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Anisotropic Scattering in Half-Space Transport Problems

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Case's singular normal mode analysis of half-space transport problems is extended to include anisotropic scattering. The cases of absorbing and nonabsorbing media are treated separately. By way of illustration, the solution to the Milne problem is obtained in each instance. It is shown that the results may always be written in terms of an appropriate X-function, and a rapidly convergent iterative scheme for evaluating the X-functions is exhibited. However, differences from the isotropic scattering problem arise: For the nonabsorbing medium the zeros of the characteristic function coalesce at infinity; for the absorbing medium, the normal mode analysis contains undetermined constants which are seen to be related to coefficients in the Laurent series of the X-function. In each case the solution is carried out with the appropriate modifications, and Case's three X-function identities are rederived.

I. INTRODUCTION

Case (*1*; this paper will be referred to as C) has provided an elegant treatment of neutron transport problems by means of a normal mode expansion in the singular eigenfunctions of the Boltzmann equation. In this paper we trace through the modifications which become necessary when a nonabsorbing medium and/or anisotropic scattering is considered. By way of illustration we obtain the solution to the Milne problem in each instance.

Consider the homogeneous steady-state neutron transport equation in plane-symmetric geometry:

$$\mu \frac{\partial}{\partial x} \Psi(x, \mu) + \Psi = \frac{c}{2} \sum_{l=0}^{N} b_l \, P_l(\mu) \int_{-1}^{1} P_l(\mu')\Psi(x, \mu') \, d\mu' \tag{1}$$

where it has been assumed that the scattering transfer function may be expanded in a finite set of $N + 1$ Legendre polynomials in the laboratory scattering angle.

$$T(\hat{\Omega} \rightarrow \hat{\Omega}') = \frac{1}{4\pi} \sum_{l=0}^{N} b_l \, P_l(\hat{\Omega}\cdot\hat{\Omega}') \qquad b_0 = 1 \tag{2}$$

and where

$$c = \frac{\sigma_s}{\sigma_s + \sigma_a} \tag{3}$$

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is the average number of neutons emerging from a collision event. Distances are measured in units of the mean free path.

Normal-mode solutions to Eq. (1) for the case of isotropic scattering ($T = 1/4\pi$), $c \neq 1$ are presented in C. Normal-mode solutions for anisotropic scattering were obtained by Mika (*2*). Generally, these solutions are of the form $\psi_\nu = e^{-x/\nu}\varphi_\nu(\mu)$ where the index ν takes on all values in the interval $(-1, 1)$ (continuum) as well as pairs of discrete values $\pm\nu_i$. The continuum $\varphi_\nu(\mu)$ are singular eigenfunctions. The discrete ν_i are the zeros of the characteristic function $\Lambda(\nu)$, which is analytic in the complex plane cut along the real axis from -1 to 1. For isotropic scattering $\Lambda(\nu)$ is given by

$$\Lambda(\nu) = 1 + \frac{c\nu}{2}\int_{-1}^{1}\frac{d\mu}{\mu - \nu} \tag{4}$$

and there is one pair of discrete roots with $\nu_0\{\text{real}/\text{imaginary}\}$ for $c \lessgtr 1$. As $c \to 1$, the roots $\pm\nu_0$ coalesce at infinity. This complication is considered in Section II.

In the normal-mode formulation, the solution to half-space problems reduces to the solution of the singular integral equation implied by the application of appropriate boundary conditions to the eigenfunction expansion of the complete distribution function $\Psi(x, \mu)$. The main step in the solution (as obtained in C) is the construction of the function $X(\nu)$, the Wiener-Hopf factorization of the quotient $\Lambda^+(\nu)/\Lambda^-(\nu)$.

Mika (*2*) has pointed out that for anisotropic scattering, the singular integral equation is complicated by the addition of a Fredholm part. This situation, which does not arise in the case treated in Section II, is treated in Section III.

In a series of lectures (*3*; this paper will be referred to as CL), Case presents several identities obeyed by the X-function and shows how these identities may be used to reduce the implicit solutions of C to tractable forms. In Section IV we generalize the identities to incorporate the scattering anisotropy of Sections II and III and present concise closed form expressions for the outgoing neutron angular density and total neutron density in each case. In Section V we convert one of the X-function identities into an integral equation with an extremely rapidly convergent iterative solution. We compute the outgoing neutron angular density for quadratically anisotropic scattering in a nonabsorbing medium, exhibiting directly the forward peaking of the current and the modification of the extrapolated end point as a function of the anisotropy parameter.

II. THE NONABSORBING MEDIUM ($c = 1$)

When $c = 1$, the transport equation (1) possesses the integral

$$J \equiv \int_{-1}^{1} P_1(\mu)\Psi(x, \mu)\, d\mu = \text{constant}. \tag{5}$$

This may be utilized to eliminate the b_1 term from the scattering function. In fact, we have the very general result, valid whenever $c = 1$. If $\Psi(x, \mu)$ satisfies (1) and $\Psi'(x, \mu)$ satisfies (1) with the scattering function given by

$$T(\hat{\Omega} \rightarrow \hat{\Omega}') = \frac{1}{4\pi} \sum_{i=0,\, i\neq 1}^{N} b_i\, P_i(\hat{\Omega}\cdot\hat{\Omega}') \tag{6}$$

then if Ψ and Ψ' represent the same current J and satisfy the same boundary condition at the interface $x = 0$,

$$\Psi(x, \mu) = \Psi'(x, \mu) + \frac{b_1 J}{2} \tag{7}$$

This result is verified directly by substitution into (1). Hence, to illustrate the effect of anisotropic scattering we choose

$$T(\hat{\Omega} \rightarrow \hat{\Omega}') = 1 + b_2 P_2(\hat{\Omega}\cdot\hat{\Omega}') \tag{8}$$

