

A Theorem on Maximal Monotonic Sets in Hilbert Space*

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1. INTRODUCTION

Let H be a Hilbert space, with real or complex scalars. For $x, y \in H$, we denote by $\langle x, y \rangle$ the *real part* of the inner product. A set $E \subset (H \times H)$ is called *monotonic* (see [1, 2]) provided that for all $(x_1, y_1), (x_2, y_2) \in E$, we have $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$, and is called *maximal* if it is not properly contained in another monotonic set.

The zero vector of H will be denoted by θ . The set $S \subset H$ will be said to *surround* θ provided every ray $R_x = \{tx : t > 0\}$ (for $x \neq \theta$) contains a point of S . The *closed* convex hull of S is denoted by $K(S)$. (By a well known theorem, $K(S)$ is also weakly closed.)

The main theorem of this paper is:

THEOREM 1. *Let E be a maximal monotonic set in $H \times H$ satisfying*

- (i) *There exists a bounded set $C \subset H$ surrounding θ such that, for any $x \in C$, there exists $y \in H$ with $(x, y) \in E$ and $\langle x, y \rangle \geq 0$.*
- (ii) *There exists a bounded set $D \subset H$ surrounding θ such that, for any $y \in D$, there exists $x \in H$ with $(x, y) \in E$ and $\langle x, y \rangle \geq 0$.*

Then for any closed linear subspace X , with orthogonal complement Y , the set $E \cap (X \times Y)$ is nonempty. Moreover, these conditions are sufficient for the set $E \cap (X \times Y) \cap [K(C) \times K(D)]$ to be nonempty.

This theorem seems to be applicable to "orthogonal projection" methods for the solution of differential equations; see the papers of F. E. Browder [3-7] for indications of these directions.

We shall call a function f with domain in H and range in H *monotonic* if the graph of f is a monotonic set, and *strongly monotonic* if there exists $c > 0$ such that for any x_1, x_2 in the domain of f ,

$$\langle x_1 - x_2, f(x_1) - f(x_2) \rangle \geq c \|x_1 - x_2\|^2.$$

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Also, f will be called *hemicontinuous* provided its restriction to line segments in H is continuous with respect to the weak topology in the range. A special case of Theorem 1 in which $X = H, Y = \{\theta\}$ is proved in an earlier paper of the writer [2] in a Banach-space context. (The writer has not yet been successful in transferring the present theorem to such a context.) In the main theorem of [2], there is a continuity-assumption which is used, however, only to guarantee the maximality of the set E .

2. TOOLS

LEMMA 1 ([1], Theorem 3). *Let H be a Hilbert space, and $E \subset H \times H$ a maximal monotonic set. Then the map $(x, y) \rightarrow x + y$ is a homeomorphism of E onto H .*

LEMMA 2 ([2], Lemma 1). *Let $f: H \rightarrow H$ be a hemicontinuous function, and $x_0, y_0 \in H$. Then sufficient for $y_0 = f(x_0)$ is: for any x ,*

$$\langle x_0 - x, y_0 - f(x) \rangle \geq 0.$$

LEMMA 3 ([5]). *Let $f: H \rightarrow H$ be a continuous, strongly monotonic function. Then f has a continuous inverse defined on all of H .*

(Lemma 3 can be easily proved from Lemmas 1 and 2 by consideration of the continuous monotonic function $[c^{-1}f(x) - x]$.)

LEMMA 4. *Let $f: H \rightarrow H$ be a continuous, strongly monotonic function. Let X and Y be orthogonal complements in H . Then there exists a unique $x \in X$ with $f(x) \in Y$.*

PROOF. Let P be the projection map of H onto X . Then $g = Pf$ is a continuous strongly monotonic function carrying X into X . By Lemma 3, $g(x) = \theta$ has a unique solution. Since $Pf(x) = \theta$, we have $f(x) \in Y$.

REMARKS. Lemma 4 is a kind of primitive form of Theorem 1, as can be seen by thinking of the set E of Theorem 1 as the graph of f . Notice that the proof of the conclusion on g depends very strongly on the assumption of continuity (or hemicontinuity) of f ; it is, in fact, the main object of this paper to show that such an assumption is not really necessary. (One may refer to a later section of this paper for the motivations.)

3. PROOF OF MAIN THEOREM

Let H be a Hilbert space. For $S \subset H$, we denote by $N_\epsilon(S)$ the closed ϵ -neighborhood of S , i.e. the set $\{x + y : x \in S, \|y\| \leq \epsilon\}$.

