A Theorem on Maximal Monotonic Sets in Hilbert Space*

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1. Introduction

Let $H$ be a Hilbert space, with real or complex scalars. For $x, y \in H$, we denote by $\langle x, y \rangle$ the real part of the inner product. A set $E \subset (H \times H)$ is called monotonic (see [1, 2]) provided that for all $(x_1, y_1), (x_2, y_2) \in E$, we have $(x_1 - x_2, y_1 - y_2) \geq 0$, and is called maximal if it is not properly contained in another monotonic set.

The zero vector of $H$ will be denoted by $0$. The set $S \subset H$ will be said to surround $\theta$ provided every ray $R_x = \{tx : t > 0\}$ (for $x \neq 0$) contains a point of $S$. The closed convex hull of $S$ is denoted by $K(S)$. (By a well known theorem, $K(S)$ is also weakly closed.)

The main theorem of this paper is:

**Theorem 1.** Let $E$ be a maximal monotonic set in $H \times H$ satisfying

(i) There exists a bounded set $C \subset H$ surrounding $\theta$ such that, for any $x \in C$, there exists $y \in H$ with $(x, y) \in E$ and $\langle x, y \rangle \geq 0$.

(ii) There exists a bounded set $D \subset H$ surrounding $\theta$ such that, for any $y \in D$, there exists $x \in H$ with $(x, y) \in E$ and $\langle x, y \rangle \geq 0$.

Then for any closed linear subspace $X$, with orthogonal complement $Y$, the set $E \cap (X \times Y)$ is nonempty. Moreover, these conditions are sufficient for the set $E \cap (X \times Y) \cap [K(C) \times K(D)]$ to be nonempty.

This theorem seems to be applicable to "orthogonal projection" methods for the solution of differential equations; see the papers of F. E. Browder [3-7] for indications of these directions.

We shall call a function $f$ with domain in $H$ and range in $H$ monotonic if the graph of $f$ is a monotonic set, and strongly monotonic if there exists $c > 0$ such that for any $x_1, x_2$ in the domain of $f$, 

$$
\langle x_1 - x_2, f(x_1) - f(x_2) \rangle \geq c \| x_1 - x_2 \|^2.
$$

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Also, \( f \) will be called \textit{hemicontinuous} provided its restriction to line segments in \( H \) is continuous with respect to the weak topology in the range. A special case of Theorem 1 in which \( X = H, Y = \{\theta\} \) is proved in an earlier paper of the writer [2] in a Banach-space context. (The writer has not yet been successful in transferring the present theorem to such a context.) In the main theorem of [2], there is a continuity-assumption which is used, however, only to guarantee the maximality of the set \( E \).

2. Tools

\textbf{Lemma 1 ([I], Theorem 3).} Let \( H \) be a Hilbert space, and \( E \subseteq H \times H \) a maximal monotonic set. Then the map \((x, y) \mapsto x + y\) is a homeomorphism of \( E \) onto \( H \).

\textbf{Lemma 2 ([2], Lemma 1).} Let \( f : H \to H \) be a hemicontinuous function, and \( x_0, y_0 \in H \). Then sufficient for \( y_0 = f(x_0) \) is: for any \( x, \)

\[ \langle x_0 - x, y_0 - f(x) \rangle \geq 0. \]

\textbf{Lemma 3 ([5]).} Let \( f : H \to H \) be a continuous, strongly monotonic function. Then \( f \) has a continuous inverse defined on all of \( H \).

(Lemma 3 can be easily proved from Lemmas 1 and 2 by consideration of the continuous monotonic function \([c^{-1}f(x) - x]\).)

\textbf{Lemma 4.} Let \( f : H \to H \) be a continuous, strongly monotonic function. Let \( X \) and \( Y \) be orthogonal complements in \( H \). Then there exists a unique \( x \in X \) with \( f(x) \in Y \).

\textbf{Proof.} Let \( P \) be the projection map of \( H \) onto \( X \). Then \( g = Pf \) is a continuous strongly monotonic function carrying \( X \) into \( X \). By Lemma 3, \( g(x) = \theta \) has a unique solution. Since \( Pf(x) = \theta \), we have \( f(x) \in Y \).

\textbf{Remarks.} Lemma 4 is a kind of primitive form of Theorem 1, as can be seen by thinking of the set \( E \) of Theorem 1 as the graph of \( f \). Notice that the proof of the conclusion on \( g \) depends very strongly on the assumption of continuity (or hemicontinuity) of \( f \); it is, in fact, the main object of this paper to show that such an assumption is not really necessary. (One may refer to a later section of this paper for the motivations.)

