JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 7, 62-63 (1963)

Remark on a Paper of Kalisch*

M. SCHREIBER

Cornell University, Ithaca, New York and
The University of Michigan, Ann Arbor, Michigan

Submitted by C. L. Dolph

In a recent paper [1] Kalisch establishes the Brodskii characterization of the invariant subspaces of the Volterra operator V defined on $L_2((0, 1); dx)$ by the relation

$$Vf(x) = \int_0^x f(t)\, dt,$$

which is the following:

If $M \subset L_2(0, 1)$ *is a closed subspace with* $VM \subset M$ *then*

$$M = L_2(a, 1) \textit{ for some } a, 0 \leq a \leq 1. \tag{1}$$

He then shows the equivalence of this theorem to the Titchmarsh convolution theorem:

If $f, g \in L_1(0, 1), f^*g = 0$ a.e. *in* $[0, 1]$, *and* 0 *is in the support*

$$\textit{of } f, \textit{ then } g = 0 \text{ a.e. } \textit{in } [0, 1]. \tag{2}$$

(The support of a function f, which we denote Spt f, is the support of the (complex) measure fdx it defines.) It is the purpose of this note to show the equivalence of (1) and (2) by a very simple argument, which is inspired by, and unifies, the several arguments given in [1].

We first remark (as does Kalisch) that (1) is equivalent to

If $0 \in \operatorname{Spt} f$ *then the closed linear space of*

$$\{V^n f \mid n = 0, 1, 2, \ldots\} \text{ is } L_2(0, 1). \tag{1'}$$

Indeed, (1) implies (1′) trivially, and given (1′) it is clear that if $VM \subset M$ then $M = L_2(a, 1)$ with $a = \operatorname{Inf} \{t \mid t = \operatorname{Inf} \operatorname{Spt} f, f \in M\}$. Write $\tilde{\varphi}(x) = \varphi(1 - x)$ for any function φ. We will show

$$(V^n f, \bar{g}) = \frac{1}{(n-1)!} \int_0^1 t^{n-1} \widetilde{f^*\tilde{g}}(t)\, dt, \qquad n = 1, 2, \ldots,$$

$$(f, \bar{g}) = \widetilde{f^*\tilde{g}}(0), f, g \in L_2. \tag{3}$$

* This work was supported by an NSF Grant.

Thus the inner products determining the "orbit" of f under the action of V are essentially the moments of a convolution, and this makes the equivalence of (1′) and (2) clear. Assuming (2), if $0 \in \operatorname{Spt} f$, and $(V^n f, \bar{g}) = 0$, $n = 1, 2, \cdots$, then $\widetilde{f^*\tilde{g}} = f^*\tilde{g} = 0$ a.e., so $\tilde{g} = \bar{g} = g = 0$ a.e. Assuming (1′), if $0 \in \operatorname{Spt} f$ and $f^*\tilde{g} = 0$, then $(V^n f, \bar{g}) = 0$, $n = 0, 1, 2, \cdots$, so that $\bar{g} = \tilde{g} = 0$ a.e. This proves (2) for $f, g \in L_2$, and the L_1 case follows immediately from the facts that Vf is continuous for $f \in L_1$ and $Vf = e^*f$ on $[0, 1]$, where e is the characteristic function of $[0, 1]$.

If we write e^n for the n-fold convolution of e with itself then, as just noted, we have $e^n{}^*f = V^n f$ on $[0, 1]$, and this is a continuous function on $[0, 1]$ for every $f \in L_2$.

The proof of (3) is then as follows.

$$(V^n f, \bar{g}) = \int_0^1 V^n f(x)\, \tilde{g}(1 - x)\, dx = (V^n f^* \tilde{g})(1) = (e^n{}^* f^* \tilde{g})(1)$$

$$= (V^n(f^*\tilde{g}))(1) = \frac{1}{(n-1)!} \int_0^1 (1 - u)^{n-1} f^*\tilde{g}(u)\, du$$

$$= \frac{1}{(n-1)!} \int_0^1 t^{n-1} \widetilde{f^*\tilde{g}}(t)\, dt, \qquad n \geq 1,$$

and

$$\widetilde{f^*\tilde{g}}(0) = f^*\tilde{g}(1) = \int_0^1 f(1 - y)\, g(1 - y)\, dy = (f, \bar{g}).$$

Reference

1. Kalisch, G. K. A functional analysis proof of Titchmarsh's theorem on convolution. *J. Math. Anal. Appl.* **5**, 176 (1962).