Then, under the *ansatz* $\psi_\nu(x, \mu) = e^{-x/\nu}\varphi_\nu(\mu)$, the transport equation becomes

$$(\nu - \mu)\varphi_\nu(\mu) = \frac{\nu}{2}\left\{\int_{-1}^{1} \varphi_\nu(\mu')\, d\mu' + b_2\, P_2(\mu) \int_{-1}^{1} P_2(\mu')\, \varphi_\nu(\mu')\, d\mu'\right\} \tag{9}$$

which implies

$$\int_{-1}^{1} \mu\varphi_\nu(\mu)\, d\mu = 0 \tag{10}$$

$$\int_{-1}^{1} \mu^2\, \varphi_\nu(\mu)\, d\mu = 0 \tag{11}$$

From (11) we obtain

$$\int_{-1}^{1} P_2(\mu')\varphi_\nu(\mu')\, d\mu' = -\frac{1}{2}\int_{-1}^{1} \varphi_\nu(\mu')\, d\mu' \tag{12}$$

Hence (9) possesses a discrete solution whenever ν is a root of

$$\Lambda(\nu) \equiv 1 - \frac{\nu}{2}\int_{-1}^{1} \frac{1 - \frac{b_2}{2}\, P_2(\mu)}{\nu - \mu}\, d\mu = 0 \tag{13}$$

$\Lambda(\nu)$ possesses no zeroes in the finite plane, but for large $|\nu|$

$$\begin{aligned}\Lambda(\nu) &= 1 - \frac{1}{2}\int_{-1}^{1}\left(1 - \frac{b_2}{2}\, P_2(\mu)\right)\left(1 + \frac{\mu}{\nu} + \frac{\mu^2}{\nu^2} + \cdots\right) d\mu \\ &\approx -\frac{1}{3\nu^2}\left(1 - \frac{b_2}{5}\right)\end{aligned} \tag{14}$$

The discrete modes associated with this double root at infinity are

$$\begin{aligned} \psi_1(x, \mu) &= \tfrac{1}{2} \\ \psi_2(x, \mu) &= \tfrac{1}{2}(x - \mu) \end{aligned} \tag{15}$$

Whenever $\nu \neq \infty$ we will choose the convenient normalization $\int_{-1}^{1} \varphi_\nu(\mu)\, d\mu = 1$. The continuum modes are

$$\varphi_\nu(\mu) = \frac{\nu}{2}\left[1 - \frac{b_2}{2} P_2(\mu)\right] \mathcal{P} \frac{1}{\nu - \mu} + \lambda(\nu)\delta(\nu - \mu) \tag{16}$$

The $\mathcal{P}$ signifies that the principal value prescription is to be applied when the factor $1/(\nu - \mu)$ appears in an integrand.

$\lambda(\nu)$ is given by

$$\lambda(\nu) = 1 - \frac{\nu}{2} \mathcal{P} \int_{-1}^{1} \frac{1 - \frac{b_2}{2} P_2(\mu)}{\nu - \mu}\, d\mu = \frac{1}{2}[\Lambda^+(\nu) + \Lambda^-(\nu)] \tag{17}$$

For future reference we note also that

$$\Lambda^+(\nu) - \Lambda^-(\nu) = \pi i \nu \left(1 - \frac{b_2}{2} P_2(\nu)\right) \tag{18}$$

By virtue of the completeness proof of C, extended by Mika to include anisotropic scattering, the general neutron angular density $\Psi(x, \mu)$ may be written

$$\Psi(x, \mu) = a\psi_1(x, \mu) + a'\psi_2(x, \mu) + \int_{-1}^{1} A(\nu)\varphi_\nu(\mu)\, e^{-x/\nu}\, d\nu \tag{19}$$

In the Milne problem we seek the distribution produced by a source deep within a half-space. The proper boundary conditions are then

(a) $\Psi(x, \mu)$ increases without bound as $x \to \infty$, but no faster than the slowest growing mode (in this case ψ_2).

(b) $\Psi(0, \mu) = 0 \quad \mu > 0$ (no incident flux). (20)

Condition (a) requires that

$$A(\nu) = 0 \qquad \nu < 0 \tag{21}$$

and we choose $a' = 1$ for normalization. This implies [cf. Eq. (10)]

$$J = \int_{-1}^{1} \mu\psi_2(x, \mu)\, d\mu = -\frac{1}{3} \tag{22}$$

Boundary condition (b) yields the integral equation

$$\frac{\mu - a}{2} = \int_0^1 A(\nu)\varphi_\nu(\mu)\, d\nu = \frac{1}{2}\left[1 - \frac{b_2}{2} P_2(\mu)\right] \mathcal{P}\int_0^1 \frac{\nu A(\nu)\, d\nu}{\nu - \mu} + \lambda(\mu)A(\mu) \tag{23}$$

for the unknown coefficients a, $A(\nu)$.

Define:

$$N(z) \equiv \frac{1}{2\pi i} \int_0^1 \frac{\nu A(\nu)\, d\nu}{\nu - z} \tag{24}$$

$N(z)$ (if a solution to (23) exists) will be analytic in the complex plane cut along the real line from 0 to 1, and vanish as $|z| \to \infty$ at least as fast as z^{-1}. Its boundary values will satisfy

$$\pi i\,[N^+(\mu) + N^-(\mu)] = \mathcal{P} \int_0^1 \frac{\nu A(\nu)\, d\nu}{\nu - \mu} \tag{25}$$

$$N^+(\mu) - N^-(\mu) = \mu A(\mu) \tag{26}$$

Conversely, if $N(z)$ possessing the required analyticity properties can be found, $A(\nu)$ will be given by Eq. (26).

