

# The Operator Theory of the Pseudo-Inverse

## II. Unbounded Operators with Arbitrary Range\*

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### I. INTRODUCTION

Part I of this paper [1] has generalized the concept of the pseudo-inverse to encompass linear bounded operators on Hilbert spaces, under the assumption that the operator's range is closed. The function-analytic approach used there supersedes a predominantly algebraic viewpoint that interpreted the pseudo-inverse in the limited context of operations with (finite) matrices [2].

It is the purpose of Part II to extend the pseudo-inverse to Hilbert space operators which need not be bounded, and which may not have a closed range. A principal tool is the representation obtained in Part I [1] only for bounded operators with closed range. For normal operators, the spectral representation enables us to present the pseudo-inverse in more explicit form.

A restricted form of the representation theorem proved in [1] is the following: let  $A$  be a linear bounded operator from the Hilbert space  $H$  into itself, and let the range  $R$  of  $A$  be closed. Then  $A$  has a representation

$$A = P_R \tilde{A}, \tag{1.1}$$

where  $P_R$  is the projection on  $R$ , and  $\tilde{A}$  is a bounded operator whose inverse (defined on all of  $H$ ) is also bounded.

The above construction leads directly to an expression for the pseudo-inverse, whose definition is

DEFINITION 1a.  $A^+$  is the pseudo-inverse of  $A$  if, for every  $y \in H$ ,

$$\inf_{z \in H} \| Ax - y \| \tag{1.2}$$

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is attained by

$$\hat{x} = A^+y; \quad (1.3)$$

if  $x' \neq \hat{x}$  also attains the infimum (1.2),

$$\|x'\| > \|\hat{x}\|. \quad (1.4)$$

It is shown in [1] that a unique  $A^+$  as just defined exists for bounded  $A$  whenever  $R$  is closed. Moreover,  $A^+$  is given by

$$A^+ = P_M \bar{A}^{-1}. \quad (1.5)$$

Here  $\bar{A}$  may be taken as any  $\bar{A}$  satisfying (1.1), and  $P_M$  is the projection on  $M$ , the subspace which is the orthogonal complement of

$$N = \{x: Ax = 0\}. \quad (1.6)$$

If  $H$  is of finite dimension,  $A^+$  always exists, and may be represented by a matrix that can be constructed explicitly, either by algebraic methods [3], or by a scheme based on the representation (1.1). If  $A$  is Hermitian,  $A^+$  takes a particularly simple form. Suppose  $A$  is written in the canonical form

$$A = C^*DC \quad (1.7)$$

where  $C$  is unitary, and  $D$  is the diagonal matrix with elements  $d_{ij} = \delta_{ij}\lambda_i$ . Then

$$A^+ = C^*D^+C, \quad (1.8)$$

in which  $D^+$  is again diagonal, with  $d_{ij}^+ = \delta_{ij}\lambda_i^{-1}$  whenever  $\lambda_i \neq 0$ , and  $d_{ij}^+ = 0$  for  $i$  such that  $\lambda_i = 0$ . Further,

$$\bar{A} = C^*\bar{D}C \quad (1.9)$$

with  $\bar{d}_{ij} = \delta_{ij}\lambda_i$  or  $\bar{d}_{ij} = \delta_{ij}$  according as  $\lambda_i \neq 0$  or  $\lambda_i = 0$ . Finally,

$$P_R = P_M = C^*FC, \quad (1.10)$$

where  $f_{ij} = \delta_{ij}$  or  $f_{ij} = 0$ , according as  $\lambda_i \neq 0$  or  $\lambda_i = 0$ .

The formulas associated with a Hermitian matrix  $A$  can easily be interpreted in terms of its spectral representation. We then ask whether the resulting expressions may not hold more generally—as indeed they do. If for a Hermitian matrix  $A$ ,

$$A = \int_X \lambda dE_\lambda, \quad (1.11)$$

(1.8) becomes

$$A^+ = \int_{X-\{0\}} \lambda^{-1} dE_\lambda. \quad (1.12)$$

One also obtains

$$\tilde{A} = \int_X \lambda dE_\lambda + E_\lambda(\{0\}). \quad (1.13)$$

and

$$P_R = E_\lambda(X - \{0\}) \quad (1.14)$$

In the above,  $E_\lambda(C)$  would denote the projection specified by

$$E_\lambda(C) = \int_C dE_\lambda, \quad (1.15)$$

which is applicable to any set  $C$  (in the complex plane) measurable with respect to the measures  $(E_\lambda x, x)$  for every  $x \in H$ , and thus includes all Borel measurable sets.

It will be seen later that the formulas (1.12), (1.13), and (1.14) (which have been deduced for the finite-dimensional case only) remain correct for unbounded normal operators; if the pseudo-inverse is suitably redefined, (1.12) holds even when  $R$  is not closed.

Applications of the theory rest upon the "best approximate solution" property of the pseudo-inverse. When, for example, an integral equation is "solved" in this manner, the ordinary solution results whenever it exists; otherwise, the solution is the "best possible" in the sense of Definition 1a. In prediction theory (with quadratic error norm), exact solutions correspond to perfect prediction, and occur only in trivial problems. The pseudo-inverse is therefore a powerful tool, which may be applied in particular to prediction of wide-sense Markov processes [3] with an infinite number of components; consideration of these is reserved for a future paper.

## II. OPERATOR REPRESENTATIONS

This section is devoted to an analysis of operator representations of the form  $A = P\tilde{A}$ , where  $P$  is a projection, and  $\tilde{A}$  is a closed invertible operator. Any linear (not necessarily bounded) operator from a Hilbert space into itself has such a representation. Moreover,  $\tilde{A}^{-1}$  is bounded and defined on all of  $H$  iff the range of  $A$  is closed; otherwise,  $\tilde{A}^{-1}$  is unbounded but its domain is dense in  $H$ . In this fashion, the representation in question may be completely characterized.

Throughout,  $H$  is a Hilbert space, and  $A$  is a closed linear operator defined

on the linear manifold  $\mathcal{D}_A \subset H$ , with range  $R = A(\mathcal{D}_A)$  again in  $H$ . Since  $R$  is a linear manifold in any case, its orthogonal complement  $S$  is a subspace, and we have  $H = \bar{R} \oplus S$ , where  $\bar{R}$  is the closure of  $R$ . The subspace  $N$  has already been defined by (1.6); its orthogonal complement will be denoted by  $M$ . A projection operator is symbolized by  $P$ , where the subscript indicates (whenever necessary) the associated subspace. The notation of (1.15) is also occasionally used to indicate projection.

The first theorem extends a result in [1] to unbounded operators. Although the theorem is stated only in the limited context of operators from a Hilbert space to itself, it is easily seen that it applies equally to linear mappings from one Banach space to another.

THEOREM 1. *Let  $A$  have the representation*

$$A = P\tilde{A} \tag{2.1}$$

where  $P$  is a projection, and  $\tilde{A}$  has a bounded inverse defined everywhere on  $H$ . Then

- (i)  $R$  is closed.
- (ii)  $P = P_R$ .
- (iii) The restriction of  $\tilde{A}$  to  $N$  is a bounded operator which provides a 1-1 mapping from  $N$  onto  $S$ .

PROOF. Evidently,  $\mathcal{D}_A = \mathcal{D}_{\tilde{A}}$ , and  $\tilde{A}^{-1}$  takes  $H$  onto  $\mathcal{D}_{\tilde{A}}$ . Hence

$$P = A\tilde{A}^{-1} \tag{2.2}$$

implies that

$$P(H) = A\tilde{A}^{-1}(H) = A(\mathcal{D}_A) = R. \tag{2.3}$$

Since the left side of (2.3) is closed,  $R$  must be closed also. Also,  $P(H) = R$  means that  $P = P_R$ . Thus, (i) and (ii) have been proved.

