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A Unified Theory of the Nonlinear Oscillations of a Cold Plasma*

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INTRODUCTION

An extensive literature has evolved since 1951 dealing with various aspects of the nonlinear oscillations of a cold plasma. Much of this literature is repititious and shows a lack of awareness of previous work by other authors. It is often also sprinkled with errors which make it difficult for the novice to develop a consistent picture of the state of the theory. By far the largest effort has been devoted to the special cases where only one independent spatial variable exists. As pointed out by Jackson [1] this results in a separation of space and time which forces the resulting oscillation to be localized in space and, as such, most of the theory cannot be thought of as applying to the real physical problem.

Even so a wide variety of methods have been used and though at first glance they appear unrelated, they all succeed via the introduction of Lagrangian coordinates. Some authors start with Newton's law of motion, others with the Eulerian equation of motion and introduce a pseudo-stream function, while still others employ the Lagrangian form of the equation of motion and Poisson's equation.

In this paper an attempt at a unified theory will be made by use of the existing theory of systems of first order partial differential equations with equal principal parts. Such systems are known to be completely equivalent to systems of ordinary differential equations (cf. [2]) and the large body of existing knowledge for systems of ordinary differential equations can be applied to the various problems. Apart from the advantages in unification achieved by casting these problems in this form, the method does permit some generalization to arbitrary orthogonal coordinate systems and suggests a natural perturbation approach to the general problem involving three independent spatial coordinates.

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Considerations of space do not permit the extension of the various spectral analyses carried out by various authors to the slightly more general situations considered here, but such extensions are clearly possibly once the reduction to systems of ordinary differential equations has been achieved.

Briefly the plan of the paper is as follows: Section I contains a short historical survey of the class of problems under discussion and while it is unquestionably incomplete, it is hoped that its inclusion will be useful for orientation; Section II is a discussion of the simplest nonstationary case involving no external magnetic field and heavy ions. The inconsistent results obtained by different authors are commented upon and corrected; Section III is concerned with a discussion of the more general case in rectangular coordinates when a dc magnetic field is allowed and the ions have finite mass. While three velocity components are allowed the restriction to one independent spatial variable is retained. In particular, the recent formula given by Wilhelmson [3] for the shift in the plasma frequency is shown to be an approximation to a closed expression; Section IV specializes the discussion of Section III to the so-called traveling wave case. In particular it contains a closed form expression corresponding to the one first given without proof by Vedenov [4] which is shown to be in error; Section V consists of a discussion of the method for general orthogonal coordinate systems subject to the restriction to one independent spatial coordinate. For simplicity this is carried out for the infinite ion case only. In particular the resulting equations are used to rederive the anharmonic oscillations found by Dawson [5] in the cylindrical and spherical geometries; Section VI considers the possibility of basing a perturbation procedure for the case of three independent spatial variables on the approach given here. In particular the general first order approximation is discussed in the presence of an external magnetic field and then the Cesari-Hale-Gambill [6] method is applied to the two-dimensional example treated by Dawson [5] in order to obtain a third order perturbation free from secular terms and to derive a formula for the shift in plasma frequency for this case. In common with Dawson's result, the first shift in the frequency of oscillation is of the order $(1/\omega_0)^2$ where ω_0 is the plasma frequency.

Thoroughout the discussion it will be assumed that only single streaming occurs so that all quantities will be single valued functions of Lagrangian coordinates.

I. Historical Survey

Apparently the first widely recognized result of the nonlinear theory is due to Akhiezer and Lubarskii [7] in 1951 (henceforth referred to as A-L). They treated a one-dimensional model and found an exact nonlinear traveling wave solution in which all quantities were dependent upon the single combination $X - V_0t$, where V_0 was a constant streaming velocity. Their work was subsequently critized by Smerd [8] in 1955 and by Dawson [5] in 1959. Smerd pointed out that the A-L solution failed to satisfy Poisson's equation and then indicated a modification without proof that did. Dawson's criticism is as follows:

"Non-linear travelling waves in a cold plasma have been found by Akhiezer and Lubarskii. Their solutions are special solutions to the non-linear equations. Actually any solution to the non-linear equations can be built up from their solutions although they did not recognize this due to the complex nature of their method. They also apparently did not realize the amplitude limitation which exists for their solutions."

The amplitude limitation referred to above occurs at the breakdown of single streaming when the electrons overtake each other. (For some indication as to what happens without this restriction, cf. Dawson [9] who calculated the effects numerically.)

Dawson then proceeded to discuss the A-L model and several generalizations of it by use of Newton's law and Gauss' theorem. In contrast, A-L and Smerd had employed the Eulerian equations.

Independently of this, Sen [10] in 1955 treated the A-L model by a different procedure in that while A-L eliminated the potential of the static electric field and the density in order to obtain an integrable velocity equation, Sen eliminated the density and velocity and obtained a nonlinear dispersion relation from Poisson's equation. While he integrated this only once, he was nevertheless able to discuss the resulting fluctuation of the electric field graphically and showed that a nonlinear jump phenomenon occurred, as in relaxation oscillations. (The dispersion relation was subsequently integrated once more by Gold [11] and Kalman [12].) In addition Sen obtained the same nonlinear dispersion relation from the traveling wave solutions to the collisionless Vlaslov equations for an initial distribution of the form of a delta function, the usual cold plasma definition.

In 1957 Vedenov [4] exhibited without proof a traveling wave solution for a slightly more general situation in which two velocity components were allowed in a rectangular frame of reference as well as a dc magnetic field normal to this plane. The restriction to the combination of the traveling wave $x' = x - V_0t$ where V_0 is a streaming velocity was assumed by him in common with A-L and his solution reduced to theirs when the magnetic field and extra velocity component were set equal to zero. This same year Sturrock [13] examined the nonsteady behavior of the A-L model by Fourier methods in a manner reminscient of the usual treatment of isotropic turbulence.

In 1958 the time dependent case was treated for the first time by Gold [14] who subsequently took into effect a dc magnetic field and collisions in the first approximation [15]. His results appear to be in error as a result of the way in which he determined the time derivative of the internal electric field. They are easily correctable, however, as will be seen in Section II. This same year Amer [16] gave a discussion of the nonlinear equation for the density fluctuations. This appears to be in error as a result of incorrect use of the total derivative (cf. the discussion of Section II).

In 1959, as mentioned earlier, Dawson [5] also treated the time varying case for rectangular, cylindrical and spherical geometries. In addition he developed a perturbation procedure for the general nonstationary three independent variables case and applied it to a two-dimensional situation (cf. the discussion of Section VI). This same year Konyukov [17] derived what we believe to be the correct equations for the density fluctuations, and once again redid the simplest time dependent problem.

In 1960 Kalman [12] criticized Sturrock's treatment in the wave-number domain and in those two papers he made an extensive study of the one-dimensional nonsteady case by introducing Lagrangian coordinates via a pseudo stream function. In particular he discussed the chain of special circumstances necessary for the excitation of a traveling wave and concluded that the solutions found by so many of the authors listed above were not physically traveling waves in that at no time did other than the initially disturbed particles become involved in the oscillation. He also discussed the formation of the shock which develops when overtaking occurs.

Subsequently, Jackson [1] re-examined Sturrock's treatment and clarified it by the use of Lagrange's implicit function theorem and the method of removing secular terms due to Krylov and Bogolyubov. He was thus able to show that no frequency shift occurred and by so correcting this error of Sturrock, he was able to invalidate the criticism made by Kalman.

Very recently Wilhelmson [3] extended the approach of Sen to the case of finite mass ions and deduced a change in the plasma frequency due to the finite mass.