LEMMA 5. Let $E \subset H \times H$ be a maximal monotonic set satisfying the hypotheses (i), (ii) of Theorem 1. Let X and Y be orthogonal complements in H . Then there exists a sequence $(p_n, q_n) \in X \times Y$ such that, for any $(x_0, y_0) \in E$ and any $\epsilon > 0$, we have for sufficiently large n :

$$(p_n, q_n) \in S(x_0, y_0, \epsilon) = \{(x, y) : x \in X, y \in Y, \langle x_0 - x, y_0 - y \rangle \geq -\epsilon, \\ x \in N_\epsilon(K(C)), y \in N_\epsilon(K(D))\}.$$

PROOF. For each fixed $n > 0$: let $E_n = \{(x, y/n) : (x, y) \in E\}$. It is trivial to show that E_n is a maximal monotonic set. Thus by Lemma 1 applied to E_n , for each $u \in H$ there exists a unique point (x, y) of E with $u = x + y/n$ and moreover, x and y/n depend continuously on u . Hence $y + x/n$ depends continuously on u . It follows that the set

$$\bar{E}_n = \{(x + y/n, y + x/n) : (x, y) \in E\}$$

is the graph of a continuous function F defined on all of H . We have, for (x_1, y_1) and (x_2, y_2) any two elements of E ,

$$\begin{aligned} & (x_1 + y_1/n) - (x_2 + y_2/n), (y_1 + x_1/n) - (y_2 + x_2/n) \\ &= \frac{\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2}{n} + (1 + 1/n^2) \langle x_1 - x_2, y_1 - y_2 \rangle \\ &\geq \|x_1 - x_2\|^2/n \end{aligned}$$

and hence F is a strongly monotonic function.

By Lemma 4, there is a point of X which is mapped into Y by F , and hence there exists $(x_n, y_n) \in E$ with $p_n = x_n + y_n/n \in X$, and $q_n = y_n + x_n/n \in Y$. Form the real part of the inner product:

$$\langle p_n, q_n \rangle = \frac{\|x_n\|^2 + \|y_n\|^2}{n} + (1 + 1/n^2) \langle x_n, y_n \rangle = 0. \quad (3.1)$$

We shall show (by an argument used in [2]) that $x_n \in K(C)$. For if not, then since C surrounds θ , $x_n \neq \theta$, and there is a point $(z_n, w_n) \in E$ with $z_n \in C$, $\langle z_n, w_n \rangle \geq 0$, $x_n = \lambda_n z_n$, $\lambda_n > 1$. Thus by monotonicity of E ,

$$\langle x_n - z_n, y_n - w_n \rangle \geq 0.$$

Substituting $\lambda_n z_n$ for x_n and cancelling $(\lambda_n - 1)$, we obtain

$$\langle z_n, y_n \rangle \geq \langle z_n, w_n \rangle \geq 0.$$

Multiplying by λ_n , we see finally $\langle x_n, y_n \rangle \geq 0$, in contradiction with Eq. (3.1). Now, by the analogous argument, $y_n \in K(D)$. Hence,

$$\|x_n/n\| \rightarrow 0 \quad \text{and} \quad \|y_n/n\| \rightarrow 0 \tag{3.2}$$

as $n \rightarrow \infty$.

It follows from relations (3.2) that $p_n = x_n + y_n/n$ and $q_n = y_n + x_n/n$ lie in $N_\epsilon(K(C))$ and $N_\epsilon(K(D))$ respectively for sufficiently large n .

It remains to be shown that for sufficiently large n ,

$$\langle x_0 - p_n, y_0 - q_n \rangle \geq -\epsilon.$$

We have

$$\begin{aligned} & \langle x_0 - (x_n + y_n/n), y_0 - (y_n + x_n/n) \rangle \\ &= \langle x_0 - x_n, y_0 - y_n \rangle + \langle y_n/n, y_0 - y_n \rangle + \langle x_0 - x_n, x_n/n \rangle + \langle y_n/n, x_n/n \rangle \\ &\geq -\|y_n/n\| \|y_0 - y_n\| - \|x_0 - x_n\| \|x_n/n\| - \|y_n/n\| \|x_n/n\| \end{aligned}$$

and the conclusion follows from relations (3.2) and the fact that $x_n \in K(C)$, $y_n \in K(D)$, with C and D bounded, so that x_n and y_n remain bounded.