To prove that the restriction of  $\tilde{A}$  to  $N$  is bounded, we first show  $\tilde{A}$  to be closed. Indeed,  $\tilde{A}^{-1}$  is bounded (and hence closed), so that  $\tilde{A} = (\tilde{A}^{-1})^{-1}$  is closed, as is its restriction to any (closed) subspace.<sup>1</sup> Now  $N \subset \mathcal{D}_A$ , and  $N$  is a subspace because  $A$  is a closed operator. Then the restriction of  $\tilde{A}$  to  $N$  is a closed operator defined on all of this subspace. By the closed graph theorem,  $\tilde{A}$  is therefore bounded on  $N$ .

It follows from (2.1) that

$$\tilde{A}(N) \subset S. \tag{2.4}$$

To show that (2.4) is actually an equality, we first prove that  $\tilde{A}(N)$  is closed,

<sup>1</sup> A subspace, as used herein, will always mean a closed linear manifold.

and then verify that  $S$  contains no nonnull vector orthogonal to  $\tilde{A}(N)$ . The boundedness of  $\tilde{A}^{-1}$  yields the existence of an  $\alpha > 0$  such that

$$\alpha \|x\| \leq \|\tilde{A}x\|, \quad x \in \mathcal{D}_{\tilde{A}}. \tag{2.5}$$

Suppose that  $\{y_n\} \in \tilde{A}(N)$  forms a Cauchy sequence. Then there exists  $\{x_n\} \in N$  such that  $Ax_n = y_n$ , and  $\{x_n\}$  is also a Cauchy sequence by (2.5). Now  $x_n \rightarrow x \in N$ , and  $y_n \rightarrow y$ . Since  $\tilde{A}$  is closed,  $y = \tilde{A}x$ , i.e.,  $y \in \tilde{A}(N)$  as was to be shown.

Let  $y \in S$  be orthogonal to  $\tilde{A}(N)$ , with  $y \neq 0$ . Because  $\tilde{A}(\mathcal{D}_{\tilde{A}}) = H$ , there is an  $x \in H$  such that

$$\tilde{A}x = y; \tag{2.6}$$

this  $x$  cannot belong to  $N$  since  $\tilde{A}x \notin \tilde{A}(N)$ . If the projection  $P_R$  is applied to both sides of (2.6), we obtain  $P_R \tilde{A}x = 0$  (since  $y \in S$ ). From (2.1),  $Ax = 0$ , so that  $x \in N$ . Hence the existence of the assumed  $y$  leads to a contradiction, and the proof of the theorem is complete.

The necessity conditions given in Theorem 1 are also sufficient. Thus, as we shall prove, if (i) is satisfied and there exists an operator as in (iii), we can find a representation (2.1). Of course, (iii) implies that  $S$  and  $N$  have the same dimension; it is this (weaker) statement that appears in the theorem.

**THEOREM 2.** *Let  $A$  have closed range  $R$ , and let  $N$  and  $S$  be of the same dimension. Then  $A$  possesses a representation*

$$A = P_R \tilde{A} \tag{2.7}$$

where  $\tilde{A}$  has a bounded inverse defined on all of  $H$ .<sup>2</sup>

**PROOF.** We call  $\hat{A}$  the restriction of  $A$  to  $M$ . This restriction has an inverse  $\hat{A}^{-1}$ , which is defined on  $R$ , and bounded there. To see that  $\hat{A}$  has an inverse, take  $x \in M \cap \mathcal{D}_A$ , and consider the possibility that  $\hat{A}x = Ax = 0$ . But then  $x \in N$ , so we have  $x = 0$ . Next, we observe that the range of  $\hat{A}$  is  $R$ . Let  $y \in R$ , and denote a corresponding element of  $\mathcal{D}_A$  by  $x$ . Now

$$x = x_1 + x_2, \quad x_1 \in M, \quad x_2 \in N. \tag{28}$$

Here  $x \in \mathcal{D}_A$ , and  $x_2 \in N \subset \mathcal{D}_A$ , so we must have  $x_1 \in \mathcal{D}_A$ . Therefore,  $y = Ax = Ax_1 = \hat{A}x_1$ , as was to be shown. Finally,  $\tilde{A}$  is closed because  $A$  is closed by hypothesis. Then  $\tilde{A}^{-1}$  is also closed, and, being defined for

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<sup>2</sup> It actually suffices that  $R$  be of the second category. The proof also applies to operators from one Banach space to another, provided that there exists a linear closed operator yielding a 1-1 mapping from all of  $N$  onto  $S$ .

all of  $R$  (a Hilbert space), is bounded. The last statement is a direct consequence of the closed graph theorem.

The construction of  $\tilde{A}$  is not unique; one which is convenient is

$$\tilde{A} = \hat{A}P_M + UP_N, \quad (2.9)$$

where  $U$  is a partial isometry, taking  $N$  onto  $S$ . Since  $P_RUP_N = \theta$  and  $P_R\hat{A}P_M = A$ , we see that (2.9) satisfies (2.1). The inverse of the specified  $\tilde{A}$  is

$$\tilde{A}^{-1} = \hat{A}^{-1}P_R + U^{-1}P_S. \quad (2.10)$$

This operator is bounded, for  $\|\tilde{A}^{-1}\| \leq \|\hat{A}^{-1}\| + 1$ . Further, the  $\tilde{A}^{-1}$  given by (2.10) satisfies the equations

$$\tilde{A}\tilde{A}^{-1} = I \quad \text{and} \quad \tilde{A}^{-1}\tilde{A}x = x \quad \text{for} \quad x \in \mathcal{D}_A, \quad (2.11)$$

as we can readily verify by direct calculation. The proof is then complete.

That  $N$  and  $S$  have the same dimension is (according to Theorem 2 and (iii)) both necessary and sufficient. As we shall see, there is no loss of generality in assuming this equality of dimension. Suppose, for example, that  $A'$  satisfies the hypotheses of Theorem 2 with the exception that the dimension of  $N' = \{x: A'x = 0\}$  is less than that of  $S$ . Then there exists a partial isometry  $U'$  from  $N'$  onto a subspace  $S' \subset S$ . Now let  $H_1 = H \oplus (S \ominus S')$ , and define  $A$  as an operator from  $H_1$  into  $H$  as follows:  $A(S \ominus S') = 0$ , and for  $x \in H_1$  such that  $P_{H_1}x \in \mathcal{D}_{A'}$ ,  $Ax = A'P_{H_1}x$ . Thus  $A$  and  $A'$  are essentially equivalent and  $A$ , a linear closed operator from  $H_1$  to  $H$ , can be shown to possess the desired representation. Detailed verification of the above assertions is routine but tedious, and follows the proof given in [1]. An analogous procedure disposes of the case that the dimension of  $S$  is less than that of  $N$ .

To complete this section, we consider operators whose range is not closed. For such operators the proof of Theorem 2 provides a construction (2.7) where  $\tilde{A}$  is once more given by (2.9). This construction assures that  $\tilde{A}$  is closed and has an inverse (2.10). The range of  $\tilde{A}$  includes  $S$  and  $R$ ; since the domain (and therefore the range) of  $\tilde{A}$  is a linear manifold, the range of  $\tilde{A}$  is dense in  $H$ .

