{ if } (u) \text{ is a tangential parameter system} \\ dx(0; h_i) \qquad \text{otherwise.} \end{matrix} \tag{3.30}$$

PROOF. By definition of the gradient, (3.23) may be written in the form

$$(g(x_0), u) = (\gamma(0), u). \tag{3.31}$$

Using the decomposition (3.26) we see that $g(x_0)$ in (3.31) may be replaced by $g_t(x_0)$ since $u \in T(x_0)$. The equation thus obtained from (3.31) holds for all u in the Hilbert space $T(x_0)$, and therefore implies (3.28) since $\gamma(0) \in T(x_0)$. (3.29) is an immediate consequence of (2.15), Theorem 3.2, and Definition 3.4.

4. On the Normals of a Bounding Hypersurface

In this section S will be always a smooth hypermanifold (Definition 3.2(γ)) which is the boundary of a bounded open connected set in the Hilbert space E. The parameter system (u) used in the local representation

$$x = x(u) \qquad x(0) = x_0 \in S \tag{4.1}$$

of S will always be normal (Definition 3.4). The main object of this section is to prove the existence of a continuous exterior normal $n(x)$, and if S is of class C'', the existence and continuity of the differential dn.

DEFINITION 4.1. Let $x_0 \in S$, and let p_0 be an element of norm 1 which is orthogonal to $T(x_0)$, the tangent space at x_0. If there exists a positive λ_0 such that all points of the segment

$$x = x_0 + \lambda p_0\,, \qquad 0 < \lambda < \lambda_0 \tag{4.2}$$

are exterior to R, then p_0 is called exterior normal; the interior normal is defined correspondingly.

THEOREM 4.1. *In every point x_0 of S there exists a unique exterior and a unique interior normal.*

PROOF. If $N_\sigma(x_0)$denotes the open ball with center x_0 and radius σ then the representation (4.1) defines for small enough σ a 1 to 1 correspondence between the points of $N_\sigma(x_0) \cap S$ and the points of a neighborhood $U_0 = U_0(\sigma)$ of the zero point of $T(x_0)$. But for a tangential parameter system (Definition 3.4) we can also assert that if θ is a positive number less than 1 then for σ small enough

$$\| r(u) | \leqslant \theta \| u \|, \qquad u \in U_0 \tag{4.3}$$

where

$$r(u) = x(u) - x_0 - u; \tag{4.4}$$

for by (4.1) and (3.11) the right member of (4.4) equals $x(u) - x(0) - dx(0; u)$ such that (4.3) for small enough U_0 follows from the definition of the differential.

From now on we assume a choice of σ for which (4.3) is true. We note that for a normal system (u) (Definition 3.4)

$$(r(u), u) = 0 \qquad u \in T(x_0). \tag{4.5}$$

Let now ρ be a positive number such that the ball

$$U_\rho(0) \subset U_0 \qquad \text{and} \qquad \rho < \sigma/3. \tag{4.6}$$

We then define the cylinder Z as the set of all points

$$z = x(u) + \lambda p_0, \qquad \| u \| < \rho, \qquad \lambda \text{ real}, \qquad | \lambda | < \sigma/3. \tag{4.7}$$

It is then easily verified that

$$Z \subset N_\sigma(x_0). \tag{4.8}$$

We now assert that Z contains a neighborhood of x_0; more specifically

$$N_\epsilon(x_0) \subset Z \qquad \text{if} \qquad 0 < \epsilon < \theta\rho/2. \tag{4.9}$$

Indeed, since $T(x_0)$ is a hyperspace any $x \in E$ is of the form

$$x = x_0 + u + \mu p_0, \qquad \mu \text{ real}, \qquad u \in T(x_0). \tag{4.10}$$

For $x \in N_\epsilon(x_0)$ we may apply (4.4) to (4.10):

$$x = x(u) - r(u) + \mu p_0 . \tag{4.11}$$

Moreover we see from (4.5) and (4.3) that

$$r(u) = \mu_1 p_0 \qquad \text{with} \qquad | \mu_1 | < \| u \| \theta, \tag{4.12}$$

and thus from (4.11) that

$$x = x(u) + \lambda p_0, \qquad \lambda = \mu - \mu_1 . \tag{4.13}$$

Now from the orthogonality of u and p_0 in the decomposition (4.10) we see that $\| u \|$ and $| \mu |$ are both majorized by $\| x - x_0 \|$. Moreover

$$| \lambda | \leqslant | \mu | + | \mu_1 | \leqslant | \mu | + \| u \| \theta$$

by using also (4.12). These inequalities and a comparison of (4.13) with (4.7) show immediately that $x \in Z$ if $x \in N_\epsilon(x_0)$.

Now x_0 is a boundary point of R. Consequently $N_\epsilon(x_0)$ contains a point $\bar{x}$ which is exterior to R. Since $\bar{x} \in N_\epsilon(x_0) \subset Z$ we may write

$$\bar{x} = x(\bar{u}) + \bar{\lambda} p_0, \qquad \| \bar{u} \| < \rho < \sigma/3, \qquad | \bar{\lambda} | < \sigma/3. \tag{4.14}$$

$\bar{\lambda}$ must be either positive or negative since $\bar{\lambda} = 0$ would imply that $\bar{x}$ is a boundary point. Suppose first that $\bar{\lambda}$ is positive. We claim then that p_0 is an exterior normal (Definition 4.1), an assertion which obviously follows from the following one: every $z \in Z$ for which the λ occurring in the representation (4.7) is positive is an exterior point. To prove this let x_1 be a point of Z

$$x_1 = x(u_1) + \lambda_1 p_0\,, \qquad \text{with} \qquad \lambda_1 > 0, \tag{4.15}$$

and for $0 \leqslant t \leqslant 1$ let

$$u_t = \bar{u}(1-t) + u_1 t, \qquad \lambda_t = \bar{\lambda}(1-t) + \lambda_1 t. \tag{4.16}$$

It is then easily verified that

$$x_t = x(u_t) + \lambda_t p_0 \in Z, \qquad \text{and} \qquad \lambda_t \geqslant \min(\bar{\lambda}, \lambda_1) > 0. \tag{4.17}$$

The fact that x_1 is an exterior point follows now by the classical method of considering the least upper bound T of those τ in the closed unit interval which have the property that for $0 \leqslant t < \tau$ the point x_t is exterior: x_T cannot be interior, for every neighborhood of x_T contains exterior points since all points x_t for $0 \leqslant t < T$ are exterior. But from (4.17) we see that $\lambda_T \neq 0$. Thus x_T is not a boundary point. Therefore x_T is an exterior point. Now the assumption $T < 1$ leads easily to a contradiction with the definition of T and the fact that the set of exterior points is open. Thus $T = 1$, and $x_1 = x_T$ is exterior.

We assumed $\bar{\lambda} > 0$. If $\bar{\lambda} < 0$ we simply have to write $(-\bar{\lambda})\,(-p_0)$ for $\bar{\lambda} p_0$ in (4.15) to see that then $-p_0$ is an exterior normal. Thus an exterior normal exists in any case. Similarly an interior normal exists.

Finally the uniqueness assertion of our theorem follows immediately from Lemma 3.1.

Definition 4.2. $p(x)$ is the exterior unit normal to S at the point x of S. With the local representation (4.1) by normal parameters (u) we write $n(u) = p(x(u))$. For $p(x_0)$ we write shortly p_0; correspondingly $n_0 = n(0)$.

Theorem 4.2. *Let S be locally uniformly of class C'' and let (4.1) be a normal local representation at the arbitrary point x_0 of S valid in the neighborhood U_0 of* 0 *in $T(x_0)$; then:* (a) *$n(u)$ is continuous*, (b) *the differential $dn(u; h)$ exists*, (c) *$dn(u; h)$ is uniformly continuous in u (cf. the definition given in Section 2).*

Proof of (a). For $i = 1, 2$ let $u_i \in U_0$, $x_i = x(u_i)$, $n_i = n(u_i)$, and

$$\Delta = n_2 - n_1\,. \tag{4.18}$$

We have to prove

$$\lim_{u_2 \to u_1} \Delta = 0. \tag{4.19}$$

Now by Section 2

$$dx(u_2; h) - dx(u_1; h) = d^2x(u_1; u_2 - u_1, h) + r(u_1, u_2 - u_1, h) \tag{4.20}$$

where r satisfies the uniform estimate

$$\| r(u_1, u_2 - u_1, h) \| \leqslant \| u_2 - u_1 \| \| h \| \epsilon \qquad \text{with} \qquad \lim_{\substack{u_2 \to u_1 \\ h \to 0}} \epsilon = 0. \tag{4.21}$$

Since by Definition 3.1 the tangent space $T(x_i)$ at x_i is the range of the linear map $dx(u_i; h)$ we have

$$(n_i, dx(u_i; h)) = 0 \qquad \text{for all} \qquad h \in T(x_0), \qquad i = 1, 2. \tag{4.22}$$

If Δ_1 is the projection of Δ on the tangent space $T(x_1)$ at x_1 then

$$\Delta = \Delta_1 + n_1(\Delta, n_1). \tag{4.23}$$

In order to estimate Δ_1 we multiply (4.20) scalar by $n_2 = n_1 + \Delta$. Using (4.22) and the fact that

$$(\Delta, dx(u_1; h)) = (\Delta_1, dx(u_1; h))$$

we have

$$(-\Delta_1, dx(u_1; h)) = (n_2, d^2x(u_1; u_2 - u_1, h)) + (n_2, r(u_1, u_2 - u_1, h)). \tag{4.24}$$

Now the map $h \to h_1$ given by $h_1 = dx(u_1; h)$ has an inverse $h = \delta(h_1)$ and maps $T(x_0)$ onto $T(x_1)$. Since, in addition, $\Delta_1 \in T(x_1)$ there is a unique $h_0 \in T(x_0)$ such that $dx(u_1; h_0) = \Delta_1$. If we set $h = h_0$ in (4.24) and observe (4.21) we see easily from the assumed uniform continuity of d^2x that for a suitable constant C_0

$$\| \Delta_1 \| \leqslant C_0 \| u_2 - u_1 \|. \tag{4.25}$$

To estimate the second term in (4.23) we note first that

$$2(\Delta, n_1) + (\Delta, \Delta) = 0 \tag{4.26}$$

as follows immediately from $1 = \| n_2 \|^2 = \| n_1 + \Delta \|^2$. On the other hand we see from the orthogonality of the decomposition (4.23) that

$$\| \Delta \|^2 = \| \Delta_1 \|^2 + (\Delta, n_1)^2. \tag{4.27}$$

Combining this equation with (4.26) we obtain for (Δ, n_1) the quadratic equation

$$(\Delta, n_1)^2 + 2(\Delta, n_1) + \| \Delta_1 \|^2 = 0. \tag{4.28}$$

Here we have to choose the solution

$$(\Delta, n_1) = -1 + \sqrt{1 - \| \Delta_1 \|^2} \tag{4.29}$$

with the positive square root. For if we set $z_1 = x(u_2) + \lambda(n_2 - \Delta_1)$, $(\lambda > 0)$, we see from (4.18) and (4.23) that $z_1 = x(u_2) + \lambda n_1(1 + (\Delta, n_1))$. If now (4.29) were true with the minus sign in front of the square root it would follow that

$$z_1 = x(u_2) - \lambda n_1 \sqrt{1 - \| \Delta_1 \|^2} \tag{4.30}$$

and that for λ small enough (say $0 < \lambda < \lambda_0$) $z_1 \in Z_1$ where Z_1 is defined with respect to x_1 in the same way as the set Z was defined by (4.7) with respect to x_0. Now since the coefficient of n_1 in (4.30) is negative and n_1 is the exterior normal a discussion analogue to the one following (4.15) would show that z_1 is an interior point of R. But because of the estimate (4.25), the point $z_1 + \lambda\Delta_1 = x(u_2) + \lambda n_2$ would for $\| u_2 - u_1 \|$ small enough also be interior to R for $0 < \lambda < \lambda_0$. But by Definition 4.1, n_2 would then be interior normal against our assumption.

We now see from (4.23), (4.25) and (4.29) that for a suitable constant C_1

$$\| \Delta \| \leqslant C_1 \| u_2 - u_1 \|. \tag{4.31}$$

Thus (a) is proved.

To prove (b) we remark first that by (4.26) and (4.31)

$$2 \, | (\Delta, n_1) | = \| \Delta \|^2 \leqslant C_1{}^2 \| u_2 - u_1 \|^2.$$

This implies that the differential of the n_1-component (n_2, n_1) of $n_2 = n(u_2)$ at $u_2 = u_1$ exists and equals 0:

$$d(n, n_1)(u; h) = 0 \qquad \text{at} \qquad u = u_1 . \tag{4.32}$$

It remains to prove the differentiability of the projection of n_2 on the tangent space $T(x_1)$ at x_1. Using (4.18) we can write (4.24) in the form

$$(-\Delta_1, dx(u_1; h) - (n_1, d^2x(u_1; u_2 - u_1, h)) = \bar{r}(u_1, u_2 - u_1, h) \tag{4.33}$$

where we have set

$$(n_2, r(u_1, u_2 - u_1, h) + (\Delta, d^2x(u_1; u_2 - u_1, h)) = \bar{r}(u_1, u_2 - u_1, h). \tag{4.34}$$

It follows from this definition for $\bar{r}$ and from (4.31) that the estimate (4.21) still holds if r is replaced by $\bar{r}$. Now there exist unique elements h_1 and $\gamma(u_1; u_2 - u_1)$ in $T(x_1)$ such that

$$(-n_1, d^2x(u_1; u_2 - u_1, h) = (\gamma(u_1; u_2 - u_1), h_1). \tag{4.35}$$

Indeed, let again $h_1 = dx(u_1\,; h)$ and let $h = \delta(h_1)$ be the inverse. Then the left member of (4.35) may be considered as a bounded linear functional defined for all points h_1 of the Hilbert space $T(x_1)$. This proves the asserted existence and uniqueness of h_1 and γ such that (4.35) holds.