We write (23) in terms of the boundary values of $N(z)$:

$$\frac{\mu}{2}(\mu - a) = N^+(\mu)\Lambda^+(\mu) - N^-(\mu)\Lambda^-(\mu) \tag{27}$$

and solve by introducing the function $X(z)$ with the following properties:

(a) $X(z)$ is analytic and nonvanishing along with its boundary values $X^{\pm}(\mu)$ in the complex plane cut from 0 to 1 along the real axis.

(b) Along the cut, the boundary values satisfy

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{\Lambda^+(\mu)}{\Lambda^-(\mu)} \tag{28}$$

The appropriate X-function is identical in form to the X-function of C.

$$X(z) = (1 - z)^{-1} X_0(z) = (1 - z)^{-1} \exp\left\{\frac{1}{2\pi i} \int_0^1 \frac{\ln\,[\Lambda^+(\mu)/\Lambda^-(\mu)]}{\mu - z}\, d\mu\right\} \tag{29}$$

We note that

$$\lim_{|z|\to\infty} zX(z) = -1 \tag{30}$$

The integral equation then becomes

$$(\mu - a)\gamma(\mu) = X^+(\mu)N^+(\mu) - X^-(\mu)N^-(\mu) \tag{31}$$

where

$$\gamma(\mu) = \frac{\mu}{2}\frac{X^+(\mu)}{\Lambda^+(\mu)} \tag{32}$$

Equation (31) is a statement of the analyticity properties of $N(z)$ sufficient to determine $N(z)$ immediately. Since $(\mu - a)\gamma(\mu)$ is the discontinuity of $X(z)N(z)$

across the cut, the function $K(z)$ defined by

$$K(z) \equiv X(z)N(z) - \frac{1}{2\pi i}\int_0^1 \frac{(\mu - a)\gamma(\mu)}{\mu - z}\, d\mu \tag{33}$$

must be an entire function. Since $K(z)$ tends to zero for large $|z|$, we must have

$$K(z) \equiv 0 \qquad \text{(Liouville's theorem)} \tag{34}$$

In fact, since $X(z)N(z)$ must vanish at least as fast as z^{-2} for large $|z|$, the function $\int_0^1[(\mu - a)\gamma(\mu)]/(\mu - z)\, d\mu$ must also have that property. This last condition yields

$$\int_0^1 (\mu - a)\gamma(\mu)\, d\mu = 0 \tag{35}$$

which determines a. In any event, $N(z)$ is given by

$$N(z) = \frac{1}{X(z)}\frac{1}{2\pi i}\int_0^1 \frac{(\mu - a)\gamma(\mu)\, d\mu}{\mu - z} \tag{36}$$

which, together with Eqs. (19), (21), (26), and (35) yield the formal solution to the Milne problem.

III. ANISOTROPIC SCATTERING $c \neq 1$

Here, to illustrate the effect of anisotropic scattering, it suffices to deal with a scattering function of the form

$$T(\hat{\Omega} \to (\hat{\Omega}')) = \frac{1}{4\pi}(1 + b_1 P_1(\hat{\Omega}\cdot\hat{\Omega}')) \tag{37}$$

In this case, with $\psi_\nu = e^{-x/\nu}\varphi_\nu(\mu)$, Eq. (1) becomes

$$(\nu - \mu)\varphi_\nu(\mu) = \frac{c\nu}{2}\left\{\int_{-1}^1 \varphi_\nu(\mu')\, d\mu' + b_1\,\mu\int_{-1}^1 \mu'\,\varphi_\nu(\mu')\, d\mu'\right\} \tag{38}$$

Integrating over $d\mu$, we find

$$\int_{-1}^1 \mu\varphi_\nu(\mu)\, d\mu = (1 - c)\nu\int_{-1}^1 \varphi_\nu(\mu)\, d\mu \tag{39}$$

Hence Eq. (38) will possess a discrete solution when ν is a root of

$$\Lambda(\nu) \equiv 1 - \frac{c\nu}{2}\int_{-1}^1 \frac{1 + b_1(1 - c)\nu\mu}{\nu - \mu}\, d\mu = 0 \tag{40}$$

We assume $c < 1$. In this case,[1] there are two real eigenvalues, $\pm\nu_0$. The

[1] See ref. 2 for general comments on the nature of the discrete spectrum.

corresponding eigenfunctions are

$$\varphi_{0\pm}(\mu) = \pm \frac{c\nu_0}{2} \frac{1 \pm b_1(1-c)\,\nu_0\,\mu}{\pm\nu_0 - \mu} \tag{41}$$

The continuum nodes are

$$\varphi_\nu(\mu) = \frac{c\nu}{2}[1 + b_1(1-c)\nu\mu]\mathcal{P}\,\frac{1}{\nu-\mu} + \lambda(\nu)\delta(\nu-\mu) \tag{42}$$

with

$$\lambda(\nu) = \tfrac{1}{2}[\Lambda^+(\nu) + \Lambda^-(\nu)] \tag{43}$$

We also have

$$\Lambda^+(\nu) - \Lambda^-(\nu) = \pi i c\nu[1 + b_1(1-c)\nu^2] \tag{44}$$

The Milne problem may be treated by writing the general solution

$$\Psi(x,\mu) = ae^{-x/\nu_0}\,\varphi_{0+}(\mu) + e^{x/\nu_0}\,\varphi_{0-}(\mu) + \int_0^1 A(\nu)\,e^{-x/\nu}\,\varphi_\nu(\mu)\,d\nu \tag{45}$$

just as in the case treated in Section II. Then condition (b) [Eq. (20)]

$$\Psi(0,\mu) = 0 \qquad \mu > 0 \tag{46}$$

implies the integral equation

$$\begin{aligned} -a\varphi_{0+}(\mu) - \varphi_{0-}(\mu) &= \int_0^1 A(\nu)\varphi_\nu(\mu)\,d\nu \\ &= \frac{c}{2}[1 + b_1(1-c)\,\mu^2]\mathcal{P}\int_0^1 \frac{\nu A(\nu)\,d\nu}{\nu-\mu} + \lambda(\mu)A(\mu) \\ &\qquad + \frac{c}{2}b_1(1-c)\,\mu\int_0^1 \nu A(\nu)\,d\nu \end{aligned} \tag{47}$$