In the proof of Theorem 1, the closure property of  $R$  led to the boundedness of  $\tilde{A}^{-1}$ . Thus the method of Theorem 1 cannot guarantee that  $\tilde{A}^{-1}$  is bounded. This is hardly surprising, for we shall prove later that  $\tilde{A}^{-1}$  must be unbounded. These results are summed up in

**THEOREM 3.** *Let  $A$  have a nonclosed range. Then  $A$  has a representation (2.7), in which  $\tilde{A}$  is a closed operator whose inverse is densely defined but unbounded.*

That  $\tilde{A}^{-1}$  is indeed unbounded follows from

**THEOREM 4.** *Let  $A$  have a nonclosed range, and suppose that  $A$  has the representation*

$$A = P\tilde{A} \tag{2.13}$$

where  $P$  is a projection, and  $\tilde{A}$  is a closed operator whose inverse is densely defined. Then  $P = P_R$ , and  $\tilde{A}^{-1}$  is unbounded.

**PROOF.** Since  $R$  is dense in  $\tilde{R}$ , (2.13) requires that  $P \geq P_R$ . Using  $P_R A = A$  and  $P_R P = P_R$ , we note that (2.13) implies

$$A = P_R \tilde{A}. \tag{2.14}$$

The combination of (2.13) and (2.14) yields

$$(P - P_R)\tilde{A} = \theta \tag{2.15}$$

on  $\mathcal{D}_{\tilde{A}}$ . But  $\tilde{A}(\mathcal{D}_{\tilde{A}})$  is dense in  $H$ , so  $P = P_R$ .

If  $\tilde{A}^{-1}$  were bounded, its domain would be all of  $H$ , for  $\tilde{A}^{-1}$  is densely defined and closed (since  $\tilde{A}$  is closed). Then, by Theorem 1,  $R$  is closed, thus contradicting the assumptions of the theorem. Hence  $\tilde{A}^{-1}$  must be unbounded, and the proof is complete.

Many plausible conjectures on further properties of the representation (2.13) or (2.14) are unfortunately false. The difficulties are partly due to the arbitrary manner in which  $A$  maps vectors into  $S$  (for  $x \in N$ ), and partly a result of the unspecified nature of the component in  $S$  for mappings of vectors in  $M \cap \mathcal{D}_A$ . It is possible, for example to construct an  $\tilde{A}$  whose inverse is bounded but not densely defined, or to exhibit—for closed  $A$ —a representation whose  $\tilde{A}$  has no closed extension. Or (2.13) may hold for closed  $\tilde{A}$ , while  $A$  does not admit of a closed extension. These and possibly other combinations of properties for  $A$  and  $\tilde{A}$  render the construction of a more comprehensive theory difficult.

Because the necessity conditions of Theorems 1 and 4 pertain only to closed  $\tilde{A}$ , there is particular interest in its closure properties. Some of these are revealed by the following two theorems.

**THEOREM 5.** *Let the closed operator  $A$  have the representation*

$$A = P\tilde{A}. \tag{2.16}$$

Then

$$A = P_R \tilde{A}, \tag{2.17}$$

and either

(a)  $\bar{A}$  is closed,

or

(b)  $\bar{A}$  possesses no closed extension.

*Remark.* Neither this nor the succeeding theorem requires  $\bar{A}$  to be invertible.

PROOF. The first assertion merely reemphasizes part of the statement of Theorem 4, and uses the part of the proof which does not depend on  $\bar{A}^{-1}$ . For the second claim, we make use of the fact that the equality sign in (2.16) implies that  $\mathcal{D}_A = \mathcal{D}_{\bar{A}}$ . Let us assume that  $\bar{A}$  is not closed. Then there exists  $\{x_n\} \in \mathcal{D}_{\bar{A}}$  such that  $x_n \rightarrow x$  and  $\bar{A}x_n \rightarrow w$ , but either  $x \notin \mathcal{D}_{\bar{A}}$  or  $\bar{A}x \neq w$ . But in fact,  $x \in \mathcal{D}_{\bar{A}}$ , so  $\bar{A}x \neq w$ ; this means that  $\bar{A}$  has no closed linear extensions.

To show  $x \in \mathcal{D}_{\bar{A}}$ , we note that (2.17) requires that

$$\bar{A}x_n = Ax_n + z_n, z_n \in S. \tag{2.18}$$

*A fortiori*,  $\{Ax_n\}$  converges because of the convergence of  $\{\bar{A}x_n\}$ . Since  $A$  is closed,  $x \in \mathcal{D}_A$ , so  $x \in \mathcal{D}_{\bar{A}}$ , as was to be proved.

THEOREM 6.<sup>3</sup> *Let  $A$  and  $\bar{A}$  both have closed linear extensions, and assume that  $A$  has a representation*

$$A \supseteq P_R \bar{A}. \tag{2.19}$$

Then

$$\bar{A} \supseteq P_R \bar{\bar{A}}. \tag{2.20}$$

PROOF: Consider any  $x \in \mathcal{D}_{\bar{\bar{A}}}$ . Either  $x \in \mathcal{D}_{\bar{A}} \subset \mathcal{D}_A$ , or there exists a sequence  $\{x_n\} \in \mathcal{D}_{\bar{A}}$  with  $x_n \rightarrow x$  and  $\bar{A}x_n \rightarrow w$ ; then  $\bar{\bar{A}}x = w$ .

We show that this  $x \in \mathcal{D}_{\bar{A}}$ , and that  $\bar{A}x = P_R \bar{\bar{A}}x$ . In order that (2.19) be satisfied, (2.18) must hold. Again, the convergence of  $\{\bar{A}x_n\}$  entails that of  $\{Ax_n\}$ , so  $x \in \mathcal{D}_{\bar{A}}$  and  $\bar{A}x = y (= \lim Ax_n)$ . One also obtains  $z_n \rightarrow z \in S$  from the convergence of  $\{\bar{A}x_n\}$ . Therefore,

$$\bar{\bar{A}}x = \bar{A}x + z, z \in S. \tag{2.21}$$

An application of  $P_R$  to both sides of (2.21) then shows that (2.20) is an equality when restricted to  $x \in \mathcal{D}_{\bar{\bar{A}}}$ .

<sup>3</sup> An alternative proof: Since  $P_R$  is bounded,  $A^* = (P_R \bar{A})^* = \bar{A}^* P_R$ . We again take the adjoint of both sides to obtain  $(A^*)^* = (\bar{A}^* P_R)^* \supseteq P_R (\bar{A}^*)^*$ , which is equivalent to (2.20).



COROLLARY. *If  $A = P_R \tilde{A}$ , then  $A \subseteq P_R \bar{\tilde{A}} \subseteq \bar{A}$ .*

*Remarks.* (a) Invertibility of  $\tilde{A}$  does not ensure that of  $\bar{\tilde{A}}$ . (b) The assertion (which strengthens the corollary) that  $A = P_R \tilde{A}$  implies  $\bar{A} = P_R \bar{\tilde{A}}$  is false.

We conclude this section with a routine comment. If  $A$  has a closed extension, the construction (2.9) yields

$$A \subseteq P_R \tilde{A}; \tag{2.22}$$

this  $\tilde{A}$  has a closed extension such that

$$\bar{A} = P_R \bar{\tilde{A}}. \tag{2.23}$$

### III. THE GENERALIZED PSEUDO-INVERSE

The principal result of this section is that every linear closed operator  $A$  defined on a linear manifold possesses a pseudo-inverse  $A^+$  given by

$$A^+ = P_M \tilde{A}^{-1}, \tag{3.1}$$

assuming only that  $N$  and  $S$  are of the same dimension. As before, it is convenient to distinguish the two cases (1)  $R$  is closed and (2)  $R$  is not closed; in the former, somewhat stronger results are obtained.

For our purposes, Definition 1a (see Section I) presents a restrictive concept of the pseudo-inverse applicable to neither unbounded operators, nor to those whose range is not closed. Clearly, the infimum indicated by (1.2) can be taken only over  $\mathcal{D}_A$ . Moreover, it cannot be expected that the infimum is attained for every  $y \in H$  when the range of  $A$  is not closed. Thus we are led to define a generalized pseudo-inverse which reduces to that of Definition 1a when  $A$  is bounded and  $R$  is closed.