Finally for the sake of completeness, attention should be called to the papers by Bernstein *et al.* [18], Nekrasov [19], Akhiezer and Polovin [20], and Wyld [21]. All of these except Wyld's are concerned with the traveling wave solution. In the first the collisionless Vlaslov's equation is used and trapping is allowed. The results are interesting in that the delta function singularities of the linearized theory are related to the nonlinear results. The paper by Akhiezer and Polovin uses the Eulerian representation in the presence of all of Maxwell's equations but in view of the above noted criticism of Kalman for the one-dimensional case, it is difficult to assess the physical significance of the results they obtain. Wyld is interested in the nearly cold plasma and computes first order effects.

II. Nonsteady Oscillations for Heavy Ions and Zero Magnetic Field

As mentioned in the introduction this problem for the cold plasma has been the most widely discussed situation. We will discuss it here once again both for motivation for the sequel and in order to compare the various approaches employed as well as correct some mistakes that have occurred in the literature. Since most authors start with the Eulerian equations in the hydrodynamic approximation we note that these are:

$$U_t + UU_x = -\frac{e}{m}E \tag{2.1}$$

$$N_t + (NU)_x = 0 \tag{2.2}$$

$$E_x = 4\pi e(N_0 - N) \tag{2.3}$$

$$\text{curl } H = 0 \tag{2.4}$$

$$E_t + 4\pi e(N_0U_0 - NU) = 0. \tag{2.5}$$

In these equations, $-e$, m, U, N, E, N_0, and U_0 are respectively the electron charge, mass, x-component of the hydrodynamic velocity, electron density, self-consistent longitudinal electric field, positive ion density, and the common ion drift velocity (which may be zero). The plasma is assumed, as usual, to be neutral when in an equilibrium state. The equations are, respectively, the equation of motion, continuity, Poisson's equation and the last two represent the decomposition of one of Maxwell's equation, since the total current is given by

$$J = 4\pi e(N_0U_0 - NU), \qquad J_+ = 4\pi eN_0U_0, \qquad J_- = 4\pi eNU. \tag{2.6}$$

If Eq. (2.3) is multiplied by U and added to (2.5), one obtains

$$E_t + UE_x = 4\pi eN_0(U - U_0) = 4\pi eN_0U - J_+. \tag{2.7}$$

Now Eq. (2.7) is consistent with Eq. (2.3) under these circumstances for, by the use of (2.3), Eq. (2.7) can be written as

$$(E_t + UE_x - 4\pi eN_0U)_x = 0. \tag{2.8}$$

Equations (2.1) and (2.7) constitute a system of first order partial differential equation with equal principal parts so that they are completely equivalent to the integration of an ordinary system of differential equations when the characteristics are introduced (cf. [2]). Here it is sufficient to introduce the single characteristic $dX/dt = U$ and we then obtain the system

$$\frac{dX}{dt} = U \tag{2.9}$$

$$\frac{dU}{dt} = \frac{-e}{m} E \tag{2.10}$$

$$\frac{dE}{dt} = 4\pi e N_0 U - J_+. \tag{2.11}$$

This system immediately implies that

$$\frac{d^2V}{dt^2} + \omega_0^2 V = \frac{e}{m} J_+; \qquad \omega_0^2 = \frac{4\pi e N_0}{m} \tag{2.12}$$

and this is the second of Konyukov's equations [17], if $u_0 = J_+ = 0$.

Equation (2.11) can be integrated at once to give

$$E = 4\pi e N_0 (X - X_0) - J_+ t + F''(t) \tag{2.13}$$

where the arbitrary function $F''(t)$ must correspond to an externallly applied electric field as noted by both Kalman [12] and Wyld [21] since Poisson's equation does not determine the dc component of the electric field and, as the alternate method of derivation of Section III will show, (2.13) can be deduced directly from Poisson's equation without use of the intermediate steps used here. One finds then successively in agreement with Kalman [12] that

$$X = X_0 + A(X_0) \sin \omega_0 t + B(X_0) \cos \omega_0 t + G(t) \tag{2.14}$$

$$U = \omega_0 [A(X_0) \cos \omega_0 t - B(X_0) \sin \omega_0 t] + G'(t). \tag{2.15}$$

To determine an expression for the density fluctuation, it is simplest to use the equation of continuity in Lagrangian form; i.e.

$$N = N_0 \frac{\partial X_0}{\partial X} = \frac{N_0}{\partial X / \partial X_0} \tag{2.16}$$

and thus one obtains, again in agreement with Kalman, that

$$N = N_0 [1 + A'(X_0) \sin \omega_0 t + B'(X_0) \cos \omega_0 t]^{-1}. \tag{2.17}$$

At this point several remarks are in order. Gold in [14] and [15] obtained, instead of (2.11), the relation

$$\frac{dE}{dt} = 4\pi e N_0 U - J_- \qquad J_- = 4\pi e N U \tag{2.18}$$

by using Poisson's equation and setting

$$\frac{dE}{dx} = \left(\frac{dE}{dt}\right)\left(\frac{dt}{dx}\right) = \left(\frac{1}{u}\right)\left(\frac{dE}{dt}\right),$$

arguing that "in general $E = E(\omega t - kx)$, but for P waves k is either zero or imaginary. Piddington points out that this corresponds to the stationary Tonks-Langmuir oscillation. (cf. PIDDINGTON, J. H., *Phil. Mag.* **46**, 1037, (1955).)" This it seems is inconsistent with the hydrodynamic approximation and explains the subsequent comments by Gold in [15] and [11] where, because of the substitution of J_- for $J_+ =$ constant, he finds it necessary to consider higher approximations where $J_- = J_-(t)$. Apart from this, he employs Eq. (2.10) and ignores the continuity equation which would be correct if he had used (2.11). Dawson [5] does use the equivalent argument correctly, modifying it only in that he invokes Gauss's law to determine the internal electric field and ignores any external electric field.

For this problem, Kalman indirectly introduced Lagrangian coordinates by use of a pseudo steam function

$$\psi_x = -\frac{N}{N_0}, \qquad \psi_t = \frac{NU}{N_0}, \qquad \psi = -X_0 \tag{2.19}$$

so that the equation of continuity (2.2) is satisfied automatically. Finally, while the above shows clearly that it is not necessary to deduce a separate equation for the density, we will give a detailed derivation of the equation obtained by Konyukov [17] in order to compare it with that deduced by Amer [16] which is believed to be in error. Konyukov's equation can be derived as follows: Differentiating (2.1) and combining it with (2.3) one obtains

$$U_{tx} + UU_{xx} + U_x^2 = \frac{\omega^2}{N_0}(N - N_0).$$

With the use of (2.9) the continuity equation can be written as

$$\frac{1}{N}\frac{dN}{dt} = \frac{N_t + UN_x}{N} = -U_x$$

so that one has

$$\frac{\partial}{\partial t}\left(\frac{1}{N}\frac{dN}{dt}\right)_t = -U_{xt}$$

and

$$U\frac{\partial}{\partial x}\left(\frac{1}{N}\frac{dN}{dt}\right) = -[UU_{xx}].$$

Thus the total derivative is given by

$$\frac{d}{dt}\left(\frac{dN}{dt/N}\right) = -\,[U_{xt} + UN_{xx}].$$

Combining this with the expression given for U_x and substituting into the first of the above equation one obtains the following equation of Konyukov:

$$N\frac{d^2N}{dt^2} - 2\left(\frac{dN}{dt}\right)^2 + \frac{\omega_0^2}{N_0}N^2(N - N_0) = 0.$$