Hence, if we define

$$N(z) \equiv \frac{1}{2\pi i}\int_0^1 \frac{\nu A(\nu)\,d\nu}{\nu - z} \tag{48}$$

and apply the formulas of Eqs. (25) and (26) we obtain

$$\begin{aligned} -a\mu\varphi_{0+}(\mu) - \mu\varphi_{0-}(\mu) &= N^+(\mu)\Lambda^+(\mu) - N^-(\mu)\Lambda^-(\mu) \\ &\qquad + \frac{c}{2}b_1(1-c)\,\mu^2\int_0^1 \nu A(\nu)\,d\nu \end{aligned} \tag{49}$$

The last term in Eq. (49) is the Fredholm contribution characteristic of anisotropic scattering. If the scattering function contained higher degree polynomials, then the Fredholm part would contain terms of the form $\mu^n \int_0^1 \nu^m A(\nu)\, d\nu$. In any case the treatment is the same—we consider the moments $\xi_m = \int \nu^m A(\nu)\, d\nu$ to be known numbers and solve (49) implicitly. Then, recognizing that $-1/(2\pi i) \cdot \xi_m$ is the mth coefficient of the Laurent expansion of $N(z)$ about infinity, we will obtain sufficient conditions to complete the solution. For the moment set

$$\alpha = \int_0^1 \nu A(\nu)\, d\nu = -2\pi i \lim_{|z| \to \infty} zN(z) \tag{50}$$

Then if we construct $X(z)$ in the same manner as in Section II and substitute in Eq. (49) we get

$$\begin{aligned} \frac{z}{c}\left(-a\varphi_{0+}(\mu) - \varphi_{0-}(\mu) - \frac{c}{2} b_1(1-c)\,\alpha\mu\right)\gamma(\mu) \equiv \gamma(\mu) f(\mu) = X^+(\mu)N^+(\mu) \\ - X^-(\mu)N^-(\mu) \end{aligned} \tag{51}$$

We note that this is of the same form as Eq. (31); hence the solution is

$$N(z) = \frac{1}{2\pi i}\frac{1}{X(z)} \int_0^1 \frac{f(\mu)\gamma(\mu)\, d\mu}{\mu - z} \tag{52}$$

The requirement that $N(z)$ must vanish as $|z| \to \infty$ yields the subsidiary condition

$$\int f(\mu)\gamma(\mu)\, d\mu = 0 \tag{53}$$

Moreover, for consistency, $N(z)$ must satisfy (50), i.e.,

$$\begin{aligned} -2\pi i \lim_{|z| \to \infty} zN(z) &= \lim_{z \to \infty} \left[\frac{-1}{zX(z)}\right]\left[-z \int_0^1 \frac{f(\mu)\gamma(\mu)\, d\mu}{1 - \mu/z}\right] \\ &\Rightarrow -\int_0^1 f(\mu)\gamma(\mu)\mu\, d\mu = \alpha \end{aligned} \tag{54}$$

These last two conditions determine the unknown parameters a, α and along with Eqs. (26), (45), and (52) comprise the formal solution to the problem.

IV. REDUCTION OF THE SOLUTIONS

In this section we show how the Milne problem solutions of the preceding two sections may be reduced to a form amenable to simple computation. This reduction rests heavily on the analyticity properties of the X-functions and follows the treatment of CL in most respects.

There are three basic identities. They are, for the case of isotropic scattering

$c \neq 1$ (from CL)

I
$$X(z) = \int_0^1 \frac{\gamma(\mu)\, d\mu}{\mu - z} \tag{55}$$

II
$$\frac{\Lambda(z)}{X(z)X(-z)} = \Lambda(\infty)(\nu_0^2 - z^2) = (1 - c)(\nu_0^2 - z^2) = \frac{1 - z^2/\nu_0^2}{X^2(0)} \tag{56}$$

III
$$X(z) = \frac{c}{2} \int_0^1 \frac{\mu\, d\mu}{X(-\mu)(\mu - z)(\nu_0^2 - \mu^2)(1 - c)} \tag{57}$$

Identity I is based on the fact that $X(z)$ is analytic in the cut plane and vanishes at infinity. Its discontinuity across the cut is given by

$$X^+(\mu) - X^-(\mu) = \frac{X^+(\mu)}{\Lambda^+(\mu)} [\Lambda^+(\mu) - \Lambda^-(\mu)] = \pi i c \mu \frac{X^+(\mu)}{\Lambda^+(\mu)} = 2\pi i \gamma(\mu) \tag{58}$$

Equation (55) then follows by applying Cauchy's theorem for $X(z)$ to the region bounded by a large circle and the contour G (Fig. 1).

Identity II is based on the fact that $1/X(z)$ is analytic in the cut plane. Equation (56) is obtained by recognizing that the defining equation (28) implies that $[\Lambda(z)]/[X(z)X(-z)]$ is continuous across the cut and must then be an entire function. This function is determined entirely from its behavior at infinity and the fact that it must vanish at $z = \pm\nu_0$. We have also used the fact that $\Lambda(0) = 1$.

Identity III is obtained by combining identities I and II.