DEFINITION 1b.  $A^+$  is a generalized pseudo-inverse of  $A$  if

(a)  $\mathcal{D}_{A^+}$  is dense in  $H$ .

(b) For every  $y \in \mathcal{D}_{A^+}$ ,

$$\inf_{x \in \mathcal{D}_A} \|Ax - y\| \tag{3.2}$$

is attained by

$$\hat{x} = A^+y; \tag{3.3}$$

(c) whenever  $x' \in \mathcal{D}_A$  also attains the infimum (for given  $y \in \mathcal{D}_{A^+}$ ),

$$\|\hat{x}\| < \|x'\| \tag{3.4}$$

unless  $\hat{x} = x'$ .

It is known [1] that the pseudo-inverse of a bounded operator with closed range (see Definition 1a) is unique. The same cannot be said for the generalized pseudo-inverse of Definition 1b, which we hereafter abbreviate GPI. Indeed, Definition 1b does not preclude various GPIs, defined on different dense sets in  $H$ . There is, however, a uniqueness theorem for GPIs, one of whose corollaries is again the uniqueness of the operator described by Definition 1a.

**THEOREM 7.** *Let  $A$  have GPIs  $A'$  and  $A''$ , and let  $\mathcal{D} = \mathcal{D}_{A'} \cap \mathcal{D}_{A''}$ . Then*

$$A'y = A''y \quad \text{for } y \in \mathcal{D}. \tag{3.5}$$

*In particular, if a GPI  $A^+$  is defined on all of  $H$ , any other GPI must be a restriction of  $A^+$ .*

**PROOF.** For  $y \in \mathcal{D}$ , both  $x' = A'y$  and  $x'' = A''y$  attain the infimum (3.2). If (3.5) is false, there is some  $y \in \mathcal{D}$  for which  $x' \neq x''$ . But since both  $A'$  and  $A''$  are GPIs,  $\|x'\| < \|x''\|$  and  $\|x''\| < \|x'\|$ , which is clearly impossible. This proves the first statement of the theorem; the second assertion is a direct consequence of the first.

For operators whose range is closed, questions of the existence and nature of a GPI are completely settled by

**THEOREM 8.** *Let  $A$  be a linear closed operator whose domain  $\mathcal{D}_A$  is a linear manifold, and whose range  $R$  is closed. Let  $N$  and  $S$  be of the same dimension. Then there exists a GPI*

$$A^+ = P_M \tilde{A}^{-1}, \tag{3.6}$$

*in which  $\tilde{A}$  may be taken as any operator having a bounded everywhere defined inverse and satisfying*

$$A = P \tilde{A}. \tag{3.7}$$

*If  $\hat{x}$  is defined by (3.3) for every  $y \in H$ , the infimum (3.2) is attained also by  $x'$  iff  $x'$  satisfies the equation*

$$A\hat{x} = Ax'. \tag{3.8}$$

*Furthermore,  $A^+$  is bounded, everywhere defined, and satisfies the equations*

$$AA^+ = P_R \quad \text{for all } y \in H, \tag{3.9}$$

*and*

$$A^+A = P_M \quad \text{for all } x \in \mathcal{D}_A. \tag{3.10}$$

**PROOF.** By Theorem 2, there is an  $\tilde{A}$  with the specified properties, so there exists at least one  $A^+$  of the form (3.6). We shall show that for any operator  $\tilde{A}$  satisfying (3.7) and the stated invertibility conditions, the corresponding  $A^+$  is a GPI defined on all of  $H$ . Then  $A^+$  is actually unique, and

does not depend on the particular  $\tilde{A}$  employed; this follows from Theorem 7.

For any  $\tilde{A}$  with everywhere defined and bounded inverse,  $A^+$  is likewise everywhere defined and bounded. Moreover, (3.7) implies that

$$A = P_R \tilde{A} \tag{3.11}$$

which yields  $A\tilde{A}^{-1} = P_R$  for any  $x \in H$ . We have shown earlier that if  $x \in \mathcal{D}_A$ , so is  $P_M x$ , and in fact,  $AP_M x = Ax$ . Since the range of  $\tilde{A}^{-1}$  is precisely  $\mathcal{D}_A$ , we have for any  $x \in H$

$$AA^+ = AP_M \tilde{A}^{-1} = A\tilde{A}^{-1} = P_R, \tag{3.12}$$

and thus (3.9) is proved.

To verify (3.10) we note that, for any  $x \in \mathcal{D}_A$

$$\tilde{A}x = Ax + z \quad z \in S \tag{3.13}$$

in order that  $\tilde{A}$  satisfy (3.11). Since  $\tilde{A}^{-1}$  is defined on all of  $H$ , we have from (3.13)

$$x = \tilde{A}^{-1}Ax + y. \tag{3.14}$$

Here  $y = \tilde{A}^{-1}z$ , and  $y \in N$  because of (iii) in Theorem 1. If  $P_M$  is now applied to both sides of (3.14), the desired result is obtained by substituting from (3.6).

It remains to show that  $A^+$  is a GPI, and that (3.8) is valid. For this purpose, define  $\hat{x}$  as in (3.3), and let  $x'$  be an arbitrary vector in  $\mathcal{D}_A$ . Then consider

$$\|Ax' - y\|^2 = \|(A\hat{x} - y) + (Ax' - A\hat{x})\|^2. \tag{3.15}$$

From (3.9) (which we just proved) we obtain  $A\hat{x} - y = (AA^+ - I)y = -P_S y$ . But  $Ax' - A\hat{x} = A(x' - \hat{x}) \in R$ , which simplifies (3.15) to

$$\|Ax' - y\|^2 = \|A\hat{x} - y\|^2 + \|Ax' - A\hat{x}\|^2. \tag{3.16}$$

This means that  $\|A\hat{x} - y\| \leq \|Ax' - y\|$  with equality iff  $Ax' = A\hat{x}$ . Thus  $A^+$  possesses property (b) of a GPI, and (3.8) is proved.

We call

$$\chi = \{x: x \in \mathcal{D}_A, Ax = A\hat{x}\}. \tag{3.17}$$

Clearly,  $\chi$  is precisely the set of elements for which the infimum (3.2) is attained.  $\chi$  is also one of the elements of the quotient space  $H/N$ , for  $x_1, x_2 \in \chi$  implies  $A(x_1 - x_2) = 0$ , i.e.,  $x_1 = x_2 \pmod N$ . Hence any  $x' \in \chi$  can be expressed in the form

$$x' = \hat{x} + z \quad z \in N, \tag{3.18}$$

where  $\hat{x}$  is taken as the representant of  $\chi$ . Now  $\hat{x} = A^+y = P_M(\tilde{A}^{-1}y)$ , so that  $\hat{x} \in M$ .

Therefore

$$\|x'\|^2 = \|\hat{x}\|^2 + \|z\|^2, \tag{3.19}$$

and consequently (3.4) holds unless  $x' = \hat{x}$ . In other words,  $\hat{x}$  satisfies condition (c) of Definition 1b, and the proof is complete.

When  $A$  has a nonclosed range, weaker results should be expected. The GPI exists, but cannot satisfy (3.9), for that would require that  $R$  be closed. (3.10) is lacking entirely, since there is no guarantee that  $R \subset \mathcal{D}_{A^+}$ . Further,  $\mathcal{D}_{A^+}$  is only dense in  $H$ , so that  $A^+$  may not be unique. Finally,  $A^+$  must be an unbounded operator when  $R$  is not closed.