PROOF OF THEOREM 1. Let ϵ stand for a real number between zero and 1 (i.e., $0 < \epsilon \leq 1$). Consider the two-parameter family of sets in $H \times H$, parametrized by $(x_0, y_0) \in E$ and ϵ :

$$S((x_0, y_0), \epsilon) = \{(x, y) : x \in X, y \in Y, \langle x_0 - x, y_0 - y \rangle \geq -\epsilon, \\ x \in N_\epsilon(K(C)), y \in N_\epsilon(K(D))\}.$$

By Lemma 5, the intersection of any finite subcollection of these sets is nonempty. Now, these sets are all subsets of $T = N_1(K(C)) \times N_1(K(D))$, which is a closed convex bounded set in $H \times H$ and hence is weakly compact. Also, $S((x_0, y_0), \epsilon)$ can be written as

$$\begin{aligned} & (X \times Y) \cap \{(x, y) : \langle x_0, y_0 \rangle - \langle x, y \rangle - \langle x_0, y \rangle \geq -\epsilon\} \\ & \cap [N_\epsilon(K(C)) \times N_\epsilon(K(D))] \end{aligned}$$

and is thus clearly a weakly closed subset of T . It follows from the finite intersection property of compact sets that the intersection of the entire family of sets is nonempty. Letting (x, y) be in this intersection, we have $x \in X, y \in Y, x \in K(C), y \in K(D)$, and: for all $(x_0, y_0) \in E$,

$$\langle x_0 - x, y_0 - y \rangle \geq 0.$$

The conclusion follows from the hypothesized maximality of E .

4. AN APPLICATION TO NONLINEAR ELECTRICAL NETWORKS

Let R be the real numbers.

Following [8], we call a maximal monotonic set in $R \times R$ a *resistor*. Let X and Y be a pair of orthogonal complementary subsets of R^n , and let E_i (for $i = 1, \dots, n$) be resistors.

THEOREM 2. *Suppose it is possible to find*

$$(x_1^0, \dots, x_n^0) \in X \quad \text{and} \quad (y_1^0, \dots, y_n^0) \in Y$$

such that the resistors $E_i^0 = \{(x, y) : (x - x_i^0, y - y_i^0) \in E_i\}$ each contain a point of the open first quadrant, and a point of the open third quadrant, in R^2 . Then there exist $(x_1, \dots, x_n) \in X$ and $(y_1, \dots, y_n) \in Y$ such that for each i , $(x_i, y_i) \in E_i$.

REMARKS. If X and Y are the current-space and voltage-drop-space of an electrical network, with "branches" numbered $1, \dots, n$, a very slightly stronger theorem (Theorem 8.1 of [8]) holds—namely, the same theorem with the weaker hypothesis that each E_i^0 contains a point of the horizontal axis and a point of the vertical axis. Another minor variant is shown in [9] by means of the Kuhn-Tucker optimality conditions of nonlinear (convex) programming. The relationship between the present "monotonicity" method and the variational method (first used by Duffin [10] for the present problem) is clear on examination of [11] and [9]. Considering that the problem has been so thoroughly studied, it would seem unnecessary to give yet another proof. Our excuse is that the present treatment seems to provide the synthesis of nonlinear electric network theory and nonlinear boundary-value problems called for by Birkhoff and Diaz in [13].

PROOF OF THEOREM 2. Let H be R^n . Let $E \subset H \times H$ be the set

$$\{((x_1, \dots, x_n), (y_1, \dots, y_n)) : \text{for each } i, (x_i, y_i) \in E_i^0\}.$$

It is easily verified that the sets E_i^0 are resistors, and that the set E is a monotonic set. Applying Lemma 1 to each set E_i^0 individually, we see that the map $(x, y) \rightarrow x + y$ carries E onto H , and then applying the lemma to E , we see that E is a *maximal* monotonic set. From the hypothesis that each E_i^0 contains points of the open first and third quadrants, the existence of the sets C and D necessary for application of Theorem 1 is easily established. Hence there exist $(\bar{x}_1, \dots, \bar{x}_n) \in X$ and $(\bar{y}_1, \dots, \bar{y}_n) \in Y$ such that for each i , $(\bar{x}_i, \bar{y}_i) \in E_i^0$. Thus

$$(x_1, \dots, x_n) = (x_1^0 + \bar{x}_1, \dots, x_n^0 + \bar{x}_n) \in X$$

and

$$(y_1, \dots, y_n) = (y_1^0 + \bar{y}_1, \dots, y_n^0 + \bar{y}_n) \in Y$$

yield the conclusion of the theorem.

A similar theorem can be stated for complex scalars, using $H = C^n$, for "resistors" in $C \times C$ of the type

$$E_i = \{(x, y) : x = z_i y + e_i\}$$

where z_i, e_i are complex constants and $\operatorname{Re} z_i > 0$. The constant z_i is of course "impedance"; Lemma 4 (or well-known results on linear dissipative operators) should be used for the existence-proof instead of Theorem 1.

V. ADDENDUM

Since this paper was written, F. E. Browder has transmitted to the writer a proof of a very closely related theorem [7] which offers the possibility of working with a reflexive Banach space rather than a Hilbert space, under slightly different hypotheses. The writer is also indebted to Browder for pointing out a redundant hypothesis in an earlier version of Theorem 1.

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