Obviously γ is linear in its second argument. To see that γ is also bounded in its second argument we have only to set $h_1 = \gamma(u_1; u_2 - u_1)$ in (4.35) and to observe that $h = \delta(h_1)$ is bounded in h_1.

Using (4.35), the equation (4.33) may now be written

$$(\rho(u_1; u_2 - u_1), h_1) = \bar{r}(u_1; u_2 - u_1\,, \delta(h_1)) \tag{4.36}$$

where

$$\rho(u_1; u_2 - u_1) = \gamma(u_1; u_2 - u_1) - \Delta_1\,. \tag{4.37}$$

If in (4.36) we set $h_1 = \rho$ we see from (4.21) with r replaced by $\bar{r}$ that

$$\lim_{u_2 \to u_1} \frac{\rho(u_1; u_2 - u_1)}{\| u_2 - u_1 \|} = 0. \tag{4.38}$$

But since Δ_1 is the projection of $\Delta = n(u_2) - n(u_1)$ on $T(x_1)$, (4.38) together with the definition (4.37) of ρ shows that $\gamma(u_1; u_2 - u_1)$ is the differential at $u = u_1$ of the projection of $n(u)$ on $T(x_1)$. This together with (4.32) shows that

$$dn(u_1; h) = \gamma(u_1; h), \qquad h \in T(x_0). \tag{4.39}$$

Proof of (c). We have from (4.35)

$$(\gamma(u_2; k) - \gamma(u_1; k), h_1) = (-n_1\,, d^2x(u_2; k, \delta(h_1)) - d^2x(u_1; k, \delta(h_1)),$$
$$k \in T(x_0), \qquad h_1 \in T(x_1).$$

If we set here $h_1 = \gamma(u_2; k) - \gamma(u_1; k)$ one sees easily from the continuity property of d^2x in its first argument (Section 2) that to every positive ϵ there corresponds an η such that

$$\| \gamma(u_2; k) - \gamma(u_1; k) \| \leqslant \| k \| \epsilon \qquad \text{if} \qquad \| u_2 - u_1 \| < \eta.$$

By (4.39) this proves (c).

Theorem 4.3. *If, in addition to the assumptions of Theorem 4.2, S is locally uniformly of class C''' then $n = n(u)$ is of class C''.*

Proof. In this proof $\epsilon(\ )$ will denote any function which tends to zero as its argument tends to zero. Thus the symbol $\epsilon(\ )$ may stand for different functions. With this notation we have to prove the existence of an $l(u; z, k) \in E$ bilinear and bounded in z and k such that

$$dn(u + z; k) - dn(u; k) - l(u; z, k) = \epsilon(z) \| z \| \| k \|, \qquad u, z, k \in T_{x_0}. \tag{4.40}$$

Now as in the proof of Theorem 4.2, $u \in T_{x_0}$ and $x_1 = x(u_1)$, $n_1 = n(u_1)$. Moreover for any $\xi \in E$ we denote by $[\xi]_1$ the projection of ξ on T_{x_1} such that

$$\xi = [\xi]_1 + n_1(\xi, n_1), \tag{4.41}$$

$$(\xi, \xi) = ([\xi]_1, [\xi]_1) + (\xi, n_1)^2. \tag{4.42}$$

We will first prove the existence of a $\gamma(u; z, k) \in T_{x_1}$, bilinear and bounded in z and k such that for $u = u_1$

$$[dn(u + z; k) - dn(u; k)]_1 - \gamma(u; z, k) = \epsilon(z) \| z \| \| k \|. \tag{4.43}$$

Secondly, we will show that for $u = u_1$

$$(dn(u + z; k) - dn(u; k), n_1) + (dn(u; z), dn(u; k)) = \epsilon(z) \| z \| \| k \|. \tag{4.44}$$

Obviously (4.43) and (4.44) together with (4.41) imply that (4.40) is satisfied with $l = \gamma + n_1(dn(u_1; z), dn(u_1; k))$, in other words that

$$d^2n(u; z, k) = \gamma(u; z, k) - n(u)\,(dn(u; z), dn(u; k)). \tag{4.45}$$

For the proof of the existence of a γ satisfying (4.43) we start from the relation

$$(dn(u; k), dx(u; h) = -\,(n(u), d^2x(u; k, h)), \qquad u, h, k \in T_{x_0} \tag{4.46}$$

obtained from (4.35) and (4.39). Replacing here u by $u + z$ (with $z \in T_{x_0}$) and subtracting the relation thus obtained from (4.46) we see by elementary calculation that

$$\begin{aligned}
&(dn(u + z; k) - dn(u; k), dx(u; h)) \\
&= -\,\{(dn(u; k), d^2x(u; z, h)) + (dn(u; z), d^2x(u; k, h)) \\
&\qquad\qquad + (n(u), d^3x(u; z, k, h)\} - (R_1 + R_2 + R_3)
\end{aligned} \tag{4.47}$$

where

$$\begin{aligned}
R_1 &= (dn(u + z; k) - dn(u; k), dx(u + z; h) - dx(u; h)) \\
&\qquad + (dn(u; k), dx(u + z; h) - dx(u; h) - d^2x(u; z, h)) \\
R_2 &= (n(u + z) - n(u) - dn(u; z), d^2x(u; k, h)) \\
R_3 &= (n(u + z) - n(u), d^2x(u + z); h, k) - d^2x(u; h, k)) \\
&\qquad + (n(u), d^2x(u + z; h, k) - d^2x(u; h, k) - d^3x(u; z, h, k)).
\end{aligned}$$

From these expressions, from our assumptions on $x(u)$ and from our previous results on $n(u)$ and its differential it is easily seen that

$$R_i = \epsilon(z) \| z \| \| h \| \| k \|, \qquad i = 1, 2, 3. \tag{4.48}$$

Let now $u = u_1$. As in the proof of Theorem 4.2 let $h_1 = dx(u_1; h)$ such that $h_1 \in T_{x_1}$ while $h, k, z \in T_{x_0}$. Since this map $h \to h_1$ has an inverse $h = \delta(h_1)$ the expression contained in $\{ \}$ at the right member of (4.47) is a bounded linear functional defined for all h_1 in the Hilbert space T_{x_1}. Therefore there exists a unique $\gamma = \gamma(u; k, z)$ in T_{x_1} such the expression in $\{ \}$ equals the scalar product of $-\gamma$ with h_1; therefore (4.47) may be written in the form

$$\big([dn(u + z; k) - dn(u; k)]_1 - \gamma(u; k, z), h_1\big) = -(R_1 + R_2 + R_3), \quad u = u_1 . \tag{4.49}$$

(4.43) follows now from (4.48) and (4.49) if we set $h = \delta(h_1)$ in (4.48) and then set h_1 equal to the first factor in the scalar product at the left member of (4.49).

We now turn to the proof of (4.44). We set

$$u_2 = u_1 + k, \qquad u_2' = u_1 + z, \qquad u_3 = u_1 + z + k, \qquad (k, z \in T_{x_0})$$

$$n_i = n(u_i) \qquad \text{for} \qquad i = 1, 2, 3, \qquad n_2' = n(u_2')$$

$$\varDelta = n_2 - n_1 , \qquad \varDelta' = n_2' - n_1 ,$$

$$\varDelta^2 = n_3 - n_2 - n_2' + n_1 = n_3 - (\varDelta + \varDelta) - n_1 . \tag{4.50}$$

Squaring $n_3 = \varDelta^2 + \varDelta + \varDelta' + n_1$, taking into account that $n_1{}^2 = n_3{}^2 = 1$, applying (4.42) to $\xi = \varDelta^2$, and finally using (4.26) we obtain by elementary calculation the following quadratic equation for $(\varDelta^2, n_1)$

$$(\varDelta^2, n_1)^2 + 2(\varDelta^2 , n_1) + 2B + R = 0 \tag{4.51}$$

where

$$B = (\varDelta, \varDelta'), \qquad R = 2(\varDelta^2, \varDelta) + 2(\varDelta^2, \varDelta') + ([\varDelta^2]_1 , [\varDelta^2]_1). \tag{4.52}$$

Obviously, $| 2B + R | < 1$ for $\| h \|$, $\| k \|$, $\| z \|$ small enough. We then conclude from (4.51) that for some positive $\theta < 1$

$$(\varDelta^2, n_1) = -1 + \sqrt{1 - (2B + R)} = -B - \frac{R}{2} - \frac{(2B + R)^2}{8(1 + \theta(2B + R))^{3/2}} \tag{4.53}$$

or, adding $(dn(u_1; k), dn(u_1; z))$ to both members

$$(\varDelta^2, n_1) + (dn(u_1; k), dn(u_1; z)) = r \tag{4.54}$$

where

$$r = (dn(u_1; k), dn(u_1; z)) - B - \left(\frac{R}{2} + \frac{2(B+R)^2}{8(1+\theta(2B+R))^{3/2}}\right). \tag{4.55}$$

We claim:

$$r = \| k \| \, \| z \| \, (\epsilon(z) + \epsilon(k)). \tag{4.56}$$

Postponing the proof of (4.56), we remark first that (4.56) together with (4.54) implies (4.44). Indeed: replacing k by τk where $0 < \tau < 1$, and dividing by τ we obtain from (4.54), (4.56), and from the definition (4.50) of Δ^2

$$\left(\frac{n(u_1 + z + \tau k) - n(u_1 + z)}{\tau} - \frac{n(u_1 + \tau k) - n(u_1)}{\tau}, n_1\right)$$
$$+ (dn(u_1; k), dn(u_1; z)) = \frac{r}{\tau} = \| k \| \, \| z \| \, (\epsilon(z) + \epsilon(\tau k)).$$

Letting τ approach zero we obtain (4.44).

We now turn to the proof of (4.56). From the definition of Δ^2 and from the mean value theorem we see that

$$\begin{aligned}\Delta^2 &= n(u_1 + k + z) - n(u_1 + k) - (n(u_1 + z) - n(u_1)) \\ &= \int_0^1 (dn(u_1 + k + tz; z) - dn(u_1 + tz; z))\, dt.\end{aligned}$$

This proves that

$$\Delta^2 = \epsilon(k) \, \| z \| \tag{4.57}$$

since dn is continuous. Moreover by symmetry

$$\Delta^2 = \epsilon(z) \, \| k \|. \tag{4.58}$$

Obviously (cf. Theorem 4.2)

$$\| l \| < \text{Const.} \, \| k \|, \qquad \| \Delta' \| < \text{Const.} \, \| z \|, \qquad \| B \| < \text{Const.} \, \| k \| \, \| z \|. \tag{4.59}$$

It follows from (4.58) and (4.59) that R and also the expression contained in { } at the right member of (4.55) is of the desired from $\| k \| \, \| z \| \, (\epsilon(z) + \epsilon(k))$. It remains to show that the same is true for the difference of the first two terms at the right member of (4.55). This however follows easily from the identity

$$\begin{aligned}B &= (n(u_1 + k) - n(u_1), n(u_1 + z) - n(u_1)) \\ &= (dn(u_1; k), dn(u_1; z)) + (n(u_1 + k) - n(u_1) - dn(u_1; k), n(u_1 + z) - n(u_1)) \\ &\qquad + (dn(u_1; k), n(u_1 + z) - n(u_1) - dn(u_1; z)).\end{aligned}$$

This finishes the proof of the existence of the second differential of $n(u)$.

This differential is given by (4.45). It remains to prove its continuity. Now from previous results concerning $n(u)$ and its first differential it is clear that the second term at the right member of (4.45) is continuous. It remains to prove the continuity of the first term, i.e., to show that

$$\gamma(u + w; z, k) - \gamma(u; z, k) = \epsilon(w) \| z \| \| k \|. \tag{4.60}$$

Now taking into account the definition of γ given in the paragraph following (4.48), the assumptions made concerning the differentials of $x(u)$ up to and including the third one, and the properties of dn and d^2n already proved one sees easily that for all $h_1 \in T_{x_1}$

$$(\gamma(u + w; k, z) - \gamma(u; k, z), h_1) = \epsilon(w) \| z \| \| k \| \| h_1 \|.$$

(4.60) now follows upon setting

$$h_1 = \gamma(u + w; k, z) - \gamma(u; k, z).$$

5. A Local Coordinate System for Points Neighboring the Hypersurface S

A smooth enough hypersurface S^{m-1} in the m-dimensional space E^m has the following property: if the point $x \in E^m$ is near enough to S^{m-1} then there exists a unique point $\bar{x} \in S^{m-1}$ such that x lies on the normal to S^{m-1} at $\bar{x}$. Consequently if (u) is a local parameter system for S at the point x_0 of S then every x in a small enough neighborhood of x_0 may be expressed in the form $x = (u, s)$ where u is the parameter point corresponding to $\bar{x}$, and where $s = \pm \| x - \bar{x} \|$. In their paper [1] Morse and van Schaack constructed, and made essential use of, such a coordinate system (u, s).

It is the object of the present section to construct such a coordinate system for hypersurfaces in the Hilbert space E. To this end we prove the following.