3. Proof of Main Theorem

Let \( H \) be a Hilbert space. For \( S \subseteq H \), we denote by \( N_\epsilon(S) \) the closed \( \epsilon \)-neighborhood of \( S \), i.e. the set \( \{x + y : x \in Cl(S), \|y\| \leq \epsilon\} \).
Lemma 5. Let $E \subset H \times H$ be a maximal monotonic set satisfying the hypotheses (i), (ii) of Theorem 1. Let $X$ and $Y$ be orthogonal complements in $H$. Then there exists a sequence $(p_n, q_n) \in X \times Y$ such that, for any $(x_0, y_0) \in E$ and any $\epsilon > 0$, we have for sufficiently large $n$:

$$(p_n, q_n) \in S(x_0, y_0, \epsilon) = \{(x, y) : x \in X, y \in Y, \langle x - x_0, y - y_0 \rangle \geq -\epsilon, x \in N_\epsilon(K(C)), y \in N_\epsilon(K(D))\}.$$

Proof. For each fixed $n > 0$: let $E_n = \{(x, y/n) : (x, y) \in E\}$. It is trivial to show that $E_n$ is a maximal monotonic set. Thus by Lemma 1 applied to $E_n$, for each $u \in H$ there exists a unique point $(x, y)$ of $E$ with $u = x + y/n$ and moreover, $x$ and $y/n$ depend continuously on $u$. Hence $y + x/n$ depends continuously on $u$. It follows that the set

$$E_n = \{(x + y/n, y + x/n) : (x, y) \in E\}$$

is the graph of a continuous function $F$ defined on all of $H$. We have, for $(x_1, y_1)$ and $(x_2, y_2)$ any two elements of $E$,

$$\langle x_1 + y_1/n, x_2 + y_2/n \rangle = \langle x_1 - x_2, y_1 - y_2 \rangle + (1 + 1/n^2) \langle x_1 - x_2, y_1 - y_2 \rangle$$

and hence $F$ is a strongly monotonic function.

By Lemma 4, there is a point of $X$ which is mapped into $Y$ by $F$, and hence there exists $(x_n, y_n) \in E$ with $p_n = x_n + y_n/n \in X$, and $q_n = y_n + x_n/n \in Y$. Form the real part of the inner product:

$$\langle p_n, q_n \rangle = \frac{\|x_n\|^2 + \|y_n\|^2}{n} + (1 + 1/n^2) \langle x_n, y_n \rangle = 0. \quad (3.1)$$

We shall show (by an argument used in [2]) that $x_n \in K(C)$. For if not, then since $C$ surrounds $\theta$, $x_n \neq \theta$, and there is a point $(z_n, w_n) \in E$ with $z_n \in C$, $\langle x_n, w_n \rangle \geq 0$, $x_n = \lambda_n z_n$, $\lambda_n > 1$. Thus by monotonicity of $E$,

$$\langle x_n - z_n, y_n - w_n \rangle \geq 0.$$

Substituting $\lambda_n z_n$ for $x_n$ and cancelling $(\lambda_n - 1)$, we obtain

$$\langle x_n, y_n \rangle \geq \langle z_n, w_n \rangle \geq 0.$$
Multiplying by \( \lambda_n \), we see finally \( \langle x_n, y_n \rangle \geq 0 \), in contradiction with Eq. (3.1). Now, by the analogous argument, \( y_n \in K(D) \). Hence,

\[
\|x_n/n\| \to 0 \quad \text{and} \quad \|y_n/n\| \to 0
\]

(3.2)
as \( n \to \infty \).

It follows from relations (3.2) that \( p_n = x_n + y_n/n \) and \( q_n = y_n + x_n/n \) lie in \( N_\epsilon(K(C)) \) and \( N_\epsilon(K(D)) \) respectively for sufficiently large \( n \).

It remains to be shown that for sufficiently large \( n \),

\[
\langle x_0 - P_n, y_0 - Q_n \rangle \geq -\epsilon.
\]

We have

\[
\langle x_0 - (x_n + y_n/n), y_0 - (y_n + x_n/n) \rangle
\]

\[
= \langle x_0 - x_n, y_0 - y_n \rangle + \langle y_n/n, y_0 - y_n \rangle + \langle x_0 - x_n, x_n/n \rangle + \langle y_n/n, x_n/n \rangle
\]

\[
\geq -\|y_n/n\| \|y_0 - y_n\| - \|x_0 - x_n\| \|x_n/n\| = \|y_n/n\| \|x_n/n\|
\]

and the conclusion follows from relations (3.2) and the fact that \( x_n \in K(C) \), \( y_n \in K(D) \), with \( C \) and \( D \) bounded, so that \( x_n \) and \( y_n \) remain bounded.