The argument given by Amer in [16] is obscure so that we will quote it:

"We consider an isotropic, indefinite plasma consisting of highly mobile electrons in a uniform background of fixed positive electric charge. The density of electrons is such that the plasma is, as a whole, neutral. If at any point there is an excess of deficiency of electrons $\rho = N - N_0$, where N is the actual and N_0 the mean electron density, then there will be an electrostatic potential ϕ in the plans." The motion of an electron, subject only to the Coulomb forces derived from ϕ, is given by

$$m\frac{d\mathbf{V}}{dt} = e \operatorname{grad} \phi \tag{1}$$

where $\mathbf{V}$ is the velocity of the electron. We can apply the principle of conservation of electric charge to both sides of (1): in its spatial form (Poisson's equation) to the right and in its temporal form (equation of continuity) to the left-hand side; then if we keep nonlinear terms, we obtain the exact equation

$$\rho'' - \frac{\rho^2}{N_0 + \rho} + \omega_0^2\rho\left(1 + \frac{\rho}{N_0}\right) = 0 \tag{2}$$

where a dot denotes derivation with respect to time and $\omega_0^2 = 4\pi\rho^2N_0/m$ is the classical plasma frequency. If in (2) we neglect nonlinear terms we find the classical equation for plasma oscillations

$$\rho'' + \omega_0^2\rho = 0.$$

In order to derive Amer's equation it appears necessary to assume that the total time derivative d/dt and the divergence operator commute but this is of course false.[1] A *spurious* derivation would be as follows:

[1] It has recently come to the author's attention that a similar conclusion was reached by H. Derfler in "The frequency of non-linear plasma oscillations," Tech. Rept. No. 104-7, Stanford Electronics Lab. (May 10, 1961), Air Force Contract AF 19(604)-5480 (May 10, 1961). This report also contains a new derivation of the frequency shift for cold spherical and cylindrical plasmas first given by Dawson [5]. This report has recently been published in *J. Electronics and Control* 11, 3 (1961).

Write the equation of continuity as

$$\rho_t + \mathbf{V} \cdot \nabla\rho = -(\rho + N_0)\,\nabla \cdot \mathbf{V}$$

from which it follows that

$$\frac{1}{\rho + N_0}\frac{d\rho}{dt} = -\nabla \cdot \mathbf{V}$$

when one sets $d\mathbf{x}/dt = \mathbf{V}$. From the equation of motion

$$\frac{d\mathbf{V}}{dt} = \frac{e}{m}\nabla\phi$$

and Poisson's equation

$$\nabla^2\phi = 4\pi e\rho$$

one then obtains

$$\nabla \cdot \frac{d\mathbf{V}}{dt} = \frac{e}{m}\nabla^2\phi = \frac{\omega^2}{N_0}\rho$$

so that if one *falsely* assumes that

$$\nabla \cdot \frac{d\mathbf{V}}{dt} = \frac{d}{dt}(\nabla \cdot \mathbf{V})$$

we would have

$$\frac{d}{dt}\left(\frac{1}{\rho + N_0}\frac{d\rho}{dt}\right) = -\frac{d}{dt}(\nabla \cdot \mathbf{V}) = -\frac{\omega_0^2}{N_0}\rho$$

or

$$\frac{d^2\rho}{dt^2} - \frac{1}{(\rho + N_0)}\frac{d\rho}{dt} + \omega_0^2\rho\left(1 + \frac{\rho}{N_0}\right) = 0$$

in agreement with Amer. Introducing Amer's notation in Konyukov's equation, it can be written as

$$\frac{d^2\rho}{dt^2} - \frac{2}{\rho + N_0}\frac{d\rho}{dt} + \omega_0^2\rho\left(1 + \frac{\rho}{N_0}\right) = 0$$

so that the two differ only in a factor of two in the middle term and both yield the usual linear equation if the nonlinear terms are neglected. In the one-dimensional case Amer's error appears to be in the neglect of the term U_x^2 or possibly the confusion between the total and partial time derivative since the latter does commute with the divergence. It is also unfortunate in that without this error it does not appear easy to derive a three-dimensional equivalent of the equation of Konyukov. The factor of two noted above completely changes the character of the solutions and those given by Amer which involve logarithmic terms appear to have no validity.

III. Oscillations with DC Magnetic Field

The analysis of the previous paragraph is readily extended to the situation where the ions are assumed of finite mass, three velocity components are allowed, and a constant external magnetic field is permitted, provided only that the restriction to one independent spatial variable is retained. Under these circumstances the previous discussion makes it clear that the usual equations

$$\frac{d\mathbf{V}_\pm}{dt} = \pm \frac{e}{m}\left[\mathbf{E} + \frac{1}{c}\mathbf{V}_\pm \times \mathbf{H}\right] \tag{3.1}$$

are being considered in the special cases where

$$\frac{d\mathbf{V}_\pm}{dt} = \frac{\partial \mathbf{V}_\pm}{\partial t} + U_\pm \frac{\partial \mathbf{V}_\pm}{\partial x}\,; \quad V_1 = U. \tag{3.2}$$

Now Eq. (3.1) is always correct and one can always make the transformation from Eulerian to Lagrangian coordinates via the characteristic equations

$$\frac{d\mathbf{X}_\pm}{dt} = \mathbf{V}_\pm \tag{3.3}$$

but unless Poisson's equation involves only one spatial coordinate the use of Lagrangian equations is not sufficient to permit an exact and complete solution of the problem as the discussion in Section VI will show. Since here the ions are assumed of finite mass, the previous technique needs some modification in order to avoid ambiguities. Perhaps the simplest approach is that of Wyld [21] who noted that the Lagrangian form of the equation of continuity

$$N_\pm = N_\pm^0 \frac{\partial X_\pm^0}{\partial X_\pm} \tag{3.4}$$

implies that Poisson's equation can be written as

$$\frac{\partial E}{\partial X} = 4\pi e N_0 \left(\frac{\partial X_+^0}{\partial X} - \frac{\partial X_-^0}{\partial X}\right) \tag{3.5}$$

so that if one sets $X = X_+ = X_-$ in order to obtain the equilibrium position of an electron and an ion which find themselves at the same point at the same time t, (3.5) can be integrated to yield:

$$E = 4\pi e N_0 (X_+^0 - X_-^0) + F(t) \tag{3.6}$$

where $F(t)$ is still arbitrary. If one now decomposes X according to

$$X = X_\pm = X_\pm^0 + X_\pm^1(t) + X_\pm^2(X_\pm^0, t) \tag{3.7}$$

Eq. (3.6) can be written as

$$E = 4\pi e N_0 [X_-^2(X_-^0, t) - X_+^2(X_+^0, t)] + \mathscr{E}_0(t) \tag{3.8}$$

where

$$\mathscr{E}_0(t) = F(t) + 4\pi e\, N_0\, [X_-'(t) - X_+'(t)] \tag{3.9}$$

can only represent the contribution of an externally applied electric field. The decomposition (3.7) implies that

$$\mathbf{V}_\pm = \mathbf{V}_\pm^1(\mathbf{t}) + \mathbf{V}_\pm^2(X_\pm^0, t) \tag{3.10}$$

so that the equation (3.2) splits into the two systems:

$$\frac{d\mathbf{V}_\pm^2}{dt} = \mp\, \omega_\pm^2 [X_+^2 - X_-^2]\, \mathbf{i} \pm \frac{e}{mc} [\mathbf{V}_\pm^2 \times \mathbf{H}] \tag{3.11}$$

$$\frac{d\mathbf{V}_\pm^1}{dt} = \pm \frac{e}{mc} [\mathbf{V}_\pm^1 \times \mathbf{H}] \pm \frac{e}{m} \mathscr{E}_0(t) \tag{3.12}$$