The generalization of these identities to include anisotropic scattering is straightforward. Define

$$h(\mu) = \frac{1}{2\pi i \mu} [\Lambda^+(\mu) - \Lambda^-(\mu)] \tag{59}$$

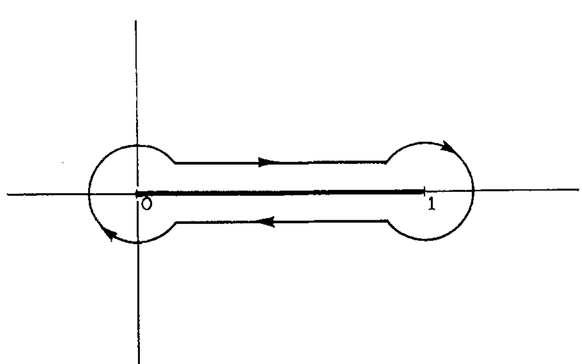

FIG. 1. The contour G

Then, with the scattering function considered in Section II, we have

$$h(\mu) = \frac{1}{2}\left\{ 1 - \frac{b_2}{2} P(\mu) \right\} \tag{60}$$

With the scattering function considered in Section III

$$h(\mu) = \frac{c}{2}\{1 + b_1(1-c)\mu^2\} \tag{61}$$

We denote these by *case A* and *case B*, respectively. Quite generally

$$X^+(\mu) - X^-(\mu) = \frac{X^+(\mu)}{\Lambda^+(\mu)} 2\pi i \mu h(\mu) = 2\pi i \frac{2}{c} \gamma(\mu) h(\mu) \tag{62}$$

Hence, applying Cauchy's theorem for $X(z)$ to the contour G (Fig. 1), we obtain

I′
$$X(z) = \frac{2}{c} \int_0^1 \frac{\gamma(\mu)h(\mu)\, d\mu}{\mu - z} \tag{63}$$

Turning to identity II, we note that the relation

II′
$$\frac{\Lambda(z)}{X(z)X(-z)} = \frac{1 - z^2/\nu_0^2}{X^2(0)} \tag{64}$$

is still valid, in case A where $\nu_0 = \infty$, as well as in case B. Now, however, $X^2(0)$ is obtained from

$$X^2(0) = \lim_{z\to\infty}\left(1 - \frac{z^2}{\nu_0^2}\right)\frac{X(z)X(-z)}{\Lambda(z)}$$

$$= \begin{cases} \dfrac{1}{\nu_0^2\Lambda(\infty)} = \dfrac{1}{\nu_0^2(1-c)\left(1 - \dfrac{cb_1}{3}\right)} & \text{Case A} \\[2ex] -\lim\limits_{z\to\infty} \dfrac{1}{z^2\Lambda(z)} = \dfrac{3}{1 - b_2/5} & \text{Case B} \end{cases} \tag{65}$$

Since $X(z)$ is positive for all negative z[cf. Eq. (29)], $X(0)$ is given by the positive square root of (65).

Finally combining Eqs. (63) and (64), we find

III′
$$X(z) = X^2(0) \int_0^1 \frac{\mu h(\mu)\, d\mu}{X(-\mu)(\mu - z)(1 - \mu^2/\nu_0^2)} \tag{66}$$

The function $H(\mu)$ defined by

$$\frac{1}{H(\mu)} = \frac{X(-\mu)(1 + \mu/\nu_0)}{X(0)} \tag{67}$$

is Chandraesekhar's H-function ("with characteristic function $h(\mu)$") (*4*).

By the use of the identities I′–III′, we may express all the results of Sections II and III in terms of the values of the X-function on the interval $(-1, 0)$ on the real axis. In fact, we show that the outgoing neutron density is of the form

$$\Psi(0, \mu) = \begin{cases} \dfrac{\alpha_1}{X(\mu)} & \mu < 0 \qquad \text{Case A} \\[2ex] \dfrac{\alpha_2 + \alpha_3 \mu}{X(\mu)(1 - \mu^2/\nu_0^2)} & \mu < 0 \qquad \text{Case B} \end{cases} \tag{68}$$

The constants α_i and other parameters appearing in our results may be written in terms of the moments of the function $\gamma(\mu)$:

$$\gamma_n = \int \mu^n \gamma(\mu)\, d\mu \tag{69}$$

Certain relations between the γ_n follow from I′. Setting $z = 0, \infty$ in (63) yields:

$$X(0) = \frac{2}{c} \int_0^1 \gamma(\mu) h(\mu) \frac{d\mu}{\mu} \tag{70}$$

$$-\lim_{z \to \infty} zX(z) = 1 = \frac{2}{c} \int_0^1 \gamma(\mu) h(\mu)\, d\mu \tag{71}$$

Moreover, since in both case A and case B $h(\mu)$ is of the form

$$\frac{2}{c} h(\mu) = \sigma(\mu^2 - \delta^2) \tag{72}$$

it follows from (63) that

$$X(\pm\delta) = \sigma \int_0^1 \gamma(\mu)(\mu \pm \delta)\, d\mu = \sigma[\gamma_1 \pm \delta\gamma_0] \tag{73}$$

or

$$X(\delta)X(-\delta) = \sigma^2[\gamma_1^2 - \delta^2\gamma_0^2] \tag{74}$$

Then, if we apply the identity II′ to this, we obtain the following further relations between γ_0 and γ_1 :

$$(4 + b_2)\gamma_0^2 - 3b_2 \gamma_1^2 = \frac{20}{5 - b_2} \qquad \text{Case A} \tag{75}$$

$$\gamma_0^2 + \gamma_1^2 b_1(1 - c) = \frac{1}{\left(-1 \dfrac{b_1 c}{3}\right)(1 + b_1(1 - c)\nu_0^2} \qquad \text{Case B} \tag{76}$$

We apply these considerations to cases A and B separately:

CASE A

The outgoing neutron angular density is given by Eq. (19) with $x = 0$

$$\Psi(0, \mu) = \frac{a - \mu}{2} + \int_0^1 A(\nu)\varphi_\nu(\mu)\, d\mu \qquad \mu < 0 \tag{77}$$