**THEOREM 9.** *Let  $A$  be a linear closed operator whose domain  $\mathcal{D}_A$  is a linear manifold, and whose range  $R$  is not closed. Let  $N$  and  $S$  be of the same dimension. Then there exists a GPI for  $A$ , and an operator*

$$A^+ = P_M \tilde{A}^{-1} \tag{3.6'}$$

*is a GPI whenever  $\tilde{A}$  is an invertible closed operator with dense range satisfying*

$$A = P \tilde{A}. \tag{3.7'}$$

*If  $\hat{x}$  is defined by (3.3) for every  $y \in \mathcal{D}_{A^+}$ , the infimum (3.2) is attained also by  $x'$  iff  $x'$  satisfies the relation*

$$A\hat{x} = Ax'. \tag{3.8'}$$

*Furthermore,  $A^+$  is unbounded and satisfies*

$$AA^+ = P_R \text{ for all } y \in \mathcal{D}_{A^+}. \tag{3.9'}$$

**PROOF.** Theorem 3 states that  $A$  possesses at least one representation (3.7') in which  $\tilde{A}$  has the properties indicated. We show that any such  $\tilde{A}$  leads to an  $A^+$  (given by (3.6')) that is a GPI. Since the domain of  $\tilde{A}^{-1}$  is dense and coincides with  $\mathcal{D}_{A^+}$ ,  $\mathcal{D}_{A^+}$  is dense as required by (a) of Definition 1b. For future reference, we also note that any  $\tilde{A}$  satisfying (3.7') and the associated invertibility conditions will have its inverse unbounded, and will yield the more convenient representation

$$A = P_R \tilde{A}, \tag{3.20}$$

as shown in Theorem 4.

A rephrasing of the proof in Theorem 8 shows that  $A^+$  is a GPI. As in that proof, we first demonstrate that  $AA^{-1}y = P_R y$  for  $y \in \mathcal{D}_{A^+}$  ( $= \mathcal{D}_{\tilde{A}^{-1}}$ ), thus verifying (3.9'). For  $y \in \mathcal{D}_{A^+}$ , we then find that we can proceed as before, obtaining (3.6) which leads to condition (b) of Definition 1b, and (3.19), from which we derive (c) of the definition. Thus,  $A^+$  is shown to be a GPI.

Although  $\tilde{A}^{-1}$  is known to be unbounded (see Theorem 4), we have yet to prove the same for  $A^+ = P_M \tilde{A}^{-1}$ . Assume the contrary, i.e., that  $A^+$  is bounded; we show that this assumption contradicts Theorem 1. To this end, define

$$A_0 = A + UP_N \tag{3.21}$$

Where  $U$  is a partial isometry, taking  $N$  onto  $S$ . Evidently,

$$A = P_R A_0. \tag{3.22}$$

We assert that

$$B = A^+ + U^{-1}P_S \tag{3.23}$$

is the inverse of  $A_0$ . Before proving this claim, we observe that  $\mathcal{D}_B = \mathcal{D}_{A^+}$  (which is dense), and that  $B$  is (by our assumption on  $A^+$ ) a bounded operator.

Since  $AU^{-1}P_S = \theta$ ,  $UP_N A^+ = \theta$ , and  $UP_N U^{-1}P_S = P_S$ , we obtain from (3.21) and (3.23)

$$A_0 B = AA^+ + P_S \text{ for } y \in \mathcal{D}_B. \tag{3.24}$$

By (3.9'),  $AA^+ = P_R$  on  $\mathcal{D}_B (= \mathcal{D}_{A^+})$ , so that from (3.24)

$$A_0 B y = y \text{ for } y \in \mathcal{D}_B; \tag{3.25}$$

hence  $B$  is a right inverse. Then one has from (3.25)

$$B A_0 (B y) = B y. \tag{3.26}$$

Now the range of  $B$  is the linear manifold generated by  $M \cap \mathcal{D}_A$  (the range of  $A^+$ ) and  $S$ ; this manifold is precisely  $\mathcal{D}_A = \mathcal{D}_{A_0}$ . Thus, for each  $x \in \mathcal{D}_{A_0}$ , there is a  $y \in \mathcal{D}_B$  such that  $x = B y$ . In other words,

$$B A_0 x = x \text{ for } x \in \mathcal{D}_{A_0} \tag{3.27}$$

is implied by (3.26) and the discussion following. Because  $B$  is a left as well as right inverse,  $B$  is an inverse in the usual sense, and we may write  $B = A_0^{-1}$ .

It is clear from the original assumption on  $A$  that  $A_0$  is closed. Then  $A_0^{-1} = B$  is likewise closed and, being bounded and densely defined, is defined on all of  $H$ . We sum up:  $A$  has a representation (3.22), where  $A_0$  has a bounded inverse  $B$  with  $\mathcal{D}_B = H$ . By Theorem 1, the range of  $A$  must then be closed, which contradicts our original hypothesis. Therefore,  $A^+$  cannot be bounded. The theorem is proved.

In what follows, we consider "best approximate solutions" of the functional equation  $Ax = y$  in the sense of Definition 1b. When  $y \in R$ , solutions  $x \in \mathcal{D}_A$  exist, but are not unique unless  $N = \{0\}$ . If  $x_0$  is any vector such that  $Ax_0 = y$ , the "best approximate solution"  $\hat{x}$  [satisfying  $A\hat{x} = y$  and (3.4)] is uniquely specified by  $\hat{x} = P_M x_0$ . This fact is not apparent from Theorem

9; if  $R \not\subset \mathcal{D}_{A^+}$  for some GPI  $A^+$ , there are  $y \in R$  for which this  $A^+$  cannot furnish a "best approximate solution". In order that the "best approximate solution" exist for each  $y \in R$ , we must then have  $R \subset \cup \mathcal{D}_{A^+}$  taken over all GPIs. However, there is a GPI for which  $R \subset \mathcal{D}_{A^+}$ , so that this GPI is adequate to obtain the "best approximate solution" for all  $y \in R$ .

COROLLARY. *There exists a GPI  $A^+$  for which  $\mathcal{D}_{A^+} \supset R$  and*

$$A^+A = P_M \text{ for all } x \in \mathcal{D}_A. \tag{3.10'}$$

PROOF. Let  $A^+$  be specified by (3.6'), where  $\tilde{A}$  is constructed as in the proof of Theorem 3. The construction (2.9) for  $\tilde{A}$  indicates that the range of  $\tilde{A}$  (domain of  $A^+$ ) is the linear manifold generated by  $S$  and  $R$ . Therefore, the application of  $\tilde{A}^{-1}$  to (3.13) yields (3.14), with  $y = \tilde{A}^{-1}z$  in  $N$  by (2.10). If  $P_M$  is applied (as before) to (3.14), (3.10') follows.

The reader may verify that whenever  $A^{-1}$  is defined on all  $H$ , the representation (2.7) is unique, with  $P_R = I$ , and  $\tilde{A} = A$ , and  $A^+ = A^{-1}$ . If  $R = H$ , but  $N$  contains nonnull vectors, the desired representation can be achieved only through the embedding process already described, the range space being appropriately enlarged. The embedding process is merely a device of convenience, for the GPI is uniquely given by

$$A^+ = \hat{A}^{-1} \tag{3.28}$$

where  $\hat{A}$  is the restriction of  $A$  to  $M$ .

The entire material of this section holds without change when  $A$  has Hilbert spaces  $H_1$  and  $H_2$  as domain and range spaces. Indeed,  $A^+$  can even be defined by (3.6) for Banach spaces, provided there exists a linear 1-1 bicontinuous operator from  $N$  onto  $S$ . The infimum (3.2) is attained by  $\hat{x} = A^+y$  if the range space is a Hilbert space, but  $\hat{x}$  need no longer satisfy (3.4). When neither domain nor range spaces are Hilbert spaces,  $A^+$  loses property (b) of Definition 1b. Easily constructed examples verify the above facts on Banach spaces; one may use the fact that, in Banach spaces, the norm of a projection may be greater than (and is never less than) unity.