While these equations can be discussed completely the algebra is tedious so that we will restrict attention in the rest of this paragraph to the special case where $(\mathbf{H} \cdot \mathbf{i}) = 0$. With this assumption (3.11) with (3.8) leads to the coupled set

$$\frac{d^3 X_\pm^2}{dt^3} \pm \omega_\pm^2 \frac{dX_\pm^2}{dt} \mp \omega_\pm^2 \frac{dX_\pm^2}{dt} + \Omega_\pm^2 \frac{dX_\pm^2}{dt} = 0 \tag{3.13}$$

where as usual the plasma frequency is given by

$$\omega_\pm^2 = \frac{4\pi e^2 N_0}{m_\pm c} \tag{3.14}$$

and the gyration frequency is given by

$$\Omega_\pm^2 = \frac{e^2}{m_\pm^2 c} (H_y^2 + H_z^2) = \frac{e^2}{m_\pm^2 c} \,|\, \mathbf{H} \,|^2 \tag{3.15}$$

The uncoupled six order equation is, with

$$\gamma_\pm^2 = \omega_\pm^2 + \Omega_\pm^2$$

$$D^2[D^4 + (\gamma_+^2 + \gamma_-^2)\, D^2 + (\gamma_+^2\gamma_-^2 - \omega_+^2\omega_-^2)]\, X_\pm^2 = 0 \qquad (3.16)$$

and this implies that the frequencies of oscillation λ are given by

$$\lambda = 0, 0,$$

and

$$\lambda = \pm\, i\alpha_\pm;\ 2\alpha_\pm^2 = -\,(\gamma_+^2 + \gamma_-^2) \pm \{(\gamma_+^2 + \gamma_-^2)^2 - 4(\gamma_+^2\gamma_-^2 - \omega_+^2 + \omega^2)\}^{1/2}$$

Since the discriminate in this last equation can be written as

$$(\gamma_+^2 + \gamma_-^2)^2 - 4(\gamma_+^2\gamma_-^2 - \omega_+^2\omega_-^2) = (\gamma_+^2 - \gamma_-^2)^2 + 4\omega_+^2\omega_-^2 > 0$$

and since

$$\gamma_+^2\gamma_-^2 - \omega_+^2\omega_-^2 = \omega_+^2\Omega_-^2 + \omega_-^2\Omega_+^2 > 0$$

it is clear that

$$|\,\gamma_+^2 + \gamma_-^2\,| > \{(\gamma_+^2 - \gamma_-^2)^2 + 4\omega_+^2\omega_-^2\}^{1/2}$$

so that the nonzero λ are all purely imaginary, which justifies setting

$$\lambda = \pm\, i\alpha_\pm.$$

The general solution is therefore of the form

$$X_\pm^2 = A_\pm + B_\pm t + C_\pm \cos \alpha_+ t + D_\pm \sin \alpha_+ t + E_+ \sin \alpha_- t + F_\pm \cos \alpha_- t. \qquad (3.17)$$

(There are of course only six independent constants involved instead of the indicated twelve but since we will not be discussing any initial value problems explicitly we will not trouble to work out their relations.)

A similar discussion for (3.12) leads to a solution of the form

$$X_\pm^1 = G_\pm + H_\pm \cos \Omega_\pm t + I_\pm \sin \Omega_\pm t + P(t) \qquad (3.18)$$

where $P(t)$ represents a particular integral corresponding to the input

$$\frac{e}{m}\,\mathscr{E}_0(t).$$

If the external magnetic field is set equal to zero, the dispersion relation for the system (3.11) reduces to

$$\lambda^4[\lambda^2 + (\omega_+^2 + \omega_-^2)] = 0$$

so that the change in the plasma frequency caused by the finite mass of the ions is given by

$$\alpha = \omega_- \left(1 + \frac{\omega_+^2}{\omega_-^2}\right)^{1/2} \cong \omega_- \left[1 + \frac{1}{2}\frac{\omega_+^2}{\omega_-^2} - \frac{1}{8}\left(\frac{\omega_+^2}{\omega_-^2}\right)^2 + \cdots\right]$$

This result agrees with that of Wilhelmson [3] to the second term but unlike his situation it is valid without the traveling wave hypothesis. Because of the nature of his approximations we have not been able to compare the higher terms but the above simple derivation indicates the correct closed form for the plasma frequency shift.

IV. The Traveling Wave Solution

If one now restricts attention to the situation in which all independent variables occur only in the combination $X' = X - V_0 t$ where V_0 represents a constant streaming velocity, the Galilean invariant equations of continuity

$$-V_0 \frac{dN_\pm}{dx'} + U_\pm \frac{dN_\pm}{dx'} + N_\pm \frac{dU_\pm}{dt} = 0 \tag{4.1}$$

can be integrated at once to yield

$$N_\pm = N_0[U_\pm^0 - V_0]/[U_\pm - V_0]. \tag{4.2}$$

Moreover, although the last term in (3.1) is not Galilean invariant it is known that the equation (3.1) is Galilean invariant to order V_0/c, where c is the velocity of light. Thus, if the solution found in Section III is written in the form

$$X = X_\pm^0 + K(X_\pm) \cos [\alpha_+ t + \phi(X_\pm^0)] + L(X_\pm^0) \cos [\alpha_- t + \psi(X_\pm^0)] + Q(t) \tag{4.3}$$

an alternate expression for the density is correctly given by

$$N_\pm = N_0 \frac{\partial X_\pm^0}{\partial X} = \frac{N_0}{[\partial X/\partial X_\pm^0]} = N_0\{1 + K'(X_\pm^0 \rho) \cos (\alpha_+ t + \phi)$$

$$+ L'(X_\pm^0) \cos (\alpha_- t + \psi) - K\phi' \sin (\alpha_+ t + \phi) - L\psi' \sin (\alpha_- t + \phi)\}^{-1}.$$

In order for a traveling wave to exist, it is necessary that these two expressions for the densities should agree. As first deduced by Kalman [12] in his simpler case this requires that

$$Q' = U_{\pm}^{0}; \qquad Q = U_{\pm}^{0} t$$

$$K' = 0; \qquad K = \text{const.}$$

$$L' = 0; \qquad L = \text{const.}$$

$$\varphi' = \frac{\alpha_+}{U_0 - V_0}; \qquad \varphi = \frac{\alpha_+ X_0}{U_0 - V_0} + \varphi_0$$

$$\psi' = \frac{\alpha_-}{U_0 - V_0}; \qquad \psi = \frac{\alpha_- X_0}{U_0 - V_0} + \psi_0.$$

If one now sets

$$k_{\pm} = \frac{\alpha_{\pm} X_0}{V_0 - U_0}, \qquad \varphi_0 = \psi_0 = 0$$

the general solution (4.5) takes the form

$$\begin{aligned} X_{\pm}' &= X_{\pm}^{0} + U_{\pm}^{0} t + K \cos(\alpha_+ t - k_+ X_0) + L \cos(\alpha_- t - k_- X_0) \\ &= X_{\pm}^{0} + U_{\pm}^{0} t + K \cos k_+ [V_0 t - X_0 - U_0 t] + L \cos k_- (V_0 t - X_0 - U_0 t). \end{aligned}$$