Since $\varphi_\nu(\mu)$ is nonsingular in this range of ν and μ this becomes

$$\begin{aligned}\Psi(0, \mu) &= \frac{a - \mu}{2} + \left[1 - \frac{b_2}{2} P_2(\mu)\right] \frac{1}{2} \int_0^1 \frac{\nu A(\nu)\, d\nu}{\nu - \mu} \\ &= \frac{a - \mu}{2} + \pi i \left[1 - \frac{b_2}{2} P_2(\mu)\right] N(\mu)\end{aligned} \tag{78}$$

or using the solution for $N(\mu)$:

$$\Psi(0, \mu) = \frac{1}{2}\left\{a - \mu + \frac{1}{X(\mu)} \int_0^1 \frac{[1 - (b_2/2)P_2(\mu)](\nu - a)\gamma(\nu)}{\nu - \mu}\, d\nu\right\} \tag{79}$$

But since

$$\begin{aligned}\left[1 - \frac{b_2}{2} P_2(\mu)\right](\nu - a) &= \left[1 + \frac{b_2}{4} - \frac{3}{4} b_2\, a\nu - \frac{3b_2}{4}\mu(a - \nu)\right](\nu - \mu) \\ &\quad - (a - \mu)\left[1 - \frac{b_2}{2} P_2(\mu)\right]\end{aligned} \tag{80}$$

our result reduces to

$$\begin{aligned}\psi(0, \mu) &= \frac{1}{2X(\mu)}\left\{\int_0^1 \gamma(\nu)\left[1 + \frac{b_2}{4} - \frac{3b}{4} a\nu\right] d\nu\right\} \\ &= \frac{1}{2X(\mu)}\left[\left(1 + \frac{b_2}{4}\right)\gamma_0 - \frac{3b_2}{4} a\gamma_1\right]\end{aligned} \tag{81}$$

We have used identity I′ and the subsidiary condition (35). Hence, using the relation (75), we get finally,

$$\Psi(0, \mu) = \frac{1}{2\gamma_0(1 - b_2/5)} \frac{1}{X(\mu)} \tag{82}$$

Similarly, we may obtain a compact expression for the total neutron density, $\rho(x)$ defined by

$$\rho(x) \equiv \int_{-1}^1 \Psi(x, \mu)\, d\mu = x + a + \int_0^1 A(\nu)e^{-x/\nu}\, d\nu \tag{83}$$

with

$$A(\nu) = \frac{N^+(\nu) - N^-(\nu)}{\nu} = \frac{\pi i\left[1 - \frac{b_2}{2} P_2(\nu)\right][N^+(\nu) - N^-(\nu)]}{\Lambda^+(\nu) - \Lambda^-(\nu)} \tag{84}$$

The numerator is the discontinuity across the cut of the function

$$\pi i\left[1 - \frac{b_2}{2} P_2(z)\right] N(z)$$

But, in view of identity I′

$$\pi i\left[1 - \frac{b_2}{2} P_2(z)\right] N(z) = \frac{1}{X(z)} [2\gamma_0(1 - b_2/5)]^{-1} + \begin{array}{r}\text{terms continuous}\\ \text{across the cut}\end{array} \tag{85}$$

Hence, using identity II′, we find

$$\begin{aligned} A(\nu) &= [2\gamma_0(1 - b_2/5)]^{-1}\left[\frac{1}{X^+(\nu)} - \frac{1}{X^-(\nu)}\right]\frac{1}{\Lambda^+(\nu) - \Lambda^-(\nu)} \\ &= -\frac{1}{6\gamma_0}\frac{1}{\Lambda^+(\nu)\Lambda^-(\nu)} X(-\nu) \end{aligned} \tag{86}$$

The function $\Lambda^+(\nu)\Lambda^-(\nu)$ may easily be related to the tabulated (*5*) function $g(c, \nu)$ where

$$g(c, \nu) = \frac{1}{\Lambda^+_{\text{iso}}(\nu)\Lambda^-_{\text{iso}}(\nu)} \tag{87}$$

CASE B

We may reduce the solution for the neutron angular density at the boundary in Case B as follows. We have

$$\Psi(0, \mu) = a\varphi_{0+}(\mu) + \varphi_{0-}(\mu) + \int_0^1 A(\nu)\varphi_\nu(\mu)\, d\nu \qquad \mu < 0 \tag{88}$$

Using the fact that

$$\varphi_\nu(\mu) = \frac{c\nu}{2}\frac{1 + b_1(1 - c)\mu\nu}{\nu - \mu} \qquad \begin{array}{l} -1 \leqslant \mu < 0 \\ 1 \geqslant \nu \geqslant 0 \quad \text{or} \quad \nu = \pm\nu_0 \end{array} \tag{89}$$

and the definition (51) of $f(\mu)$, this becomes

$$\begin{aligned} \Psi(0, \mu) &= -\frac{c}{2} f(\mu) + \frac{c}{2}\left\{\int_0^1 \frac{\nu A(\nu)(1 + b_1(1 - c)\mu\nu)\, d\nu}{\nu - \mu}\right. \\ &\quad \left. + \int_0^1 \nu A(\nu) b_1(1 - c)\mu\, d\nu\right\} = -\frac{c}{2} f(\mu) + \pi i c[1 + b_1(1 - c)\mu^2] N(\mu) \end{aligned} \tag{90}$$

$f(\mu)$ may be written

$$f(\nu) = a\nu_0 \frac{1 + b_1(1-c)\nu^2}{\nu - \nu_0} - \nu_0 \frac{1 + b_1(1-c)\nu^2}{\nu + \nu_0} - [a\nu_0 - \nu_0 + \alpha] b_1(1-c)\nu \tag{91}$$

Also

$$\nu f(\nu) = a\nu_0^2 \frac{1 + b_1(1-c)\nu^2}{\nu - \nu_0} + \nu_0^2 \frac{1 + b_1(1-c)\nu^2}{\nu + \nu_0} + [a\nu_0 - \nu_0 + \alpha] - \alpha[1 + b_1(1-c)\nu^2] \tag{92}$$