Theorem 5.1. *Let S be a hypersurface in the Hilbert space which satisfies the assumptions of Theorem 4.2. Let $x_0 \in S$, and suppose that the normal representation (4.1) is valid for*

$$\| u \| < u_0 , \qquad u(0) = x_0 . \tag{5.1}$$

Then there exist two positive numbers $\sigma_0 < u_0$ and ρ_0 of the following property: to each x in the ball $V_{\rho_0}(x_0)$ there corresponds a unique couple $(u, s) = (u(x), s(x))$ with

$$\| u \|^2 + s^2 < \sigma_0^2, \qquad u \in T(x_0), \qquad s \text{ a real number}, \tag{5.2}$$

such that

$$x = \bar{x} + sp(\bar{x}) \tag{5.3}$$

where $\bar{x}$ is the point of S which under the representation (4.1) is the image of the parameter point $u = u(x)$, and where p is as in Definition 4.2. Moreover $u = u(x)$ and $s = s(x)$ are of class C'.

PROOF. By Definition 4.2, $p(\bar{x}) = n(u)$. Therefore (5.3) is equivalent to

$$\xi = \bar{\xi} + sn(u), \qquad \xi = x - x_0\,, \qquad \bar{\xi} = \bar{x} - x_0\,. \tag{5.4}$$

Now since $T(x_0)$ is a hyperspace with unit normal n_0 every $z \in E$ has the unique orthogonal decomposition

$$z = u + n_0(z, n_0), \qquad u \in T(x_0). \tag{5.5}$$

Applying this to $z = \bar{\xi}(u)$ and observing that u is the projection on $T(x_0)$ of $\bar{\xi} = \bar{x} - x_0$ the parameter system (u) being normal we see that (5.4) is equivalent to

$$\xi = u + n_0(\bar{\xi}(u), n_0) + sn(u). \tag{5.6}$$

We now consider the couple $y = (u, s)$ as an element of the Hilbert space Π_0 which is the product of $T(x_0)$ with the real line, the norm of y being defined by $\| y \| = \sqrt{\| u^2 \| + s^2}$. Since $\bar{\xi}(u)$ and $n(u)$ are given functions of u we may define

$$G(\xi, y) = \xi - u - n_0(\bar{\xi}(u), n_0) - sn(u). \tag{5.7}$$

We see then that our theorem is equivalent to the following statement: there exist positive $\sigma_0 < u_0$ and ρ_0 such that to every ξ with $\| \xi \| < \rho_0$ there corresponds one and only one $y = y(\xi)$ with $\| y \| < \sigma_0$ which satisfies the equation

$$G(\xi, y(\xi) = 0, \tag{5.8}$$

and this $y(\xi)$ is of class C'.

We recall that the following conditions (H_1)-(H_3) are sufficient for this statement to be true [5, p. 150]:

(H_1) $G(0, 0) = 0$.

(H_2) G is of class C' as function of the couple (ξ, y).

(H_3) The differential $d_yG(0, 0; \eta)$ of G with respect to y at $\xi = y = 0$ is nonsingular, i.e., has a bounded inverse.

We proceed to verify these conditions. (H_1) follows from (5.7) by inspection if one observes that $y = 0$ is equivalent to $u = s = 0$ and that

$$\bar{\xi}(0) = x_0 - x_0 = 0.$$

Due to the fact that $G(\xi, y)$ depends linearly on ξ it will for the proof of (H_2) be sufficient to show that $d_yG(\xi, y; \eta)$ exists and that to each $\epsilon > 0$ there corresponds a $\delta > 0$ such that

$$\| d_yG(\xi + \tilde{\xi}, y + \tilde{y}; \eta) - dG_y(\xi, y; \eta) \| < \epsilon \| \eta \| \quad \text{for} \quad \| \tilde{\xi} \|, \| \tilde{y} \| < \delta. \tag{5.9}$$

Now setting $\eta = (v, \sigma)$ we see from (5.7) that

$$d_yG(\xi, y; \eta) = -v - n_0(d\xi(u; v), n_0) - sdn(u; v) - \sigma n(u). \tag{5.10}$$

Keeping in mind the definition of continuity of a differential (Section 2) and the fact that $\xi(u)$ is of class C'' and that dn is continuous one concludes easily from (5.10) that (5.9) holds.

To verify (H_3) we notice that by Theorem 3.2, $d\xi(0; v) = d\bar{x}(0; v) = v$. Since, moreover, v is orthogonal to n_0 we see from (5.10) that

$$dG(0, 0; \eta) = -(v + \sigma n_0). \tag{5.11}$$

Obviously, this proves (H_3).

DEFINITION 5.1. The Hilbert space Π_0 defined in the lines directly following (5.6) is called the normal local coordinate space at x_0. The components u, s of the point $y = (u, s) \in \Pi_0$ corresponding by Theorem 5.1 uniquely to the point $x \in V_{\rho_0}(x_0)$ are called the normal local coordinates (at x_0) of x.

6. Critical Points and Tangentially Critical Points

Let R be a bounded connected domain in the Hilbert space E whose boundary S is smooth (Definition 3.2(γ)). Let $f(x)$ be a real valued function defined and of class C' in a bounded connected domain R_2 which contains $\bar{R}$ and whose boundary S_2 has a positive distance from the boundary S of R.

DEFINITION 6.1. A point x_0 of R is called a critical point of f if

$$g(x_0) = 0 \tag{6.1}$$

where $g(x) = \operatorname{grad} f(x)$.

DEFINITION 6.2. A bounded linear operator

$$d : E \to E$$

is called nonsingular if $d(h) = 0$ implies $h = 0$. Otherwise d is called singular. A bounded bilinear form $q(h, k)$ is called degenerate if there exists a $k_0 \neq 0$, such that $q(h, k_0) = 0$ for all h and if the statement obtained by interchanging h and k is also true. If no such k_0 exists g is called nondegenerate. If the bilinear form q is symmetric then the quadratic from $q(h, h)$ is called nondegenerate if $q(h, k)$ is nondegenerate.

REMARK. If $l(h)$ is a bounded linear symmetric operator then it is easily seen that the bilinear form $(l(h), k)$ is degenerate if and only if l is singular.

DEFINITION 6.3. A critical point x_0 of f is called nondegenerate if f is of class C'' in some neighborhood of x_0 and if the second differential $d^2f(x_0; h, k$ is nondegenerate as bilinear form in h and k.

The following lemma is an immediate consequence of Lemma 2.2:

LEMMA 6.1. *The critical point x_0 of f is nondegenerate if and only if the differential $l(x; k) = dg(x; k)$ exists in some neighborhood of x_0 and if $l(x_0; k)$ as linear operator in k is nonsingular.*

THEOREM 6.1. *Let x_0 be a nondegenerate critical point of f. We assume that the differential $l(h)$ of the gradient g of f at $x = x_0$ has a bounded every where defined inverse. Then x_0 is an isolated critical point, i.e. there exists a neighborhood $V = V_\rho(x_0)$ of x_0 such that x_0 is the only critical point of f in V.*

Theorem 6.1 is obviously a consequence of the following

LEMMA 6.2.[4] *Let g be a map defined in some neighborhood of the point x_0 which is a zero of g. Suppose that g has a differential $l = l(h)$ at $x = x_0$ which has a bounded inverse λ. Let m and μ denote the norm of l and λ resp. Then there exists a positive η such that*

$$2m \| x - x_0 \| \geqslant \| g(x) \| \geqslant \| x - x_0 \| (2\mu)^{-1} \qquad \text{for} \qquad \| x - x_0 \| < \eta. \tag{6.2}$$

PROOF. Since by assumption $g(x_0) = 0$ we have by definition of the differential

$$g(x) = g(x) - g(x_0) = l(h) + R(x_0; h), \qquad h = x - x_0,$$

where

$$\lim_{h \to 0} \frac{R(x_0, h)}{\| h \|} = 0. \tag{6.3}$$

[4] In this lemma g, is not necessarily a gradient. The proof given is essentially the same as the one given for the finite dimensional case in [10, p. 477].

Therefore

$$\| l(h) \| + \| R(x_0 , h) \| \geqslant \| g(x) \| \geqslant \| l(h) \| - \| R(x_0; h) \|. \tag{6.4}$$

Since by definition of m and μ the inequalities

$$\| h \| m \geqslant \| l(h) \| \geqslant \| h \| \mu^{-1}$$

hold, (6.2), for small enough η, follows easily from (6.3) and (6.4).

DEFINITION 6.4. Let $x_0 \in S$, and let $\phi_0(u)$ be defined as in (3.21). Then x_0 is called a tangentially critical point of f if $u = 0$ is a critical point of ϕ_0. The tangentially critical point x_0 is called non degenerate if $u = 0$ as critical point of ϕ_0 is nondegenerate. (Application of the chain rules (2.2) and (2.4) to two admissible parameter systems shows the invariance of these definitions if one notes that the second term at the right member of (2.4) equals 0 at a critical point.)

LEMMA 6.3. *Let the assumptions of Definition 3.5 be satisfied. Then:* (a) *$x_0 \in S$ is a tangentially critical point of f if and only if $g_t(x_0) = 0$;* (b) *if x_0 is a tangentially critical point then x_0 is nondegenerate if and only if $d\, grad\, \phi_0(0; v)$ as map $T(x_0) \to T(x_0)$ is nonsingular.*

PROOF. (a) is an immediate consequence of Lemma 3.4, and (b) follows from Lemma 6.1 (applied to ϕ_0 instead of f).

LEMMA 6.4. *In addition to the assumptions of the preceding lemma we suppose that f has no critical points on S. Then at any tangentially critical point $x_0 \in S$ we have*

$$g(x_0) = g_n(x) \neq 0, \tag{6.5}$$

i.e., the gradient at such a point x_0 is normally directed.

This lemma follows immediately from the preceding one in conjunction with (3.26).

7. THE MODIFICATION $\tilde{f}$ OF f AND ITS CRITICAL POINTS

Let R, S, R_2, S_2, and f be as described in the first paragraph of Section 6. Moreover S is suppsed to be of class C''. Then there obviously exists a positive number s_1 such that if x_0 is an arbitrary point of S, and if p_0 denotes the unit exterior normal to S at x_0.

$$x = x_0 + p_0 s \in R_2 \qquad \text{for} \qquad 0 \leqslant s < s_1 . \tag{7.1}$$

Let now

$$x = \bar{x}(u), \qquad \bar{x}(0) = x_0 , \tag{7.2}$$

be the representation of S in a neighborhood of x_0 by the normal parameter system (u) (Definition 3.4). The local normal coordinate system (u, s) (Definition 5.1) is then valid in the spherical neighborhood $V_\rho(x_0)$ of x_0 if ρ is small enough. We now make the new assumption that ρ can be chosen independent of x_0. Then positive numbers s_1 and δ independent of x_0 exist such that, in addition to (7.1) the representation

$$x = \bar{x}(u) + sn(u) \qquad \text{holds for} \qquad \| u \| < \delta, \qquad 0 \leqslant s < s_1 . \tag{7.3}$$

We denote by R_1 the union of R and all points (7.1) obtained as x_0 varies over S, and by S_1 the boundary of R_1.

The function f (see Section 6) is defined in R_2. We now make a number of additional assumption about f.

ASSUMPTION (A). f is of class C'', and its second differential is uniformly bounded, i.e., there exists a constant N such that

$$| d^2f(x; h, k) | \leqslant N \| h \| \| k \| \qquad \text{for all} \qquad x \in R_2 . \tag{7.4}$$

LEMMA 7.1. *Let x and h be such that $x + th \in R_2$ for $0 \leqslant t \leqslant 1$. Let $g = \operatorname{grad} f$. Then:*

(a)
$$\| g(x + h) - g(x) \| \leqslant N \| h \|. \tag{7.5}$$

where N is the constant appearing in (7.4). (b) *g is of class C' and $dg(x; h)$ as linear operator in h is symmetric.*

PROOF (a) By definition of the gradient, the scalar product of $g(x + h) - g(x)$ with an arbitrary element k of E may be written as $df(x + h; k) - df(x; k)$, and by the mean value theorem [6] this difference equals

$$\int_0^1 d^2f(x + th; h, k)\, dt.$$

Applying (7.4) to the integrand we see that

$$\| (g(x + h) - g(x), k) \| \leqslant N \| h \| \| k \| \qquad \text{for all} \qquad k. \tag{7.6}$$

This inequality implies (7.5) as is seen immediately upon setting

$$k = g(x + h) - g(x)$$

in (7.6).

(b) By Lemma 2.2, $l(x; h) = dg(x; h)$ exists and

$$(l(x + \xi; h) - l(x; h), k) = d^2f(x + \xi; k, h) - d^2f(x; k, h).$$

From assumption (A) we see that the norm of this difference equals $\epsilon(\xi) \| h \| \| k \|$. Setting $k = l(x + \xi; h) - l(x; h)$ we obtain

$$\| l(x + \xi; h) - l(x; h) \| = \epsilon(\xi) \| h \|$$

which proves the continuity of $l = dg$. The symmetry of dg follows also from Lemma 2.2.

ASSUMPTION (B). f has no critical points on S.

ASSUMPTION (C). $g(x) = x + G(x)$ where G is completely continuous.