This clearly is of the form

$$X_{\pm}' = Y_{\pm}^{0} + J(Y_{\pm}^{0})$$

and thus suitable for the application of Lagrange's theorem [22] if one sets

$$X_{\pm}' = X_{\pm} - V_0 t \qquad \text{and} \qquad Y_{\pm}^{0} = - \; V_0 t + X_{\pm}^{0} + U_{\pm}^{0} t.$$

From this it follows that

$$Y_{\pm}^{0} = X_{\pm} - V_0 t - \sum_{1}^{\infty} \frac{(-1)^n}{N!} \frac{\partial^{n-1}}{\partial X^{n-1}} \{J[(X_{\pm} - V_0 t)]\}^n$$

so that a traveling wave does exist under these circumstances, with a density variation given by

$$N_{\pm} = N_0 \Big[\sum_{0}^{\infty} \frac{(-1)^n}{N!} \frac{\partial^n}{\partial X^n} [J(X_{\pm} - V_0 t)]^n.$$

While we have not been able to obtain a nonparametric representation for the traveling wave solution for the finite ion case, the above argument implies that if the mass of the ions is assumed infinite, then the general solution will be of the form

$$X = X_0 + U_0 t + A \cos(\gamma_- t - k_0 X_0 + \phi_0); \qquad k_0 = \frac{X_0}{V_0 - U_0}$$

where the still unknown constants can be evaluated for a given situation by use of appropriate conditions. (Cf. [17] and [12].) This implies that

$$U - U_0 = - A\gamma_- \sin(\gamma_- t - k_0 X_0 + \phi_0)$$

so that

$$\frac{(X - X_0 - U_0 t)}{A^2} + \frac{(U - U_0)^2}{A^2 \gamma_-^2} = 1$$

or that

$$X = X - X_0 t \pm \frac{1}{\gamma_-} \sqrt{A^2\gamma_-^2 - (U - U_0)^2} \tag{4.5}$$

$$\gamma_- t - k_0 X_0 + \phi_0 = \arcsin \frac{U - U_0}{A\gamma_-}. \tag{4.6}$$

Elimination of X_0 from (4.5) and (4.6) yields the nonparametric expression for the traveling wave

$$X - V_0 t = \pm \frac{1}{\gamma_-} \sqrt{A^2\gamma_-^2 - (U - U_0)^2} + \frac{\phi_0}{k_0} - \frac{1}{k_0} \arcsin \frac{U_0 - U}{A\gamma_-} \tag{4.7}$$

so that the only effect of the magnetic field is to replace the plasma frequency ω by $(\omega^2 + \Omega^2)^{1/2}$. The expression (4.7) does not agree with that found by Vedenov [4] for any choice of the constants (A) and (ϕ_0) nor is it directly obtainable by substituting $X' = X - V_0 t$ into the Eulerian form of the force equation (3.1) if the magnetic field is assumed nonzero. We therefore conclude that the extra phase factor contained in the solution of Vedenov is in error and due to incorrect treatment of the Galilean transformation involved in the standing wave solution. The physical significance of the traveling wave solution has already been amply discussed by Kalman [12] and as his discussion shows the failure of this type of disturbance to involve other than the initially disturbed particles makes it of doubtful use for most problems.[1]

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[1] When the ions are assumed fixed and the magnetic field assumed zero, E. A. Jackson has established the stability of the nonlinear traveling wave for spatially bounded perturbations. Cf. "Stability of non-linear traveling waves in a cold plasma," Matt-53, Project Matterhorn, Princeton Univ., Oct. 1960.

V. Nonlinear Unsteady Oscillations in Orthogonal Coordinates

Provided that the restriction of one independent spatial variable is retained, most of the discussion of Section III can be extended to the case of a general orthogonal coordinate system. In particular for this situation one obtains a system of ordinary differential equations which is completely equivalent to the original problem although in general it is not integrable in closed form.

Any of the methods so far discussed can be used here and in particular the method of Section III will generalize for both finite and infinite ion masses provided only that one uses the correct generalization of the equation of continuity in Lagrangian form. While we will not explicitly carry out this procedure, since, for simplicity, we will assume the ions to be of infinite mass and use our first method, we note that the correct form can be found in Truesdell [23]; namely, if the coordinate system has an arclength given by

$$dS^2 = h_\alpha^2\, dX^\alpha\, dX^\alpha; \qquad \sqrt{g} = \pi_{\alpha=1}^{3} h_\alpha$$

and the Lagrangian coordinates have as their metric

$$(dS_{\pm}^{0})^2 = h_{\pm\alpha}^{0^2}\, dX_{\pm 0}^{\alpha}\, dX_{\pm 0}^{\alpha}; \qquad \sqrt{g_{\pm}^{0}} = \pi_{\alpha=1}^{3} h_{\pm\alpha}^{0}$$

the equations of continuity are the following:

$$N_{\pm} \frac{\sqrt{g}}{\sqrt{g_{\pm}^{0}}} \det \left| \frac{\partial X^\alpha}{\partial X_{\pm 0}^{\beta}} \right| = N_0.$$

Instead we shall use the equation of continuity for the electrons in contravariant form

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial X'} (\sqrt{g}\, V^1 N) + \frac{\partial N}{\partial t} = 0 \tag{5.1}$$

and the equation of Poisson in contravariant form:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial X'} (E^1 \sqrt{g}) = -\, 4\pi e (N - N_0). \tag{5.2}$$

Solving this for N_0, inserting into (5.1), and integrating, one finds, as in Section I, that

$$\frac{\partial}{\partial t} (\sqrt{g}\, E^1) + V^1 \frac{\partial}{\partial X'} (\sqrt{g}\, E^1) - 4\pi e N_0 \sqrt{g}\, V^1) = F'(t) \tag{5.3}$$

where $F'(t)$ is arbitrary.

The contravariant form of the equations of motion are in general

$$\frac{\partial V^i}{\partial t} + V^k \left\{ \frac{\partial V^j}{\partial X^k} + \Gamma^j_{\alpha k} V^\alpha \right\} = \frac{-e}{m} \left\{ E^i + \epsilon^{ijk} \frac{h_j^2 V^j H_k}{c\sqrt{g}} \right\} . \quad (5.4)$$

For a constant magnetic field and independent variables X^1 and t, these reduce to the following:

$$\frac{\partial V^1}{\partial t} + V^1 \left\{ \frac{\partial V^1}{\partial X^1} + \frac{1}{h_1} \left[\frac{\partial h_1}{\partial X^1} V^1 + \frac{\partial h_1}{\partial X^2} V^2 + \frac{\partial h_3}{\partial X^3} V^3 \right] \right\}$$

$$+ V^2 \left\{ \frac{1}{h_1} \frac{\partial h_1}{\partial X^2} V^1 - \frac{h_2}{h_1^2} \frac{\partial h_2}{\partial X^1} V^2 \right\}$$

$$+ V^3 \left\{ \frac{1}{h_1} \frac{\partial h_1}{\partial X^3} V^1 - \frac{h_3}{h_1^2} \frac{\partial h_3}{\partial X^1} V^3 \right\} = \frac{-e}{m} \left[E^1 + \frac{\epsilon^{\alpha 1 \beta} h_\alpha^2 V^\alpha H_\beta}{c} \right] \quad (5.5)$$

$$\frac{\partial V^2}{\partial t} + V^1 \left\{ \frac{\partial V^2}{\partial X^1} + \frac{1}{h_2} \frac{\partial h_2}{\partial X'} V^2 - \frac{h_1}{h_2^2} \frac{\partial h_1}{\partial X^2} V^1 \right\}$$