Thus, the subsidiary conditions (53) and (54), along with the identity I′, yield

$$a\nu_0 X(\nu_0) - \nu_0 X(-\nu_0) - [a\nu_0 - \nu_0 + \alpha] b_1(1-c)\gamma_1 = 0 \tag{93}$$

$$a\nu_0^2 X(\nu_0) + \nu_0^2 X(-\nu_0) + [a\nu_0 - \nu_0 + \alpha]\gamma_0 = 0 \tag{94}$$

These may be combined to give

$$a = \frac{X(-\nu_0)}{X(\nu_0)} \frac{d(-\bar{\nu}\nu_0)}{d(\bar{\nu}\nu_0)} \tag{95}$$

Where we have abbreviated[2]

$$d(\nu_1\nu_2) = 1 + b_1(1-c)\nu_1\nu_2 \tag{96}$$

$$\bar{\nu} = \frac{\int \nu\gamma(\nu)\, d\nu}{\int \gamma(\nu)\, d\nu} = \frac{\gamma_1}{\gamma_0} \tag{97}$$

From (90) and (52) we have

$$\Psi(0, \mu) = \frac{c}{2}\left\{ (1 + b_1(1-c)\mu^2) \frac{1}{X(\mu)} \int_0^1 \frac{f(\nu)\gamma(\nu)\, d\nu}{\nu - \mu} - f(\mu) \right\} \tag{98}$$

Using (91) for $f(\nu)$ and carrying out the integration by partial fractions with the aid of identity I′ gives

$$\frac{1}{X(\mu)} \int_0^1 \frac{f(\nu)\gamma(\nu)\, d\nu}{\nu - \mu} = \frac{a\nu_0(\nu_0 + \mu)X(\nu_0) + \nu_0(\nu_0 - \mu)X(-\nu_0)}{\nu_0^2 - \mu^2} \frac{1}{X(\mu)} - \frac{a\nu_0(\nu_0 + \mu) + \nu_0(\nu_0 - \mu)}{\nu_0^2 - \mu^2} - [a\nu_0 - \nu_0 + \alpha] \frac{b_1(1-c)}{X(\mu)} \int_0^1 \frac{\nu\gamma(\nu)\, d\nu}{\nu - \mu} \tag{99}$$

[2] We will write all results in terms of the d's in order to make evident the correspondence with the results of CL where $d \equiv 1$.

With this result, along with the relations (93) and (94), the entire expression (98) collapses to

$$\Psi(0, \mu) = \frac{c\nu_0^2 X(-\nu_0)}{[\nu_0^2 - \mu^2]X(\mu)} \frac{d(\nu_0^2)}{d(\nu_0 \bar{\nu})} d(-\bar{\nu}\mu) \tag{100}$$

For the neutron density $\rho(x)$, we write:

$$\rho(x) = ae^{-x/\nu_0} + e^{x/\nu_0} + \int_0^1 A(\nu)e^{-x/\nu} d\nu \tag{101}$$

with a given by (95). To obtain $A(\nu)$ we proceed as before:

$$A(\nu) = \frac{N^+(\nu) - N^-(\nu)}{\nu} = \frac{\pi i c[1 + b_1(1 - c)\nu^2][N^+(\nu) - N^-(\nu)]}{\Lambda^+(\nu) - \Lambda^-(\nu)} \tag{102}$$

The numerator is the discontinuity across the cut of the function

$$\pi i c[1 + b_1(1 - c)z^2]N(z) = \frac{c}{2X(z)} [1 + b_1(1 - c)z^2] \int_0^1 \frac{f(\mu)\gamma(\mu)\, d\mu}{\mu - z} \tag{103}$$

The right-hand side is simplified through the use of (93), (94), and (99) to give

$$\begin{aligned} \pi i c[1 + b_1(1 - c)z^2]N(z) = &-\frac{c}{2} \frac{[a\nu_0 - \nu_0 + \alpha]}{X(z)} [1 + b_1(1 - c)z^2] \\ &\cdot \left\{ \frac{\gamma_0 - b_1(1 - c)\gamma_1 z}{\nu_0^2 - z^2} + b_1(1 - c) \int_0^1 \frac{\nu\gamma(\nu)\, d\nu}{\nu - z} \right\} \\ &+ \text{terms continuous across the cut.} \end{aligned} \tag{104}$$

But since

$$(1 + b_1(1 - c)z^2)\nu = (1 - b_1(1 - c)\nu z)(\nu - z) + (1 + b_1(1 - c)\nu^2)z \tag{105}$$

the result reduces (using identity I$'$) to

$$\begin{aligned} \pi i c[1 + b_1(1 - c)z^2]N(z) = &\frac{-c[a\nu - \nu_0 + \alpha]}{zX(z)} [\gamma_0 - b_1(1 - c)\gamma_1 z] \\ &\cdot \left\{ \frac{1 + b_1(1 - c)\nu_0^1}{\nu_0^2 - z^2} \right\} + \text{terms continuous across the cut.} \end{aligned} \tag{106}$$

Also, (identity II$'$)

$$\frac{1}{X(z)(\nu_0^2 - z^2)} = \frac{1}{\nu_0^2\Lambda(z)} \frac{X(z)}{X^2(0)} \tag{107}$$

and since $X(-z)$ is continuous across the cut, we obtain finally from (102)

$$\begin{aligned} A(\nu) = &\frac{c}{2} [a\nu_0 - \nu_0 + \alpha](\gamma_0 - b_1(1 - c)\gamma_1 \nu)(1 + b_1(1 - c)\nu_0^2) \\ &\cdot \frac{X(-\nu)}{\nu_0^2 X^2(0)\Lambda^+(\nu)\Lambda^-(\nu)} \end{aligned} \tag{108}$$