LEMMA 7.2. *In addition to satisfying the conditions imposed previously on s_1 (cf. (7.3)), this number can be chosen such that there are no critical points of f in $R_1 - R$.*

PROOF. By assumption (B), $g(\bar{x}) \neq 0$ for $\bar{x} \in S$. It is well known that this together with assumption (C) implies the existence of a constant m such that

$$\| g(\bar{x}) \| > m > 0 \qquad \text{for all} \qquad \bar{x} \in S. \tag{7.7}$$

We now subject s_1 to the new condition $s_1 < m/2N$ where N is as in (7.5). We claim that then $g(x) \neq 0$ for $x \in R_1 - R$. Indeed for such x, (7.3) holds; moreover all points of the segment between x and $\bar{x} = \bar{x}(u)$ on the line determined by $n(u)$ lie in R_1. Applying Lemma 7.1 to these points we obtain from (7.7) and (7.5)

$$\| g(x) \| \geqslant \| g(\bar{x}) \| - \| g(\bar{x}) - g(x) \| \geqslant m - Ns \geqslant m - Ns_1 \geqslant m/2.$$

LEMMA 7.3. *There exists a positive constant M such that*

$$\| g(x) \| < M/2 \qquad \textit{for all} \qquad x \in \bar{R}_2. \tag{7.8}$$

PROOF. Since $\bar{R}_2$ is bounded the statement that g is bounded on $\bar{R}_2$ is (on account of assumption (C)) equivalent to the statement that G is bounded on $\bar{R}_2$. The latter statement is obvious from the complete continuity of G.

DEFINITION 7.1. Let s_1, R_1 be as in Lemma 7.2, let M be as in Lemma 7.3, and let s_0 be a number satisfying

$$0 < s_0 < s_1. \tag{7.9}$$

For $x \in R_1$ the modification $\tilde{f}$ of f is then defined as follows:

$$\tilde{f}(x) = \begin{matrix} f(x) & \text{for} & x \in R \\ f(x) + Ms^3/3s_0^2 & \text{for} & x \in R_1 - R \end{matrix} \tag{7.10}$$

where the relation between x and s is given in (7.3).

REMARK. In what follows s_0 and s_1 will repeatedly be subject to new conditions. It is then understood that all conditions imposed previously are still satisfied. In particular (7.9) will always hold.

DEFINITION 7.2. If $x \in R_1 - R$ is given by (7.3) we set

$$\psi(u, s) = g(x), \qquad \psi_n(u, s) = g_n(x) = n(u)\,(g(x), n(u)),$$
$$\psi_t(u, s) = g_t(x) = g(x) - g_n(x). \tag{7.11}$$

$\tilde{\psi}_n(u, s) = \tilde{g}_n(x)$ and $\tilde{\psi}_t(u, s) = \tilde{g}_t(x)$ are defined correspondingly for $\tilde{\psi} = \tilde{g} = \operatorname{grad} \tilde{f}$. (Note that for $x = x_0$, i.e, $u = 0$, Definition (7.11) agrees with the one given by (3.26).)

DEFINITION 7.3. The negative part S^- of S is defined by

$$S^- = \{x \in S \mid (g(x), p(x)) < 0\}$$

where $p(x)$ is defined in Definition 4.2.

If $x_0 \in S^-$ is a nondegenerate tangentially critical point of f (Definition 6.4) then (6.5) holds on account of assumption B). Moreover

$$(g(x_0), p(x_0)) = (g(x_0), n(0)) < 0, \qquad x \in S^-. \tag{7.12}$$

The object of the present section is to construct to such an x_0 a unique critical point x^* of $\tilde{f}$ in $R_1 - R$, i.e., a point x^* which satisfies the equation

$$\tilde{g}(x) = 0. \tag{7.13}$$

By Definition 7.2 this equation is equivalent to the two equations

$$\text{(a)} \quad \psi_t(u, s) = 0, \qquad \text{(b)} \quad \psi_n(u, s) = 0. \tag{7.14}$$

We consider first (7.14a). By the use of Theorem 3.2 it is easily seen from (7.10) that

$$\tilde{g}(x) = g(x) + Mn(u)\, s^2/s_0^2. \tag{7.15}$$

Therefore $\tilde{g}_t = g_t$ and we will show that there are positive numbers s_1 and

δ (independent of s_0) such that there exists one and only one continuous $u = u(s)$ satisfying

$$\psi_t(u(s), s) = 0, \qquad u(0) = 0, \qquad 0 \leqslant s \leqslant s_1, \qquad \| u \| < \delta. \tag{7.16}$$

It will be sufficient to verify that the conditions (H_1), (H_2), (H_3) of the implicit function theorem listed (in different notation) in the proof of Theorem 5.1 are satisfied.

(H_1). By assumption x_0 is tangentially critical. Therefore by Lemma 6.3 and Definition 7.2, $0 = g_t(x_0) = \psi_t(0, 0)$.

(H_2). By Lemma 7.1, $g(x)$ is of class C'. Therefore the same is true for $\psi_t(u, s)$ as is seen immediately from Definition 7.2 (together with an application of the chain rule).

(H_3). We have to prove that the differential $d_u\psi_t(u, s; v)$ of ψ_t with respect to u is nonsingular at $u = 0$. Now the tangentially critical point x_0 is by assumption nondegenerate. Therefore we see from Lemma 6.3 that the differential of grad $\phi_0(u)$ is nonsingular at $u = 0$. Consequently the nonsingularity of $d_u\psi_t$ follows from

Lemma 7.4. *With the above assumptions and notations*

$$d_u\psi_t(0, 0; v) = d \operatorname{grad} \phi_0(0; v). \tag{7.17}$$

Proof. With x given by (7.3) let

$$\phi(u, s) = f(x) \qquad \text{such that} \qquad \phi(u, 0) = \phi_0(u). \tag{7.18}$$

Then by the chain rule

$$d_u\phi(u, s; v) = df(x; dx) \tag{7.19}$$

where

$$dx = d\bar{x}(u; v) + s\, dn(u; v), \tag{7.20}$$

or, by definition of the gradient,

$$(\operatorname{grad}_u \phi(u, s), v) = (g(x), dx). \tag{7.21}$$

Here grad_u indicates that the gradient operation is to be taken for constant s. Now noting that $(n, dn) = 0$, we see from (7.20) that dx is orthogonal to n. Therefore we may replace $g(x)$ by $g_t(x)$ (see (7.11) in (7.21). If we apply to the equation thus obtained the operation d_u we see that

$$(d_u \operatorname{grad}_u \phi(u, s; v_1), v) = (d_u g_t(x; d_1x), dx) + (g_t(x), d^2x) \tag{7.22}$$

where $d_1(x)$ is obtained by replacing v by v_1 in (7.20), and where

$$d^2x = d_u{}^2\bar{x}(u, s; v, v_1) + s\, d^2n(u; v, v_1). \tag{7.23}$$

We now set $u = s = 0$, i.e. $x = x_0$. Using (3.11) and the second part of (7.18) we obtain from (7.22)

$$(d\,\mathrm{grad}\,\phi_0\,(0; v_1), v) = (d_u g_t(x_0; v_1), v) + (g_t(x_0), d^2x). \tag{7.24}$$

Now the second term of the right member of this equation vanishes on account of Lemma 6.3 since x_0 is tangentially critical. It follows from the Definition 7.2 of ψ_t that (7.24) implies (7.17).

This finishes the proof of the existence of a continuous $u = u(s)$ satisfying (7.16). As already remarked this $u(s)$ satisfies also (7.14). We now turn to (7.14b) and prove

LEMMA 7.5. *As before, let $x_0 \in S^-$ be a nondegenerate tangentially critical point, and let s and $u(s)$ be as in (7.16). Then there exists at least one $s = s^*$ satisfying*

$$\text{(a)}\quad \tilde{\psi}_n(u(s^*), s^*) = 0, \qquad \text{(b)}\quad 0 < s^* < s_0 < s_1 . \tag{7.25}$$

PROOF. From (7.15) and the definitions of ψ_n, $\tilde{\psi}_n$ we see that $\tilde{\psi}_n = \psi_n$ for $s = 0$ and it follows immediately from (7.12) that the scalar product of the left member of (7.25a) with $n(u(s))$ is negative for $s = 0$. Our lemma now follows from the following Lemma 7.6 which implies that the scalar product mentioned is positive for $s \geqslant s_0$.

LEMMA 7.6. *If x is given by (7.3) and if M is a constant for which (7.8) holds then*

$$(\tilde{g}(x), p(\bar{x})) \geqslant M/2 \qquad \text{for} \qquad s_1 \geqslant s \geqslant s_0 . \tag{7.26}$$

PROOF. From (7.15) and (7.8) we see that

$$(\tilde{g}(x), p(\bar{x})) = \frac{Ms^2}{s_0^{\,2}} + (g(x), p(\bar{x})) > \frac{Ms^2}{s_0^{\,2}} - \frac{M}{2}$$

which obviously proves (7.26) in the range indicated.

Actually s^* is unique if s_0 is small enough. More precisely we have

THEOREM 7.1. *With the notations and under the assumptions of the last three lemmas there exists a positive $s' < s_1$ of the following property: if*

$$0 < s_0 < s' \tag{7.27}$$

then there exists a unique positive $s^* < s_1$ such that

$$x^* = \bar{x}(u(s^*) + s^* n(u(s^*)) \tag{7.28}$$

is a critical point of $\tilde{f}$, i.e., satisfies (7.13). Moreover there exists a positive number $\theta_0 < 1$ which may depend on s_1 but not on s_0 such that

$$0 < \theta_0 s < s^* < s_0 < s_1 . \tag{7.29}$$

The theorem is an obvious consequence of the following Lemma 7.7 in conjunction with our previous results.

LEMMA 7.7. *There exists a positive s' and a positive $\theta_0 < 1$ such that for all s_0 satisfying (7.27)*

$$(\psi(u(s), s), n(u(s))) < 0 \qquad \textit{for} \qquad 0 \leqslant s \leqslant s_0\theta_0 \tag{7.30}$$

and

$$\frac{d}{ds}(\psi(u(s), s), n(u(s)) > 0 \qquad \textit{for} \qquad \theta s_0 \leqslant s \leqslant s_1 . \tag{7.31}$$

PROOF. If x is given by (7.3) we see from scalar multiplication of (7.15) by $n(u)$ that

$$(\tilde{g}(x), n(u)) = (g(x), n(u)) + Ms^2/s_0^2. \tag{7.32}$$

We now set $u = u(s)$ where $u(s)$ is the function satisfying (7.16). Then (g, n) is a continuous function of s which is negative for $s = 0$ since $x = x_0$ for $s = 0$ and $x_0 \in S^-$. Consequently there exists a positive s' such that this function is negative and therefore has a negative maximum $-m$ in the closed interval $0 \leqslant s \leqslant s'$. The right member of (7.32) is then less than $-m + Ms^2/s_0^2$. This latter quantity is in the interval considered a monotone increasing function of $\theta = s/s_0$ which is negative for $\theta = 0$, and there will be a positive $\theta_0 < 1$ such that it will still be negative for $0 \leqslant \theta \leqslant \theta_0$. On account of the definition (7.2) of ψ this obviously proves (7.30).

For the proof of (7.31) we again set $u = u(s)$ and differentiate (7.32) with respect to s

$$\frac{d}{ds}(\tilde{g}, n) = \frac{2sM}{s_0^2} + \frac{d}{ds}(g, n) \geqslant \frac{2sM}{s_0^2} - \left| \frac{d}{ds}(g, n) \right| .$$

Now for $\theta_0 s_0 \leqslant s \leqslant s_1$ and for $0 < s_0 < s''$ the right member of our inequality is not smaller than

$$\frac{2M\theta_0}{s''} - \operatorname*{Max}_{0 \leqslant s \leqslant s_1} \left| \frac{d}{ds}(g, n) \right| .$$

Obviously we may choose s'' so small that this expression is positive. This proves (7.31) if with a change in our notation we write s' instead of min (s', s'').

8. The Indices of the Critical Points of $\tilde{f}$

In this section x_0 will always denote a nondegenerate tangentially critical point of f situated on S^- (Definition 7.3), and x^* will be the critical point of $\tilde{f}$ corresponding to x_0 by Theorem 7.1. The main object will be the proof of Theorems 8.1 and 8.2.

THEOREM 8.1. *The critical point x^* of $\tilde{f}$ is nondegenerate, and if r^* denotes its index then*

$$r^* = r_0 \tag{8.1}$$

where r_0 is the index of x_0 as tangentially critical point of f (Definition 6.4). It is assumed that the assumptions (A), (B), (C), of Section 7 are satisfied as well as the assumptions (D) and (E) stated following the proof of Lemma 8.4.

We will recall the definition of "index" presently. First we need some preparatory lemmas.

LEMMA 8.1. *Let $q(x, x)$ be a bounded symmetric bilinear form defined on the Hilbert space E. Then there exists a unique direct orthogonal decomposition of E into linear subsets*

$$E = E^- \dot{+} E^0 \dot{+} E^+ \tag{8.2}$$

such that for $x^- \in E^-$, $x^0 \in E^0$, $x^+ \in E^+$

$$q(x^-, x^0) = q(x^-, x^+) = q(x^0, x^+) = 0 \tag{8.3}$$

$$q(x^0, x^0) = 0, \qquad q(x^+, x^+) > 0; \qquad q(x^-, x^-) < 0; \tag{8.4}$$

for the validity of the two inequalities in (8.4) it is of course assumed that x^- and x^+ are nonzero elements of E^- and E^+ resp.

This is well known, see [11, Theorem 7.1].

LEMMA 8.2. *The bounded quadratic form $q(x, x)$ is nondegenerate (Definition 6.2) if and only if E^0 consists only of the zero element such that the direct decomposition (8.2) reduces to*

$$E = E^- \dot{+} E^+, \qquad E^+, E^- \textit{ closed.} \tag{8.5}$$

PROOF. (a) Suppose that q is nondegenerate. Let $y^0 \in E^0$, and let $x = x^- + x^0 + x^-$ be an arbitrary element of E decomposed according to (8.2). Using (8.3) and (8.4) we see that

$$q(y^0, x) = q(y^0, x^-) + q(y^0, x^0) + q(y^0, x^+) = 0 \qquad \text{for all} \qquad x.$$

By definition of nondegeneracy of q this implies that $y^0 = 0$.