$$+ \frac{V^2}{h^2} \left\{ \frac{\partial h_2}{\partial X^2} V^1 + \frac{\partial h_2}{\partial X_1^2} V^2 + \frac{\partial h_3}{\partial X^3} V^3 \right\}$$

$$+ V^3 \left\{ \frac{1}{h_2} \frac{\partial h_2}{\partial X^3} V^2 - \frac{h_3}{h_2^2} \frac{\partial h_3}{\partial X^2} V^3 \right\} = \frac{-e}{mc} [\epsilon^{\alpha 2 \beta} h_\alpha^2 V^\alpha H_\rho] \quad (5.6)$$

$$\frac{\partial V^3}{\partial t} + V^1 \left\{ \frac{\partial V^3}{\partial X^1} + \frac{1}{h_3} \frac{\partial h_3}{\partial X^1} V^3 - \frac{h_1}{h_3^2} \frac{\partial h_1}{\partial X^3} V^1 \right\}$$

$$+ V^2 \left\{ \frac{1}{h_3} \frac{\partial h_3}{\partial X^2} V^3 - \frac{h_2}{h_3^2} \frac{\partial h_2}{\partial X^3} V^2 \right\}$$

$$+ \frac{V^3}{h_3} \left\{ \frac{\partial h_3}{\partial X^1} V^1 + \frac{\partial h_3}{\partial X^2} V^2 + \frac{\partial h_3}{\partial X^3} V^3 \right\} = \frac{-e}{mc} [\epsilon^{\alpha 3 \beta} h_\alpha^2 V^\alpha H_\beta]. \quad (5.7)$$

Now the equations (5.3), (5.5), (5.6), and (5.7) constitute a system of first order partial differential equations with equal principal parts involving the single characteristic

$$\frac{dX^1}{dt} = V^1 \quad (5.8)$$

since the first two terms of them are of the form:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V^1 \frac{\partial}{\partial X^1} \tag{5.9}$$

The resulting system cannot be integrated in closed form in general even in Lagrangian coordinates. In order to show that this formulation includes the special cases treated by Dawson [5], we specialize first to cylindrical coordinates with $H_{3'} = H_z = H$, $V^3 = 0$ and

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1, \quad X' = r, \quad X^2 = \theta, \quad X^3 = z$$

thereby obtaining

$$\frac{d^2 r}{dt^2} = r\left(\frac{d\theta}{dt}\right)^2 - \omega^2 \frac{(r^2 - r_0^2)}{\partial r} - r\frac{d\theta}{dt}\frac{He}{mc}$$

$$\frac{d^2\theta}{dt^2} = \frac{-2}{r}\frac{dr}{dt}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{He}{mc},$$

since Poisson's equation yields

$$E' = 4\pi e N_0 \frac{(r^2 - r_0^2)}{2r}.$$

For spherical coordinates with the same assumption on H and

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta, \quad X' = r, \quad X^2 = \theta, \quad X^3 = \phi.$$

the above reduces to

$$\frac{d^2 r}{dt^2} = r\left(\frac{d\theta}{dt}\right)^2 - \omega^2\left(\frac{r^3 - r_0^3}{3r^2}\right) - r\frac{dr}{d\theta}\frac{He}{mc}$$

$$\frac{d^2\theta}{dt^2} = -2r\frac{dr}{dt}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{He}{mc},$$

since Poisson's equation yields

$$E^1 = 4\pi e N_0 \frac{(r^3 - r_0^3)}{3r^2}$$

For simplicity we have taken the external electric field to be zero throughout. If one now sets the magnetic field to zero, assumes only one velocity component, and sets $r = r_0 + R$, the above sets reduce to the equations for an anharmonic oscillator discussed by Dawson [5] for the cylindrical and spherical cases respectively.

VI. The Three-Dimensional Cold Plasma

Dawson [5] proposed an iteration method for investigating the behavior of cold plasma in three-dimensional space and applied it to calculate second and third order effects for a simple case. His method led to the appearance of secular terms in the third approximation which he correctly said could be interpreted as changes in the frequency, amplitude, and phase of the oscillation but he made no attempt to derive such changes explicitly.

The method to be developed here is not only much easier to derive in view of our previous discussion but it results in equations suitable for treatment by the now highly developed method of Cesari *et al.* [6]. The convergence of this method, unlike the method of Krylov and Bogolyubov [6] used by Jackson [1] to show that no frequency shift occurred in the one-dimensional case, can be rigorously established in many cases, and in addition it was specifically designed to apply to systems of higher order than the second where the Krylov and Bogolyubov method does not permit enough degrees of freedom.

After the discussion of the zero-order system with a constant external magnetic field, the Cesari, Hale, and Gambill method will be applied to the same situation treated by Dawson in order to obtain an expression for the frequency shift in the third approximation.

The basic equation may be written as follows: (Note that the effect to first order of collisional damping could be included as in Gold [15] by adding a term $\nu\,\mathbf{V}$ to (6.1) where ν is the collision frequency.)

$$\frac{d\mathbf{V}}{dt} = \frac{-e}{m}\left[\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{H}\right] \tag{6.1}$$

$$N_t + \nabla \cdot [N\mathbf{V}] = 0 \tag{6.2}$$

$$\nabla \times \mathbf{H} = 0 \tag{6.3}$$

$$\nabla \times \mathbf{E} = 0 \tag{6.4}$$

$$\nabla \cdot \mathbf{E} = 4\pi e(N_0 - N) \tag{6.5}$$

$$\mathbf{E}_t + 4\pi e(N_0\mathbf{V}_0 - N\mathbf{V}) = 0. \tag{6.6}$$

As is well known (cf. the discussion of Section II), the equation of continuity can be derived from Maxwell's equation so that it will not be used explicitly in the sequel nor will (6.4) since it merely implies that the total electric field is governed by Poisson's equation. As in Section II, multiply (6.5) by $\mathbf{V}$ and add to (6.6) in order to obtain

$$\mathbf{E}_t + \mathbf{V}(\nabla \cdot \mathbf{E}) = 4\pi e N_0(\mathbf{V} - \mathbf{V}_0). \tag{6.7}$$

This may be written as

$$\frac{d\mathbf{E}}{dt} = \mathbf{E}_t + (\mathbf{V} \cdot \nabla)\,\mathbf{E} = (\mathbf{V} \cdot \nabla)\,\mathbf{E} - \mathbf{V}(\nabla \cdot \mathbf{E}) + 4\pi e N_0(\mathbf{V} - \mathbf{V}_0) \qquad (6.8)$$

where we have introduced the characteristics

$$\frac{d\mathbf{X}}{dt} = \mathbf{V}. \qquad (6.9)$$

In the first order approximation, the system (6.1) and (6.8) is linear and is given by:

$$\frac{d\mathbf{V}_0}{dt} = \frac{-e}{m}\left[\mathbf{E}_0 + \frac{\mathbf{V}_0}{c} \times \mathbf{H}\right] \qquad (6.10)$$

$$\frac{d\mathbf{E}_0}{dt} = 4\pi e N_0 \mathbf{V}_0. \qquad (6.11)$$

This system is readily solved and after some tedious algebra it leads to the following vector differential equations ($D = d/dt$):

$$(D^2 + \omega_0^2)\,[D^4 + (2\omega_0^2 + \Omega^2)\,D^2 + \omega^4]\,\mathbf{V}_0 = 0$$

$$\omega_0^2 = \frac{4\pi e N_0}{m}, \qquad \Omega^2 = \frac{e^2}{m^2 c}\,|\,H\,|^2. \qquad (6.12)$$

This implies that the possible frequencies of oscillation are $\pm\,\omega_0$ and $\pm\,\lambda_\pm$ where

$$2\lambda_\pm^2 = (2\omega_0^2 + \Omega^2) \pm \sqrt{4\omega_0^2\Omega^2 + \Omega^4}. \qquad (6.13)$$

The general solution for the displacement is of the form

$$\mathbf{X} = \mathbf{X}_0 + \mathbf{A}(\mathbf{X}_0)\cos\,[\omega_0 t + \phi(\mathbf{X}_0)]$$
$$+ \mathbf{B}(x_0)\cos\,[\lambda_+ t + \psi(\mathbf{x}_0)] + \mathbf{C}(x_0)\cos\,[\lambda_- t + \psi(\mathbf{x}_0)] \qquad (6.14)$$

where, it must be remembered there are only six independent functions in the set $\mathbf{A}(\mathbf{X}_0)$, $\phi(\mathbf{X}_0)$ etc.