**Proof of Theorem 1.** Let \( \epsilon \) stand for a real number between zero and 1 (i.e., \( 0 < \epsilon \leq 1 \)). Consider the two-parameter family of sets in \( H \times H \), parametrized by \( (x_0, y_0) \in E \) and \( \epsilon \):

\[
\mathcal{S}((x_0, y_0), \epsilon) = \{(x, y) : x \in X, y \in Y, \langle x_0 - x, y_0 - y \rangle \geq -\epsilon, x \in N_\epsilon(K(C)), y \in N_\epsilon(K(D))\}.
\]

By Lemma 5, the intersection of any finite subcollection of these sets is non-empty. Now, these sets are all subsets of \( T = N_\epsilon(K(C)) \times N_\epsilon(K(D)) \), which is a closed convex bounded set in \( H \times H \) and hence is weakly compact. Also, \( \mathcal{S}((x_0, y_0), \epsilon) \) can be written as

\[
(X \times Y) \cap \{(x, y) : \langle x_0, y_0 \rangle - \langle x, y_0 \rangle - \langle x_0, y \rangle \geq -\epsilon \}
\]

\[
\cap [N_\epsilon(K(C)) \times N_\epsilon(K(D))]
\]

and is thus clearly a weakly closed subset of \( T \). It follows from the finite intersection property of compact sets that the intersection of the entire family of sets is nonempty. Letting \( (x, y) \) be in this intersection, we have \( x \in X, y \in Y, x \in K(C), y \in K(D) \), and: for all \( (x_0, y_0) \in E \),

\[
\langle x_0 - x, y_0 - y \rangle \geq 0.
\]

The conclusion follows from the hypothesized maximality of \( E \).
Let $R$ be the real numbers.

Following [8], we call a maximal monotonic set in $R \times R$ a resistor. Let $X$ and $Y$ be a pair of orthogonal complementary subsets of $R^n$, and let $E_i$ (for $i = 1, \cdots, n$) be resistors.

**Theorem 2.** Suppose it is possible to find

$$\left( x_0^1, \cdots, x_0^n \right) \in X \quad \text{and} \quad \left( y_0^1, \cdots, y_0^n \right) \in Y$$

such that the resistors $E_i^0 = \{(x, y) : (x - x_i^0, y - y_i^0) \in E_i\}$ each contain a point of the open first quadrant, and a point of the open third quadrant, in $R^2$. Then there exist $(x_1, \cdots, x_n) \in X$ and $(y_1, \cdots, y_n) \in Y$ such that for each $i$, $(x_i, y_i) \in E_i$.

**Remarks.** If $X$ and $Y$ are the current-space and voltage-drop-space of an electrical network, with "branches" numbered 1, $\cdots$, $n$, a very slightly stronger theorem (Theorem 8.1 of [8]) holds—namely, the same theorem with the weaker hypothesis that each $E_i^0$ contains a point of the horizontal axis and a point of the vertical axis. Another minor variant is shown in [9] by means of the Kuhn-Tucker optimality conditions of nonlinear (convex) programming. The relationship between the present "monotonicity" method and the variational method (first used by Duffin [10] for the present problem) is clear on examination of [11] and [9]. Considering that the problem has been so thoroughly studied, it would seem unnecessary to give yet another proof. Our excuse is that the present treatment seems to provide the synthesis of nonlinear electric network theory and nonlinear boundary-value problems called for by Birkhoff and Díaz in [13].

**Proof of Theorem 2.** Let $H$ be $R^n$. Let $E \subset H \times H$ be the set

$$\{((x_1, \cdots, x_n), (y_1, \cdots, y_n)) : \text{for each } i, (x_i, y_i) \in E_i^0\}.$$ 

It is easily verified that the sets $E_i^0$ are resistors, and that the set $E$ is a monotonic set. Applying Lemma 1 to each set $E_i^0$ individually, we see that the map $(x, y) \mapsto x + y$ carries $E$ onto $H$, and then applying the lemma to $E$, we see that $E$ is a maximal monotonic set. From the hypothesis that each $E_i^0$ contains points of the open first and third quadrants, the existence of the sets $C$ and $D$ necessary for application of Theorem 1 is easily established. Hence there exist $(\bar{x}_1, \cdots, \bar{x}_n) \in X$ and $(\bar{y}_1, \cdots, \bar{y}_n) \in Y$ such that for each $i$, $(\bar{x}_i, \bar{y}_i) \in E_i^0$. Thus

$$(x_1, \cdots, x_n) = (x_1^0 + \bar{x}_1, \cdots, x_n^0 + \bar{x}_n) \in X$$
and

\[(y_1, \ldots, y_n) - (y_1^0 + \bar{y}_1, \ldots, y_n^0 + \bar{y}_n) \in Y\]

yield the conclusion of the theorem.

A similar theorem can be stated for complex scalars, using \(H = \mathbb{C}^n\), for \(\text{"resistors" in } C \times C\) of the type

\[E_i = \{(x, y) : x = z_i y + e_i\}\]

where \(z_i, e_i\) are complex constants and \(\text{Re } z_i > 0\). The constant \(z_i\) is of course \(\text{"impedance"; Lemma 4 (or well-known results on linear dissipative operators) should be used for the existence-proof instead of Theorem 1.}\)

V. ADDENDUM

Since this paper was written, F. E. Browder has transmitted to the writer a proof of a very closely related theorem [7] which offers the possibility of working with a reflexive Banach space rather than a Hilbert space, under slightly different hypotheses. The writer is also indebted to Browder for pointing out a redundant hypothesis is an earlier version of Theorem 1.

REFERENCES