(b) suppose E^0 consists only of the zero element. Let now $y \in E$ be such that $q(y, x) = 0$ for all x. We have to prove that $y = 0$. Now under our assumption the decomposition (8.2) reduces to $y = y^- + y^+$ with $y^- \in E^-$, $y^+ \in E^+$. Setting $x = y^+$ and using (8.3) we see that $0 = q(y, y^+) = q(y^+, y^+)$. By (8.4) this implies that $y^+ = 0$. In the same way we see that $y^- = 0$. Thus $y = 0$.

DEFINITION 8.1. The index $r = r(q)$ of the nondegenerate quadratic form q is the dimension of E^- in (8.5) (thus $r = \infty$ is not excluded) (cf. [11, p. 546]).

DEFINITION 8.2. The index $r = r(\tilde{x}) = r(\tilde{x}, f)$ of the nondegenerate critical point $\tilde{x}$ of the real valued function $f(x)$ is the index of $d^2f(\tilde{x}; h, h)$ as quadratic form in h.

LEMMA 8.3. *The index of Definition 8.2 is invariant under a transformation $x = x(y)$ of class C'' which has an inverse of class C'', i.e., if $\tilde{x} = x(\tilde{y})$ and $\phi(y) = f(x(y))$ and if $\tilde{x}$ is a nondegenerate critical point of f then $\tilde{y}$ is a nondegenerate critical point of ϕ, and $r(\tilde{x}, f) = r(\tilde{y}, \phi)$.*

PROOF. That $\tilde{y}$ is a critical point of ϕ follows from the chain rule (2.2) (with $f_1 = \phi$). Thus $d\phi(\tilde{y}; \eta) = 0$ for all η. Therefore we conclude from (2.4) that

$$d^2f(\tilde{x}; h^1, h^2) = d^2\phi(\tilde{y}; \eta^1, \eta^2),$$

$$\eta^i = dy(x; h^i), \qquad h^i = dx(\tilde{y}; \eta^i), \qquad i = 1, 2. \tag{8.6}$$

This obviously proves our assertion that $\tilde{y}$ is nondegenerate. Finally for the proof of the equality of the indices it is certainly sufficient to show that

$$r(\tilde{x}, f) \leqslant r(\tilde{y}, \phi). \tag{8.7}$$

Now (8.7) is trivially true if $r(\tilde{x}, f) = 0$. Let then m be a positive integer for which $r(\tilde{x}, f) \geqslant m$. Then $E^-(f)$ contains m linearly independent elements $h_1, h_2, \cdots, h_m$ if $E^-(f)$ denotes the space E^- in the direct sum (8.5) corresponding to the quadratic form $q(h, h) = d^2f(\tilde{x}; h, h)$. Now (8.7) will be proved if we can show that $E^-(\phi)$ has at least m linearly independent elements where the definition of $E^-(\phi)$ is analogous to the one of $E^-(f)$. Since the linear map $k = dy(\tilde{x}; h)$ is nonsingular the elements $k_i = dy(\tilde{x}; h_i)$ are linearly independent. Now $k_i = k_i^- + k_i^+$ where $k_i^- \in E^-(\phi)$ and $k_i^+ \in E^+(\phi)$. We claim that the k_i^- are linearly independent. Otherwise we would have $\sum_1^m \alpha_i k_i^- = 0$ for some α_i not all zero. We set $k^+ = \sum_1^m \alpha_i k_i^+$, $h = \sum_1^m \alpha_i h_i$.

Then obviously $k^+ = dy(\tilde{x}; h)$ and h is a nonzero element of $E^-(f)$. Therefore we see from (8.6) that

$$0 > d^2f(\tilde{x}; h, h) = d^2\phi(\tilde{y}; k^+, k^+) \geqslant 0$$

a contradiction.

LEMMA 8.4.[5] *Let $l(h) = h + L(h)$ where $L(h)$ is a symmetric completely continuous operator. We assume moreover that $l(h)$ is nonsingular. Then no eigenvalue of l vanishes and the quadratic form $q(h, h) = (l(h), h)$ is nondegenerate. Its index r is finite and equals the number of negative eigenvalues of l (each counted according to its multiplicity).*

PROOF. Let e_i $(i = 1, 2, \cdots)$ be a full orthonormal system of eigenelements of L corresponding to the eigenvalues $\lambda_i \neq 0$ of L. There may be only a finite number of λ_i, otherwise $\lim_{i\to\infty} \lambda_i = 0$. In any case they are all real and only a finite number of e_i belong to the same eigenvalue. Moreover

$$l(h) = \sum_i \lambda_i(h, e_i)\, e_i\,, \tag{8.8}$$

and there exists an $h_0 \in E$ (which may be zero) such that

$$h = h_0 + \sum_i (h, e_i)\, e_i\,, \qquad (h_0\,, e_i) = 0 \qquad \text{for} \qquad i = 1, 2, \cdots . \tag{8.9}$$

It is well known that all this is a consequence of the complete continuity and symmetry of L. See, e.g., [12, pp. 231, 232]. Now $\mu_i = 1 + \lambda_i$ are the eigenvalues $\neq 1$ of the operator l, and $\lim_{i\to\infty} \mu_i = \lim_{i\to\infty} (1 + \lambda_i) = 1$ if there are infinitely many. Moreover all μ_i are different from 0 since otherwise $l(h)$ would be singular against assumption. The nonsingularity assumption also implies that $q(h, h)$ is nondegenerate; see the Remark following Definition 6.2. Now if we add (8.8) and (8.9) we obtain easily for any $k \in E$

$$(l(h), k) = (h_0\,, k) + \sum_i \mu_i(h, e_i)\,(k, e_i)$$

$$q(h, h) = (l(h), h) = \| h_0 \|^2 + \sum_i \mu_i(h, e_i)^2. \tag{8.10}$$

It is clear from (8.10) that the unique space E^- in the decomposition (8.5) consists only of the zero element if no μ_i is negative and otherwise is spanned

[5] This lemma is essentially contained in [9, p. 81]. However, in that paper a Hilbert space, in agreement with the older terminology, was assumed to be separable. No such assumption is made in the present paper.

by the e_i belonging to negative μ_i. But since $\mu = 1$ is the only possible limit point for the μ_i there are at most a finite number of such e_i and E^- is finite dimensional. By Definition 8.1 this proves the lemma.

From now on we will always assume that the assumptions (A), (B), and (C) of Section 7 are satisfied. Then $l(x; h) = dg(x; h)$ exists and is continuous (Lemma 7.1). We now add the following two assumptions

ASSUMPTION (D).

$$l(x; h) = dg(x; h) = h + L(x; h) \tag{8.11}$$

where L as linear operator in h is completely continuous.

ASSUMPTION (E). The representation (7.2) of $S : x = \bar{x}(u)$ is locally uniformly of class C', C'', C''' (Section 2). Moreover $d^2\bar{x}(u; h, k)$ is completely continuous in h and in k in the sense that every bounded sequence of points k contains a subsequence k^α $(\alpha = 1, 2, \cdots)$ such that to every positive ϵ there corresponds an integer α_0 such that

$$\| d^2\bar{x}(u; h, k^\alpha - k^\beta) \| < \| h \| \epsilon \qquad \text{for} \qquad \alpha, \beta > \alpha_0 , \tag{8.12}$$

and such the corresponding inequality holds if the roles of h and k are interchanged.

LEMMA 8.5. *If f satisfies assumptions (A)-(D) and if $\tilde{x}$ is a nondegenerate critical point of f then the index $r(\tilde{x}, f)$ (Definition 8.2) equals the number of negative eigenvalues of $dg(\tilde{x}; h)$ as operator in h.*

PROOF. By Lemma 2.2

$$d^2f(x; h^1, h^2) = (dg(x; h^1), h^2). \tag{8.13}$$

It is immediately verified from the assumptions made that the quadratic form obtained from (8.13) by setting $h^1 = h^2 = h$ satisfies the assumptions of Lemma 8.4 if we identify l with dg (cf. Lemma 6.1). Thus Lemma 8.5 follows from Lemma 8.4.

Let now x_0 be as in Theorem 8.1. In order to utilize Lemma 8.5 for the proof of this theorem we introduce in a suitable neighborhood N_0 of x_0 the normal local coordinate system (u, s) of Definition 5.1. For every point x in N_0 we have then the unique representation (cf. (7.3))

$$x = \bar{x}(u) + s\, n(u), \qquad u \in T_{x_0} , \qquad s \text{ real}, \qquad \| u \| < \delta. \tag{8.14}$$

On the other hand since T_{x_0} is a hyperspace every $y \in E$ has the unique representation

$$y = sn_0 + u, \qquad u \in T_{x_0} , \qquad s \text{ real}, \qquad n_0 = n(0), \qquad n_0 \perp T_{x_0}. \tag{8.15}$$

Therefore for some neighborhood V of the point $y = 0$ we have the 1-1 maps

$$y = y(x), \qquad x = x(y) : N_0 \leftrightarrow V \tag{8.16}$$

mapping the point x given by (8.14) into the point y given by (8.15) and vice versa. We recall that the points y given by (8.15) if given the norm

$$\| y \|_0 = \sqrt{s^2 + \| u \|^2}$$

form the local normal coordinate space Π_0 of Definition 5.1.

In slight modification of the notation used in (7.18) and (7.11) we write

$$\phi(y) = f(x(y)), \qquad \psi(y) = g(x(y)) \tag{8.17}$$

and use a corresponding notation for $\tilde{f}$ and $\tilde{g}$. Moreover we set

$$\gamma(y) = \operatorname{grad} \phi(y), \qquad d\gamma(y; \eta) = \lambda(y; \eta) \tag{8.18}$$

where

$$\eta = n_0 \sigma + v, \qquad v \in T_{x_0}, \qquad \sigma \text{ real}, \tag{8.19}$$

and define $\tilde{\gamma}$, $\tilde{\lambda}$ correspondingly. We have then from (7.10)

$$\tilde{\phi}(y) = \phi(y) + \frac{Ms^3}{3s_0^2}, \qquad \tilde{\gamma}(y) = \gamma(y) + \frac{n_0 M s^2}{s_0^2} \tag{8.20}$$

$$\tilde{\lambda}(y; \eta) = \lambda(y; \eta) + \frac{2n_0 M s \sigma}{s_0^2}.$$

We now consider the critical point x^* of $\tilde{f}$ mentioned in the first paragraph of Section 8. Let y^* be the point corresponding to x^* under the mapping (8.16). We will need the fact that at this point assumption (D) is still valid in the "y-system," i.e., we need

LEMMA 8.6.

$$\tilde{\lambda}(y^*, \eta) = \eta + \tilde{\Lambda}(y^*; \eta), \qquad y^* = y(x^*) \tag{8.21}$$

where $\tilde{\Lambda}(y^*; \eta)$ as operator on η is symmetric and completely continuous.

PROOF. We use the following matrix notation: if η is given by (8.19) and if $m = m(\eta) = m(\sigma n_0) + m(v)$ is a bounded linear map $E \to E$ we write

$$m = m(\eta) = \begin{pmatrix} m_1(\sigma n_0) & m_1(v) \\ m_2(\sigma n_0) & m_2(v) \end{pmatrix} \tag{8.21a}$$

where the indices 1 and 2 denote projection on n_0 and T_{x_0} resp. Then from (8.20)

$$\tilde{\lambda}(y; \eta) = \begin{pmatrix} 2n_0 M s \sigma s_0^2 + \lambda_1(y; \sigma n_0) & \lambda_1(y; v) \\ \lambda_2(y; \sigma n_0) & \lambda_2(y; v) \end{pmatrix}. \tag{8.22}$$

Now all elements of this matrix are bounded linear maps in σn_0 or in v; moreover except for $\lambda_2(y; v)$ they have either finite dimensional range or finite dimensional domain and are therefore completely continuous. In particular the map in the upper left corner may be written as $I_1 + c_1(y; \sigma n_0)$ where I_1 denotes the identity map on the space spanned by n_0 and where c_1 is completely continuous in its second argument. Consequently for the proof of (8.21) it remains to show that

$$\lambda_2(y; v) = v + \Lambda_2(y; v), \qquad y = y^* \tag{8.23}$$

where Λ_2 is a completely continuous map $T_{x_0} \to T_{x_0}$. Now since x^* is a critical point of $\tilde{f}$, (8.6) holds with f and ϕ replaced by $\tilde{f}$ and $\tilde{\phi}$ resp., and with $\tilde{x} = x^*$, $\tilde{y} = y^*$, and (cf. (8.19), (8.14)) with

$$\eta^i = \sigma^i n_0 + v^i, \qquad h^i = d\tilde{x}(u^*; v^i) + s\,dn(u^*; v^i) + \sigma^i n(u^*), \qquad i = 1, 2 \tag{8.24}$$

where $v^i \in T_{x_0}$, σ^i real, $u^* = u(s^*)$ with s^* determined by (7.28). Now by Lemma 2.2 we see from Definitions 8.11 and 8.18 that (8.6) (with the modifications indicated) may be written in the form

$$(\tilde{\lambda}(y^*; \eta^1), \eta^2) = (l(x^*, h^1), h^2). \tag{8.25}$$

We now set $\sigma^2 = 0$ such that $\eta^2 = v^2$. We claim that then $h^2 \in T_{\tilde{x}(u^*)}$. Indeed $d\tilde{x}(u^*; v^2) \in T_{\tilde{x}(u^*)}$ by definiton, and $dn(u^*; v^2) \in T_{\tilde{x}(u^*)}$ since $dn(u^*; v^2)$ is orthogonal to the normal $n(u^*)$ at $\tilde{x}(u^*)$. Thus our assertion follows from (8.24). Moreover from (8.25)

$$(\lambda_2(y^*; \eta^1), v^2) = (\tilde{\lambda}(y^*; \eta^1), v^2) = (\tilde{l}(x^*; h^1), h^2). \tag{8.26}$$

We will now prove: *if* $\eta^i = v^i$ *such that* h^i *is given by (8.24) with* $\sigma^i = 0$ *then*

$$h^i = v^i + C(u^*, v^i), \qquad i = 1, 2 \tag{8.27}$$

where C *is completely continuous in* v^i.