#### IV. NORMAL OPERATORS

In Section I, we presented the representation and pseudo-inverse formulas for a Hermitian matrix. These expressions, when written in terms of the spectral representation, suggest generalizations to normal operators. These generalizations are indeed valid, and provide explicit formulations for the pseudo-inverse. Most of the results of Sections II and III could also be proved succinctly for normal operators (only), using the spectral representa-

tion; in the interest of brevity, we leave this as an exercise for the reader.

To furnish a proper setting for the remainder of this section, we shall state (without proof) a number of properties of the normal operator.<sup>4</sup> An operator  $B$  is normal if it is linear, closed, densely defined on a linear manifold, and satisfies

$$BB^* = B^*B. \tag{4.1}$$

In this definition, (4.1) may be replaced by the requirement that  $B$  and  $B^*$  have a common domain, with

$$\| Bx \| = \| B^*x \| \tag{4.2}$$

for all  $x \in \mathcal{D}_B (= \mathcal{D}_{B^*})$ . It is customary to analyze the (also normal) operator  $B - \lambda I$  for all complex  $\lambda$ ; instead, we shall consider  $A = B - \lambda I$  for fixed but arbitrary  $\lambda$ . The change from  $B$  to  $A$  is effected by a mere translation of the spectrum, and involves no loss in generality. Consider now the origin with respect to the spectrum of  $A$ . *Iff*  $\{0\}$  belongs to the resolvent set,  $A$  possesses a bounded inverse defined on all  $H$ . *Iff*  $\{0\}$  is in the continuous but not the point spectrum,  $R$  is dense, and  $A$  has an unbounded inverse. Under either of these conditions, the representation of Section II can hold only with  $P_R = I$  and  $\tilde{A} = A$ . One also obtains  $A^+ = A^{-1}$ .

In the more interesting case,  $\{0\}$  is in the point spectrum. The eigenmanifold corresponding to  $\{0\}$  is precisely  $N$ ; hence  $N$  contains nonnull vectors *iff* the origin belongs to the point spectrum. Since (for normal operators) the eigenmanifold is also the subspace orthogonal to  $R$ , we obtain

$$N = S \quad \text{and} \quad M = \tilde{R}. \tag{4.3}$$

From (4.3) and  $P_R A = A P_M$  follows

$$P_R A = A P_R, \tag{4.4}$$

where the equality sign indicates that  $x \in \mathcal{D}_A$  *iff*  $P_M x \in \mathcal{D}_A$ , that is,  $P_R A$  and  $A P_R$  have the same domain. That  $M$  is reduced by  $A$  is a consequence of (4.4). Finally, (4.2) implies that  $A$  and  $A^*$  both have the same null space, so that  $\tilde{R}$  must also be the same for both.

Whether  $R$  is closed depends entirely on the continuous spectrum:  $R$  is closed *iff*  $\{0\}$  does not belong to the continuous spectrum [4, p. 54]. This fact could be deduced (if desired) from the representation for normal operators to be derived presently, in combination with Theorems 1 and 4; however, it is too well known to merit one more derivation.

The representation

$$A = P_R \tilde{A} \tag{4.5}$$

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<sup>4</sup> For these and other results summarized here, see [4].

raises questions of independent interest when  $A$  and/or  $\tilde{A}$  are normal operators. If  $A$  [ $\tilde{A}$ ] is normal, must  $\tilde{A}$  [ $A$ ] be normal? Under what conditions of normality does  $P_R$  commute with  $\tilde{A}$  [does  $\tilde{R}$  reduce  $\tilde{A}$ ] ? The first query is easily answered: the restriction of  $\tilde{A}$  from  $N$  to  $S$  need only be invertible, so that  $\tilde{A}$  may be non-normal while  $A$  is normal. The remaining problems are of greater difficulty, and will be answered in a series of theorems. It will be seen that the commutativity of  $P_R$  and  $\tilde{A}$  plays a crucial role.

**THEOREM 10.** *Let  $A$  have a representation (4.5), where  $\tilde{A}$  is an invertible normal operator and  $P_R$  commutes with  $\tilde{A}$ . Then  $A$  is normal and*

$$A\tilde{A} = \tilde{A}A. \tag{4.6}$$

**PROOF.** Let  $\tilde{A}$  have the spectral representation

$$\tilde{A} = \int_x \lambda dF_\lambda. \tag{4.7}$$

Then for any  $x \in \mathcal{D}_A$ ,

$$Ax = P_R\tilde{A}x = \tilde{A}(P_Rx) = \int_x \lambda d(F_\lambda P_Rx). \tag{4.8}$$

By virtue of our hypothesis,  $\{F_\lambda\}$  and  $P_R$  commute, so that  $\{F_\lambda P_R\}$  is a family of projections—in fact, a spectral family on  $\tilde{R}$ . Thus the restriction of  $A$  to  $R$  is normal. Since  $A = P_R\tilde{A} = \tilde{A}P_R$  and  $A^* = \tilde{A}^*P_R$ ,  $Ax = A^*x = 0$  for  $x \in S$ . Therefore  $\|Ax\| = \|A^*x\|$  for all  $x \in \mathcal{D}_A (= \mathcal{D}_{A^*})$ .

Let  $x$  be such that

$$\int_x |\lambda|^4 d\|F_\lambda P_Rx\|^2 < \infty, \tag{4.9}$$

which assures that  $x \in \mathcal{D}_{\tilde{A}} (= \mathcal{D}_{\tilde{A}})$ . For arbitrary  $y \in H$

$$(\tilde{A}Ax, y) = \int_x \lambda d(F_\lambda Ax, y) = \int_x \lambda d(F_\lambda P_R\tilde{A}x, y). \tag{4.10}$$

On the other hand,

$$(A\tilde{A}x, y) = \int_x \lambda d(F_\lambda P_R[\tilde{A}x], y), \tag{4.11}$$

so that  $(\tilde{A}Ax, y) = (A\tilde{A}x, y)$ . Although the proof is finished, we may wish to complete the computation of  $\tilde{A}A$ . Using

$$(F_\lambda P_R\tilde{A}x, y) = (Ax, F_\lambda y) = \int_c \alpha d(F_\alpha P_Rx, y), \tag{4.12}$$



where  $C$  is the Borel set (in the plane) associated with  $F_\lambda$ , we obtain from the Radon-Nikodym theorem

$$(\tilde{A}Ax, y) = \int_X \lambda^2 d(F_\lambda P_{R^X}, y). \tag{4.13}$$

It is incorrect to conclude from the foregoing proof—even when  $H$  is finite dimensional—that  $P_R$  has a representation in terms of  $\{F_\lambda\}$ , i.e.,

$$P_R = \int_X r(\lambda) dF_\lambda. \tag{4.14}$$

If, however, (4.14) is valid,  $r(\cdot)$  must be an indicator function, so that  $P_R = F_\lambda(C)$  for some Borel set  $C$  of the plane. From this argument and (4.8) follows

$$A = \int_C \lambda dF_\lambda. \tag{4.15}$$

If  $\tilde{A}$  has a simple spectrum, it is known [5, Section 75] that (4.14) is true. On the other hand, suppose that  $\lambda_0$  is an eigenvalue associated with a multi-dimensional eigenmanifold, of which  $\tilde{R}$  is a proper subspace. Then  $P_R$  commutes with  $F_\lambda(\{\lambda_0\})$ , hence with  $\{F_\lambda\}$ , and therefore with  $\tilde{A}$ . If now  $P_R = F_\lambda(C)$ , we must have  $\lambda_0 \in C$  [since  $F_\lambda(X - \{\lambda_0\})$  is orthogonal to  $P_R$ ], which is impossible. In other words, (4.14) need not hold if the spectral multiplicity of  $\tilde{A}$  is greater than unity.