While the Cesari-Hale-Gambill method could be applied to the case of an external magnetic field, we will for simplicity limit considerations to the case where $\mathbf{H} = 0$. When this is true, $\mathbf{E}$ is readily eliminated from (6.1) and (6.8) and $\mathbf{V}$ satisfies the following equation in which $\theta = \omega_0 t$ and $\epsilon = 1/\omega_0$:

$$\frac{d^2\mathbf{V}}{d\theta^2} + \mathbf{V} = \epsilon\left[(\mathbf{V} \cdot \nabla)\,\frac{d\mathbf{V}}{d\theta} - \mathbf{V}\left(\nabla \cdot \frac{d\mathbf{V}}{d\theta}\right)\right]. \qquad (6.15)$$

In this equation the terms involving the operator "∇" must be viewed as computed with respect to the Eulerian variables and then expressed in terms of the Lagrangian variables. That is, any perturbation process will introduce a new metric at each successive stage via an arc length

$$dS^2 = \sum dX^\alpha \, dX^\alpha = \left(\sum \frac{\partial X^\alpha}{\partial X_0^\beta} \, dX_0^\beta\right) \sum \left(\frac{\partial X^\alpha}{\partial X_0^\beta} \, dX_0^\beta\right) = \sum g_{\beta\gamma} \, dX_0^\beta \, dX_0^\gamma$$

in terms of which the necessary tensor quantities can be expressed by the usual formulas for general coordinate systems. Since at the nth stage the vector $\mathbf{X}$ will in principle be found in the form $\mathbf{X} = \mathbf{f}(\mathbf{X}_0, t, \epsilon)$ the above transition between Eulerian and Lagrangian coordinates will involve terms in ϵ in the denominator of the terms involving the operator "∇." When expansion techniques are employed they result in terms of higher order than the approximation in which they appear and must therefore be ignored. This point will be illustrated specifically for the example treated by Dawson.

In order to explain the Cesari-Hale-Gambill method briefly, let the components of $\mathbf{V}$ be deonted by V_k, $k = 1, 2, 3$ and set

$$2iV_j = y_{2j-1} + y_{2j} \qquad j = 1, 2, 3 \tag{6.16}$$

$$2\frac{dV_j}{d\theta} = y_{2j-1} - y_{2j} \tag{6.17}$$

$$f_j = f_j\left[\frac{y_1 + y_2}{2i}, \frac{y_1 - y_2}{2}, \cdots\right] = \left[(\mathbf{V} \cdot \nabla)\frac{d\mathbf{V}}{d\theta} - \frac{d\mathbf{V}}{d\theta} \nabla \cdot \mathbf{V}\right]. \tag{6.18}$$

With these substitution, (6.15) takes the form

$$\frac{dy_{2j-1}}{d\theta} = iy_{2j-1} + \epsilon f_j \tag{6.19}$$

$$\frac{dy_{2j}}{d\theta} = -iy_{2j} - \epsilon f_j \qquad j = 1, 2, 3. \tag{6.20}$$

The zeroth order approximation is assumed to be

$$x_{0,2j-1} = a_j e^{i\tau_j\theta} \tag{6.23}$$

$$x_{0,2j} = -\bar{x}_{0,2j-1} = -\bar{a}_j e^{-i\tau_j\theta} \tag{6.24}$$

where the bar denote complex conjugation and each a_j is a nonzero complex constant, which at this point may also be thought of as undetermined.

To obtain the higher order terms, one sets,

$$\begin{aligned} y_{2j-1} &= x_{0,2j-1}(t) + \epsilon x_{1,2j-1}(t) + \epsilon^2 x_{2,2j-1}(t) + \cdots \\ y_{2j} &= x_{0,2j}(t) + \epsilon x_{1,2j}(t) + \epsilon^2 x_{2,2j}(t) + \cdots \end{aligned} \tag{6.25}$$

and lets s_{mj} denote the coefficient of ϵ^{m-1} in the expansion of f_j. The mean values are next introduced by the definition

$$a_j S_{mj} = \frac{1}{2\pi} \int_0^{2\pi} s_{mj} d\theta \tag{6.26}$$

and the successive approximation defined by

$$\begin{aligned} x_{0,2j-1} &= a_j e^{i\tau_j\theta} \\ \frac{dx_{1,2j-1}}{d\theta} &= i\tau_j x_{1,2j-1} + s_{1j} - S_{1j} x_{0j2j-1} \\ \frac{dx_{2,2j-1}}{d\theta} &= i\tau_j x_{2,2j-1} + (s_{2j} - S_{2j} x_{0,2j-1}) - S_{1j} x_{1,2j-1} \end{aligned} \tag{6.27}$$

and

$$x_{k,2j} = -\bar{x}_{k,2j-1}. \tag{6.28}$$

If one performs the obvious operations on (6.27), for example, one obtains:

$$\frac{d}{d\theta}[x_{0,2j-1} + \epsilon y_{1,2j-1} + \cdots] = i\tau_j[x_{0,2j-1} + \epsilon x_{1,2j-1} + \cdots]$$

$$+ \epsilon[s_{1j} + \epsilon s_{2j} + \epsilon^2 s_{3j} + \cdots] - \epsilon[S_{1j}x_{0,2j-1} + \epsilon(S_{2j}x_{0,2j-1} + S_{1j}x_{1,2j-1}) + \cdots] + \cdots. \tag{6.29}$$

or

$$\begin{aligned} \frac{dy_{2j-1}}{d\theta} &= i\tau_j y_{2j-1} - \epsilon[S_{1j}x_{02j-1} + \cdots] + \epsilon f_j \\ &= i\tau_j y_{2j-1} - \epsilon F^{y}_{j2j-1} + \epsilon f_j \end{aligned} \tag{6.30}$$

where

$$F_j = S_{1j} + \epsilon S_{2j} + \epsilon^2 S_{2j} + \cdots.$$

Now the (auxiliary) system (6.30) will clearly reduce to (6.19) if and only if the (determining) equations

$$i\tau_j - \epsilon F_j = i \tag{6.31}$$

have a solution. Equation (6.20) is treated in a similar fashion.