PROOF. Since $d\tilde{x}(0; v) = v$ (Definition 3.4) we have

$$d\tilde{x}(u^*, v^i) = v^i + \int_0^1 d^2\tilde{x}(tu^*; v^i, tu^*)\,(1 - t)\,dt, \tag{8.28}$$

and by assumption (E) the integral is completely continuous. Inspection of (8.24) shows that to complete the proof of (8.27) we have to prove that $dn(u^*; v^i)$ is completely continuous in v^i. For this purpose we start from

(4.46) with $k = v^1$, $h = v^2$ and with x replaced by $\bar{x}$. We thus obtain for any bounded sequence of elements v^{1j} ($j = 1, 2, \cdots$) in T_{x_0}

$$(dn(u^*; v^{1k} - v^{1j}), dx(u^*; v^2) = (n(u^*), d^2x(u^*; v^{1j} - v^{1k}, v^2). \tag{8.29}$$

Combining this equality with (8.12) in assumption (E) we see that there exists a subsequence w^j of the sequence v^{1j} such that to each positive ϵ there corresponds an integer j_0 such that

$$| (dn(u^*; w^k - w^j), d\bar{x}(u^*; v^2) | < \epsilon \| v^2 \| < \epsilon b \| dx(u^*; v^2) \|, \qquad j, k > j_0 \tag{8.30}$$

where b denotes the norm of the inverse of $d\bar{x}(u^*; v^2)$. Now both factors of the scalar product in (8.30) are in the hyperspace $T_{\bar{x}(u^*)}$; moreover since the second factor is nonsingular we can choose v^2 in such a way that the two factors are equal. We thus obtain from (8.30)

$$\| dn(u^*; w^k - w^j) \| < b\epsilon, \qquad k, j > j_0 .$$

This proves the completely continuity of $dn(u^*; v^1)$, and the proof of (8.27) is finished.

Now $C(u^*; v)$ is a completely continuous operator in v defined on the hyperplane T_{x_0}. Setting $C(u^*; v + \sigma n_0) = \sigma n_0 + C(u^*; v)$ we obtain a completely continuous operator defined in the whole space. Its adjoint C^* is also completely continuous. Keeping this in mind and using (8.27) we see that

$$(\tilde{l}(x^*; h^1), h^2) = (\tilde{l}(x^*; h^1) + C^*\tilde{l}(x^*; h^1), v^2).$$

If we substitute this expression for the right member of (8.26) we conclude that

$$\lambda_2(y^*; v^1) = [\tilde{l}(x^*; h^1) + C^*(\tilde{l}(x^*; h^1)]_2$$

where, as above, the subscript 2 denotes projection on T_{x_0}. Obviously the second term in the bracket is completely continuous in virtue of the fact that C^* is completely continuous. Therefore the proof of (8.23) will be completed when it is shown that

$$\tilde{l}(x^*; h^1) = v^1 + C^1(v^*; v^1), \qquad C^1 \text{ completely continuous in } v^1. \tag{8.31}$$

Now by definition $l = dg$, and we see from (7.15), (8.11), and (8.27) that

$$\tilde{l}(x^*; h^1) = v^1 + C(u^*; v^1) + L(x^*; h^1) + Ms^2dn(u^*; v^1)/s_0^2.$$

This proves (8.31) since C and dn are completely continuous by previous results, and since by assumption (D) the operator L is completely continuous in h^1 and therefore also in v^1.

This finishes the proof of the complete continuity of the operator $\tilde{\Lambda}$ in (8.21). It remains to prove its symmetry. Now by Lemma 2.2 and by the definitions involved we have

$$(\tilde{\lambda}(y; \eta^1)\, \eta^2) = d^2\tilde{\phi}(y; \eta^1, \eta^2). \tag{8.32}$$

Since the second differential at the right member of (8.32) is symmetric in η^1 and η^2 this equality obviously implies the symmetry of the operator $\tilde{\lambda}$ which in turn implies the symmetry of $\tilde{\Lambda}$.

Lemma 8.6 just proved deals with the critical point $x = x^*$ of $\tilde{f}$ (or $y = y^*$ of $\tilde{\phi}$). Theorem 8.1 to be proved connects this critical point with the nondegenerate tangentially critical point x_0 of f. We now turn to the consideration of the latter. We recall that by Definition 6.4, $u = 0$ is a nondegenerate critical point of the function $\phi_0(u)$ defined on the Hilbert space T_{x_0} and given by (3.21) with x to be replaced by $\bar{x}$. The number r_0 of Theorem 8.1 is then by definition the index of the nondegenerate quadratic form $d^2\phi_0(0; v, v)$, $v \in T_{x_0}$. We need various auxiliary lemmas.

LEMMA 8.7.

$$d^2\phi_0(u; v^1, v^2) = (\lambda_2(u; v^1), v^2), \qquad u, v^1, v^2 \in T_{x_0}, \tag{8.33}$$

where λ is defined in (8.18) and where as before the subscript 2 denotes projection on T_{x_0}.

PROOF. It follows from (3.21) (with x replaced by $\bar{x}$), from (8.17) and from (8.14), (8.15) that $\phi_0(u) = \phi(u)$, $(u \in T_{x_0})$. We see therefore from (8.18) that

$$d\phi_0(u; v^2) = d\phi(u; v^2) = (\gamma(u); v^2) = (\gamma_2(u), v^2).$$

Differentiating again we obtain

$$d^2\phi_0(u; v^1, v^2) = (d\gamma_2(u; v^1), v^2). \tag{8.34}$$

Now the Definition (8.18) of λ shows that $\lambda_2(u; v^1) = (d\gamma(u; v^1))_2$. Therefore (8.34) implies (8.33) provided that

$$d\gamma_2(u; v) = (d\gamma(u; v))_2, \qquad v \in T_{x_0}. \tag{8.35}$$

PROOF OF (8.35). From Lemma 2.2

$$d^2\phi(y; v, v^1) = (d\gamma(y; v), v^1) = ((d\gamma(y; v)_2, v^1). \tag{8.36}$$

On the other hand

$$d\phi(y; v^1) = (\gamma(y), v^1) = (\gamma_2(y), v^1).$$

Therefore

$$d^2\phi(y; v^1, v) = (d\gamma_2(y; v), v^1). \tag{8.37}$$

Comparison of (8.36) and (8.37) proves (8.35).

LEMMA 8.8. *Let x and y be given by (8.14) and (8.15) respectively, and let $\phi(y)$, $\gamma(y)$, $\lambda(y; \eta)$ be as in (8.17), (8.18). Then there exists a neighborhood V of $y = 0$ and a positive constant N_1 such that for all $y \in V$ with $s > 0$ and all η given by (8.24) the following statements hold*

(a) $|\, d^2\phi(y; \eta^1, \eta^2)\,| < N_1 \| \eta_1 \| \| \eta_2 \|$.

(b) $\| \lambda(y; \eta^1) \| < N_1 \| \eta_1 \|$.

(c) *$\gamma(y)$ is uniformly of class C'.*

PROOF. (a) From (2.4) (with the roles of x and y interchanged) we have

$$d^2\phi(y; \eta^1, \eta^2) = d^2f(x; h^1, h^2) + df(x; d^2x)$$

where $h^i = dx(y; \eta^i)$ is given by (8.24) (with u^* replaced by u), and where consequently

$$d^2x = d^2\bar{x}(u; v^1, v^2) + \sigma^2 dn(u; v^1) + s d^2n(u; v^1, v^2) + \sigma^1 dn(u; v^2).$$

(a) follows now obviously from (7.4) and Theorems 4.2 and 4.3.

(b) $|\, (\lambda(y; \eta^1), \eta^2) = |\, d^2\phi(y; \eta^1, \eta^2)\,| < N_1 \| \eta_1 \| \eta_2 \|$ by (a). Now set $\eta^2 = \lambda^2(y; \eta^1)$.

(c) follows from (a) in the same way as Lemma 7.1(b) follows from (7.4).

LEMMA 8.9. *As before let the subscripts 1 and 2 denote projection on n_0 and T_{x_0} respectively. Then a necessary and sufficient condition for the map m defined in (8.21a) to be symmetric is that for any $\eta \in E$, $(\eta_1, m_1(\eta_2)) = (m_2(\eta_1), \eta_2)$, and that $m_2(\eta_2)$ is symmetric.*

We omit the proof which consists of a direct verfication made obvious by the fact that n_0 is orthogonal to T_{x_0} and that $m_1(\eta_1)$ is a map of a one dimensional space into itself.

LEMMA 8.10. *Let*

$$\lambda^0(y; \eta) = \begin{pmatrix} \sigma n_0 & 0 \\ 0 & \lambda_2(y; v) \end{pmatrix}, \quad \sigma > 0 \tag{8.38}$$

where η and v are as in (8.19). Then

(a) *$\lambda^0(y; \eta)$ is symmetric.*

(b) *$\lambda^0(0; \eta) - \eta$ is completely continuous.*

(c) $\lambda^0(0; \eta)$ *is not singular (or, what is the same, does not admit the eigenvalue* 0*), and the number of negative eigenvalues of* $\lambda^0(0; \eta)$ *(counted according to their multiplicities) equals the number* r_0 *defined in the statement of Theorem 8.1.*

(d) *There exists a positive number* p *such that the assertions made in* (c) *concerning* $\lambda^0(0; \eta)$ *are still true for* $\lambda^0(y; \eta)$ *if* $\| y \| < p$.

PROOF. (a) We see from (8.20) that $\lambda_2 = \tilde{\lambda}_2$. It follows then from (8.32) that $\lambda_2(y; \eta)$ is symmetric. This obviously proves (a).

(b) It is sufficient to prove that $\lambda_2(0; v) - v$ is completely continuous. We see from the chain rule (2.4) that

$$d^2f(x_0; h^1, h^2) + df(x_0; d^2x) = d^2\phi(0; v^1, v^2) \tag{8.39}$$

where the relation between h^i and v^i is obtained from (8.24) by replacing u^* by 0 and by setting $\sigma^i = s = 0$. Using also (3.11) we obtain therefore

$$h^i = d\bar{x}(0; v^i) = v^i. \tag{8.40}$$

Moreover we know that $dn(0; v)$ is orthogonal to $n(0) = n_0$ for all $v \in T_{x_0}$. Therefore the left member in (4.46) equals 0 for $u = 0$, and we infer from this equation that $d^2\bar{x}(0; v^1, v^1) \in T_{x_0}$. Since by (7.23) (with $s = 0$)

$$d^2x(0; v^1, v^2) = d^2\bar{x}(0; v^1, v^2)$$

we see that

$$df(x_0; d^2x) = (g(x_0), d^2\bar{x}) = (g_t(x_0), d^2\bar{x}).$$

But since x_0 is a tangentially critical point, $g_t(x_0) = 0$ by Lemma 6.3. Thus the differential of first order in (8.39) vanishes, and using (8.40) we see that

$$d^2f(x_0; v^1, v^2) = d^2\phi(0; v^1, v^2) = d^2\phi_0(0; v^1, v^2) \tag{8.41}$$

since ϕ_0 is the restriction of ϕ to T_{x_0}. Using Lemma 2.2 to rewrite the left member of (8.41), and Lemma 8.7 to rewrite the right member we obtain the equality $(l(x_0; v^1), v^2) = (\lambda_2(0; v^1), v^2)$ from which we conclude that $l_2(x^0; v^1) = \lambda_2(0; v^1)$. Thus on account of assumption (D) (see (8.11)) $\lambda_2(0, v^1) - v^1$ is completely continuous; this obviously finishes the proof of (b).

(c) By assumption the point $y = u = 0$ is a nondegenerate critical point of ϕ_0. Therefore the bilinear form at the left member of (8.33) is (for $u = 0$) nondegenerate, and we conclude from (8.33) that $\lambda_2(0; v)$, and consequently $\lambda^0(0; \eta)$, is nonsingular. Moreover r_0 is by definition the index of the quadratic form $d^2\phi_0(0; v, v)$. Therefore we see from (8.33) in conjunction with Lemma 8.4 that r_0 equals the number of negative eigenvalues of $\lambda^2(0; v)$. This finishes the proof of (C) since inspection of the matrix (8.38) shows that the eigenvalues of $\lambda^0(0; \eta)$ are those of $\lambda_2(0; v)$ augmented by the positive eigenvalue 1.