Whenever  $A$  has the representation (4.15),  $C$  is defined only up to an equivalence with respect to the spectral family  $\{F_\lambda\}$ , while its definition on the complement of the spectrum of  $\tilde{A}$  is a matter of indifference. It is easy to see that  $A$  is bounded iff a compact representative of  $C$  exists. In any case,  $\{F_\lambda\}$  may be arbitrarily modified outside the intersection of any  $C$  with the spectrum  $\tilde{A}$ ; therefore, a given noninvertible  $A$  has a multiplicity of representations (4.5), in which  $\tilde{A}$  may be bounded or unbounded if  $A$  is bounded, and must be unbounded if  $A$  is not bounded.

Theorem 10 leads one to conjecture that for an  $A$  represented by (4.5) with  $\tilde{A}$  normal,  $A$  is also normal iff  $P_R$  commutes with  $\tilde{A}$ . The truth of this statement is the result of

**THEOREM 11.** *Let  $A$  have a representation (4.5) in which both  $A$  and  $\tilde{A}$  are normal. Then  $P_R$  commutes with  $\tilde{A}$ , and in fact,*

$$P_R \tilde{A} = \tilde{A} P_R \quad x \in \mathcal{D}_A. \tag{4.16}$$

**PROOF.** In order that  $\tilde{A}$  satisfy (4.5)

$$\tilde{A}x = Ax + z \quad z \in S. \tag{4.17}$$

It is clear that  $z$  is linearly related to  $x$  and exists for every  $x \in \mathcal{D}_A$ ; this defines a linear operator  $B$  with  $\mathcal{D}_B \supset \mathcal{D}_A$  such that

$$z = Bx. \quad (4.18)$$

It is then possible to write

$$\tilde{A}x = Ax + Bx \quad x \in \mathcal{D}_A, \quad (4.19)$$

and we may observe that

$$\|\tilde{A}x\|^2 = \|Ax\|^2 + \|Bx\|^2. \quad (4.20)$$

The representation (4.5) leads to the alternative form

$$A^* = \tilde{A}^*P_R \quad \text{for } x \in \mathcal{D}_A, \quad (4.21)$$

in which equality holds because  $P_R$  is bounded. Consequently,

$$\|A^*x\| = \|\tilde{A}^*x\| \quad \text{for } x \in \mathcal{D}_A \cap M. \quad (4.22)$$

Suppose now that  $BP_R \neq \theta$  on  $\mathcal{D}_A$ . Then for some  $x \in \mathcal{D}_A \cap M$ ,

$$\|\tilde{A}x\| > \|Ax\| \quad (4.23)$$

from (4.20). Since  $A$  and  $\tilde{A}$  are both normal, (4.22) and (4.23) contradict (4.2). Therefore

$$BP_R = \theta \quad \text{for } x \in \mathcal{D}_A. \quad (4.24)$$

However, it is also true from (4.18) and the fact that  $z \in S$  that  $P_R B = 0$ . Likewise,  $P_R A = A = AP_M = AP_R$  (recall  $M = \bar{R}$ ). Thus  $A$  and  $B$  both commute with  $P_R$  for all  $x \in \mathcal{D}_A$ . Reference to (4.19) then completes the proof.

Of somewhat different character is the next theorem, which relates the normality of  $A$  to its commutativity with  $\tilde{A}$ .

**THEOREM 12.** *Let  $A$  have a representation (4.5), where  $A$  is closed and densely defined, and  $\tilde{A}$  is an invertible normal operator. Assume  $H$  to be separable, and suppose that  $A$  commutes with every bounded operator that commutes with  $\tilde{A}$ . Then  $A$  is normal, and*

$$P_R \tilde{A} = \tilde{A} P_R. \quad (4.25)$$

*In terms of the spectral representation (4.7) for  $\tilde{A}$ ,  $P_R = E_\lambda(C)$  for some Borel set  $C$ , and  $A$  has a spectral representation (4.15).*

PROOF. Let (4.7) be the spectral representation for  $A$ . Then [4, p. 64]

$$A = \int_X f(\lambda) dF_\lambda \tag{4.26}$$

where  $f(\cdot)$  satisfies for each  $x \in \mathcal{D}_A$

$$\int_X |f(\lambda)|^2 d\|F_\lambda x\|^2 < \infty. \tag{4.27}$$

This already proves that  $A$  is normal. We have proved earlier that  $A\tilde{A}^{-1} \subseteq P_R$ , the domain on the left hand side being that of  $\tilde{A}^{-1}$ , which is densely defined. Indeed,

$$\tilde{A}^{-1} = \int_X \lambda^{-1} dF_\lambda, \tag{4.28}$$

so that

$$A\tilde{A}^{-1} = \int_X \lambda^{-1} f(\lambda) dF_\lambda = P_R \quad \text{for } x \in \mathcal{D}_{\tilde{A}^{-1}} \tag{4.29}$$

i.e., on a dense set. Thus the projection  $P_R$  has a representation in terms of the spectral family  $\{F_\lambda\}$ , which means that  $\lambda^{-1}f(\lambda)$  is an indicator function with respect to a Borel set in the plane. We therefore have  $P_R = F_\lambda(C)$ , and  $f(\lambda) = \lambda$  or  $0$  according as  $\lambda \in C$  or  $\lambda \notin C$ , and so (4.26) takes on the form (4.15). Moreover,  $P_R$  is a member of the spectral family for  $\tilde{A}$ , and hence  $P_R\tilde{A} \subseteq \tilde{A}P_R$ . We show that  $P_R\tilde{A}$  and  $\tilde{A}P_R$  have the same domain. In fact,  $M = \tilde{R}$  because  $A$  is normal, and  $x \in \mathcal{D}_{\tilde{A}}$  is a statement equivalent to  $P_M x \in \mathcal{D}_{\tilde{A}}$ . This completes the proof.

We observe again that, in general,  $\tilde{A}$  cannot be expressed in terms of the spectral family for  $A$ , even when  $A$  and  $\tilde{A}$  are both normal. Nevertheless, for any normal  $A$  the representation (4.5) can be written so that the  $\tilde{A}$  appearing therein is a function of  $A$ , and can be expressed in terms of the spectral family of the latter. There results a convenient and explicit expression for  $\tilde{A}$ , and likewise a direct formula for the GPI (for  $A$ ) defined on  $R$ .

**THEOREM 13.** *Let  $A$  be normal, with spectral representation*

$$A = \int_X \lambda dE_\lambda. \tag{4.30}$$

Then

$$P_R = E_\lambda(X - \{0\}) \tag{4.31}$$

and we may write the representation (4.5) with  $\tilde{A}$  given by

$$\tilde{A} = \int_X \lambda dE_\lambda + E_\lambda(\{0\}). \tag{4.32}$$

One version of the GPI is

$$A^+ = \int_{X-\{0\}} \lambda^{-1} dE_\lambda, \quad (4.33)$$

and this  $A^+$  has the property  $\mathcal{D}_{A^+} \supset R$ . Denote the spectrum of  $A$  by  $\Lambda$ . If  $R$  is closed

$$d = \inf_{x \in \Lambda - \{0\}} |x| \quad (4.34)$$

is greater than zero, and then

$$\|A^+\| = d^{-1}. \quad (4.35)$$

PROOF. From (4.30),  $Ax = 0$  iff  $x$  belongs to the subspace associated with the projection  $E_\lambda(\{0\})$ . This projection must then be  $P_N$ , and so the complementary projection  $P_R = I - P_N$  is given by (4.31). When  $P_R$  is applied to (4.32), it is seen that  $A = P_R \tilde{A}$  because  $P_R A = A$  and

$$P_R E_\lambda(\{0\}) = \theta.$$

The inverse of  $\tilde{A}$  is given by

$$\tilde{A}^{-1} = \int_{X-\{0\}} \lambda^{-1} dE_\lambda + E_\lambda(\{0\}), \quad (4.36)$$

which is easily checked because the two terms in (4.32) and (4.36) represent the reduction of the operators in question by subspaces  $M(= \bar{R})$  and  $N(= S)$ .