The relations (93) and (94) yield the concise final form:

$$A(\nu) = -\frac{d(-\bar{\nu}\nu)\, d(\nu_0^2)}{d(\bar{\nu}\nu_0)} \frac{cX(-\nu_0)X(-\nu)}{\Lambda^+(\nu)\Lambda^-(\nu)X^2(0)} \tag{109}$$

V. NUMERICAL COMPUTATION

It only remains to obtain numerical values for the different X-functions. In principle these may be obtained to any desired accuracy from identity III′. That is, the nth approximation is given by

$$X_n(z) = X^2(0) \int_0^1 \frac{\mu h(\mu)\, d\mu}{X_{n-1}(-\mu)(\mu - z)(1 - \mu^2/\nu_0^2)} \tag{110}$$

with

$$X_0(z) = X(0) \tag{111}$$

A better starting point for the iterative scheme is obtained by introducing an inhomogeneous term. That is, we use the identity

$$\text{III}'' \qquad X(z) = X(0) + zX^2(0) \int_0^1 \frac{h(\mu)\, d\mu}{X(-\mu)(\mu - z)(1 - \mu^2/\nu_0^2)} \tag{112}$$

instead of III′. This is obtained in the same manner as III′ except that the function $[X(z) - X(0)]z^{-1}$ [having the same analyticity properties as $X(z)$] is substituted for $X(z)$ in the derivation of Eq. (63). The first approximation is obtained by a simple integration. Successive approximations require the integrals be computed numerically.

The most satisfactory scheme is obtained by approximating $X(z)$ by a rational function at the start. The function

$$\Omega(z) \equiv \left(\frac{1}{X(0)} - z\right) X(z)$$

is analytic in the cut plane. Moreover,

$$\Omega(0) = \Omega(\infty) = 1 \tag{113}$$

and we expect (subject to verification) that $\Omega(z)$ will be slowly varying in the interval $(-1, 0)$. The function $[\Omega(z) - 1]z^{-1}$ is analytic in the cut plane and vanishes at infinity; thus we may apply Cauchy's theorem to the contour G (Fig. 1):

$$\frac{\Omega(z) - 1}{z} = \frac{1}{2\pi i} \int_0^1 \frac{\Omega^+(\mu) - \Omega^-(\mu)}{\mu(\mu - z)}\, d\mu \tag{114}$$

But from identity II′ we have

$$\frac{\Lambda(z)}{\Omega(z)\Omega(-z)} = \frac{1 - z^2/\nu_0^2}{1 - z^2X^2(0)} \tag{115}$$

or,

$$\Omega^{+}(\mu) - \Omega^{-}(\mu) = \frac{1 - \mu^2 X^2(0)}{(1 - \mu^2/\nu_0^2)\Omega(-\mu)} [\Lambda^{+}(\mu) - \Lambda^{-}(\mu)] \tag{116}$$

Thus,

$$\Omega(z) = 1 + z \int_0^1 \frac{[1 - \mu^2 X^2(0)]h(\mu)\, d\mu}{(1 - \mu^2/\nu_0^2)(\mu - z)\Omega(-\mu)} \tag{117}$$

This method yields extremely accurate results, the second approximation differing from the first usually only in the fourth or fifth significant figure. Since the first approximation does not require numerical integration, it is a simple matter to obtain the X-function by this method.

TABLE I

THE OUTGOING NEUTRON ANGULAR DENSITY $\frac{3}{2}\psi(0, \mu)$ FOR THE NONABSORBING MEDIUM WITH SCATTERING LAW $T = (1/4\pi)(1 + bP_2(\hat{\Omega}\cdot\hat{\Omega}'))$

$-\mu$	b				
	0	0.2	0.5	1.0	1.8
0	0.43304	0.42755	0.41904	0.40399	0.37649
0.1	0.54034	0.53632	0.53001	0.51837	0.49565
0.2	0.62801	0.62483	0.61986	0.61057	0.59169
0.3	0.71114	0.70890	0.70496	0.69780	0.68287
0.4	0.79191	0.79017	0.78754	0.78242	0.77127
0.5	0.87152	0.87042	0.86886	0.86564	0.85843
0.6	0.95003	0.94953	0.94824	0.94783	0.94474
0.7	1.0277	1.0277	1.0283	1.0289	1.0302
0.8	1.1052	1.1059	1.1074	1.1099	1.1150
0.9	1.1833	1.1846	1.1870	1.1914	1.2014
1.0	1.2592	1.2608	1.2644	1.2707	1.2844

TABLE II

THE ASYMPTOTIC END-POINT x_0 FOR THE NONABSORBING MEDIUM WITH SCATTERING LAW $T = (1/4\pi)[1 + bP_2(\hat{\Omega}\cdot\hat{\Omega}')]$

b	x_0
0.0	−0.7104
0.2	−0.7108
0.5	−0.7114
1.0	−0.7125
1.8	−0.7148

Table I shows the outgoing angular density $\psi(0, \mu)$ for Case A for 5 different values of the anisotropy parameter b. Actually in order to compare results with those of reference (4) we plot $\frac{3}{2}\psi(0, \mu)$. In Chandrasekhar's notation $\frac{3}{2}\psi(0, -\mu) = I(0, \mu)/F$, a quantity plotted in reference (4) for the values $b = 0, 0.5$.

Lastly, we consider the "extrapolated end-point," i.e., that value of x for which the discrete part of the total density extrapolates to zero. In case A, the discrete part of the density is simply

$$\rho_{\text{dis}}(x) = x + a \tag{118}$$

Thus the extrapolated end-point is

$$X_0 = -a = -\frac{\gamma_1}{\gamma_0} \tag{119}$$

These values are shown in Table II.

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