(d) We will assume that $\lambda_2(0; v)$ has at least one negative eigenvalue. It will be clear from the proof in this case how to treat the even simpler situation where there is no negative eigenvalue. Let then $\bar{\mu}_1, \bar{\mu}_2, \cdots, \bar{\mu}_r$ $(1 \leqslant r \leqslant r_0)$ be the distinct negative eigenvalues of $\lambda_2(0; v)$, and let δ be a positive number which if there are positive eigenvalues is smaller that the smallest positive eigenvalue. Moreover for $\rho = 1, 2, \cdots, r$ let i_ρ be an open interval with center $\bar{\mu}_\rho$ where we suppose that the closures of these intervals are disjoint and that they have a positive distance from 0. (The existence of such δ and such i_ρ follows from (a) and (b).) Now there exists a positive ϵ with this property: if $m_2(y)$ is a linear symmetric bounded operator on T_{x_0} which in the uniform operator topology differs from $\lambda_2(0; v)$ by less than ϵ, i.e., for which

$$\| m_2(v) - \lambda_2(0; v) \| < \epsilon \| v \| \tag{8.42}$$

then the following two statements are true:

(i) Each interval i_ρ contains as many eigenvalues of $m_2(v)$ (counted according to their multiplicities) as the multiplicity of $\bar{\mu}_\rho$ as eigenvalue of $\lambda_2(0; v)$ indicates.

(ii) The complement of $\bigcup_1^r i_\rho$ with respect to the interval $(-\infty, \delta)$ (having a positive distance from the spectrum of $\lambda_2(0; v)$) contains no eigenvalue of $m_2(v)$.

(The existence of an ϵ satisfying (i) follows from a classical theorem by Rellich [13, Satz 5], and the existence of an ϵ such that (ii) is true follows from the "continuity" of the spectrum, see [14, Theorem 5, 2.3]).

Now by Lemma 8.8(c) the inequality (8.42) is satisfied with

$$m_2(v) = \lambda_2(y; v) = (d\gamma(y; v))_2$$

for small enough $\| y \|$.

Therefore for such y the operator $\lambda_2(y; v)$ is nonsingular and has as many negative eigenvalues as $\lambda_2(0, v)$. This proves (d).

LEMMA 8.11. *Let α be a positive number and let*

$$\lambda^\alpha(y; \eta) = \begin{pmatrix} \sigma n_0 + \alpha^2 \lambda_1(y; \sigma n_0) & \alpha \lambda_1(y; v) \\ \alpha \lambda_2(y; \sigma n_0) & \lambda_2(y; v) \end{pmatrix}. \tag{8.43}$$

Then:

(a) *λ^α is symmetric (as operator in η).*

(b) *There exist positive numbers α_0 and p_0 such that for $0 < \alpha < \alpha_0$ and $\| y \| < p_0$ the negative part of the spectrum of the operator $\lambda^\alpha(y; \eta)$ consists of r_0 negative eigenvalues, and 0 is not a point (or limit point) of the spectrum.*

PROOF. (a) By (8.32), $\tilde{\lambda}(y; \eta)$ is symmetric. The validity of (a) follows now from a comparison of $\tilde{\lambda}$ in the form (8.22) with (8.43) and from the fact that the condition for symmetry given in Lemma 8.9 is necessary and sufficient.

(b) We restrict y to the requirement $\| y \| < p_0$ and $y \in V$ where p_0 is as in Lemma 8.10(d) and where V is the neighborhood defined in Lemma 8.8. Now $| \sigma |$ and $\| v \|$ are both majorized by $\| \eta \|$, and $\| \lambda_i \| \leqslant \| \lambda \|$ for $i = 1, 2$. Using this fact and assuming $0 < \alpha < 1$ we obtain easily from Lemma 8.8(b) the uniform estimate

$$\| \lambda^\alpha(y; \eta) - \lambda^0(y; \eta) \| < 3N_1 \alpha \| \eta \|.$$

Let now δ be as in the proof of Lemma 8.10(d). We know from that lemma that the part of the spectrum of $\lambda^0(y; \eta)$ lying in $(-\infty, \delta)$ consists exactly of r_0 eigenvalues (counted according multiplicity) and that all of these are negative. It is therefore obvious that the construction and argument used in the proof of Lemma 8.10(d) serves also to prove the (b) — part of the present lemma.

We now return to the critical point x^* of Theorem 8.1. This critical point of $\tilde{f}$ is given by (7.28). It follows from (8.14) and (8.15) that

$$y^* = s^* n_0 + u(s^*) \tag{8.44}$$

is the corresponding critical point of $\tilde{\phi}$.

COROLLARY TO LEMMA 8.11. *Let s_0 be as in Definition 7.10 of $\tilde{f}$. Let*

$$\alpha^* = \frac{s_0}{\sqrt{2Ms^*}}. \tag{8.45}$$

Then there exists a positive $\bar{s}_0$ such that for $0 < s_0 < \bar{s}_0$ the operator $\lambda^{\alpha^*}(y^*; \eta)$ has the spectral properties described in Lemma 8.11.

PROOF. Let α_0 and p_0 be as in Lemma 8.11(b). We have to show that for small enough s_0

$$\text{(i)} \quad \| y^* \| < p_0, \qquad \text{(ii)} \quad 0 < \alpha^* < \alpha_0.$$

Now by (7.29), $s^* < s_0$. Since moreover $u(s)$ is continuous and $u(0) = 0$ (see(7.16)), (i) follows immediately from (8.44). For the proof of (ii) we have only to note that by (8.45) and (7.29), $0 < \alpha^* < \theta s_0/2M\theta_0$.

In order to connect the spectral properties of the operator $\lambda^{\alpha^*}(y^*;\eta)$ of the preceding corollary with those of the operator $\tilde{\lambda}(y^*; \eta)$ (see (8.22)) we prove

LEMMA 8.12. *Let*

$$m^\beta(\eta) = \begin{pmatrix} \beta^2 m_1(\eta_1) & \beta m_1(\eta_2) \\ \beta m_2(\eta_1) & m_2(\eta_2) \end{pmatrix}$$

with $\eta_1 = \sigma n_0$, $\eta_2 = v$, $\eta = \eta_1 + \eta_2$ *such that the* $m(\eta)$ *of (8.21a) is* $m^1(\eta)$. *Then:*

(i) *If* m^1 *is symmetric then* m^β *is symmetric for all* β.

(ii) *If* $m^1 = \eta + M^1(\eta)$ *where* M^1 *is completely continuous then*

$$m^\beta(\eta) = \eta + M^\beta(\eta)$$

where M^β *is completely continuous for all* β.

(iii) *If* m^1 *is nonsingular then* m^β *is nonsingular for all* $\beta \neq 0$.

(iv) *Under the assumptions made on* m^1 *in* (i), (ii), (iii) *let* r^β *be the number of negative eigenvalues of* m^β. *(By* (i) *and* (ii) r^β *is finite.) Then for all positive* β

$$r^\beta = r^1. \tag{8.46}$$

Proof. (i) is an immediate consequence of Lemma 8.9.

(ii) Follows from the considerations contained in the first paragraph following (8.22) together with the fact that the element in the right lower corner of the matrix for m^β is independent of β.

(iii) If $\eta = \eta_1 + \eta_2$ annihilates m^β then $\bar{\eta} = \beta\eta_1 + \eta_2$ annihilates m^1 and vice versa. Moreover $\eta = 0$ if and only if $\bar{\eta} = 0$ since the decomposition of η (and $\bar{\eta}$) is direct.

(iv) It follows from (iii) and the Remark following Definition 6.2 that the quadratic form

$$q^\beta(\eta, \eta) = (m^\beta(\eta), \eta) \tag{8.47}$$

is nondegenerate, and by Lemma 8.4 its index equals r^β. r^β is therefore the dimension of E_β^- if $E = E_\beta^- + E_\beta^+$ is the direct decomposition of E corresponding to q (see (8.5)), and the assertion (8.46) is equivalent to the assertion that E_β^- and E_1^- have the same dimension. We now show

$$r_\beta \geqslant r_1 . \tag{8.48}$$

On account of the orthogonality of n_0 and T_{x_0} it is easily verified that

$$q^\beta(\eta, \eta) = q^1(\bar{\eta}, \bar{\eta}), \qquad \begin{matrix} \eta = \eta_1 + \eta_2 \\ \bar{\eta} = \beta\eta_1 + \eta_2 \end{matrix} . \tag{8.49}$$

Now if $r_1 = 0$ then (8.48) is trivial. Therefore we assume that r_1 is positive. We know that because of (ii), r_1 is finite. Let then $\bar{\eta}^\rho = \bar{\eta}_1{}^\rho + \bar{\eta}_2{}^\rho$, $\rho = 1, 2, \cdots, r_1$ be a base for E_1^-, and let $\eta = \beta^{-1}\bar{\eta}_1{}^\rho + \bar{\eta}_2{}^\rho$. Since by definition of E_1^-, $q^1(\bar{\eta}^\rho, \bar{\eta}^\rho) < 0$ we see from (8.49) that

$$q^\beta(\eta^\rho, \eta^\rho) < 0. \tag{8.50}$$

Let now $\eta^\rho = (\eta^\rho)^- + (\eta^\rho)^+$, $(\eta^\rho)^- \in E_\beta^-$, $(\eta^\rho)^+ \in E_\beta^+$. Since $q^\beta((\eta^\rho)^+, (\eta^\rho)^+) \geqslant 0$ we see from (8.50) and (8.3) that

$$q^\beta(\eta^{\rho-}, \eta^{\rho-}) \leqslant q^\beta(\eta^{\rho-}, \eta^{\rho-}) + q^\beta(\eta^{\rho+}, \eta^{\rho+}) = q^\beta(\eta^\rho, \eta^\rho) < 0.$$

Thus the $\eta^{\rho-}$ are nonzero elements of E_β^-. For the proof of (8.48) it remains to show that they are linearly independent. We will show first that the η^ρ are linearly independent: suppose that

$$0 = \sum_1^r \gamma_\rho \eta^\rho = \sum_1^r \gamma_\rho \eta_1{}^\rho + \sum_1^r \gamma_\rho \eta_2{}^\rho;$$

this implies $\sum_1^r \gamma_\rho \eta_1 = 0$, $\sum_1^r \gamma_\rho \eta_2 = 0$. If we multiply the first of these equations by β and add the result to the second one we obtain $\sum_1^r \gamma_\rho \bar{\eta}_\rho = 0$. This proves the linear independence of the η_ρ since the $\bar{\eta}_\rho$ are linearly independent by assumption.

To prove the linear independence of the $\eta^{\rho-}$ suppose that

$$\sum_1^r \delta_\rho \eta^{\rho-} = 0, \qquad \sum_1^r \delta_\rho{}^2 \neq 0. \tag{8.51}$$

Let now $\eta = \sum \delta_\rho \eta^\rho$. Then $\eta \neq 0$ because of the linear independence of the η^ρ just proved. Moreover it follows from (8.51) that $\eta = \sum \delta\rho\, \eta^{\rho+}$. Therefore η is a nonzero element of $E_\beta{}^+$ and

$$q^\beta(\eta, \eta) > 0. \tag{8.52}$$

On the other hand it is immediately verified that $\bar{\eta} = \sum \delta\rho \bar{\eta}^\rho$ where $\bar{\eta} = \beta\eta_1 + \eta_2$. Consequently $\bar{\eta} \in E_1^-$. Moreover $\bar{\eta} \neq 0$ since the $\bar{\eta}_\rho$ are linearly independent by definition. Therefore $q^1(\bar{\eta}, \bar{\eta}) < 0$. But this inequality is, on account of (8.49), in contradiction with (8.52).

Proof of Theorem 8.1. We first claim that the map $\lambda^{\alpha *}(y^*; \eta)$ of the corollary to Lemma 8.11 satisfies the assumptions made on m^1 in (i), (ii) and (iii) of Lemma 8.12. The assumptions contained in (i) and (iii) follow from Lemma 8.11. Moreover (8.23) obviously implies that $\lambda_2(y^*; v) - v$ is completely continuous, and the reasoning contained in the paragraph preceding (8.23) completes the proof of the assumption of (ii).

We now conclude from Lemmas 8.12, 8.11, and its Corollary: for no $\beta \neq 0$ is the operator

$$m^\beta(\eta) = \begin{pmatrix} \beta^2(\sigma n_0 + (\alpha^*)^2 \lambda_1(y^*; \sigma n_0) & \beta\alpha^*\lambda_1(y^*; v) \\ \beta\alpha^*\lambda_2(y^*; \sigma n_0) & \lambda_2(y^*; v) \end{pmatrix}$$

singular, and it has for all positive β exactly r_0 negative eigenvalues. This proves our theorem since $m^{\beta}(\eta)$ and $\lambda(y^*; \eta)$ become identical for $\beta = 1/\alpha^*$ as is seen immediately from (8.22) and (8.45).

Before stating Theorem 8.2, the main result of this section, we introduce

ASSUMPTION (F). f has on S at most a finite number of tangentially critical points, i.e. (Lemma 6.3), of zeros of the tangential component g_t of the gradient of f. They are all nondegenerate, and $\| g_t \|$ is bounded away from zero on any closed set on S containing none of the zeros of g_t.

THEOREM 8.2. *Let $x_1^-, x_2^-, \cdots, x_q^-$ denote the tangentially critical points on S^- (Definition 7.3), and let $r_1^-, r_2^-, \cdots, r_q^-$ be their respective indices. Then there exist two positive numbers $s_0 < s_1$ such that* (i) *on the boundary S_1 of R_1 the gradient field $\tilde{g}$ of $\tilde{f}$ is exteriorly directed, and* (ii) *$\tilde{f}$ has in $R_1 - R$ exactly q critical points $x_1^*, x_2^*, \cdots, x_q^*$ (cf. Definition 7.1); moreover these critical points are nondegenerate, and if r_k^* is index of x_k^* then*

$$r_k^* = r_k, \qquad k = 1, 2, \cdots, q. \tag{8.53}$$

PROOF. (i) follows from (7.26). We proceed to prove (ii). It is clear from Theorem 8.1 and Theorem 7.1 that there exist positive numbers $s_0' < s_1'$, and disjoint neighborhoods $\overline{W}_1^-, \overline{W}_2^-, \cdots, \overline{W}_q^-$ relative to S of the points $x_1^-, x_2^-, \cdots, x_q^-$ respectively such that: (i) for each $k = 1, 2, \cdots, q$ the normal local coordinate system at x_k^- (Section 5) is valid for $\bar{x} \in \overline{W}_k^-$ and $0 \leqslant s \leqslant s_1'$; (ii) if s_0, s_1 are positive numbers such that $s_0 < s_0'$, $s_1 < s_1'$, $s_0 < s_1$ then the "cylinder" Z_k^- of points $(\bar{x}, s)$ with $\bar{x} \in \overline{W}_k^-$, $0 \leqslant s \leqslant s_1$ contains exactly one critical point x_k^* of $\tilde{f}$, and this critical point is nondegenerate and its index r_k^* satisfies (8.53).