Specializing to $j = 1, 2$ and writing Dawson's initial linearized solutions as

$$X = X_0 + \epsilon A \sin K_1 X_0 \sin(\omega_0 t + \alpha) \tag{6.32}$$

$$Y = Y_0 + \epsilon B \sin K_2 Y_0 \sin(\omega_0 t + \alpha). \tag{6.33}$$

The corresponding initial values of the velocity components are

$$U_1^0 = \epsilon A \omega_0 \sin K_1 X_0 \cos \alpha \tag{6.34}$$

$$V_1^0 = \epsilon B \omega_0 \sin K_2 Y_0 \cos \beta. \tag{6.35}$$

It is a simple matter to verify that the following choices are consistent with these, Eq. (6.16), (6.17) and (6.23), (6.24),

$$\begin{aligned} \epsilon &= \frac{1}{\omega_0} \\ a_1 &= A \sin K_1 X_0 e^{i(\alpha+\pi/2)} \\ a_2 &= B \sin K_2 Y_0 e^{i(\beta+\pi/2)}. \end{aligned} \tag{6.36}$$

Next note that, for example,

$$\frac{\partial}{\partial x} a_1 = \frac{\partial}{\partial x_0} a_1 \frac{\partial x_0}{\partial x} = \frac{\partial a_1/\partial x_0}{\partial x/\partial x_0} = \frac{\partial a_1/\partial x_0}{1 + \epsilon(\partial G/\partial x_0)} \cong \frac{\partial a_1}{\partial x_0}, \text{ etc.}$$

since at any stage the solution for x, y will be of the form:

$$X = X_0 + \epsilon G(X_0, Y_0, t, \epsilon)$$

$$Y = Y_0 + \epsilon H(X_0, Y_0, t, \epsilon)$$

Since the nonlinear behavior of the above system will be close to that of the linear approximation, $\tau_1 = \tau_2 = \tau$ is the only rational choice for the auxilary system (6.30). One now finds that the first order means S_{11}, S_{12} are zero so that there is no shift in the plasma frequency to the order ϵ and that there are no amplitude restrictions to this order of approximation. After a straightforward but tedious computation, one finds that the determining equations (cf. Eq. (6.31)) take the form

$$\begin{aligned} \tau &= 1 + \frac{\epsilon^2}{12\tau} K_2^2 B^2 \cos(2 K_2 Y_0) \\ &= 1 + \frac{\epsilon^2}{12\tau} K_2^2 A^2 \cos(2 K_2 X_0). \end{aligned}$$

In reality these expressions are the imaginary parts of the determining equations and the situation here is special in that the determining equations are purely imaginary instead of complex. Thus, instead of having four equations which must be compatible for a periodic solution to exist for certain specific amplitudes (cf. the equation of Van der Pol), a family of periodic solutions is possible provided only that the amplitudes satisfy the relation

$$K_2^2B^2 \cos (2\, K_2 Y_0) = K_1^2 A^2 \cos (2\, K_1 X_0)$$

in this approximation. It should be noted that this result is independent of the initial phase angles α, β. The shift in the plasma frequency to this order is then given by

$$\tau = 1 + \frac{\epsilon^2}{12} K_1^2 A^2 \cos (2\, K_1 X_0).$$

This is similar to the result found by Sturrock [13] who showed by another means that steady free oscillations at the plasma frequency will not persist for a cold plasma if it is multidimensional. This of course implies that the electron velocity will eventually become multivalued.

The second order terms in the velocity components in this approximation take the relatively simple form

$$U_2 = - ABK_2 \sin (K_1 X_0) \cos (K_2 Y_0) \{\tfrac{1}{6} \cos [2\omega_0 t + \alpha + \beta + O(\epsilon^2)] + \tfrac{1}{2} \sin [\alpha - \beta + 0(\epsilon^2)]\}.$$

$$V_2 = - ABK_1 \cos (K_1 X_0) \sin (K_2 Y_0) \{\tfrac{1}{6} \cos [2\omega_0 t + \alpha + \beta + O(\epsilon^2)] + \tfrac{1}{2} \sin (\alpha - \beta + O(\epsilon^2)]\}.$$

The corresponding third order terms are quite involved and will not be given explicitly.

A similar analysis applied to the three dimensional situation (6.15) would also show that the plasma frequency is inherently stable to order $\epsilon^2 = 1/\omega_0^2$.

The stability of the family of periodic solutions determined above has not been established and much theoretical work still remains to be done before the Ceasari method used here can be mathematically justified. It is a convergent and rigorous process for ordinary differential systems under many conditions but the corresponding theory for partial differential equations is still in a primitive state.

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REFERENCES

1. JACKSON, E. A. Nonlinear oscillations of a cold plasma. *Phys. Fluids* **3**, 831 (1960).
2. COURANT-HILBERT. "Methoden Mathematischen Physik. Vol. II, p. 117. Springer, Berlin, 1937.
3. WILHELMSON, H. Stationary non-linear plasma oscillations. *Phys. Fluids* **4**, 335 (1961).
4. VEDENOV, A. A. Concerning some solutions of the equations of a hydrodynamic plasma. *Soviet Phys. JETP* **6**, 1165 (1958).
5. DAWSON, J. M. Nonlinear oscillations of a cold plasma. *Phys. Rev.* **113**, 383 (1959).
6. CESARI, L. Asymptotic behavior and stability problems in ordinary differential equations. *Ergeb. Math.* **16** (1954). Springer, Berlin.
7. AKHIEZER, L., AND LUBARSKII, G Ya. On the nonlinear theory of electron plasma oscillations. *Doklady Akad. Nauk SSSR* **80**, 193 (1951).
8. SMERD, S. F. Non-linear plasma oscillations and bursts of solar radio emission. *Nature* **175**, 297 (1955).
9. DAWSON, J. M. Breaking of finite amplitude waves. *Proc. Conf. Plasma Oscillations*, June 1959, Indianapolis, Indiana.
10. SEN, H. K. Nonlinear theory of a space charge wave in moving interacting electron beams with application to solar radio noise. *Phys. Rev.* **97**, 849 (1955).
11. GOLD, L. Current voltage behavior in a plasma. *J. Electronics and Control* **5**, 432 (1958).
12. KALMAN, G. Non-linear oscillations and non-stationary flow in a zero temperature plasma. *Ann. Phys. (NY)* **10**, 29 (1960).
13. STURROCK, P. A. Non-linear effects in electron plasmas. *Proc. Roy. Soc.* **A 242**, 277 (1957).
14. GOLD, L. Non-linear phenomenological theory of plasma oscillations. *J. Electronics and Control* **4**, 219 (1958).
15. GOLD, L. Oscillations in a plasma with oriented (dc) magnetic field. *J. Electronics and Control* **5**, 409 (1958).
16. AMER, S. Non-linear plasma oscillations. *J. Electronics and Control* **5**, 105 (1958).
17. KONYUKOV, M. V. Non-linear Lagmuir oscillations in a plasma. *Soviet Phys. JETP* **10**, 570 (1960).
18. BERNSTEIN, I. B., GREENE, J. M., AND KRUSKAL, M. D. Exact non-linear plasma oscillations. *Phys. Rev.* **103**, 546 (1957).
19. NEKRASOV, F. M. On the non-linear theory of stationary processes in a electron plasma. *Soviet Phys. JETP* **11**, 170 (1960).
20. AKHIEZER, A. I., AND POLOVIN, R. V. Theory of wave motion of an electron plasma. *Soviet Phys. JETP* **3**, 696 (1956).
21. WYLD, H. W. Non-linear plasma oscillations. Space Tech. Lab. Rept. R-60-0000-GR 303 (1960).
22. WHITTAKER, AND WATSON. "Modern Analysis," p. 132. MacMillan, New York, 1945.
23. TRUESDELL, C. The mechanical foundations of elasticity and fluid dynamics. *J. Rational Mech. and Anal.* **1**, 125 (1952).
24. STURROCK, P. A. Generalizations of the Lagrange expansion with application to physical problems. *J. Math. Phys.* **1**, 405 (1960).