It was shown in Section III that  $A^+ = P_M \tilde{A}^{-1}$  is a GPI for  $A$ , with  $\mathcal{D}_{A^+} = \tilde{A}(\mathcal{D}_A)$ . The  $\tilde{A}^{-1}$  used here is that given by (4.36). Since the restrictions of  $A$  and  $\tilde{A}$  to  $M$  are the same,  $\tilde{A}(\mathcal{D}_A) \supset R$ ; this proves the assertion following (4.33). The actual computation of  $A^+$ , based on the spectral representations for  $\tilde{A}^{-1}$  and  $P_R$ , yields

$$A^+ \subseteq \int_{X-\{0\}} \lambda^{-1} dE_\lambda \quad (4.37)$$

but as the domain of the right hand side is identical with the domain of  $\tilde{A}^{-1}$ , (4.37) assumes an equality sign, and becomes (4.33).

When  $R$  is closed,  $\{0\}$  is an isolated point of the spectrum, and the  $d$  defined by (4.34) is nonzero.  $A^+$  is then bounded as indicated by (4.35) because the essential supremum of  $|\lambda^{-1}|^2$  with respect to the spectral measures  $\|E_\lambda x\|^2$  taken, for all  $x \in H$ , over  $X - \{0\}$ , is precisely  $d^{-2}$ . For non-closed  $R$ ,  $\{0\}$  is a point of the continuous spectrum, and  $d = 0$ .

V. COMPACT NORMAL OPERATORS

For compact (completely continuous) normal operators, the results of the previous sections can be further specialized. There is no need to provide proofs, since the expressions to appear are merely applications of Theorem 13, using the spectral characteristics peculiar to compact operators.

Let  $A$  be a compact normal operator in the functional equation

$$Ax - \gamma x = y \quad y \in H. \tag{5.1}$$

If  $\gamma$  (a complex number) is in the resolvent set of  $A$ , the solution of (5.1) is of no interest and will not be discussed explicitly. Similarly, we pay no particular attention to spectra consisting only of a finite number of points; in that case,  $M$  is of finite dimension, or equivalently, the range of  $A$  is closed.

The spectral representation for  $A$  will be written

$$A = \int_X \lambda dE_\lambda, \tag{5.2}$$

and we shall adopt the following auxiliary notation:  $\gamma_1, \gamma_2, \dots$  are the points of the spectrum for  $A$ , arranged in order of decreasing modulus. With each  $\gamma_j$  is associated a finite number  $G_j$  of eigenvectors  $e_{j1}, e_{j2}, \dots, e_{jG_j}$ . Finally,  $M_j, N_j, \bar{R}_j$ , and  $S_j$  are the subspaces for

$$B_j = A - \gamma_j I \tag{5.3}$$

which correspond to  $M, N, \bar{R}$ , and  $S$  for  $A$ .

In terms of these conventions, consider now (5.1) with  $\gamma = \gamma_j$ . Since  $\gamma_j$  is an isolated point of the spectrum of  $A$ ,  $R_j$  is closed, and a GPI (called  $B_j^+$ ) is defined on all  $H$ . We may write

$$B_j = \sum_k (\gamma_k - \gamma_j) E_\lambda(\{\gamma_k\}) - \gamma_j E_\lambda(\{0\}) \tag{5.4}$$

and

$$P_{R_j} = E_\lambda(X - \{\gamma_j\}). \tag{5.5}$$

As we know,  $B_j$  has a representation  $B_j = P_{R_j} \tilde{B}_j$ , in which  $\tilde{B}_j$  may be taken as

$$\tilde{B}_j = \sum_k (\gamma_k - \gamma_j) E_\lambda(\{\gamma_k\}) - \gamma_j E_\lambda(\{0\}) + E_\lambda(\{\gamma_j\}). \tag{5.6}$$

A direct calculation verifies

$$\tilde{B}_j^{-1} = \sum_{k \neq j} (\gamma_k - \gamma_j)^{-1} E_\lambda(\{\gamma_k\}) + E_\lambda(\{\gamma_j\}) - \gamma_j^{-1} E_\lambda(\{0\}) \tag{5.7}$$

where  $\tilde{B}_j^{-1}$  is bounded, having norm

$$\|\tilde{B}_j^{-1}\| = \max(1, |\gamma_j|^{-1}, \sup_{k \neq j} [|\gamma_k - \gamma_j|^{-1}]) < \infty. \tag{5.8}$$

The norm is actually attained by every  $x \in H$  lying in the proper subspace (depending on which of the three terms in (5.8) yields the maximum).

The (unique) GPI is specified by  $B_j^+ = P_{R_j} \tilde{B}_j^{-1}$ , and may be obtained from (5.5) and (5.7), yielding

$$B_j^+ = \sum_{k \neq j} (\gamma_k - \gamma_j)^{-1} E_\lambda(\{\gamma_k\}) - \gamma_j^{-1} E_\lambda(\{0\}). \tag{5.9}$$

The norm of  $B_j^+$  is then

$$\|B_j^+\| = \max(|\gamma_j|^{-1}, \sup_{k \neq j} [|\gamma_k - \gamma_j|^{-1}]). \tag{5.10}$$

To compute  $x = B_j^+ y$ , it is convenient to write  $y$  in the form

$$y = \sum_m \sum_{n=1}^{G_m} (y, e_{mn}) e_{mn} + E_\lambda(\{0\}) y, \tag{5.11}$$

Whence

$$x = \sum_{m \neq j} (\gamma_m - \gamma_j)^{-1} \sum_{n=1}^{G_m} (y, e_{mn}) e_{mn} - \gamma_j^{-1} E_\lambda(\{0\}) y. \tag{5.12}$$

The case  $\gamma = 0$  in (5.1) leads to a GPI which is unbounded. We shall construct a GPI  $A^+$  so chosen that  $\mathcal{D}_{A^+} \supset R$ . This time

$$P_R = E_\lambda(X - \{0\}) \tag{5.13}$$

and the  $\tilde{A}$  leading to the desired  $A^+$  is

$$\tilde{A} = \sum_j \gamma_j E_\lambda(\{\gamma_j\}) + E_\lambda(\{0\}); \tag{5.14}$$

then

$$A^+ = \sum_j \gamma_j^{-1} E_\lambda(\{\gamma_j\}) \tag{5.15}$$

and, for  $y$  specified as in (5.11),

$$x = \sum_m \gamma_m^{-1} \sum_{n=1}^{G_m} (y, e_{mn}) e_{mn} \tag{5.16}$$

whenever  $y \in \mathcal{D}_{A^+}$ . We show that  $\mathcal{D}_{A^+} \supset R$ . In fact,

$$A = \sum_j \gamma_j E_\lambda(\gamma_j) \quad (5.17)$$

so that the proof (showing  $R \subset \tilde{A}(\mathcal{D}_A)$ ) is identical with that of Theorem 13. The range of  $\tilde{A}(=\mathcal{D}_{A^+})$  is given more explicitly as the set of  $y \in H$  for which

$$\sum_m \sum_{n=1}^{G_m} |(y, e_{mn}) \gamma_m^{-1}|^2 < \infty. \quad (5.18)$$

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