Let now x_0 be one of the tangentially critical points on S in which g has the direction of the exterior normal n_0. Then

$$g(x_0) = \| g(x_0) \| n_0 . \tag{8.54}$$

Let (u, s) be the normal local coordinate system at x_0 such that the representation (7.3) holds in some neighborhood of x_0. Since $n(u)$ and $g(x)$ are then continuous functions of s and u and since $n(0) = n_0$ and $x = x_0$ correspond to $s = 0$, $u = 0$ it follows from (8.54) that in some neighborhood of x_0.

$$(n(u), g(x)) > (n_0 , g(x_0)/2 = \| g(x_0) \|/2 > 0. \tag{8.55}$$

Therefore by (7.15)

$$\| \tilde{g}(x) \| \geqslant (\tilde{g}(x), n(u)) = (g(x), n(u)) + Ms^2/s_0^2 \geqslant (g(x), n(u)) > \| g(x_0) \|/2 > 0. \tag{8.56}$$

Thus there is no critical point of $\tilde{f}$ in some neighborhood of x_0. Therefore we can assert: if $x_1^+, x_2^+, \cdots, x_p^+$ are the tangentially critical points of $\tilde{f}$ on S

at which g has the direction of the exterior normal then there exists a positive number s_1'' and disjoint neighborhoods W_j^+ of x_j with respect to S ($j = 1, 2, \cdots, p$) such that the cylinder Z_j^+ consisting of the points $(\bar{x}, s)$ with $\bar{x} \in W_j^+$ and $0 \leqslant s \leqslant s_1''$ contains no critical point of $\tilde{f}$. Obviously we may suppose that the neighborhoods W_k^- are disjoint from the neighborhoods W_j^+.

Finally let

$$S' = S - \bigcup_1^q W_k^- - \bigcup_1^p W_j^+.$$

Then S' is a closed subset of S in which $g_t \neq 0$ (Lemma 6.3), and by assumption (F) there exists a positive constant m such that

$$\| g_t(x_0) \| \geqslant 2m > 0 \qquad \text{for all} \qquad x_0 \in S'. \tag{8.57}$$

Now by (7.5), $g(x)$ is uniformly continuous in R_2. This together with (8.57) implies that there exists a constant s_1''' such that

$$\begin{aligned} \| g_t(x_0 + sn_0) \| &\geqslant \| g_t(x_0) \| - \| g_t(x_0 + sn_0) - g_t(x_0) \| \\ &\geqslant \| g_t(x_0) \| - \| g(x_0 + sn_0) - g(x_0) \| \geqslant m, \qquad 0 \leqslant s \leqslant s_1'''. \end{aligned} \tag{8.58}$$

Since g_t is orthogonal to $n(0) = n_0$ we see from (7.15) that

$$g_t(x_0 + sn_0) = \tilde{g}_t(x_0 + sn_0),$$

and from (8.58) that $\| \tilde{g}_t(x_0 + sn_0) \| > 0$; thus $sx_0 + sn_0$ is not a critical point of $\tilde{f}$ for $0 \leqslant s \leqslant s_1'''$.

It is clear that $s_1 = \operatorname{Min}(s_1', s_1'', s_1''')$ satisfies the requirements of Theorem 8.2.

MODIFICATION OF THEOREM 8.2. *Let S^+ be the subset of S on which $g(x)$ is exteriorly directed. Let $x_1^+, x_2^+, \cdots, x_p^+$ denote the tangentially critical points on S^+, and let $r_1^+, r_2^+, \cdots, r_p^+$ denote their respective indices. Denote by $\tilde{\tilde{f}}$ the function obtained from $\tilde{f}$ (see (7.10)) by replacing s by $-s$, and let $\tilde{\tilde{g}} =$ grad $\tilde{\tilde{f}}$. Then there exist two positive numbers $s_0 < s_1$ such that;* (i) *on the boundary S_1 of R_1, $\tilde{\tilde{g}}(x)$ is interiorly directed, and* (ii) *$\tilde{\tilde{f}}$ has exactly p critical points $\tilde{x}_1^*, \tilde{x}_2^*, \cdots, \tilde{x}_p^*$ in $R_1 - R$; these are nondegenerate and*

$$\tilde{r}_j^* = r_j^+ + 1, \qquad j = 1, 2, \cdots, p, \tag{8.60}$$

where $\tilde{r}^*$ denotes the index of the critical point $\tilde{x}_j^*$.

Except for obvious changes the proof of this modification is the same as the one given for Theorem 8.2 in its original form. Suffice it to indicate why (8.53) becomes (8.60) in the modified case: in the original proof the number of negative eigenvalues of the linear map (8.22) was (for s_0 small enough) shown to be equal to the number of negative eigenvalues of the map (8.38) a number which obviously equals the number of negative eigenvalues of $\lambda_2(y; v)$. In the modified case (8.22) has to be replaced by the map obtained from (8.22) by replacing s by $-s$ in the upper left corner element. Correspondingly (8.38) has to be modified by writing a minus sign in front of the upper left corner element. Obviously the number of negative eigenelements of the modified map is by one greater than the number of negative eigenelements of $\lambda_2(y; v)$.

REMARK. If S consists of a finite number of disjoint components it is possible to use Theorem 8.2 on some components in its original form and on the remaining ones in the modified form (see Section 9).

9. The Analogue of the Morse Relation (1.6b)

In this section we use domains $R \in E$ of the following special form

$$R = V_0 + \bigcup_{i=1}^{k} V_i \tag{9.1}$$

where for $i = 1, 2, \cdots, k$, V_i is the open ball with center a_i and radius ρ_i, and V_0 is the ball $\| x \| < \rho_0$. The closures $\bar{V}_i$ of V_i are supposed to be disjoint and to be subsets of V_0. S_i is the boundary of V_i, and S_0 that of V_0.

It is easy to see that the boundary S of R is of class C''' in the sense of Definition 3.2(α)-(δ), and that S satisfies assumption (E) (Section 8). To verify these statements it is obviously sufficient to verify them for one component of S, say S_0. Now for $x_0 \in S$ and $u \in T_{x_0}$

$$x(u) = u + \frac{x_0}{\rho_0} \sqrt{\rho_0{}^2 - (u, u)}, \qquad u = x - \frac{x_0}{\rho_0{}^2} (x, x_0)$$

is a parameter representation of S in a neighborhood of x_0. Moreover

$$dx(u; h) = h - \frac{x_0}{\rho_0} \frac{(u, h)}{\rho_0{}^2 - (u, u)},$$

$$d^2x(u; h, k) = - \frac{x_0}{\rho_0} \frac{(k, h)(\rho_0{}^2 - (u, u)) + (u, h)(u, k)}{(\rho_0{}^2 - (u, u))^{3/2}}.$$

These formulas show that the above statements are true and that dx and d^2x are uniformly locally continuous (Section 2); computation shows the same to be true for d^3x. The expression for $dx(u; h)$ shows also that $dx(0; h) = h$. Thus our parameter system (u) is not only tangential but also normal in the sense of Definition 3.4.

Before stating the next theorem we recall that for j an integer the jth Morse number of a nondegenerate critical point of index r equals $\delta_r{}^j$ where $\delta_r{}^j$ is the Kronecker symbol.[6] If $c_1, c_2, \cdots, c_l$ are the critical points of f in R we denote by r_i the index of c_i and by $m_i{}^j$ the jth Morse number of the critical point c_i. Then $M^j = \sum_{i=1} m_i{}^j$, the number of critical points of index j of f in R, is called the jth Morse number of f in R.

The following Theorem 9.1 is the analogue of the relation (1.6b), and Theorem 9.2 is the analogue of (1.11).

THEOREM 9.1. *Let $f(x)$ be defined in the region R given by (9.1). We assume that f satisfies all conditions stated in the earlier sections except that assumption (F) (Section 8) is replaced by assumption (F′): in no point of S_0 has $g(x)$ the direction of the normal interior to R, and on no point of $\bigcup_1^k S_i$ has $g(x)$ the direction of the normal exterior to R. Then $M_j = 0$ except for a finite number of j-values, and*

$$\sum_{j=0} (-1)^j M^j = 1 - k. \tag{9.2}$$

PROOF. For $\alpha = 0, 1, 2, \cdots, k$ let χ_α be the characteristic of the vectorfield $g(x)$ on S_α,[7] and for $i = 1, 2, \cdots, l$ let d_i be the Leray-Schauder index [15, p. 54] of c_i as zero of $g = \operatorname{grad} f$. Then [2; Satz 5]

$$\sum_{i=1}^{l} d_i = \chi_0 - \sum_{\beta=1}^{k} \chi_\beta . \tag{9.3}$$

Now by assumption (F′) in no point of S_α has $g(x)$ the direction of the normal directed towards the interior of the ball V_α bounded by S_α. Therefore [2; Satz 7*b*)] $\chi_\alpha = 1$, and (9.3) becomes

$$\sum_{i=1}^{l} d_i = 1 - k. \tag{9.4}$$

[6] This is classical in the finite dimensional case. For a proof in the Hilbert space case see, e.g., [4, pp. 254-255]. Note that the assumption of separability made there is superfluous as is seen if in the proof one uses Lemma 8.4 of the present paper instead of Lemma 6.4 of [4].

[7] For the definition of the characteristic of a vectorfield see [10, p. 478] for the finite dimensional case, and [2, Section 3] for the Banach space case.

On the other hand $d_i = (-1)^{r_i}$ [9, Theorem 5.1]. Consequently from (9.3):

$$\sum_{i=1}^{l} d_i = \sum_{i=1}^{l} (-1)^{r_i} = \sum_{i=1}^{l} \sum_{j=0} (-1)^j \delta_{r_i}^j = \sum_{i=1}^{l} \sum_{j=0} (-1)^j m_i^j$$

$$= \sum_{j=0} (-1)^j \sum_{i=1}^{l} m_i^j = \sum_{j=0} (-1)^j M^j. \tag{9.5}$$

Thus (9.2) follows from (9.4).

THEOREM 9.2. *The assumptions and notations for this theorem are the same as for the preceding one except that we reinstate the original assumption (F). In addition let M^{j-} be the number of tangentially critical points of index j on S_0 in which $g(x)$ has the direction of the normal interior to R, and let M^{j+} be the number of tangentially critical points of index j on $\bigcup_{i=1}^{k} S_i$ in which $g(x)$ has the direction of the normal exterior to R. Then*

$$\sum_{j=0} (-1)^j (M^j + M^{j-} - M^{j+}) = 1 - k. \tag{9.6}$$

PROOF. Let

$$R_1 = V_0' - \bigcup_{i=1}^{k} V_i'$$

where V_0' and V_i' are balls concentric to V_0 and V_i respectively while the respective radii ρ_0' and ρ_i' satisfy $\rho_0' > \rho_0$, $\rho_i' < \rho_i$. We now define

$$\begin{aligned} \bar{f}(x) &= f(x) \qquad \text{for} \qquad x \in R \\ &= \tilde{f}(x) \qquad\qquad\quad x \in V_0' - V_0 \\ &= \tilde{\tilde{f}}(x) \qquad\qquad\quad x \in V_i' - V_i', \qquad i = 1, 2, \cdots, k, \end{aligned}$$

where $\tilde{f}$ and $\tilde{\tilde{f}}$ are as in Theorem 8.2 and in the modification of Theorem 8.2. It follows from these theorems that (for $\rho_0' - \rho_0$ and $\rho_i - \rho_i'$ small enough) on S_0' the gradient $\bar{g}$ of $\bar{f}$ is exteriorly directed with respect to R_1, or what is the same with respect to V_0' while on S_i' $(i = 1, 2, \cdots, k)$ $\bar{g}$ is interiorly directed with respect to R_1 or what is the same exteriorly with respect to V_i'. Therefore we can apply Theorem 9.1 to R_1 and it follows from (9.2) that

$$\sum_{j=0} (-1)^j \bar{M}^j = 1 - k \tag{9.7}$$

where $\bar{M}^j$ is the number of critical points of index j of f in R_1. Let now $\bar{M}^{j-}$ and $\bar{M}^{j+}$ be the number of critical points of index j in $V_0' - V_0$ and in $\bigcup_{i=1}^k (V_i - V_i')$ respectively. Then (for $\rho_0' - \rho_0$ and $\rho_i - \rho_i'$ small enough) we see from (8.53) that $\bar{M}^{j-} = M^{j-}$, and from (8.60) that $\bar{M}^{j+} = M^{(j-1)+}$ where M^{-1+} is defined to mean zero. Since the number of critical points of index j of f in R is the same as that of f, namely M^j, we see that

$$\bar{M}^j = M^j + M^{j-} + M^{(j-1)+}.$$

This equality together with (9.7) obviously implies (9.6).

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